Symbolic conversion of holonomic functions to hypergeometric type power series

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Abstract

A term a_n is m-fold hypergeometric, for a given positive integer m, if the ratio a_{n+m}/a_n is a rational function over a field $\mathbb{K}$ of characteristic zero. We establish the structure of holonomic recurrence equations, i.e. linear and homogeneous recurrence equations having polynomial coefficients, that have m-fold hypergeometric term solutions over $\mathbb{K}$, for any positive integer m. Consequently, we describe a new algorithm, say **mfoldHyper**, that extends the algorithms by Petkovšek (1992) and van Hoeij (1998) which compute a basis of hypergeometric ($m = 1$) term solutions of holonomic recurrence equations to the more general case of m-fold hypergeometric terms.

Given a Laurent-Puiseux series

$$\sum_{n=n_0}^{\infty} a_n(z-z_0)^{n/k} \ (a_n \in \mathbb{K}, k \in \mathbb{N}, n_0 \in \mathbb{Z}), \tag{1}$$

where k denotes the corresponding Puiseux number, the most descriptive data to obtain (1) is a “closed-form” expression of the n^{th} coefficient (or simply coefficient) a_n. We generalize the concept of hypergeometric type power series introduced by Koepf (1992), by considering linear combinations of Laurent-Puiseux series whose coefficients are m-fold hypergeometric terms. Thanks to **mfoldHyper**, it is possible to have a complete algorithm to decide on the conversion of holonomic functions to hypergeometric type power series provided that the coefficients are m-fold hypergeometric term solutions of the underlying recurrence equation. Indeed, in such a case, it turns out that every linear combination of power series with m-fold hypergeometric term coefficients, for finitely many values of m, is detected.

This paper is accompanied by implementations in the Computer Algebra Systems (CAS) Maxima 5.44.0 and Maple 2021. These can be downloaded at `http://www.mathematik.uni-kassel.de/~bteguia/FPS_webpage/FPS.htm`.

Keywords: m-fold hypergeometric term; holonomic differential equation; holonomic recurrence equation; hypergeometric type power series

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1. Introduction

The applicability of complex analysis is essentially restricted to analytic functions, since they easily allow both differentiation and integration. These functions are represented by power series with positive radius of convergence. Power series are used to represent orthogonal polynomials (see e.g. Koepf and Schmersau (1998)); in combinatorics, generating functions are power series (Stanley (2011)); in dynamical systems, algebraic properties of power series involve most of the constructions (see Lubin (1994)); we can also enumerate commutative algebra and algebraic geometry (Brewer (2014)), (Zariski and Samuel, 1960, Chapter VII). It is therefore important to know the exact general coefficient or formula of a power series. We build an algorithm to find the power series representation of a given holonomic (also called D-finite) function, whenever the closed-form expressions of the series coefficients are hypergeometric terms. Thus, we are not considering complex functions as abstract objects defined in a certain domain and its range, but instead as differentiable objects which satisfy a linear homogeneous differential equation with polynomial coefficients, and that we can manipulate symbolically to find the n^{th} term of its Taylor coefficients. Moreover, by the unique power series characterization, this approach does not only lead to the verification of known identities, but also to the discovery of new ones. Generally, let $\mathbb{K}$ be a field of characteristic zero[1].

Definition 1 (m-fold hypergeometric term)**.** *A term a_n is said to be m-fold hypergeometric, for a positive integer m, if the term ratio $r(n) := a_{n+m}/a_n$ is a rational function over $\mathbb{K}$. When $m = 1$ one talks about a hypergeometric term. When used without specifying the value of m, m-fold hypergeometric term denotes a term with this property, i.e. such an m exists.*

Given a holonomic expression f, our first interest is to describe an algorithm that decides by computation whether the power series of f is a linear combination of Laurent-Puiseux series of the form

$$\sum_{n=n_0}^{\infty} a_n(z-z_0)^{n/k} \; (a_n \in \mathbb{K}, k \in \mathbb{N}, n_0 \in \mathbb{Z}), \tag{2}$$

where a_n is an m-fold hypergeometric term. More often n^{th} coefficient will be substituted by coefficient and n^{th} term by term when there is no ambiguity. By computing power series we mean finding a formula that can be used to generate truncated series of any order.
In 1992, Koepf published an algorithmic approach for computing power series (see Koepf (1992)). The algorithm was implemented in the computer algebra systems (CAS) Maple (Maplesoft (2020)) and Mathematica (Wolfram (2003)). Three types of functions were considered:

t1 The *two-term recurrence relation* type [2] which corresponds to expressions leading to linear recurrence equations of the form

$$Q_n a_{n+m} + P_n a_n = 0, \tag{3}$$

where Q_n, P_n are polynomials in $\mathbb{K}[n]$.

[1]Mostly $\mathbb{K} := \mathbb{Q}(\alpha_1, \ldots, \alpha_N)$ is the field of rational functions in several variables
[2]Originally called hypergeometric type but we avoid this calling since we are redefining this terminology.

t2 The *exp-like* type corresponding to expressions leading to linear recurrence equations with constant coefficients in $\mathbb{K}$.

t3 The *rational* type which corresponds to functions having a derivative which is rational over $\mathbb{K}(z)$.

All gathered in the Maple package `FormalPowerSeries` could already recover many power series formulas. However, using an algorithm that computes m-fold hypergeometric term solutions of so-called holonomic recurrence equations, that is linear homogeneous recurrence equations with polynomial coefficients, we define a general-purpose approach that finds representations for any of these three type of input functions. The type t1 is obviously covered according to the definition of an m-fold hypergeometric term. In this context, the need of our approach is more justified by the input functions of type t2 and t3 for which commonly used techniques complicate the result with coefficients over larger algebraic extension fields (Teguia Tabuguia (2020a)). Let us consider two examples for expansion about $z_0 = 0$[3]. Our Maple implementation is the ModuleApply[4] of our package of name `FPS`. Using the current Maple implementation for *exp-like* type expressions to compute the power series of $f(z) := (\sin(z) + \cos(z))^3$ one gets

```
> convert((sin(z)+cos(z))^3,FormalPowerSeries,method=
  exponential)
```

$$\sum_{k=0}^{\infty} \frac{\left(\left(\frac{3}{4} + \frac{3\,\mathrm{I}}{4}\right)(-\mathrm{I})^k + \left(\frac{3}{4} - \frac{3\,\mathrm{I}}{4}\right)\mathrm{I}^k - \left(\frac{1}{4} + \frac{\mathrm{I}}{4}\right)(3\,\mathrm{I})^k + \left(-\frac{1}{4} + \frac{\mathrm{I}}{4}\right)(-3\,\mathrm{I})^k\right) z^k}{k!}, \tag{4}$$

which is given in a simpler form by our Maple implementation as presented below.

```
> FPS((sin(z)+cos(z))^3,z,n)
```

$$\sum_{n=0}^{\infty}\left(-\frac{(-1)^n\,(9^n-3)\,z^{2n}}{2\,(2n)\,!}\right) + \left(\sum_{n=0}^{\infty}\frac{3\,(-1)^n\,(9^n+1)\,z^{2n+1}}{2\,(2n+1)\,!}\right). \tag{5}$$

This is due to the fact that the needed power series coefficients can be written as a linear combinations of 2-fold hypergeometric term solutions of a holonomic recurrence equation satisfied by the Taylor coefficients of $f(z)$.
Similarly, the partial fraction decomposition of a given rational function cannot always lead to "useful" power series representations. The result obtained by Maple for $1/\big((1-3z^2)\cdot(4-z^3)\big)$ extends to several pages and will not be displayed here. In contrast, using our Maple implementation one gets

```
> FPS(1/(1-3*z^2)/(4-z^3),z,n)
```

$$\left(\sum_{n=0}^{\infty}\frac{108\,3^n z^{2n}}{431}\right) + \left(\sum_{n=0}^{\infty}\frac{9\,3^n z^{2n+1}}{431}\right) + \sum_{n=0}^{\infty}\left(-\frac{4^{-n} z^{3n}}{1724}\right) + \sum_{n=0}^{\infty}\left(-\frac{9\,4^{-n} z^{3n+1}}{431}\right) + \sum_{n=0}^{\infty}\left(-\frac{3\,4^{-n} z^{3n+2}}{1724}\right). \tag{6}$$

For the latter example, the result shows that 2-fold and 3-fold hypergeometric terms were computed prior to the representation.
Algorithms to compute hypergeometric term ($m = 1$) solutions of holonomic recurrence equations were developed by Marko Petkovšek and Mark van Hoeij (see Petkovšek (1992), Van Hoeij

[3]Throughout this paper we mainly give representations about $z_0 = 0$ since the case of arbitrary z_0 deduces easily.
[4]A Maple command to use package names as Maple procedures.

(1999), Cluzeau and van Hoeij (2006)). The complexity of Petkovšek's algorithm is exponential in the degree of the leading and the trailing polynomial coefficients. Mark van Hoeij used a different approach and got a much more efficient algorithm for the same purpose. Indeed, he considered the local behavior of solution terms, which naturally decreases the complexity by reducing the number of candidates since hypergeometric term solutions are built from some factors of the leading and the trailing polynomial coefficients. Van Hoeij implemented his algorithm in Maple as `LREtools[hypergeomsols]`. An equivalent algorithm that is part of our main approach is described in (Teguia Tabuguia (2020b)).
The current Maple `convert/FormalPowerSeries` command implements Koepf's original approach followed by an invocation of van Hoeij's algorithm to solve recurrence equations which are not one of the three type of input functions t1, t2, and t3 listed above. This is the reason why Maple can handle the power series of $\exp(z) + \ln(1 + z)$ without treating $\exp(z)$ and $\ln(1 + z)$ separately. It should be noted that we are focusing on algorithms which do not check or analyze the form of their inputs, because this loses many aspects of a general-purpose approach. A simple example is $\sin(z)^2 + \cos(z)^2$ whose power series expansion is 1, and yet if we treat $\sin(z)^2$ and $\cos(z)^2$ separately, we get a different result (see Teguia Tabuguia and Koepf (2021b)). That being mentioned, one can see the limits of the following Maple results

```
> convert(ln(1+z)+sin(z),FormalPowerSeries,method=
  hypergeometric)
```

$$\sum_{k=0}^{\infty}\left(-\frac{(-1)^{k+1}}{k+1}+\frac{\mathrm{I}\,(-\mathrm{I})^{k+1}}{2\,(k+1)\,!}-\frac{\mathrm{I}\,\mathrm{I}^{k+1}}{2\,(k+1)\,!}\right)z^{k+1}, \tag{7}$$

```
> convert(arctan(z)+1/(1+z),FormalPowerSeries,method=
  hypergeometric)
```

$$1+\left(\sum_{k=0}^{\infty}\left((-1)^{k+1}+\frac{\mathrm{I}\,(-\mathrm{I})^{k+1}}{2\,(k+1)}-\frac{\mathrm{I}\,\mathrm{I}^{k+1}}{2\,(k+1)}\right)z^{k+1}\right), \tag{8}$$

whose desired representations are obtained using our `FPS` as follows.

```
> FPS(ln(1+z)+sin(z),z,n)
```

$$\left(\sum_{n=0}^{\infty}\frac{(-1)^n\,z^{n+1}}{n+1}\right)+\left(\sum_{n=0}^{\infty}\frac{(-1)^n\,z^{2n+1}}{(2n+1)\,!}\right), \tag{9}$$

```
> FPS(arctan(z)+1/(1+z),z,n)
```

$$\left(\sum_{n=0}^{\infty}(-1)^n\,z^n\right)+\left(\sum_{n=0}^{\infty}\frac{(-1)^n\,z^{2n+1}}{2n+1}\right). \tag{10}$$

Indeed, the algorithms by Petkovšek and van Hoeij might only find hypergeometric term solutions in $\overline{\mathbb{Q}}$, which in certain cases can be equivalent to m-fold hypergeometric term solutions in $\mathbb{Q}$ for some $m > 1$. This brings us back to the issue of algebraic extension fields encountered for the *exp-like* and *rational* functions by the methods in (Koepf (1992)).
Note, however, that simplifying power series formulas by avoiding algebraic extension fields is just a consequence of our general-purpose algorithm. With this new approach, we are able to compute several power series that Maple, Mathematica and Maxima cannot find at the time of writing this article. We are in collaboration with Maplesoft and the Maxima developers for the integration of our packages into their systems. It is easy to build examples for which Maple's current `convert/FormalPowerSeries` misses the results.

```
> convert(arcsin(z)+cos(z),FormalPowerSeries)
```

$$\arcsin(z) + \cos(z), \tag{11}$$

```
> convert(exp(z^3)+ln(1+z^2),FormalPowerSeries)
```

$$e^{z^3} + \ln\left(z^2 + 1\right). \tag{12}$$

The above Maple failures rely on the incapacity of van Hoeij's algorithm to detect m-fold ($m > 1$) hypergeometric term solutions of holonomic recurrence equations. Indeed, the Taylor coefficients of $\arcsin(z) + \cos(z)$ satisfies the recurrence equation

$$\begin{aligned} RE1 :=& -n\left(n^3 - 10n^2 + 21n - 22\right)a(n) + (n-4)^2\, a(n-4) \\ &+ (n-2)\left(n^3 - 11n^2 + 39n - 41\right)a(n-2) + 2\,(n+1)\,(n+2)\left(n^2 + 4n - 1\right)a(n+2) \\ &- 2\,(n+1)\,(n+2)\,(n+3)\,(n+4)\,a(n+4) = 0. \end{aligned} \tag{13}$$

Using `LREtools[hypergeomsols]` to solve this recurrence equation yields

```
> LREtools:-hypergeomsols(RE1,a(n),{},output=basis)
```

$$\left[\frac{I^n}{\Gamma(n+1)}, \frac{(-I)^n}{\Gamma(n+1)}\right], \tag{14}$$

which shows that the coefficient of the power series of $\arcsin(z)$ is missed! Similarly, for $\exp(z^3)+\ln(1+z^2)$, `LREtools[hypergeomsols]` misses the power series coefficient of $\exp(z^3)$. We have the recurrence equation (say `RE2`)

$$\begin{aligned} RE2 :=& -9\,(n-9)^2\, a(n-9) - 18\,(n-8)\,(n-7)\,a(n-7) + 3\,(n-15)\,(n-6)^2\, a(n-6) \\ &- 9\,(n-5)\,(n-7)\,a(n-5) + 6\,(n-4)\left(n^2 - 17n + 63\right)a(n-4) \\ &+ 3\,(n-6)\,(n-3)^2\, a(n-3) + 3\,(n-2)\,(n-4)\,(n-9)\,a(n-2) \\ &+ 2\,(n-1)\,(n-4)\,(2n-7)\,a(n-1) + (n-1)\,(n-2)\,(n+1)\,a(n+1) = 0, \end{aligned} \tag{15}$$

for which van Hoeij's implementation gives

```
> LREtools:-hypergeomsols(RE2,a(n),{},output=basis)
```

$$\left[\frac{I^n}{n}, \frac{(-I)^n}{n}\right]. \tag{16}$$

We use the name **mfoldHyper** to denote our algorithm to compute m-fold hypergeometric term solutions of holonomic recurrence equations. It is implemented in our packages under the same name (`mfoldHyper`). Using `mfoldHyper` to solve `RE1` over the rationals yields

```
> FPS:-mfoldHyper(RE1,a(n))
```

$$\left[\left[2, \left\{\frac{(-1)^n}{(2n)\,!}, \frac{n!^2\, 4^n}{n^2\,(2n)\,!}\right\}\right]\right], \tag{17}$$

which are the needed coefficients to compute the power series of $\arcsin(z) + \cos(z)$. Therefore `FPS` finds the representation

```
> FPS(arcsin(z)+cos(z),z,n)
```

$$\left(\sum_{n=0}^{\infty} \frac{(-1)^n\, z^{2n}}{(2n)\,!}\right) + \left(\sum_{n=0}^{\infty} \frac{(2n)\,!\, 4^{-n} z^{2n+1}}{(2n+1)\, n!^2}\right). \tag{18}$$

In the same way, `mfoldHyper` finds the needed coefficients which `FPS` used to determine the power series of $\exp(z^3) + \ln(1 + z^2)$.

```
> FPS:-mfoldHyper(RE2,a(n))
```

$$\left[\left[2, \left\{\frac{(-1)^n}{n}\right\}\right], \left[3, \left\{\frac{1}{n!}\right\}\right]\right]; \tag{19}$$

```
> FPS(exp(z^3)+ln(1+z^2),z,n)
```

$$\left(\sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+2}}{n+1}\right) + \left(\sum_{n=0}^{\infty} \frac{z^{3n}}{n!}\right). \tag{20}$$

The solutions over $\overline{\mathbb{Q}}$ as those computed with van Hoeij's algorithm can be obtained with `mfoldHyper` by adding `complex` as a third argument. However this is only used in `FPS` when no solution is found in $\mathbb{Q}$. Although we will investigate the efficiency for computing holonomic differential equations by the method of ansatz with undetermined coefficients, we do not intend to always use the recurrence equation of least order since **mfoldHyper** is not restricted by the order of the recurrence equation. For more references on dealing with holonomic functions see (Zeilberger (1990), Salvy and Zimmermann (1994), Koepf (1997), Kauers and Paule (2011)).

Definition 2 (Hypergeometric type series and functions (see Teguia Tabuguia and Koepf (2021a))**.** *For an expansion around $z_0 \in \mathbb{K}$, a series $s(z)$ is said to be of hypergeometric type if it can be written as*

$$s(z) := T(z) + \sum_{j=1}^{J} s_j(z), \; s_j = \sum_{n=n_{j,0}}^{\infty} a_{j,n}(z-z_0)^{n/p_j}, \tag{21}$$

where n is the summation variable, $T(z) \in \mathbb{K}[z, 1/z, \ln(z)]$, $n_{j,0} \in \mathbb{Z}$, $J, p_j \in \mathbb{N}$, and $a_{j,n}$ is an m_j-fold hypergeometric term.
A hypergeometric function is a function that can be expanded as a hypergeometric type power series. $T(z)$ is called the Laurent polynomial part of the expansion, and the p_j's are its Puiseux numbers.

We often say that hypergeometric type power series are linear combinations of Laurent-Puiseux series having m-fold hypergeometric term coefficients, although this may ignore $T(z)$ in Definition 2. Nevertheless $T(z)$ must be taken into account while computing linear combinations in order to use appropriate starting points for m-fold hypergeometric terms. Indeed, it happens that evaluations of hypergeometric terms of the series coefficients start at some non-zero integers ($n_{j,0}, j = 1, \ldots, J$ in (21)) determined from the degree of $T(z)$. Several results are missed by Maple 2021 because of this step of calculating the linear combination. E.g. $(z + z^2 + 1)e^z + (z^3 + 3)\ln(1 + z)$ and $1 + z + z^2 + z^3 \cdot \arctan(z)$ (see Teguia Tabuguia (2020a)). Another step to take into account is the computation of Puiseux numbers. This is usually tackled by the so-called Frobenius method in the literature. However, we use a different strategy to make sure that Puiseux numbers appear as in (21) unlike the Frobenius method which forces them to appear additively (z^{n+r} $r \in \mathbb{Q}$) in the power of the indeterminate (see Ryabenko (2000)).
Our results are in line with the work of Anna Ryabenko (see Ryabenko (2002)) which uses an extended version of Petkovšek's algorithm (see Petkovšek and Salvy (1993)) for m-fold hypergeometric terms (called m-hypergeometric sequences there) to find formal hypergeometric type solutions of holonomic differential equations (see also Abramov (2000)). Another independent development from which m-fold hypergeometric terms are considered can be found in (Horn et al. (2012)).

The CAS Maxima was the main system used when our results were being developed. Note that the current Maxima command to find power series representations is `powerseries`. This command is based on a pattern matching instead of (symbolic) algorithmic model, which explains some "bad" results obtained with this command for non-trivial input functions (see Teguia Tabuguia (2020a)). In the following sections of the paper, most computations will be given using Maxima.
The overview of the paper is as follows. In order to make the description of our algorithm for computing power series self-contained, in the next section we define an equivalent approach to the method of ansatz with undetermined coefficients for computing holonomic differential equations to have a higher efficiency. The deduction of recurrence equations is straightforward after this step.
Section 3 is devoted to our most important result, this is the description of **mfoldHyper** which is our algorithm to find a basis of m-fold hypergeometric term solutions of holonomic recurrence equations.
In Section 4 we present the details of our algorithm to compute hypergeometric type power series. We will see in this section how our algorithm handles the Puiseux numbers, the Laurent polynomial part, and the starting points involved in a given hypergeometric type series expansion.

2. Computing holonomic equations

This section is about computing holonomic differential equations (DE) from given holonomic expressions and use them to deduce holonomic recurrence equations (RE) satisfied by their power series coefficients prior to computing m-fold hypergeometric term solutions. One of the most prominent implementations for dealing with holonomic equations is the Maple `gfun` package (see Salvy and Zimmermann (1994)). However, the method used by `gfun` (see approach 3 in Table 1) to compute DEs is different from that of the ansatz with undetermined coefficients as developed in (Koepf (1992)). We present a variant of the latter which finds the same DEs with less computations.

2.1. Idea of the method

Let $\mathbb{K}$ be a field of characteristic zero, and $(A_0, A_1, \ldots, A_{N-1}) \in \mathbb{K}(z)^N$, $N \in \mathbb{N}$ such that a holonomic function f satisfies

$$\mathcal{F}\left(f, f', \ldots, f^{(N-1)}, f^{(N)}\right) = f^{(N)} + A_{N-1} \cdot f^{(N-1)} + \cdots + A_1 \cdot f + A_0 f = 0. \qquad (22)$$

Assume that $(e_1, e_2, \ldots, e_l)$ is a basis of linearly independent functions that spans all the summands appearing in the expansion of the derivatives $f, f', \ldots, f^{(N)}$ over $\mathbb{K}$. Thus each derivative $f^{(j)}$, $j \in \mathbb{N}_{\geqslant 0}$ $\left(f^{(0)} = f\right)$ can be seen as a vector in the linear space $\langle e_1, e_2, \ldots, e_l\rangle$. For example, if we consider $f(z) = \cos(z) + \sin(z)$, it is clear that the DE sought is equivalent to a linear combination of the derivatives of $f(z)$ which equals zero, with coefficients in $\mathbb{K}(z)$, expanded in the basis $(\cos(z), \sin(z))$.
Since

$$\mathcal{F}\left(f, f', \ldots, f^{(N-1)}, f^{(N)}\right) = 0 \iff -f^{(N)} = A_0 \cdot f + A_1 \cdot f' + \cdots + A_{N-1} \cdot f^{(N-1)}, \qquad (23)$$

we can write in matrix representation

$$-f^{(N)} = \left[f, f', \ldots, f^{(N-1)}\right]_{(e_1, e_2, \ldots, e_l)} (A_0, A_1, \ldots, A_{N-1})^T. \qquad (24)$$

Therefore, we realize that seeking for a holonomic DE of order N satisfied by $f(z)$ is equivalent to finding a basis in a $\mathbb{K}(z)$-linear space where the system

$$\left(f^{(N)}(z), f^{(N-1)}(z), \ldots, f'(z), f(z)\right)$$

is linearly dependent. The idea is to construct such a basis while computing each derivative of $f(z)$ and their components. Thus, in each iteration step N, if all the $N+1$ derivatives are expanded in the same basis, we try to solve the resulting linear system. The advantage of this approach is the remember effect of computations from the first N iterations, which induces important time savings.

2.2. Description of the method

Let $f(z)$ be a holonomic function expanded as

$$f(z) = f_1(z) + f_2(z) + \cdots + f_{l_0}(z), \tag{25}$$

with $f_i(z)/f_j(z) \notin \mathbb{K}(z),\ 1 \leqslant i \neq j \leqslant l_0$. We mention that the rationality of $f_i(z)/f_j(z)$ cannot always be decided. However, we are only interested in collecting the rational coefficients appearing in the expansion of $f(z)$. That is why our implementations use other mathematical tools to recognize functions like $\cos(z)\tan(z)$[5] as holonomic functions.

From (25), we can state that $f(z)$ is a vector in the basis $E_0 = (e_1, e_2, \ldots, e_{l_0})$ where $e_i = f_i$. Then we compute the first derivative of $f(z)$, and we get the following two possibilities:

- either $f'(z)$ is expressed as a linear combination in E_0, which means that there exist $\alpha_{1,i} = \alpha_{1,i}(z) \in \mathbb{K}(z), i = 1, \ldots, l_0$ such that

$$f'(z) = \alpha_{1,1}e_1 + \alpha_{1,2}e_2 + \ldots + \alpha_{1,l_0}e_{l_0}. \tag{26}$$

  For hyperexponential functions we have $\alpha_{1,1} = \cdots = \alpha_{1,l_0} \in \mathbb{K}(z)$ and the DE is immediately deduced. But if $f'(z)$ and $f(z)$ are linearly independent, then we know that all derivatives can be expanded in E_0.

- Or $f'(z)$ is not expanded in E_0, which means that E_0 has to be augmented and there exist $\alpha_{1,i} \in \mathbb{K}(z), i = 1, \ldots, l_0$ and an integer $l_1 > l_0$ such that

$$f'(z) = \alpha_{1,1}e_1 + \alpha_{1,2}e_2 + \ldots + \alpha_{1,l_0}e_{l_0} + e_{l_0+1} + \ldots + e_{l_1}. \tag{27}$$

  Observe here that the new basis is $E_1 = (e_1, \ldots, e_{l_1})$ with $e_{l_0+1}, \ldots, e_{l_1}$ corresponding to independent basis elements brought by $f'(z)$. And also $\alpha_{1,i}$, $i \leqslant l_0$ could be zero.

To give a general view of the algorithm, let us assume that $f(z)$ satisfies a DE of order $N \geqslant 1$. By repeating the same analysis done for the first derivative, one obtains the following configuration

$$\begin{aligned}
&f(z) = e_1 + \ldots + e_{l_0}\\
&f'(z) = \alpha_{1,1}e_1 + \cdots + \alpha_{1,l_0}e_{l_0} + e_{l_0+1} + \cdots + e_{l_1}\\
&f''(z) = \alpha_{2,1}e_1 + \cdots + \alpha_{2,l_0}e_{l_0} + \alpha_{2,l_0+1}e_{l_0+1} + \cdots + \alpha_{2,l_1}e_{l_1} + e_{l_1+1} + \ldots + e_{l_2}\\
&\ldots\\
&f^{(N-1)}(z) = \alpha_{N-1,1}e_1 + \cdots + \alpha_{N-1,l_{N-2}}e_{l_{N-2}} + e_{l_{N-2}+1} + \cdots + e_{l_{N-1}}\\
&f^{(N)} = \alpha_{N,1}e_1 + \cdots + \alpha_{N,l_{N-1}}e_{l_{N-1}},
\end{aligned}$$

[5]The tangent function is usually not encoded as $\sin(z)/\cos(z)$, and this fact is also ignore by the implemented differentiation.

with positive integers $l_0 \leqslant l_1 \leqslant \ldots \leqslant l_{N-1}$, and $\alpha_{i,j} \in \mathbb{K}(z),\ i = 1,\ldots,N,\ j = 1,\ldots,l_{i-1}$. The final basis considered is $E_{N-1} = (e_1,\ldots,e_{l_{N-1}})$. In each iteration N, the algorithm keeps the components $\alpha_{N,i}$, the augmented basis and the current derivative. The components are kept in a matrix form, say H, and at the N^{th} iteration we have

$$H = \begin{bmatrix} 1 & \cdots & 1 & 0 & \cdots & 0 & 0 & \cdots & \cdots & 0 & \cdots & 0 \\ \alpha_{1,1} & \cdots & \alpha_{1,l_0} & 1 & \cdots & 1 & 0 & \cdots & \cdots & 0 & \cdots & 0 \\ \alpha_{2,1} & \cdots & \alpha_{2,l_0} & \cdots & \cdots & \alpha_{2,l_1} & 1 & \cdots & \cdots & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \alpha_{N-1,1} & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \alpha_{N-1,l_{N-2}} & 1 & \cdots & 1 \\ \alpha_{N,1} & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \alpha_{N,l_{N-2}} & \alpha_{N,l_{N-2}+1} & \cdots & \alpha_{N,l_{N-1}} \end{bmatrix}. \tag{28}$$

This matrix contains all needed information to find the holonomic DE sought. Indeed the polynomial coefficients of the differential equation sought are the components of the unique vector solution of the matrix system

$$A \cdot v = b,$$

with

$$A = \begin{bmatrix} 1 & \alpha_{1,1} & \alpha_{2,1} & \ldots & \alpha_{N-1,1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & \alpha_{1,l_0} & \alpha_{2,l_0} & \ldots & \alpha_{N-1,l_0} \\ 0 & 1 & \alpha_{2,l_0+1} & \ldots & \alpha_{N-1,l_0+1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 1 \end{bmatrix}, \text{ and } b = -\begin{bmatrix} \alpha_{N,1} \\ \alpha_{N,2} \\ \vdots \\ \alpha_{N,l_{N-1}} \end{bmatrix}. \tag{29}$$

From the matrix H, b is defined as the negative (note the minus in front) of the transpose of the last row of H, and A is the transpose of H deprived of its last row.

2.3. Examples

1. $f(z) = \sin(z)+z\cos(z)$. We have two linearly independent terms over $\mathbb{Q}(z)$ ($\sin(z)/(z\cos(z)) \notin \mathbb{Q}$), and we can write

$$f(z) = e_1 + e_2,$$

with $e_1 = \sin(z)$ and $e_2 = z\cos(z)$. Computing the first derivative gives

$$f'(z) = -z\sin(z) + 2\cos(z) = -z \cdot e_1 + \frac{2}{z} \cdot e_2.$$

At this step we have

$$H = \begin{bmatrix} 1 & 1 \\ -z & \frac{2}{z} \end{bmatrix},$$

and we get the system $\begin{bmatrix} 1 \\ 1 \end{bmatrix} v = \begin{bmatrix} z \\ -\frac{2}{z} \end{bmatrix}$, which has no solution $v \in \mathbb{Q}(z)$ (considered as a one-dimensional vector space). Now we compute the second derivative, and we get

$$f''(z) = -3\sin(z) - z\cos(z) = -3 \cdot e_1 - e_2.$$

H becomes

$$H = \begin{bmatrix} 1 & 1 \\ -z & \frac{2}{z} \\ -3 & -1 \end{bmatrix},$$

which gives the system

$$\begin{bmatrix} 1 & -z \\ 1 & \frac{2}{z} \end{bmatrix} v = \begin{bmatrix} 3 \\ 1 \end{bmatrix}, \ v \in \mathbb{Q}(z)^2,$$

and we get the solution

$$\left\{ \left(\frac{6+z^2}{2+z^2}, \frac{-2z}{2+z^2} \right) \right\}. \tag{30}$$

The differential equation sought is therefore

$$(2+z^2)f''(z) - 2zf'(z) + (6+z^2)f = 0. \tag{31}$$

2. $f(z) = \arctan(z)$. We have only one term so $e_1 = \arctan(z)$. For the first derivative

$$f'(z) = \frac{1}{1+z^2} = 0 \cdot e_1 + e_2,$$

where $e_2 = 1/(1+z^2)$. Since the basis has been augmented there is no system to be solved, and at this step we have

$$H = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

The second derivative gives

$$f''(z) = -\frac{2z}{(1+z^2)^2} = 0 \cdot e_1 - \frac{2z}{1+z^2} \cdot e_2,$$

and we get

$$H = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & -\frac{2z}{1+z^2} \end{bmatrix}$$

which produces the system

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} v = \begin{bmatrix} 0 \\ \frac{2z}{1+z^2} \end{bmatrix}, \ v \in \mathbb{Q}(z)^2.$$

We get $v = \big(0, 2z/(1+z^2)\big)$, hence the holonomic DE

$$(1+z^2)f''(z) + 2zf'(z) = 0. \tag{32}$$

3. $f(z) = \exp(z) + \log(1+z) = e_1 + e_2$, with $e_1 = \exp(z)$ and $e_2 = \log(1+z)$. The first derivative yields

$$f'(z) = \exp(z) + \frac{1}{1+z} = e_1 + 0 \cdot e_2 + e_3,$$

with $e_3 = 1/(1+z)$. Since a new term is added to the basis, the next step is to compute the second derivative

$$f''(z) = \exp(z) - \frac{1}{(1+z)^2} = e_1 + 0 \cdot e_2 - \frac{1}{(1+z)} \cdot e_3.$$

No term is added to the basis. We try to solve the resulting system. At this stage

$$H = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & -\frac{1}{1+z} \end{bmatrix},$$

and we get the system

$$\begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} v = \begin{bmatrix} -1 \\ 0 \\ \frac{1}{1+z} \end{bmatrix}, \; v \in \mathbb{Q}(z)^2,$$

which has no solution. We move on and compute the third derivative

$$f^{(3)}(z) = \exp(z) + \frac{2}{(1+z)^3} = e_1 + 0 \cdot e_2 + \frac{2}{(1+z)^2} \cdot e_3.$$

Thus

$$H = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & -\frac{1}{1+z} \\ 1 & 0 & \frac{2}{(1+z)^2} \end{bmatrix},$$

and we obtain the system

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & -\frac{1}{1+z} \end{bmatrix} v = \begin{bmatrix} -1 \\ 0 \\ -\frac{2}{(1+z)^2} \end{bmatrix},$$

whose solution in $\mathbb{Q}(z)^3$ is

$$\left\{ \left(0, -\frac{3+z}{(1+z)(2+z)}, -\frac{-1+2z+z^2}{(1+z)(2+z)} \right) \right\}. \tag{33}$$

Therefore we get the holonomic DE

$$(1+z)(2+z)f^{(3)}(z) - (-1+2z+z^2)f''(z) - (3+z)f'(z) = 0. \tag{34}$$

We implemented this algorithm as `HolonomicDE(f,y(z))` in our Maple and Maxima packages. It computes a holonomic DE in terms of the dependent variable `y(z)` for a function f of the variable z. Our Maple implementation will be available in future releases of Maple under the command `DEtools[FindODE]`. Table 1 shows an efficiency gain of this method (Approach 1) on the original one implemented in the current `DEtools[FindODE]` (Approach 2). The third column presents the timings of `gfun[holexprtodiffeq]` (Approach 3) for the same computations.

As expected, after trying several other examples, it appears that `HolonomicDE` generally gives better timings than the current `FindODE`. The current `holexprtodiffeq` does not really

Table 1: Comparison of timings for computing holonomic DEs

$f(z)$	CPU time		
	Approach 1	Approach 2	Approach 3
$(\arccos(z^2) - \arcsin(z^3))^7$	219.594	294.047	∞
$\sin(z)^6 \arcsin(z)^3$	71.672	347.078	748.610
$\arctan(z)^2 + \sin(z)^4 + \log(1+z)^5$	0.500	3.359	1.203
$\exp(z^{11}+3)\cos(z) + \log(1+z^7)$ $+ \cos(z)^3 \sinh(z)^5$	161.578	140.562	5706.468
$\arccos(z)^{13} + \sinh(z)^{19}$	53.063	295.750	609.813
$(3z + 5z^7 + 11z^{13})\log(1+z^3+z^7)$ $+ \operatorname{arctanh}(z)\cos(z)^5$	7.984	31.500	2.703

compare to the other approaches since it also seeks for inhomogeneous DEs which often lead to holonomic DEs of higher orders (last row, Table 1). The first row in Table 1 indicates 6 hours of computation without result for `holexprtodiffeq`. We also mention that some holonomic functions are not recognized as such by `holexprtodiffeq`, an example is $\cos(k \cdot \arccos(z)), k \in \mathbb{N}$. These are some reasons why we rather use the method of this section.

Holonomic recurrence equations are easily deduced from holonomic DEs by expanding them and rewriting powers into polynomials and derivatives into shifts (implemented as `FindRE`) (Koepf (1992), Salvy and Zimmermann (1994)). However, although the described method often finds the differential equation of lowest order, this process does not guarantee finding recurrence equations of lowest order. Another approach of getting lower-order recurrences is applying the Almkvist-Zeilberger (see Zeilberger (1990) and Almkvist and Zeilberger (1990)) algorithm to the Cauchy integral $\oint f(z)/z^{n+1}dz$. The choice of `FindRE` is justified in Subsection 4.2.

3. Algorithm mfoldHyper

Let $\mathbb{K}$ be a field of characteristic zero. We consider the generic holonomic recurrence equation

$$P_d(n)a_{n+d} + P_{d-1}(n)a_{n+d-1} + \cdots + P_0(n)a_n = 0, \tag{35}$$

$P_d(n), \ldots, P_0(n) \in \mathbb{K}[n], P_d(n) \cdot P_0(n) \neq 0$.

By definition, a term a_n is said to be m-fold hypergeometric (see Definition 1), if there exists a rational function $r(n) \in \mathbb{K}(n)$ such that

$$r(n) = \frac{a_{n+m}}{a_n}. \tag{36}$$

In this section, we present an algorithm to compute a basis of m-fold hypergeometric term solutions of (35).

At first glance, one should remark that m-fold hypergeometric terms have rational functions as the ratio of terms with index difference equal to m. Consequently, if we can find a way to transform this property to the simple one of hypergeometric term then by iterative computation of hypergeometric terms ($m = 1$), we are done. However, such a transformation cannot be valid

for all recurrences given that the indices of the indeterminate term a_n in the equation must be linear in n with slope 1.
From the characterization (36) one can deduce that for $0 \leqslant j \leqslant m-1$ the following is valid

$$r(m \cdot n + j) = \frac{a_{m\cdot(n+1)+j}}{a_{m\cdot n+j}}. \tag{37}$$

Therefore instead of considering the representation (36) one could rather see an m-fold hypergeometric term with m related rational functions as defined in (37). The m rational functions define m linearly independent m-fold terms. In some sense, although hypergeometric terms does not form a linear space, one could see the computation of a_n as the determination of a vector in the basis $(a_{mn}, a_{mn+1}, \dots, a_{mn+m-1})$, which of course appears more difficult than computing a_{mn}. Observe that for a fixed $m \in \mathbb{N}$ and $j \in [\![0, m-1]\!]$, if we compute an m-fold hypergeometric term solution of (35) with ratio $r(m \cdot n + j)$, then we can repeat the same computation by updating the value of j in order to get the other $m-1$ terms of this sub-basis[6] of solutions. This is one difference of our approach with previous ones which try to find $r(n)$ directly as a right-factor using a shift of order m. In our case, we concentrate on the interlacing sub-terms $r(m \cdot n + j)$, $j \in [\![0, m-1]\!]$ to find $(a_{mn}, a_{mn+1}, \dots, a_{mn+m-1})$ instead of a_n. Without loss of generality, since the computations of the a_{mn+j}, $j \in [\![0, m-1]\!]$ are similar, we only detail the algorithm for finding the first element a_{mn} ($j = 0$) of the basis. We will say that the basis is given in an incomplete form when only first elements of m-fold hypergeometric term (m fixed) bases are returned. This is how our implementation of algorithm **mfoldHyper** returns outputs, but it also gives the possibility to complete the basis of solutions once the corresponding values of m are already known.
The following lemma gives a condition on the order of a given holonomic RE for its m-fold hypergeometric term solutions to be computable over a given field $\mathbb{K}$. This lemma can relate to ((Hendricks and Singer, 1999, Theorem 5.1)) and (Abramov (2000)), but with completely different perspectives as shown in the proofs.

Lemma 3. *Let h_n be an m-fold hypergeometric term, for a fixed $m \in \mathbb{N}$. Assume*

$$\forall u \in \mathbb{N},\ u < m,\ \text{ there is no rational function } r_u(n) \in \mathbb{K}(n)\ :\ h_{u+n} = r_u(n) h_n. \tag{38}$$

Then there is no holonomic recurrence equation over $\mathbb{K}$ of order less than m satisfied by h_n.

Proof. Let h_n be an m-fold hypergeometric term such that

$$h_{n+m} = r(n) \cdot h_n \iff Q_m(n) \cdot h_{n+m} + Q_0(n) \cdot h_n = 0, \tag{39}$$

where $Q_m(n), Q_0(n) \in \mathbb{K}[n]$ and $r(n) = -Q_0(n)/Q_m(n) \in \mathbb{K}(n)$.
Suppose that h_n satisfies a holonomic recurrence equation of order less than m. Then there exists an equation of the form

$$P_{m-1} a_{n+m-1} + P_{m-2} a_{n+m-2} + \cdots + P_1 a_{n+1} + P_0 a_n = 0, \tag{40}$$

with polynomials $P_j = P_j(n) \in \mathbb{K}[n]$, $j \in [\![0, m-1]\!]$, and $P_0(n) \neq 0$, satisfied by h_n.

[6]For each fixed m corresponding to an m-fold hypergeometric term we have a basis, and the basis of all m-fold hypergeometric term solutions is the collection of these bases.

- If P_0 is the only non-zero polynomial in the equation then h_n is zero, which is a contradiction by definition.
- We assume that at least one other polynomial factor in the equation is non-zero. Then h_n satisfying (40) yields the following equation after substitution of n by $m \cdot n$

  $$P_{m-1}(mn)h_{mn+m-1} + P_{m-2}(mn)h_{mn+m-2} + \cdots + P_1(mn)h_{mn+1} + P_0(mn)h_{mn} = 0. \quad (41)$$

  By assumption (38), we know that $\forall u \in \mathbb{N},\ u < m$, h_n is not a u-fold hypergeometric term. So the holonomic recurrence equation of lowest order over $\mathbb{K}$ satisfied by h_n is

  $$Q_m(n) \cdot a_{n+m} + Q_0(n) \cdot a_n = 0,$$

  which is a two-term recurrence relation whose basis of m-fold hypergeometric term solutions is

  $$(h_{mn+m-1}, h_{mn+m-2}, \ldots, h_{mn+1}, h_{mn}). \quad (42)$$

  Therefore (41) cannot hold since its left-hand side is a linear combination of linearly independent terms with respect to $\mathbb{K}(n)$, which implies that all the polynomial coefficients must be zero. Therefore we get a contradiction.

□

Remark 4.
Observe that the linear independence with respect to $\mathbb{K}(n)$ *of the elements of the basis* (42) *used in the proof of Lemma 3 can be interpreted in the following different way. Since* $h_n \neq 0$, h_n *satisfying* (40) *yields the following identity after dividing* (40) *by* h_n

$$-P_0 = P_{m-1}\frac{h_{n+m-1}}{h_n} + P_{m-2}\frac{h_{n+m-2}}{h_n} + \ldots + P_1\frac{h_{n+1}}{h_n}. \quad (43)$$

By assumption (38)*, we know that* $\forall u \in \mathbb{N},\ u < m$*, the ratio* $\frac{h_{n+u}}{h_n}$ *is not a rational function over* $\mathbb{K}(n)$*. So, each non-zero term on the right-hand side of* (43) *is not rational over* $\mathbb{K}$*. This does not necessarily imply the non-rationality of the whole right-hand side. However by the linear independence of the elements of the basis* (42) *one can assume that this holds. Similar arguments will be used to conclude the proof of Theorem 7.*

More generally, any shift of a holonomic recurrence equation of order less than m does not have m-fold hypergeometric term solutions. In (Abramov (2000)) the property of Lemma 3 defines primitive m-hypergeometric sequences.
Checking the hypothesis of Lemma 3 is an important task for the algorithm. Fortunately, this can be done iteratively. Indeed for a fixed field $\mathbb{K}$, if we have already looked for u-fold hypergeometric term solutions for integers $u < m$, then we can proceed to the computation of m-fold hypergeometric terms knowing that m is less than the order of the recurrence equation under consideration. Thus this iterative algorithm is terminating.

Definition 5 (m-fold holonomic recurrence equation)**.** *A holonomic recurrence equation is said to be m-fold holonomic,* $m \in \mathbb{N}$*, if it has at least two non-zero polynomial coefficients and the difference between every pair of indices in the equation is a multiple of m. Choosing* 0 *as the trailing term order gives the general form*

$$P_d(n) \cdot a_{n+md} + P_{d-1}(n) \cdot a_{n+m(d-1)} + \cdots + P_1(n) \cdot a_{n+m} + P_0(n) \cdot a_n = 0,. \quad (44)$$

so that $P_d \cdot P_0 \neq 0$.

Assume an m-fold holonomic RE with representation (44) is given. The basis of m-fold hypergeometric term solutions of (44) can be computed by means of change of variable and computation of hypergeometric terms.
Having an m-fold hypergeometric term a_n starting with a_0 (by shift it is always possible to define the initial value by a_0), then from (36) the next value computed from a_0 is a_m, and afterwards a_{2m} $\ldots, a_{km}, \ldots$. Thus setting $s_n = a_{mn}$ implies that all values computed from s_0 have their indices corresponding to multiples of m for a_n. Moreover, since $a_{m\cdot(n+1)}/a_{m\cdot n} = s_{n+1}/s_n \in \mathbb{K}(n)$, s_n is a hypergeometric term whose formula is the same as that of a_{mn}. Therefore we can update (44) accordingly to compute its m-fold hypergeometric terms as hypergeometric terms. This analysis shows that the change of variable

$$\begin{cases} m \cdot k = n \\ s_k = a_{m\cdot k} \end{cases}, \tag{45}$$

transforms (44) to a 1-fold holonomic RE for hypergeometric term solutions s_k satisfying

$$\frac{s_{k+1}}{s_k} = \frac{a_{mk+m}}{a_{mk}} = r(mk). \tag{46}$$

The resulting RE is

$$P_d(mk) \cdot s_{k+d} + P_{d-1}(mk) \cdot s_{k+(d-1)} + \cdots + P_1(mk) \cdot s_{k+1} + P_0(mk) \cdot s_k = 0. \tag{47}$$

For $j \in [\![0, m-1]\!]$, a similar analysis leads to the following change of variable

$$\begin{cases} m \cdot k + j = n, \\ s_k = a_{m\cdot k+j} \end{cases} \quad (0 \leqslant j \leqslant m-1). \tag{48}$$

Since an arbitrary holonomic RE is not necessarily m-fold holonomic, $m \in \mathbb{N}_{\geqslant 2}$, the most interesting part is how to solve non-m-fold holonomic REs.
Without using the shift that rewrites an m-fold holonomic RE in the form (44), its general representation is given by

$$P_d a_{n+k+md} + P_{d-1} a_{n+k+m(d-1)} + \cdots + P_0 a_{n+k} = 0, \tag{49}$$

where $k \in [\![0, m-1]\!]$.
Consider the three following 3-fold holonomic REs

$$\begin{aligned} RE1 &: \; P_{1,3} \cdot a_{n+7} + P_{1,2} \cdot a_{n+4} + P_{1,1} \cdot a_{n+1} = 0, \\ RE2 &: \; P_{2,4} \cdot a_{n+11} + P_{2,3} \cdot a_{n+8} + P_{2,2} \cdot a_{n+5} + P_{2,1} \cdot a_{n+2} = 0, \\ RE3 &: \; P_{3,4} \cdot a_{n+13} + P_{3,3} \cdot a_{n+10} + P_{3,2} \cdot a_{n+7} + P_{3,1} \cdot a_{n+4} = 0. \end{aligned} \tag{50}$$

- The difference between an index of the indeterminate (a_n) in $RE1$ and another index in $RE2$ is never divisible by 3. In this case we say that $RE1$ and $RE2$ are 3-fold distinct.
- The difference between an index of the indeterminate (a_n) in $RE1$ and another index in $RE3$ is always a multiple of 3. In this case we say that $RE1$ and $RE3$ are 3-fold equivalent.

More generally we have the following definitions.

Definition 6. *Let $m \in \mathbb{N}$,*

$$RE_1 \; : \; P_{d_1} a_{n+k_1+md_1} + P_{d_1-1} a_{n+k_1+m(d_1-1)} + \cdots + P_{0_1} a_{n+k_1} = 0, \tag{51}$$

and

$$RE_2 \; : \; P_{d_2} a_{n+k_2+md_2} + P_{d_2-1} a_{n+k_2+m(d_2-1)} + \cdots + P_{0_2} a_{n+k_2} = 0 \tag{52}$$

be two m-fold holonomic recurrence equations.

- *We say that RE_1 and RE_2 are m-fold distinct holonomic equations if $k_2 - k_1$ is not divisible by m.*
- *We say that RE_1 and RE_2 are m-fold equivalent holonomic equations if $k_2 - k_1$ is divisible by m.*

An immediate consequence of these definitions is that linear combinations of m-fold equivalent holonomic REs always give m-fold holonomic recurrence equations whereas linear combinations of m-fold distinct holonomic REs are never m-fold holonomic. For example, summing $RE1$ and $RE3$ from (50) yields a 3-fold holonomic RE, whereas summing $RE1$ and $RE2$ yields a non-3-fold holonomic RE.

We can now give the fundamental theorem of this paper from which algorithm **mfoldHyper** is deduced.

Theorem 7 (Structure of holonomic REs having m-fold hypergeometric term solutions)**.** *Let $m \in \mathbb{N}$, $\mathbb{K}$ a field of characteristic zero, and h_n be an m-fold hypergeometric term which is not u-fold hypergeometric over $\mathbb{K}$ for all positive integers $u < m$. Then h_n is a solution of a given holonomic recurrence equation, if that equation can be written as a linear combination of m-fold holonomic recurrence equations that have h_n as solution of each of the m-fold distinct holonomic recurrence equations involved.*

Proof. Let h_n be an m-fold hypergeometric term solution of the recurrence equation

$$P_d a_{n+d} + P_{d-1} a_{n+d-1} + \cdots + P_0 a_n = 0, \;\; d > m, \;\; P_d \cdot P_0 \neq 0. \tag{53}$$

It suffices to show that for any non-zero term $P_j a_{n+j}$ in (53), there exists another term, say $P_i a_{n+i}$, such that m divides $j - i$. Indeed, by summing m-fold holonomic REs we are sure that for each term appearing on the left-hand side of the sum there must exist another term whose index differs from the one of that term by a multiple of m.

We proceed by contradiction. Assume there exists a non-zero term $P_j a_{n+j}$ in (53) such that any other term $P_i a_{n+i}$, $i \neq j$ does not verify that m divides $j - i$. Since h_n is a non-zero solution, we can divide the equation by h_{n+j} and write

$$-P_j = \sum_{\substack{i=0\\ i\neq j}}^{d} P_i \cdot \frac{h_{n+i}}{h_{n+j}}. \tag{54}$$

The situation is now two-fold:

- For i verifying $|i-j| < m$, for each corresponding $P_i \cdot h_{n+i}/h_{n+j}$ on the right-hand side of (54), the fact that m does not divide $j - i$ implies that $h_{n+i}/h_{n+j} \notin \mathbb{K}(n)$ since by assumption h_n is an m-fold hypergeometric term over $\mathbb{K}$ that is not u-fold hypergeometric for all integers $u < m$. Therefore $P_i \cdot h_{n+i}/h_{n+j} \notin \mathbb{K}(n)$.

- For i verifying $|i-j|> m$, for each corresponding $P_i \cdot h_{n+i}/h_{n+j}$ on the right-hand side of (54), we have two possibilities:
  - either $h_{n+i}/h_{n+j} \notin \mathbb{K}(n)$ and we have the same conclusion as in the previous case;
  - or $h_{n+i}/h_{n+j} \in \mathbb{K}(n)$, but in this case since m does not divide $j-i$, this implies that h_n is not an m-fold hypergeometric term and we get a contradiction.

Thus the identity (54) has all terms on its right-hand side not belonging to $\mathbb{K}(n)$. From the argument in Remark 4 one shows that this leads to a contradiction by a similar argument as in the proof of Lemma 3. Indeed, we should get an identity between

$$\sum_{\substack{i=0\\ i\neq j}}^{d} P_i \cdot \frac{h_{n+i}}{h_{n+j}} \notin \mathbb{K}(n),$$

and $P_j(n) \in \mathbb{K}[n] \subset \mathbb{K}(n)$.
For the second part of the theorem, since the multiplication of a holonomic recurrence equation by a polynomial does not affect the computation of its solutions, a linear combination of m-fold holonomic REs can always be considered as a sum of m-fold holonomic REs. Therefore it is enough to show that an m-fold hypergeometric term solution of a sum of m-fold holonomic recurrence equations is a solution of each of the involved m-fold distinct holonomic recurrences. The sum of M m-fold holonomic recurrence equations, $M \in \mathbb{N}$, can be written as

$$\sum_{j=1}^{M} RE_j(a_n) = \sum_{j=1}^{M} \left(P_{d_j} a_{n+k_j+md_j} + P_{d_j-1} a_{n+k_j+m(d_j-1)} + \cdots + P_{0_j} a_{n+k_j} \right) = 0, \tag{55}$$

where $k_j \in [\![0, m-1]\!]$, and $P_{d_j} \cdot P_{0_j} \neq 0$, $j \in [\![1, M]\!]$.
If $M = 1$, then (55) is an m-fold holonomic recurrence equation and h_n is an m-fold hypergeometric term solution of it.
We assume now that $M \geqslant 2$ and that there are at least two m-fold distinct holonomic recurrence equations in (55). Note that if the M m-fold holonomic REs are m-fold equivalent then the situation is similar to the case $M = 1$ since every linear combination of m-fold equivalent holonomic REs is an m-fold holonomic RE.
Now suppose that h_n is not solution of RE_{j_1} in (55), $j_1 \in [\![1, M]\!]$, then given that $\sum_{j=1}^{M} RE_j(h_n) = 0$, there must be another m-fold holonomic recurrence equation RE_{j_2}, $j_2 \in [\![1, M]\!]$, m-fold distinct with RE_{j_1} such that $RE_{j_2}(h_n) \neq 0$. Without loss of generality, we consider that RE_{j_2} is the only m-fold holonomic RE with these properties. Of course, if $RE_{j_1}(h_n) \neq 0$ and $RE_{j_1}(h_n) + RE_{j_2}(h_n) = 0$ then $RE_{j_2}(h_n) \neq 0$. Thus, we have

$$\begin{cases} RE_{j_1}(h_n) \neq 0 \\ RE_{j_2}(h_n) \neq 0 \\ RE_{j_1}(h_n) + RE_{j_2}(h_n) = 0 \end{cases}. \tag{56}$$

The fact that the m-fold holonomic recurrence equations RE_{j_1} and RE_{j_2} are m-fold distinct implies that $k_{j_1} - k_{j_2}$ is not a multiple of m.
Using (56), after substitution of h_n in the sum of the equations and division by $h_{n+k_{j_1}+d_{j_1}m}$, we deduce that

$$-P_{d_{j_1}} = \sum_{e_{j_1}=0_{j_1}}^{d_{j_1}-1} P_{e_{j_1}} \frac{h_{n+k_{j_1}+e_{j_1}m}}{h_{n+k_{j_1}+d_{j_1}m}} + \sum_{e_{j_2}=0_{j_2}}^{d_{j_2}} P_{e_{j_2}} \frac{h_{n+k_{j_2}+e_{j_2}m}}{h_{n+k_{j_1}+d_{j_1}m}} = S_{j_1} + S_{j_1,j_2}, \tag{57}$$

which is equivalent to

$$-P_{d_{j_1}} - S_{j_1} = S_{j_1,j_2}. \tag{58}$$

All terms in S_{j_1} belong to $\mathbb{K}(n)$ since h_n is an m-fold hypergeometric term and the corresponding index differences

$$n + k_{j_1} + e_{j_1}m - (n + k_{j_1} + d_{j_1}m) = m \cdot (e_{j_1} - d_{j_1})$$

are multiples of m. However, for S_{j_1,j_2} the index differences

$$n + k_{j_2} + e_{j_2}m - (n + k_{j_1} + d_{j_1}m) = k_{j_2} - k_{j_1} + m \cdot (e_{j_2} - d_{j_1})$$

are not multiples of m. Therefore by the same argument used in the first part of the proof we deduce that $S_{j_1,j_2} \notin \mathbb{K}(n)$. Thus for (58) to hold we must have $-P_{d_{j_1}} - S_{j_1} \in \mathbb{K}(n)$ and $S_{j_1,j_2} \notin \mathbb{K}(n)$, which is a contradiction. □

From this theorem, given $m \in \mathbb{N}$, we can compute a basis of m-fold hypergeometric term solutions of a given holonomic RE by splitting it into the sum of m-fold distinct holonomic REs that we transform into 1-fold holonomic REs, solve by computing hypergeometric terms, and select the linearly dependent among them. Note that this step can be done differently as suggested in (Abramov (2000), pages 7-9). We proceed in this way to avoid the computation of greatest common divisors which can be inefficient in terms of implementation. Also, we do not use the substitution method because we have observed that it is generally more difficult to verify that a term is solution of a holonomic RE when factorials and Pochhammer symbols are involved. To compute hypergeometric terms we use the variant of van Hoeij's algorithm (Cluzeau and van Hoeij (2006), Van Hoeij (1999)) described in (Teguia Tabuguia (2020b)) which apart from computing them efficiently, gives them in normal forms that facilitate checking of linear dependency (mostly simple computation of ratios) and evaluations at natural integers.
Note that the computation of m-fold hypergeometric term solutions of an RE in the form (49) is done after rewriting it into the form (44).
Let us take as an example the holonomic RE satisfied by the Taylor coefficients of $\exp(z) + \cos(z)$.

```
(%i1) FindRE(cos(z)+exp(z),z,a[n]);
```

(%o1) $(1 + n) \cdot (2 + n) \cdot (3 + n) \cdot a_{n+3} - (1 + n) \cdot (2 + n) \cdot a_{n+2} + (1 + n) \cdot a_{n+1} - a_n = 0$

This is a linear combination of two 2-fold distinct holonomic REs, namely

$$RE1: \ (1 + n) \cdot (2 + n) \cdot (3 + n) \cdot a_{n+3} + (1 + n) \cdot a_{n+1} = 0,$$

and

$$RE2: \ (-1 - n) \cdot (2 + n) \cdot a_{n+2} - a_n = 0.$$

Only $RE1$ has to be rewritten since its trailing term is not of order 0. This yields

$$RE11: \ n \cdot (1 + n) \cdot (2 + n) \cdot a_{n+2} + n \cdot a_n = 0.$$

From this one easily sees that the given holonomic RE has 2-fold hypergeometric term solutions since we get two two-term recurrence relations that are linearly dependent:

$$-n \cdot RE2 = RE11.$$

Remember that there is no need to use all the m changes of variable of (48) because as we explained earlier, once one succeeds in computing a basis of m-fold hypergeometric term solutions corresponding to the representation (37) for a fixed $j \in [\![0, m-1]\!]$, the other ones can be computed in a similar way. This will be used for power series computations in order to consider all possible linear combinations.

Note that even though the algorithm proceeds by iteration up to the order of the given RE, more often the number of cases to be considered is much smaller than the order of the given RE. For example, the recurrence equation

```
(%i1) RE:FindRE(sin(z^3)^3,z,a[n]);
```

(%o1) $(n-8)\cdot(n-5)\cdot(n-2)\cdot(1+n)\cdot a_{n+1} + 90\cdot(n-8)\cdot(n-5)\cdot a_{n-5} + 729\cdot a_{n-11} = 0$

is a 2-fold, 3-fold and 6-fold holonomic RE of order 12. It is straightforward to see that all the other cases do not lead to a solution since the recurrence equation cannot be written as a sum of m-fold distinct holonomic REs for $m \notin \{1,2,3,6\}$.

The steps of algorithm **mfoldHyper** can be defined as follows.

Algorithm 1 mfoldHyper: compute m-fold hypergeometric term solutions of holonomic REs of order $d \in \mathbb{N}$

Input: A holonomic recurrence equation

$$P_d a_{n+d} + P_{d-1} a_{n+d-1} + \cdots + P_0 a_n = 0, \quad d > m, \; P_d \cdot P_0 \neq 0 \tag{59}$$

Output: A basis (incomplete form) of m-fold hypergeometric term solutions of (59).

1. Set $H = \{\}$.
2. Use the algorithm in (Teguia Tabuguia (2020b)) to find the basis, say H_1, of all hypergeometric term solutions of (59). If $H_1 \neq \emptyset$, then add $[1, H_1]$ to H.
3. For $2 \leqslant m \leqslant d$ do:
   (a) Extract the following m-fold holonomic recurrence equations from (59) and construct the system

$$\begin{cases} P_0(n)\cdot a_n + P_m(n)\cdot a_{n+m} + \cdots + P_{m\cdot\lfloor\frac{d}{m}\rfloor}(n)\cdot a_{n+m\cdot\lfloor\frac{d}{m}\rfloor} = 0 \\ P_1(n)\cdot a_{n+1} + P_{m+1}(n)\cdot a_{n+m+1} + \cdots + P_{m\cdot\lfloor\frac{d}{m}\rfloor+1}(n)\cdot a_{n+m\cdot\lfloor\frac{d}{m}\rfloor+1} = 0 \\ \cdots \\ P_{m-1}(n)\cdot a_{n+m-1} + P_{2m-1}(n)\cdot a_{n+2m-1} + \cdots + P_{m\cdot\lfloor\frac{d}{m}\rfloor+m-1}(n)\cdot a_{n+m\cdot\lfloor\frac{d}{m}\rfloor+m-1} = 0 \end{cases}, \tag{60}$$

   assuming $P_j(n) = 0$ for $j > d$.

   (b) If there exists a holonomic RE with only one non-zero polynomial coefficient in (60), then stop and go back to step 3.(a) for $m+1$.

   (c) Shift all the m-fold holonomic recurrence equations in (60) so that the order of the trailing term equals 0.

   (d) Apply the change of variable (45) to each m-fold holonomic recurrence equation.

Algorithm 1 mfoldHyper

3. (e) Compute a basis of hypergeometric term solutions s_k as defined in (45) for (47) of each resulting holonomic recurrence equation using (Teguia Tabuguia (2020b)).
   (f) Construct the set H_m of hypergeometric terms which are each linearly dependent to some terms in each of the m computed bases in step 3.(d).
   (g) If $H_m \neq \emptyset$ then add $[m, H_m]$ in H.
4. Return H.

We implemented **mfoldHyper** in Maxima as `mfoldHyper(RE,a[n],[m,j])`[7], by default `[m,j]` is an empty list. In that default case each list of m-fold hypergeometric term solutions, say $[m, [h_{1,m}, h_{2,m}, \ldots]]$, contains "closed-forms" of hypergeometric terms corresponding to $j = 0$ in (37). Once we know that there are some m-fold hypergeometric term solutions for particular $m \in \mathbb{N}$, the algorithm can be called as `mfoldHyper(RE,a[n],m,j)` for $0 \leqslant j < m$ to get the complete basis of solutions. Algorithm **mfoldHyper** will appear in future releases of Maple as `LREtools[mhypergeomsols]`.
Let us present some examples. We hide the recurrence equations for space saving purposes. All these computations can be done with our package `FPS` currently available as third-party Maxima package on Github.

```
(%i1) RE:FindRE(atan(z)+exp(z),z,a[n])$
(%i2) mfoldHyper(RE,a[n]);
```

$$(\%o2) \quad \left[\left[1, \left\{\frac{1}{n!}\right\}\right], \left[2, \left\{\frac{(-1)^n}{n}\right\}\right]\right]$$

For algebraic extension fields the syntax is `mfoldHyper(RE,a[n],K)`, for the two possible values `K=C` or `K=Q` (default value for rationals). To ask for specific m-fold hypergeometric term solutions the syntax is `mfoldHyper(RE,a[n],K,m,j)`.

```
(%i3) RE:FindRE(log(1+z+z^2)+cos(z),z,a[n])$
(%i4) mfoldHyper(RE,a[n],C);
```

$$(\%o4) \quad \left[\left[1, \left\{\frac{\left(\frac{-1-\sqrt{3}\cdot i}{2}\right)^n}{n}, \frac{\left(\frac{\sqrt{3}\cdot i-1}{2}\right)^n}{n}, \frac{(-i)^n}{n!}, \frac{(-1)^{\frac{n}{2}}}{n!}\right\}\right], \left[2, \left\{\frac{(-1)^n}{(2\cdot n)!}\right\}\right]\right]$$

where the obtained 2-fold hypergeometric term is the coefficient of the hypergeometric type series of $\cos(z)$.

```
(%i5) declare(q1,constant)$
(%i6) declare(q2,constant)$
(%i7) RE:FindRE(1/((q1-z^2)*(q2-z^3)),z,a[n])$
(%i8) mfoldHyper(RE,a[n],C);
```

[7] The brackets around `m,j` means optional arguments

(%o8) $\left[\left[1,\left\{\left(-\sqrt{\frac{1}{q1}}\right)^n,\left(\frac{1}{q1}\right)^{\frac{n}{2}},\left(\frac{\left(\sqrt{3}\cdot i-1\right)\left(\frac{1}{q2}\right)^{\frac{1}{3}}}{2}\right)^n,\left(-\frac{\left(\sqrt{3}\cdot i+1\right)\left(\frac{1}{q2}\right)^{\frac{1}{3}}}{2}\right)^n,\left(\frac{1}{q2}\right)^{\frac{n}{3}}\right\}\right],\left[2,\left\{\left(\frac{1}{q1}\right)^n\right\}\right],\left[3,\left\{\left(\frac{1}{q2}\right)^n\right\}\right]\right]$

For these previous examples, the current Maple `convert/FormalPowerSeries` yields complicated power series representations because the above m-fold hypergeometric terms, $m \geqslant 2$, are not found. Next we compute the power series coefficients of some expressions for which `convert/FormalPowerSeries` misses representations.

```
(%i9)  RE:FindRE(exp(z^2)+cos(z^2),z,a[n])$
(%i10) mfoldHyper(RE,a[n]);
```

(%o10) $\left[\left[2,\left\{\frac{1}{n!}\right\}\right],\left[4,\left\{\frac{(-1)^n}{(2\cdot n)!}\right\}\right]\right]$

```
(%i11) RE:FindRE(cosh(z^3)+sin(z^2),z,a[n])$
(%i12) mfoldHyper(RE,a[n]);
```

(%o12) $\left[\left[3,\left\{\frac{1}{n!},\frac{(-1)^n}{n!}\right\}\right],\left[4,\left\{\frac{(-1)^n}{(2\cdot n)!}\right\}\right],\left[6,\left\{\frac{1}{(2\cdot n)!}\right\}\right]\right]$

```
(%i13) RE:FindRE(asin(z^2)^2+acos(z),z,a[n])$
(%i13) mfoldHyper(RE,a[n]);
```

(%o13) $\left[\left[2,\left\{\frac{4^n\cdot n!^2}{n^2\cdot(2\cdot n)!}\right\}\right],\left[4,\left\{\frac{4^n\cdot n!^2}{n^2\cdot(2\cdot n)!}\right\}\right]\right]$

```
(%i14) RE:FindRE(sqrt(sqrt(8*z^3+1)-1)+sqrt(7+13*z^4),z,a[n])$
(%i15) mfoldHyper(RE,a[n]);
```

(%o15) $\left[\left[3,\left\{\frac{\left(\frac{1}{4}\right)_n\cdot\left(\frac{3}{4}\right)_n\cdot(-8)^n\cdot 4^n}{(4\cdot n-1)\cdot(2\cdot n)!}\right\}\right],\left[4,\left\{\frac{4^{-4-n}\cdot(-13)^n\cdot(2\cdot n)!}{(2\cdot n-1)\cdot 7^n\cdot n!^2}\right\}\right]\right]$

```
(%i16) RE:FindRE(sin(z^3)^3,z,a[n])$
(%i17) mfoldHyper(RE,a[n]);
```

(%o17) $\left[\left[6,\left\{\frac{(-9)^n}{(2\cdot n)!},\frac{(-1)^n}{(2\cdot n)!}\right\}\right]\right]$

Let us now use our implementation for the computation of a specific representation of m-fold hypergeometric term solutions. In this case the user has to specify a value for m and j with $j \in [\![0, m-1]\!]$.

```
(%i18) RE:FindRE(asin(z)^2+log(1+z^5),z,a[n])$
(%i19) mfoldHyper(RE,a[n],5,0);
```

(%o19) $\left\{\frac{(-1)^n}{n}\right\}$

```
(%i20) mfoldHyper(RE,a[n],5,3);
```

$$(\%o20) \quad \left\{ \frac{(-1)^n}{(5 \cdot n + 3)} \right\}$$

```
(%i21) mfoldHyper(RE,a[n],2,1);
```

$$(\%o21) \quad \left\{ \frac{(2 \cdot n)\,!}{(2 \cdot n + 1) \cdot 4^n \cdot n!^2} \right\}$$

We mention that the existence of m-fold hypergeometric term solutions of a holonomic RE satisfied by the Taylor coefficients of a given expression does not necessarily guarantee that this expression represents a hypergeometric type function. For example, $\arctan(z) \cdot \cos(z)$ yields a recurrence equations satisfied by the coefficients of $\cos(z)$.

```
(%i22) RE:FindRE(atan(z)*cos(z),z,a[n])$
(%i23) mfoldHyper(RE,a[n]);
```

$$(\%o23) \quad \left[\left[2, \left\{ \frac{(-1)^n}{(2n)\,!} \right\}\right]\right]$$

However, we know that the coefficient must be different. In the next section, by finding the linear combination of hypergeometric type power series we will be able to decide using some initial values whether a potential coefficient is the correct one.

4. Hypergeometric type power series

We defined hypergeometric type functions and series in Definition 2. The presence of $\ln(z)$ in the expansion is justified by the solution of the underlying holonomic DE (see Kauers and Paule (2011)). The definition in (Koepf (1992)) reduces to the case $T(z) = 0$ and $J \leqslant m$, where m is the unique type[8] encountered in Definition 2. With this new definition, we can define the type of the series (21) as the tuple $(m_1, m_2, \ldots, m_J)$. However, we do not compute the coefficients as they appear in (21), but instead for powers of the form $z^{m_j \cdot n + i}$, $0 \leqslant i < m_j$ which is more suitable for the coefficients computed using **mfoldHyper**.
We recall the necessary steps towards hypergeometric type representations (see Teguia Tabuguia and Koepf (2021b)). Having a holonomic function f:

1. Find a holonomic RE satisfied by the power series coefficients of f (see Section 2);
2. Compute a basis of m-fold hypergeometric term solutions of that RE using **mfoldHyper** (see Section 3);
3. If there are solutions, use initial values to find the linear combination of the resulting hypergeometric type power series that corresponds to the power series expansion of f, if such a linear combination is valid.

Regarding Puiseux series, we generalize the idea given in ((Gruntz and Koepf, 1995, Section 5)). We will see that computing Puiseux numbers p_j's appearing in (21) reduces to finding a number p which can be defined as the Puiseux number of the corresponding hypergeometric type series. Once p is found, we use the substitution $h(z) = f(z^p)$ to convert the situation to that of Laurent

[8] Originally the type was used to denote the value of m for an m-fold hypergeometric term coefficient.

series, and finally divide the general power of the indeterminate z in the obtained power series representation of $h(z)$ by p to get the expansion sought. This is an intermediate step between the first and the second step above.
The targeted representations have the form

$$f(z) = T(z) + F(z), \tag{61}$$

where $T(z) \in \mathbb{K}[z, \frac{1}{z}, \ln(z)]$ is a Laurent polynomial in the variable z with coefficients in $\mathbb{K}[\ln(z)]$, and $F(z)$ is a linear combination of hypergeometric type series. We mention that $T(z)$ is not uniquely determined but its determination is made more precise by Lemma 8 and Algorithm 2.

4.1. Finding the Puiseux number

For this part we ignore the Laurent polynomial part since it only undergoes through the transformations that need to be done. It is enough to suppose that $f(z) = F(z)$ is the sum of two hypergeometric type series given as

$$F(z) := \sum_{n=0}^{\infty} s_{1_n} z^{(m_1 \cdot n + j_1)/p_1} + \sum_{n=0}^{\infty} s_{2_n} z^{(m_2 \cdot n + j_2)/p_2}, \tag{62}$$

where $m_i, p_i \in \mathbb{N}$, $j_i \in [\![0, m_i - 1]\!]$, s_{i_n} is an m_i-fold hypergeometric term corresponding to $j = j_i$ in the representation (37), $i \in \{1, 2\}$. For simplicity, we also assume that p_1 and p_2 are co-prime. This is to avoid the use of more variables since in particular this assumption implies that the least common multiple of p_1 and p_2 is $\text{lcm}(p_1, p_2) = p_1 \cdot p_2$. Substituting z by $z^{\text{lcm}(p_1,p_2)}$ in (62) gives

$$F(z^{\text{lcm}(p_1,p_2)}) = \sum_{n=0}^{\infty} s_{1_n} z^{(m_1 \cdot n + j_1)\cdot p_2} + \sum_{n=0}^{\infty} s_{2_n} z^{(m_2 \cdot n + j_2)\cdot p_1} \tag{63}$$

$$= \sum_{n \in p_2 \cdot (m_1 \cdot \mathbb{N}_{\geqslant 0} + j_1)} a_{1_{\frac{n}{p_2}}} z^n + \sum_{n \in p_1 \cdot (m_2 \cdot \mathbb{N}_{\geqslant 0} + j_2)} a_{2_{\frac{n}{p_1}}} z^n, \tag{64}$$

where a_{i_n} is obtained from s_{i_n} by the change of variable (48), $i \in \{1, 2\}$.
Observe that in (63) the powers of the indeterminate z are integers. In general, the right-hand side in (62) always gives a representation with integer powers when we substitute z by z^μ, for any positive multiple μ of $\text{lcm}(p_1, p_2)$. Thus determining the positive integers p_1 and p_2 may reduce to finding a positive multiple μ of $\text{lcm}(p_1, p_2)$ so that we can compute the power series of $f(z^\mu)$ and substitute z by $z^{1/\mu}$ in the obtained representation to get the one of $f(z)$.
By the general representation (36) of an m-fold hypergeometric term, we know that there exist rational functions $r_1(n)$ and $r_2(n)$ such that

$$a_{1_{n+m_1}} = r_1(n) \cdot a_{1_n} \text{ and } a_{2_{n+m_2}} = r_2(n) \cdot a_{2_n},$$

for the coefficients in (64). Therefore we can write

$$a_{1_{\frac{n}{p_2}+m_1}} = r_1\left(\frac{n}{p_2}\right) \cdot a_{1_{\frac{n}{p_2}}} \text{ and } a_{2_{\frac{n}{p_1}+m_2}} = r_2\left(\frac{n}{p_1}\right) \cdot a_{2_{\frac{n}{p_1}}}. \tag{65}$$

where $\frac{n}{p_1}$ and $\frac{n}{p_2}$ are not necessarily integers.
To compute the holonomic recurrence equation of smallest order for the m_1-fold and the m_2-fold hypergeometric terms $a_{1_{\frac{n}{k_2}}}$ and $a_{2_{\frac{n}{k_1}}}$, one needs the smallest integer p such that $p \cdot \frac{n}{p_2} \in \mathbb{N}$

and $p \cdot \frac{n}{p_1} \in \mathbb{N}$. Thus $p = \operatorname{lcm}(p_1, p_2)$ and the obtained holonomic RE is of course compatible with the one computed using `FindRE` for the input function $f(z)$. From (65), substituting n by $\operatorname{lcm}(p_1, p_2) \cdot n = p_1 \cdot p_2 \cdot n$ yields

$$a_{1_{p_1 \cdot n+m_1}} = r_1 (p_1 \cdot n) \cdot a_{1_{p_1 \cdot n}} \text{ and } a_{2_{p_2 \cdot n+m_2}} = r_2 (p_2 \cdot n) \cdot a_{2_{p_2 \cdot n}}. \tag{66}$$

Since $a_{1_{p_1 \cdot n+m_1}}$ and $a_{2_{p_2 \cdot n+m_2}}$ are, respectively, m_1-fold and m_2-fold hypergeometric term solutions of a holonomic RE satisfied by the power series coefficients of $f(z)$, by algorithm **mfoldHyper** we know how such terms are computed using an algorithm to compute the hypergeometric terms s_{i_n} such that

$$\frac{s_{i_{n+1}}}{s_{i_n}} = \frac{a_{i_{n+m_i}}}{a_{i_n}} = r_i(p_i \cdot n),\ i \in \{1, 2\}.$$

By Petkovšek's (see Petkovšek (1992)) algorithm we know that ratios of hypergeometric term solutions are built from monic factors of the trailing and leading polynomial coefficients of the recurrence equation. This implies in particular that some zeros and poles of $r_i(p_i \cdot n)$ are the roots of the shifted[9] trailing and leading polynomial coefficients, $i \in \{1, 2\}$. Therefore by computing the least common multiple of all the trailing and leading polynomial coefficient rational root denominators we must obtain a multiple of $\operatorname{lcm}(p_1, p_2)$.
For example, the power series coefficient of $\exp(z^{3/4}) + \sin(\sqrt{z})$ satisfy the recurrence equation

```
(%i1) FindRE(exp(sqrt(z))+exp(-z^(1/3)),z,a[n]);
```

(%o1)
$$-576\,(n+1)\,(n+2)\,(2n+3)\,(3n+4)\,(3n+5)\,a_{n+2} \\ -24\,(n+1)\left(10494n^4+5247n^3-3806n^2-2041n-174\right)a_{n+1} \\ +2\left(39366n^5-177147n^4+321624n^3-210377n^2+41242n+396\right)a_n \\ -a_{n-1}\left(19683n^3-81648n^2+112995n-56710\right)-729a_{n-2}=0$$

Therefore we deduce the Puiseux number $\operatorname{lcm}(1, 2, 3, 3) = 6$. Indeed the factors $(n+1)$ and $(n+2)$ have both root denominators equal to 1, $(2 \cdot n + 3)$ has root denominator equal to 2, and $(3 \cdot n + 4)$ and $(3 \cdot n + 5)$ have both root denominators equal to 3. After substitution the new holonomic RE is free of Puiseux numbers.

```
(%i2) FindRE(exp(z^3)+exp(-z^2),z,a[n]);
```

(%o2)
$$2\,(n-1)\,(n+1)\,a_{n+1}+3\,(n-3)\,n\,a_n+4a_{n-1}\,(n-1)-9a_{n-3}\,(n-3)-6a_{n-2} \\ -12a_{n-4}-18a_{n-5}=0$$

For the rest of this section we assume that the Puiseux number is 1.

4.2. Computing the Laurent polynomial part

Some m-fold hypergeometric terms may not be defined at certain integer values. We show how to extract the part of the series expansion which cannot be deduced from the hypergeometric type part. The first thing to notice is that the Laurent polynomial part $T(z)$ has a sequence of coefficients that satisfies the recurrence equation of $f(z)$ in (61). $T(z)$ is viewed as a series with finitely many non-zero coefficients. The following lemma gives an idea of how to extract $T(z)$ from the series expansion of $f(z)$.

[9] Integer shift used in Petkovšek's algorithm, see also Lemma 8

Lemma 8. *Let $\mathbb{K}$ be a field of characteristic zero, $N, M \in \mathbb{Z}, N \geqslant M$, $T(z) \in \mathbb{K}[z, \frac{1}{z}]$ be a Laurent polynomial of degree N and lowest non-zero monomial degree M. The coefficients of $T(z)$ satisfy the holonomic recurrence equation*

$$P_d a_{n+d} + P_{d-1} a_{n+d-1} + \ldots + P_0 a_n = 0, \tag{67}$$

$d \in \mathbb{N}, P_j \in \mathbb{K}[n], j \in [\![0, d]\!], P_d \cdot P_0 \neq 0$, if N is a root of P_0 and M is a root of $P_d(n-d)$.

Proof. Suppose that the coefficients of $T(z)$ satisfy (67). Since $T(z)$ has finitely many non-zero coefficients we can write

$$T(z) = \sum_{n\in\mathbb{Z}} c_n z^n,$$

where $c_n = 0$ for $n \in \mathbb{Z} \setminus [\![M, N]\!]$. Saying that the coefficients of $T(z)$ satisfy (67) is equivalent to say that the sequence $(c_n)_{n\in\mathbb{Z}}$ is a sequence solution of (67). Given that (67) is valid for all integer indices, observe that by substituting a_n by c_n in (67) for sufficiently large positive or negative integers, all terms on the left-hand side of (67) vanish.
Furthermore, we can make a substitution such that either the trailing or the leading term does not necessarily vanish. Indeed, since $c_n = 0$ for $n \in \mathbb{Z} \setminus [\![M, N]\!]$, substituting a_n by c_n in (67) for $n = N$ yields

$$P_0(N) c_N = 0,$$

and therefore using the assumption $c_N \neq 0$ we deduce that $P_0(N) = 0$. Similarly, substituting a_n by c_n in (67) for $n = M - d$ gives

$$P_d(M-d) c_M = 0,$$

and therefore as $c_M \neq 0$ by assumption, it follows that $P_d(M-d) = 0$. □

Algorithmically, we proceed as follows.

Algorithm 2 Computing $T(z)$ and a possible starting point of $F(z)$ in (61)

Input: An expression f whose series coefficients satisfy the holonomic recurrence equation

$$P_d a_{n+d} + P_{d-1} a_{n+d-1} + \ldots + P_0 a_n = 0, \tag{68}$$

$d \in \mathbb{N}, P_j \in \mathbb{K}[n], j \in [\![0, d]\!], P_d \cdot P_0 \neq 0$,

Output: $T(z)$ and a starting point N_0 for $F(z)$ for the representation (61) of f.

1. Compute the minimum integer roots M of $P_d(n-d)$ and the maximum integer root N of $P_0(n)$.
2. If N does not exist then set $T(z) := 0$ and set $N_0 := M$.
3. If N does exist then set $T(z) := Taylor(f(z), z, 0, N)$[10]and set $N_0 := N + 1$.
4. Return $[T(z), N_0]$.

[8]Truncated series of order N at 0.

Remark 9. *Note that generally when $T(z) = 0$ and $F(z)$ starts at 0, $N = M = 0$ and 0 is not necessarily a root of the trailing polynomial coefficient. For example*

```
(%i1) FindRE(exp(z),z,a[n]);
```

$$(\%o1) \qquad (1+n)\cdot a_{n+1} - a_n = 0$$

whose trailing polynomial coefficient has no root. In this case $T(z) = 0$ or does not exist. Here $M = 0$ is a root of the leading polynomial coefficient (shifted by -1) which represents the starting point of the series expansion of exp(z)*. Note that* `FindRE` *does not cancel the common factors after rewriting DEs into REs and this is essential for computing the Laurent polynomial part. These factors contain necessary information to determine the first non-zero coefficient of the series expansion sought. More precision for finding the starting point N_0 is given in Remark 12.*

Similar results were discussed in (Abramov et al. (2000)) by extending the finding of formal series solutions of holonomic DEs to the skew Laurent-Polynomial ring in the case of rational coefficients.
Note, however, that the returned $T(z)$ is generally a polynomial from which the exact Laurent polynomial part can be subtracted. Our Maxima package has the code `LPolyPart(f,z)` that implements Algorithm 2.

```
(%i1) LPolyPart(asech(z),z);
```

$$(\%o1) \qquad [\log(2) - \log(z), 1]$$

```
(%i2) LPolyPart(sin(z)/z^5,z);
```

$$(\%o2) \qquad [0, -5]$$

```
(%i3) LPolyPart(cos(4*acos(z)),z);
```

$$(\%o3) \qquad [8\cdot z^4 - 8\cdot z^2 + 1, 5]$$

In the latter example the linear combination of the corresponding hypergeometric type series yields 0 so that one finally gets the known Chebychev polynomial $\cos(4\arccos(z)) = 8\cdot z^4 - 8\cdot z^2 + 1$.

4.3. Finding the linear combination

Since we have an idea of how to determine $T(z)$ and the starting point N_0, we can now explain how to find $F(z)$ in the representation (61). The importance of this part lies in constructing a consistent linear system for determining the coefficients of the linear combination sought. This step is all the more important than the previous one because the fact that the RE is not irreducible in most cases makes its order inappropriate for the initial values. We construct that system by using the integers m of m-fold hypergeometric term solutions and the degree of the Laurent polynomial part.
Let

$$\begin{aligned} \mathcal{H} &:= \Big[[1,\{h_{n,1},\ldots,h_{n,l_1}\}], \big[m_1, \{h_{m_1n,1},\ldots,h_{m_1n,l_{m_1}}\}\big],\ldots,\big[m_\mu,\{h_{m_\mu n,1},\ldots,h_{m_\mu n,l_{m_\mu}}\}\big]\Big] \\ &= \Big[[1,S_{1,0}],[m_1,S_{m_1,0}],\ldots,\big[m_\mu,S_{m_\mu,0}\big]\Big] \end{aligned} \qquad (69)$$

for integers $1 < m_1 < \cdots < m_\mu$ be the non-empty basis of m-fold hypergeometric term solutions of a holonomic RE satisfied by the series coefficients of $f(z)$. The constant l_m is the number of m-fold hypergeometric terms in $\mathcal{H}$ $m \in \{1, m_1, \ldots, m_\mu\}$. Our approach to find the representation (61) for the power series of $f(z)$ goes as follows.

Algorithm 3 Computing hypergeometric type series

Input: $f(z)$, the recurrence equation, say *RE* computed by `FindRE`, the incomplete basis of m-fold hypergeometric term solutions of *RE*, say $\mathcal{H}$, computed by **mfoldHyper**, $T(z)$ and N_0 computed by Algorithm 2.

Output: The representation (61) of $f(z)$.

1. Find the other m-fold symmetric terms associated to each m-fold hypergeometric term in $\mathcal{H}$ for $m \in \{m_1, \ldots, m_\mu\}$. For that purpose one calls our implementation `mfoldHyper(RE,a[n],m,j)` of Algorithm 1 for $j = 1, \ldots, m-1$, $m \in \{m_1, \ldots, m_\mu\}$. This allows to build the sets

$$S_m := \{S_{m,0}, S_{m,1} \ldots, S_{m,m-1}\}, \tag{70}$$

for $m \in \{1, m_1, \ldots, m_\mu\}$, where

$$S_{m,j} := \left\{h_{mn+j,1}, h_{mn+j,2}, \ldots, h_{mn+j,l_m}\right\},\ 0 \leqslant j \leqslant m-1. \tag{71}$$

2. Compute $i_{m,j} = \left\lceil \frac{N_0 - j}{m} \right\rceil$ for $j = 0, \ldots, m-1$, $m \in \{m_1, \ldots, m_\mu\}$.
3. Set

$$\mathcal{N} = N_0 + \left(\sum_{m \in \{1, m_1, \ldots, m_\mu\}} l_m - 1 \right) \cdot \operatorname{lcm}(1, m_1, \ldots, m_\mu) + m_\mu - 1 \tag{72}$$

4. Compute $\gamma_{m,j} = \left\lfloor \frac{\mathcal{N} - j}{m} \right\rfloor$, $j = 0, \ldots, m-1$, $m \in \{m_1, \ldots, m_\mu\}$.
5. Let $\alpha_{m,j,k} \in \mathbb{K}$, $m \in \{1, m_1, \ldots, m_\mu\}$, $j = 0, \ldots, m-1$, $k = 1, \ldots, l_m$ be some unknown constants and define

$$I(z) := \sum_{m \in \{1, m_1, \ldots, m_\mu\}} \sum_{j=0}^{m-1} \sum_{k=1}^{l_m} \alpha_{m,j,k} \sum_{n=i_{m,j}}^{\gamma_{m,j}} h_{mn+j,k} z^{mn+j}. \tag{73}$$

6. Solve the linear system resulting from the equation

$$I(z) + T(z) - Taylor(f(z), z, 0, \mathcal{N}) = 0, \tag{74}$$

for the unknown $\left(\alpha_{m,j,k}\right)^T_{m \in \{1, m_1, \ldots, m_\mu\},\ 0 \leqslant j \leqslant m-1,\ 1 \leqslant k \leqslant l_m} \in \mathbb{K}^{\sum_{m \in \{1, m_1, \ldots, m_\mu\}} l_m \cdot m}$.

7. If there is no solution then stop and return FALSE. No linear combination exists in this case.

Algorithm 3 Computing hypergeometric type series

8. If there is a solution then set all parameters of dependency to 0 (if there are some). This gives the choice of the linear combination. We denote by $\alpha'_{m,j,k}$ the resulting value found for $\alpha_{m,j,k}$, $m \in \{1, m_1, \ldots, m_\mu\}$, $j = 0, \ldots, m-1$, $k = 1, \ldots, l_m$.
9. For each $S_m, m \in \{1, m_1, \ldots, m_\mu\}$ construct the term

$$S'_m := \sum_{S_{m,j} \in S_m} \left(\sum_{h_{mn+j,k} \in S_{m,j}} \alpha'_{m,j,k} h_{mn+j,k} \right) z^{mn+j-i_{m,j}} \tag{75}$$

$$:= \sum_{j=0}^{m-1} \left(\sum_{k=1}^{l_m} \alpha'_{m,j,k} h_{mn+j,k} \right) z^{mn+j-i_{m,j}} \tag{76}$$

10. Return $T(z) + \sum_{m \in \{1, m_1, \ldots, m_\mu\}} \sum_{n=0}^{\infty} S'_m$.

The correctness of this algorithm depends on whether the solution of the linear system in step 6 has enough equations to determine the possible coefficients of the linear combination sought. This is established by the following lemma.

Lemma 10. *In Algorithm* 3, $\mathcal{N}$ *given in* (72) *is a valid integer for which the series expansion of order* $\mathcal{N}$ *of* $f(z)$ *suffices for determining the linear combination sought.*

Proof. The computation is similar for any integer N_0, therefore we assume that $N_0 = 0$. The number of unknowns in each equation is $q = \sum_{m \in \{1, m_1, \ldots, m_\mu\}} l_m$. The aim is to find $\mathcal{N}$ such that $Taylor(f(z), z, 0, \mathcal{N})$ in Algorithm 3 step 6 yields a linear system with at least q equations with q unknowns each. Of course, the minimal value of $\mathcal{N}$ is an integer that verifies

$$\mathcal{N} = m_1 \cdot x_1 = m_2 \cdot x_2 = \cdots = m_\mu \cdot x_\mu,$$

for some positive integers $x_1, x_2, \ldots, x_\mu$, since we have to find q equations that correspond to the q first coincidences of

$$z^{m_1 \cdot n},\ z^{m_2 \cdot n},\ \ldots, z^{m_\mu \cdot n}.$$

The second coincidence is reached at the expansion of order $\mathrm{lcm}(m_1, \ldots, m_\mu)$, therefore by induction we deduce that for any positive integer p, the p^{th} coincidence is reached at the expansion of order $(p-1) \cdot \mathrm{lcm}(m_1, \ldots, m_\mu)$. Hence we finally get

$$\mathcal{N} = (q-1) \cdot \mathrm{lcm}(m_1, \ldots, m_\mu) = \left(\sum_{m \in \{1, m_1, \ldots, m_\mu\}} l_m - 1 \right) \cdot \mathrm{lcm}(m_1, \ldots, \cdot m_\mu) + m_\mu - 1$$

where $m_\mu - 1$ is added to get similar coincidences with some

$$z^{m_1 \cdot n + j},\ z^{m_2 \cdot n + j},\ \ldots, z^{m_\mu + j \cdot n},$$

with $j \in [\![1, m_1]\!]$. □

Remark 11. *In* (72) *we use* $\mathrm{lcm}(1, m_1, \ldots, m_\mu)$ *because it allows to recover the order* $(l_1 - 1) \cdot 1$ *when there are only hypergeometric terms* ($\mu = 0$)*.*

Remark 12. *The value of the starting point N_0 is made more precise after taking into account the maximum integer among the zeros and poles of the primitive m-fold hypergeometric terms (see Petkovšek and Salvy (1993)). This is a prior step for Algorithm 3 to work as expected. For example, for $f(z) = z^{10}\exp(z) + z$, Algorithm 2 finds the Laurent polynomial part $T(z) = z$, and **mfoldHyper** finds the hypergeometric term*

$$h_n := \left\{\frac{(n-9)\,(n-8)\,(n-7)\,(n-6)\,(n-5)\,(n-4)\,(n-3)\,(n-2)\,(n-1)\,n}{n\,!}\right\}. \tag{77}$$

*However, to find the value of $c \in \mathbb{Q}$ such that $f(z) = T(z) + c\sum_{n=0}^{\infty} h_n z^n$, evaluations should start at $n = 10$ since $h_n = 0$ for $n = 2, 3, \ldots, 9$. Thus, from the m-fold hypergeometric terms computed by **mfoldHyper**, one can find the needed N_0 and adapt $T(z)$ accordingly. Finally, a reduced hypergeometric type representation is obtained after subtracting the terms of $T(z)$ that came from the hypergeometric type part. This is how our algorithm handles the following examples after shifting the indices and simplifying the coefficients.*

```
(%i1) FPS(z+z^10*exp(z),z,n)
```

$$(\%o1)\quad \left(\sum_{n=0}^{\infty} \frac{z^{n+10}}{n\,!}\right) + z,$$

```
(%i2) FPS(z+z^5*exp(z)+z^10*cos(z),z,n)
```

$$(\%o2)\quad \left(\sum_{n=0}^{\infty} \frac{(-1)^n\, z^{2(n+5)}}{(2n)\,!}\right) + \left(\sum_{n=0}^{\infty} \frac{z^{n+5}}{n\,!}\right) + z.$$

Theorem 13. *The power series representation of every hypergeometric type function whose n^{th} term series coefficients are m-fold hypergeometric term solutions of a holonomic recurrence equation (as computed using `FindRE`[11]) that they satisfy is detected by Algorithm 3.*

Proof. Due to Theorem 7, Algorithms 1, 2, 3 and Lemmas 8, 10, provided the recurrence equation computed using a similar approach to `FindRE`. □

One should remark that the class of holonomic functions considered in Theorem 13 covers all those mentioned in (Abramov et al. (2000)).
As already used many times, the command `FPS(f(z),z,n,[z0])` of our Maxima package computes the power series representation of `f(z)` at the point of expansion `z0`$\in \mathbb{C}$ (if given or 0 otherwise) with the index variable `n` by combining `FindRE`, `mfoldHyper`; implementations of Algorithms 2 and 3, and the one to compute Puiseux numbers. All this leads to an algorithm to compute hypergeometric type series from holonomic functions whenever possible.

```
(%i1) FPS(exp(sqrt(z))+exp(-z^(1/3)),z,n);
```

$$(\%o1)\quad \left(\sum_{n=0}^{\infty} \frac{z^{\frac{n}{2}}}{n\,!}\right) + \sum_{n=0}^{\infty} \frac{(-1)^n\, z^{\frac{n}{3}}}{n\,!}$$

```
(%i2) FPS(z*cos(z^(3/2)) + asin(z^(1/3))^2,z,n);
```

$$(\%o2)\quad \left(\sum_{n=0}^{\infty} \frac{(-1)^n\, z^{3n+1}}{(2n)\,!}\right) + \sum_{n=0}^{\infty} \frac{2\, 4^n\, (n+1)\,!^2\, z^{\frac{2(n+1)}{3}}}{(n+1)^2\, (2\,(n+1))\,!}$$

[11]This is to make sure that cancellation of common factors is avoided.

```
(%i3) FPS(log(1+sqrt(z)+z+z^(3/2)),z,n);
```

$$(\%o3)\quad \left(\sum_{n=0}^{\infty} \frac{(-1)^n z^{n+1}}{n+1}\right) + \sum_{n=0}^{\infty} \frac{(-1)^n z^{\frac{n+1}{2}}}{n+1}$$

Some examples with non-zero Laurent polynomial parts (see Subsection 4.2).

```
(%i4) FPS(1+z+z^2+z^3*atan(z),z,n);
```

$$(\%o4)\quad \left(\sum_{n=0}^{\infty} \frac{(-1)^n \cdot z^{2\cdot n}}{2\cdot n - 3}\right) + z + \frac{4}{3}$$

```
(%i5) FPS(sin(z)^2+cos(z)^3,z,n);
```

$$(\%o5)\quad \left(\sum_{n=0}^{\infty} -\frac{(-(-9)^n - 3\cdot(-1)^n + 2\cdot(-1)^n\cdot 4^n)\cdot z^{2\cdot n}}{4\cdot(2\cdot n)\,!}\right) + \frac{1}{2}$$

```
(%i6) FPS(1+2*z+3*z^2+5*z^3+exp(asinh(z)),z,n);
```

$$(\%o6)\quad \left(\sum_{n=0}^{\infty} -\frac{(-1)^n\ (2n)\,!\, z^{2n}}{(2n-1)\ 4^n\ n\,!^2}\right) + 5z^3 + 3z^2 + 3z + 1$$

Let us use other points of expansion.

```
(%i7) FPS(sin(2*z)+cos(z),z,n,%pi/2);
```

$$(\%o7)\quad -\sum_{n=0}^{\infty} \frac{(-1)^n\cdot(1+2\cdot 4^n)\cdot\left(z-\frac{\pi}{2}\right)^{1+2\cdot n}}{(2\cdot n+1)\cdot(2\cdot n)\,!}$$

Definition 2 extends to asymptotic expansion at infinity.

```
(%i8) FPS(atan(z)+exp(-z),z,n,inf);
```

$$(\%o8)\quad \frac{\pi}{2} - \sum_{n=0}^{\infty} \frac{(-1)^n z^{-2n-1}}{2n+1}$$

The current Maple `convert/FormalPowerSeries` misses the initial term $\pi/2$ for the latter example. The computations easily extend over $\mathbb{K}[\log(z)]$ as shown below.

```
(%i9) FPS(log(z)*exp(z),z,n);
```

$$(\%o9)\quad \log(z) \sum_{n=0}^{\infty} \frac{z^n}{n\,!}$$

Our Maple implementation will replace the `FormalPowerSeries` package in future releases of Maple. The link to download our Maple and Maxima packages is `http://www.mathematik.uni-kassel.de/~bteguia/FPS_webpage/FPS.htm`. Note, furthermore, that our `FPS` package also contains implementation for dealing with non-holonomic power series like the series of $\tan(z)$, $z/(\exp(z)-1)$, etc. A demonstration of the latter result will be presented at the Maple conference 2021.

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