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Research paper

Inner bounds for the extreme zeros of ${}_3F_2$ hypergeometric polynomials

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Zeilberger's celebrated algorithm finds pure recurrence relations (w. r. t. a single variable) for hypergeometric sums automatically. However, in the theory of orthogonal polynomials and special functions, contiguous relations w. r. t. several variables exist in abundance. We modify Zeilberger's algorithm to generate unknown contiguous relations that are necessary to obtain inner bounds for the extreme zeros of orthogonal polynomial sequences with ${}_3F_2$ hypergeometric representations. Using this method, we improve previously obtained upper bounds for the smallest and lower bounds for the largest zeros of the Hahn polynomials and we identify inner bounds for the extreme zeros of the Continuous Hahn and Continuous Dual Hahn polynomials. Numerical examples are provided to illustrate the quality of the new bounds.

Without the use of computer algebra such results are not accessible. We expect our algorithm to be useful to compute useful and new contiguous relations for other hypergeometric functions.

Keywords: Orthogonal polynomials, extreme zeros, bounds for zeros, recurrence relations

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1. Introduction

A sequence of real polynomials $\{p_n\}_{n=0}^{\infty}$, where p_n is of exact degree n, is orthogonal with respect to a positive measure $\mu(x) > 0$ on an interval (a, b), if the scalar product

$$\langle p_m, p_n \rangle = \int_a^b p_m(x) p_n(x) d\mu(x) = 0, \quad m \neq n.$$

If $\mu(x)$ is absolutely continuous, then it can be represented by a real weight function $w(x) > 0$ so that $d\mu(x) = w(x)\, dx$. If $\mu(x)$ is discrete with support in $\mathbb{N}_{\geq 0}$, then it can be represented by a discrete weight $w(x) \geqq 0$ ($x \in \mathbb{N}_{\geqq 0}$) and the scalar product is given by

$$\langle p_m, p_n \rangle = \sum_{x=0}^{\infty} p_m(x)\, p_n(x)\, w(x)\ .$$

The aim of this paper is to find an upper bound for the smallest and a lower bound for the largest zero of the following families of orthogonal polynomials [1]:

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- Hahn polynomials: discrete weight $w(x) = \binom{\alpha+x}{x}\binom{\beta+N-x}{N-x}$ in $\{0, 1, \ldots, N\}$ when $\alpha, \beta > -1$ and

$$Q_n(x; \alpha, \beta, N) = {}_3F_2\left(\begin{matrix} -n, n+\alpha+\beta+1, -x \\ \alpha+1, -N \end{matrix}\middle| 1\right).$$

- Continuous Hahn polynomials: continuous weight $w(x) = \Gamma(a+ix)\Gamma(b+ix)\,\Gamma(c-ix)\Gamma(d-ix)$ in the interval $(-\infty, \infty)$ when the real parts of a, b, c and d are positive and $c = \bar{a}$, $d = \bar{b}$ and

$$p_n(x; a, b, c, d) = i^n \frac{(a+c)_n(a+d)_n}{n!} {}_3F_2\left(\begin{matrix} -n, n+a+c+b+d-1, a+ix \\ a+c, a+d \end{matrix}\middle| 1\right).$$

- Continuous Dual Hahn polynomials: continuous weight

$$w(x) = \left|\frac{\Gamma(a+ix)\Gamma(b+ix)\Gamma(c+ix)}{\Gamma(2ix)}\right|^2$$

  in the interval $(0, \infty)$, where a, b, c are positive, except possibly for a pair of complex conjugates with positive real parts and

$$\tilde{S}_n(x^2; a, b, c) = {}_3F_2\left(\begin{matrix} -n, a+ix, a-ix \\ a+b, a+c \end{matrix}\middle| 1\right).$$

Here,

$$\Gamma(z) = \int_0^\infty t^{z-1}e^{-t}dt$$

denotes the Gamma function,

$${}_pF_q\left(\begin{matrix}\alpha_1, \alpha_2, \ldots, \alpha_p \\ \beta_1, \beta_2, \ldots, \beta_q\end{matrix}\middle| x\right) = \sum_{k=0}^{\infty} \frac{(\alpha_1)_k \cdot (\alpha_2)_k \cdots (\alpha_p)_k}{(\beta_1)_k \cdot (\beta_2)_k \cdots (\beta_q)_k} \frac{x^k}{k!}$$

denotes the hypergeometric series and $(a)_k = a(a+1)\cdots(a+k-1)$ denotes the shifted factorial (Pochhammer symbol), as usual.

If $\{p_n\}_{n=0}^{\infty}$ is a sequence of monic orthogonal polynomials with zeros $x_{n,1} < x_{n,2} < \cdots < x_{n,n}$, then it satisfies a three-term recurrence relation

$$p_n(x) = (x - B_n)p_{n-1}(x) - C_n p_{n-2}(x), \tag{1}$$

where B_n and C_n do not depend on x, $p_{-1} \equiv 0$, $p_0(x) = 1$ and $C_n > 0$. Furthermore, each open interval with endpoints at successive zeros of p_n contains exactly one zero of p_{n-1}. Stieltjes [2, Theorem 3.3.3] extended this interlacing property by proving that if $m < n-1$, provided p_m and p_n are co-prime (i.e., they do not have any common zeros), there exist m open intervals with endpoints at successive zeros of p_n, each of which contains exactly one zero of p_m. Beardon [3, Theorem 5] provided additional insight into the Stieltjes interlacing process by proving that for every $m < n-1$, if p_m and p_n are co-prime, there exists a real polynomial S_{n-m} of degree $n-m-1$ in x, whose real simple zeros, together with those of p_m, interlace with the zeros of p_n.

This phenomenon will be called *completed Stieltjes interlacing,* of which a direct consequence is that the zeros of the polynomial S_{n-m} act as *inner* bounds for the extreme zeros (i.e., upper (lower) bounds for the smallest (largest) zeros) of the polynomial p_n, e.g., when $m = n - 2$, it follows directly from (1) that $x_{n,1} < B_n < x_{n,n}$.

Iterating (1) leads to

$$\begin{aligned} p_n(x) &= x^n - \left(\sum_{i=1}^{n} B_i\right) x^{n-1} + \dots \\ &= \prod_{i=1}^{n}(x - x_{n,i}) \\ &= x^n - \left(\sum_{i=1}^{n} x_{n,i}\right) x^{n-1} + \dots, \end{aligned}$$

from which another (trivial) inner bound for the extreme zeros of p_n can be deduced:

$$x_{n,1} < \frac{1}{n}\sum_{i=1}^{n} x_{n,i} = \frac{1}{n}\sum_{i=1}^{n} B_i < x_{n,n}. \tag{2}$$

In order to find more precise inner bounds for the extreme zeros of a polynomial in an orthogonal sequence, the study of completed Stieltjes interlacing of zeros of *different* orthogonal sequences, where the different sequences are obtained by integer shifts of the *parameters* of the appropriate polynomials, can be helpful. This was done for the Gegenbauer, Laguerre and Jacobi polynomials in [4], the Meixner and Krawtchouk polynomials in [5] and the Pseudo-Jacobi polynomials in [6]. Mixed three-term recurrence relations satisfied by the polynomials under consideration and obtained from the connection between the appropriate polynomials, their hypergeometric representations, as well as contiguous function relations satisfied by these polynomials, are used to obtain these bounds and a Maple computer package [7] for computing contiguous relations of exclusively ${}_2F_1$ series is helpful in this regard.

Different Maple routines, however, are necessary to obtain similar identities for polynomial sequences that lie on the ${}_3F_2$ and ${}_4F_3$ planes of the Askey scheme. Zeilberger [8] developed a powerful method for proving identities for hypergeometric series and we apply Zeilberger's algorithm (command `sumrecursion` of the Maple package `hsum.mpl` accompanying [9]) to generate the three-term recurrences of the ${}_3F_2$ hypergeometric families under consideration. Variations of Zeilberger's algorithm were given by [10, 11]. Some of the recurrences obtained are very complicated and for this reason we do not consider the ${}_4F_3$ families (Wilson and Racah polynomials) in this paper. However, in principal the method can be extended to the ${}_4F_3$ case which will be considered in a forthcoming paper. For a recent study on the properties of the zeros of these polynomials, we refer the reader to [12].

Zeilberger's algorithm is much more flexible as was shown by examples in [9]. Similarly as the commands `sumdiffeq` [9, Session 10.5] and `sumdiffrule` [9, Session 10.7] are variations of `sumrecursion`, by changing the setting in the computation, one can write Maple routines to compute the identities necessary to obtain our results. We also refer the reader to [13] where an application of Zeilberger's algorithm is used to find three-term recurrence equations for Hahn polynomials and other discrete orthogonal families.

In this paper, we provide inner bounds for the extreme zeros of different sequences of the Hahn, continuous Hahn and continuous dual Hahn polynomials. In the Hahn case, we compare the quality of our newly found bounds with results obtained in [14]. An intensive study on the location of the zeros of the Hahn polynomials is made in [15] and lower (upper) bounds for the smallest (largest) zeros of Hahn polynomials are provided in [14, 16]. The monotonicity of the zeros of Hahn polynomials with respect to α and β, as determined by Markov's monotonicity theorem (cf. [17]), is useful and is, together with the monotonicity of the zeros with respect to N (cf. [15]), clearly illustrated in our tables. No previous results on inner bounds for the extreme zeros of the Continuous Hahn and Continuous Dual Hahn polynomials are known. The weight function of the Continuous Dual Hahn polynomials is even and monotonicity properties of all the zeros of orthogonal polynomials associated with an even weight function is discussed in [18]. Recently in [19], upper and lower bounds for the zeros of Gram polynomials are provided in terms of the zeros of Legendre polynomials.

The following result provides the conditions necessary for the mixed three-term recurrence relations to hold and will be used to prove our results.

THEOREM 1.1 *[5] Let $\{p_n\}_{n=0}^{\infty}$ be a sequence of polynomials orthogonal on the (finite or infinite) interval (a,b) with respect to $d\mu(x)>0$. Let $k\in\mathbb{N}_{\geq 0}$ be fixed and suppose $\{g_{n,k}\}_{n=0}^{\infty}$ is a sequence of polynomials orthogonal with respect to $\sigma_k(x)d\mu(x)>0$ on (a,b), where $\sigma_k(x)$ is a polynomial of degree k, that satisfies*

$$(x-B_n)p_{n-1}(x)=a_{k-2}(x)p_n(x)+A_n\sigma_k(x)g_{n-2,k}(x),\ n\in\mathbb{N}_{\geqq 1}, \tag{3}$$

with $g_{-1,k}=0$, A_n, B_n, a_{-1}, a_{-2} constants and a_{k-2} a polynomial of degree $k-2$ defined on (a,b) whenever $k\in\{2,3,\dots\}$. Then

(i) $k\in\{0,1,2,3,4\}$*;*
(ii) the $n-1$ real, simple zeros of $(x-B_n)g_{n-2,k}$ interlace with the zeros of p_n and B_n is an upper bound for the smallest, as well as a lower bound for the largest zero of p_n if $g_{n-2,k}$ and p_n are co-prime;
(iii) if $g_{n-2,k}$ and p_n are not co-prime,
  (a) they have one common zero that is equal to B_n and this common zero cannot be the largest or smallest zero of p_n;
  (b) the $n-2$ zeros of $g_{n-2,k}(x)$ interlace with the $n-1$ non-common zeros of p_n;
  (c) B_n is an upper bound for the smallest as well as a lower bound for the largest zero of p_n. ■

All relevant contiguous relations that we need in the next sections were computed automatically using the Maple package `hsum.mpl` accompanying [9] and procedures that are specifically adapted for each family using the corresponding hypergeometric representation. These computations with their complete computations and results, that build the heart of our approach, can be downloaded from `http://www.mathematik.uni-kassel.de/~koepf/Publikationen`. In the given article, however, we have only included the necessary bounds deduced from the much more complicated contiguous relations since the full relations do not contribute to our results. The computations of the zeros in our tables can also be found in the above-mentioned Maple file.

2. Inner bounds for the extreme zeros of Hahn polynomials

We refer the reader to [20] for more information on discrete orthogonal polynomials and the difference equations they satisfy. For $\alpha,\beta>-1$, the parameter shifted Hahn polynomials $Q_n(x;\alpha+k,\beta+m,N)$ are orthogonal at the points $x\in\{0,1,2,\dots,N\}$ with respect to

$$\binom{\alpha+k+x}{x}\binom{\beta+m+N-x}{N-x} = \frac{(x+\alpha+1)_k(-x+\beta+N+1)_m}{(\alpha+1)_k(\beta+1)_m}w(x) = \sigma_{k,m}(x)w(x) > 0$$

and together with $Q_n(x;\alpha,\beta,N)$, they satisfy the mixed three-term recurrence relations

$$\begin{aligned}&(x - B_n^{\alpha+k,\beta+m})Q_{n-1}(x;\alpha,\beta,N)\\ &\quad = a_{k+m-2}(x)Q_n(x;\alpha,\beta,N) + A_n\sigma_{k,m}(x)Q_{n-2}(x;\alpha+k,\beta+m,N),\end{aligned} \tag{4}$$

where a_0, a_1, A_n and $B_n^{\alpha+k,\beta+m}$ are constants and $a_{k+m-2}(x)$ is a polynomial of degree $k+m-2$ for $k+m \in \{2,3,\dots\}$.

From Theorem 1.1 it follows that the mixed three-term recurrence relations (4) only exist if $k+m \in \{0,1,2,3,4\}$ and each of the points $B_n^{\alpha+k,\beta+m}, k+m \in \{0,1,2,3,4\}$, is an upper bound for the smallest as well as a lower bound for the largest zero of $Q_n(x;\alpha,\beta,N)$. Moreover, the relations that involve the largest possible parameter difference, are found to be particularly useful to obtain sharp bounds.

When $k=2$ and $m=0$, we obtain the equation

$$\begin{aligned}&\frac{\left(N\alpha+\alpha^2+\alpha\beta+\alpha n+\beta n+n^2+N+2\alpha-n+2\right)(N-n+1)}{(n+1+N+\alpha+\beta)(-\alpha-\beta-2n)}Q_n(x;\alpha,\beta,N)\\ &= \left(x - B_n^{\alpha+2,\beta}\right)Q_{n-1}(x;\alpha,\beta,N) + \frac{(n-1)(n+\beta-1)(x+1+\alpha)_2}{(\alpha+1)_2(n+1+N+\alpha+\beta)}Q_{n-2}(x;\alpha+2,\beta,N),\end{aligned} \tag{5}$$

where

$$B_n^{\alpha+2,\beta} = \frac{N\alpha-\alpha n+\beta n+n^2+N+\alpha-\beta-3n+2}{\alpha+\beta+2n}. \tag{6}$$

The weight function w satisfies the symmetry property $w(\alpha,\beta,x) = w(\beta,\alpha,N-x)$ from which the symmetry relation [21]

$$(\alpha+1)_nQ_n(N-x;\alpha,\beta,N) = (-1)^n(\beta+1)_nQ_n(x;\beta,\alpha,N)$$

can be proved and we can deduce that if x is a zero of $Q_n(x;\alpha,\beta,N)$, then $N-x$ will be a zero of $Q_n(x;\beta,\alpha,N)$. Likewise, the extra interlacing points obtained from the mixed three-term recurrence relations satisfied by $Q_n(x;\alpha,\beta,N), Q_{n-1}(x;\alpha,\beta,N)$ and $Q_{n-2}(x;\alpha+m,\beta+k,N)$, are

$$B_n^{\alpha+m,\beta+k} = N - B_n^{\beta+k,\alpha+m} \tag{7}$$

for all values of k and m in $\mathbb{N}_{\geqq 0}$ such that $k+m \in \{0,1,2,3,4\}$. Thus, from (6) and (7) we obtain the bound

$$B_n^{\alpha,\beta+2} = \frac{N\alpha-\alpha n+\beta n-n^2+2Nn-N+\alpha-\beta+3n-2}{\alpha+\beta+2n}. \tag{8}$$

We note that

$$B_n^{\alpha+2,\beta} - B_n^{\alpha,\beta+2} = -\frac{2(n-1)(N-n+2)}{\alpha+\beta+2n} < 0$$

for $\alpha, \beta > -1$ and $n \in \{1, 2, \ldots, N\}$. Moreover, from [17, Theorem 7.1.1] it follows that the zeros of the Hahn polynomials increase with α and decrease with β and it is clear that

$$0 < x_{n,1} < B_n^{\alpha+2,\beta} < B_n^{\alpha,\beta+2} < x_{n,n} < N,$$

for all values of $\alpha, \beta > -1$ and $n \in \{1, 2, \ldots, N\}$. However, the point $B_n^{\alpha+4,\beta}$ ($B_n^{\alpha,\beta+4}$) will be a more precise upper (lower) bound for the smallest (largest) zero of $Q_n(x; \alpha, \beta, N)$ obtained by this method. Because of the complexity of the relations from which the latter two bounds can be found, we do not include them in this paper, but they can be downloaded from `http://www.mathematik.uni-kassel.de/~koepf/Publikationen`. In order to show the accuracy of these bounds, we include them in our examples:

$$\begin{aligned}
B_n^{\alpha+4,\beta} = \Big(& n^6 + 3(\alpha+\beta+1)\,n^5 + (N\alpha + 3\alpha^2 + 8\alpha\beta + 3\beta^2 + N + 4\alpha + 8\beta - 1)\,n^4 \\
&+ (\alpha+\beta+1)\,(2N\alpha + \alpha^2 + 6\alpha\beta + \beta^2 + 2N - 2\alpha + 6\beta - 7)\,n^3 \\
&+ (N^2\alpha^2 + N\alpha^3 + 4N\alpha^2\beta + N\alpha\beta^2 - \alpha^4 + 2\alpha^3\beta + 5\alpha^2\beta^2 + 2\alpha\beta^3 \\
&+ 3N^2\alpha + N\alpha^2 + 9N\alpha\beta + N\beta^2 - 12\alpha^3 - \alpha^2\beta + 7\alpha\beta^2 + 2\beta^3 \\
&+ 2N^2 - 3N\alpha + 5N\beta - 42\alpha^2 - 15\alpha\beta - \beta^2 - 3N - 56\alpha - 12\beta - 24)\,n^2 \\
&+ (\alpha+\beta+1)(N^2\alpha^2 + 2N\alpha^2\beta - \alpha^4 + \alpha^2\beta^2 + 3N^2\alpha - 2N\alpha^2 + 5N\alpha\beta \\
&- 10\alpha^3 - 3\alpha^2\beta + \alpha\beta^2 + 2N^2 - 6N\alpha + 3N\beta - 34\alpha^2 - 9\alpha\beta - \beta^2 - 4N \\
&- 46\alpha - 6\beta - 20)\,n + (\alpha+1)_2(N\alpha + 3N + \alpha - \beta + 4)\,(\alpha + 2 + N + \beta)_2\Big) \Big/ \\
&\Big((2N + 3\alpha + \beta + 8)\,n^4 + 2\,(\alpha+\beta+1)\,(2N + 3\alpha + \beta + 8)\,n^3 \\
&+ (2N^2\alpha + 8N\alpha^2 + 8N\alpha\beta + 2N\beta^2 + 7\alpha^3 + 13\alpha^2\beta + 7\alpha\beta^2 + \beta^3 + 4N^2 \\
&+ 30N\alpha + 10N\beta + 41\alpha^2 + 48\alpha\beta + 13\beta^2 + 30N + 79\alpha + 39\beta + 52)\,n^2 \\
&+ (\alpha+\beta+1)(2N^2\alpha + 6N\alpha^2 + 4N\alpha\beta + 4\alpha^3 + 6\alpha^2\beta + 2\alpha\beta^2 + 4N^2 \\
&+ 26N\alpha + 6N\beta + 27\alpha^2 + 24\alpha\beta + 3\beta^2 + 28N + 60\alpha + 22\beta + 44)\,n \\
&+ (\alpha+1)_2\,(\alpha+\beta+2)\,(\alpha + 2 + N + \beta)_2\Big) \qquad (9)
\end{aligned}$$

and $B_n^{\alpha,\beta+4}$ can be obtained from (7).

In [14, Lemma 9], the following inner bounds for the extreme zeros of the Hahn polynomials are provided for $\alpha \geqq \beta > -1$ or $\alpha \leqq \beta \leqq -N-1$:

$$x_{n,1} < \frac{(n+\alpha)(N-n+1)}{\alpha+\beta+1} \qquad (10)$$

$$< \frac{N(\alpha+n) + (\beta+n)(n-1)}{\alpha+\beta+2n} < x_{n,n}. \qquad (11)$$

In Tables 1 to 4 we compare these bounds, together with the bounds obtained from relations (6) and (7), and the bound in (9), to the actual values of the extreme zeros. In each case, the more precise bound is printed in bold. The precise zeros were computed using *Maple* with up to 100 digits numerical precision.

Remark 2.1 By using appropriate limiting processes, inner bounds for the extreme zeros of other classical polynomial systems can be obtained from the bounds obtained in this paper. When we,

Table 1. Comparison of bounds for the extreme zeros of $Q_5(x; 10, 2, N)$ for different values of N.

N	$x_{5,1}$	$B_5^{\alpha+4,\beta}$ in (9)	$B_5^{\alpha+2,\beta}$ in (6)	Bound in (10)	Bound in (11)	$B_5^{\alpha,\beta+2}$ from (8)	$B_5^{\alpha,\beta+4}$ from (7)	$x_{5,5}$
5	0.1659	1.4108	1.5909	**0.6818**	**4.6818**	2.3182	2.7262	4.9975
10	1.5604	**3.4837**	4.0909	4.0909	**8.0909**	6.6364	7.8550	9.9130
50	15.8455	**20.4292**	24.0909	31.3636	35.3636	41.1818	**46.1970**	47.8746
100	34.2895	**41.7837**	49.0909	65.4545	69.4545	84.3636	**93.0772**	94.9150
500	182.5365	**212.9863**	249.0909	338.1820	342.1820	429.8182	**466.2995**	470.6930

Table 2. Comparison of bounds for the extreme zeros of $Q_5(x; \alpha, 2, 30)$ for different values of α.

α	$x_{5,1}$	$B_5^{\alpha+4,\beta}$ in (9)	$B_5^{\alpha+2,\beta}$ in (6)	Bound in (10)	Bound in (11)	$B_5^{\alpha,\beta+2}$ from (8)	$B_5^{\alpha,\beta+4}$ from (7)	$x_{5,5}$
-0.5	0.2966	0.7131	3.2174	n/a	n/a	22.0000	25.0283	26.3038
5	5.0673	**7.5314**	10.5882	15.2941	19.2941	23.2941	**26.7850**	28.3414
10	8.5443	**11.9184**	14.0909	17.7273	21.7273	23.9090	**27.2727**	29.0000
50	18.3546	**21.4496**	21.7742	23.0645	27.0645	25.2581	**27.3696**	29.9025
200	23.2194	**24.7356**	24.7642	25.1415	**29.1415**	25.7830	26.6385	29.9985

Table 3. Comparison of bounds for the extreme zeros of $Q_5(x; 10.5, \beta, 30)$ for different values of β.

β	$x_{5,1}$	$B_5^{\alpha+4,\beta}$ in (9)	$B_5^{\alpha+2,\beta}$ in (6)	Bound in (10)	Bound in (11)	$B_5^{\alpha,\beta+2}$ from (8)	$B_5^{\alpha,\beta+4}$ from (7)	$x_{5,5}$
-0.5	11.2191	**14.0779**	15.6500	20.1500	24.1500	26.4500	**29.0938**	29.9141
5	6.9636	**10.7099**	13.1373	15.8039	19.8039	21.6078	**24.8851**	27.6465
10	5.0265	**9.0092**	11.6393	13.2131	17.2131	18.7213	**21.4190**	25.3339
50	1.0722	5.2996	7.3050	n/a	n/a	10.3688	10.9749	15.6847
200	0.0657	4.2096	5.0567	n/a	n/a	6.0363	6.1001	8.8014

Table 4. Comparison of bounds for the extreme zeros of $Q_{100}(x; 3, -0.5, N)$ for different values of N.

N	$x_{100,1}$	$B_{100}^{\alpha+4,\beta}$ in (9)	$B_{100}^{\alpha+2,\beta}$ in (6)	$B_{100}^{\alpha,\beta+2}$ from (7)	$B_{100}^{\alpha,\beta+4}$ from (8)	$x_{100,100}$
1 000	0.0361	5.8021	65.9531	947.908	995.2478	999.9999
10 000	7.9491	11.6801	243.7308	9925.6864	9 999.1817	9 999.6205
100 000	96.2846	115.4888	2021.5086	99 703.4642	99 994.0984	99 994.2872
500 000	489.3568	578.8144	9922.7432	498 715.8099	499 969.9708	499 970.4528

for example, let $\alpha = b - 1$, $\beta = N(1-c)c^{-1}$ and $N \to \infty$ in the definition of the Hahn polynomials, we obtain the Meixner polynomials [1, Equation (9.5.15)]. Similarly, by making the same substitution in (8), we obtain

$$\lim_{N\to\infty} -\frac{(bc+c+n-nc-1)\,N+bc+c-2\,nc+n^2c-ncb}{N\,(c-1)-2\,nc-bc+c} = \frac{bc+(n-1)(1-c)}{1-c},$$

which is the inner bound obtained for the extreme zeros of the Meixner polynomial $M_n(x,b;c)$ by shifting b by two units, using the same method [5, Theorem 3.1].

3. Inner bounds for extreme zeros of Continuous Hahn polynomials

Let $n \in \mathbb{N}_{\geq 0}$. The continuous Hahn polynomials are orthogonal on $\mathbb{R}$ if the real parts of a, b, c and d are positive and $c = \bar{a}$, $d = \bar{b}$, and these conditions force us to simultaneously shift both parameters a and c, as well as b and d. We will denote the real and imaginary parts of z by $\mathrm{Re}(z)$ and $\mathrm{Im}(z)$ respectively. The parameter shifted polynomial $p_n(x; a+k, b+m, c+k, d+m)$, which is orthogonal on $\mathbb{R}$ with respect to

$$\begin{aligned}\Gamma(a+k+ix)\Gamma(b+m+ix)\Gamma(c+k-ix)\Gamma(d+m-ix)\\ &= (a+ix)_k(b+ix)_m(c-ix)_k(d-ix)_m\, w(a,b,c,d,x)\\ &= \sigma_{k,m}(x)\, w(a,b,c,d,x) > 0,\end{aligned}$$

together with the polynomial $p_n(x;a,b,c,d)$, satisfy the mixed three-term recurrence relations

$$\begin{aligned}&(x-B_n(k,m))p_{n-1}(x;a,b,c,d)\\ &= a_{k+m-2}(x)p_n(x;a,b,c,d) - d_n\sigma_{k,m}(x)p_{n-2}(x;a+k,b+m,c+k,d+m),\end{aligned}$$

where $a_{k+m-2}(x)$ is a polynomial of degree $k+m-2$ when $k+m \in \{2,3,\dots\}$, and $a_0,\ a_1,\ d_n$ and $B_n(k,m)$ are constants. From Theorem 1.1 we deduce that each point $B_n(k,m)$ such that $k+m \in \{0,1,2,3,4\}$ will be an upper (lower) bound for the smallest (largest) zero of $p_n(x;a,b,c,d)$.

The mixed three-term recurrence relations that involve parameter shifts $a+1, c+1$ and $b+1, d+1$, are given here:

$$\begin{aligned}&(x-B_n(1,0))P_{n-1}(x;a,b,c,d) \qquad (12)\\ &= \frac{(a+c)n}{a+b+c+d+2n-2}P_n(x;a,b,c,d) + \frac{b+d+n-2}{a+b+c+d+2n-2}\sigma_{1,0}(x)P_{n-2}(x;a+1,b,c+1,d),\end{aligned}$$

$$\begin{aligned}&(x-B_n(0,1))P_{n-1}(x;a,b,c,d) \qquad (13)\\ &= \frac{(b+d)n}{a+b+c+d+2n-2}P_n(x;a,b,c,d) + \frac{a+c+n-2}{a+b+c+d+2n-2}\sigma_{0,1}(x)P_{n-2}(x;a,b+1,c,d+1)\end{aligned}$$

and by letting $a = p+iq = \bar{c}$ and $b = r+is = \bar{d}$ where $p,q,r,s \in \mathbb{R}$, $p,r>0$, we have

$$\begin{aligned}\sigma_{1,0}(x) &= (a+ix)(c-ix) = p^2+(q+x)^2 > 0,\\ \sigma_{0,1}(x) &= (b+ix)(d-ix) = r^2+(s+x)^2 > 0\end{aligned}$$

and the bounds are given by

$$B_n(1,0) = i\,\frac{ab-cd+(n-1)(a-c)}{a+b+c+d+2n-2} = -\frac{q(r+n-1)+ps}{p+r+n-1} \qquad (14)$$

$$B_n(0,1) = i\,\frac{ab-cd+(n-1)(b-d)}{a+b+c+d+2n-2} = -\frac{s(p+n-1)+qr}{p+r+n-1}. \qquad (15)$$

Furthermore, we see that if $q \le s$, i.e., $\mathrm{Im}(a) \le \mathrm{Im}(b)$,

$$B_n(0,1) - B_n(1,0) = \frac{(n-1)(q-s)}{p+r+n-1} \le 0.$$

From Theorem 1.1 we know that the points $B_n(1,0)$ and $B_n(0,1)$, obtained from (12) and (13) respectively, are both upper (lower) bounds for the smallest (largest) zero of $P_n(x;a,b,c,d)$.

Furthermore, we observe that

$$\lim_{n\to\infty} B_n(1,0) = -\operatorname{Im}(a) \quad \text{and} \quad \lim_{n\to\infty} B_n(0,1) = -\operatorname{Im}(b)$$

and the bounds are thus less sharp for larger values of n, and, when $\operatorname{Im}(a) = \operatorname{Im}(b)$, $B_n(1,0) = B_n(0,1) = -\operatorname{Im}(a)$ for all values of n.

In Table 5 we show the quality of these bounds, together with bounds obtained by the parameter shifts $a+2, c+2$ and $b+2, d+2$, which are, in fact, more precise bounds and can be obtained in the same manner as shown above. The recurrence equation providing the bound

$$B_n(0,2) = -\frac{2ps(n+r) + s(n-1)(n+2r) + 2qr(r+1)}{n(n+2p+2r-1) + 2r(p+r)} \tag{16}$$

is much more complicated and can be downloaded from `http://www.mathematik.uni-kassel.de/~koepf/Publikationen`. Because of time and memory constraints, Maple cannot provide the recurrence equation with the bound

$$B_n(2,0) = -\frac{2qr(n+p) + q(n-1)(n+2p) + 2ps(p+1)}{n(n+2p+2r-1) + 2p(p+r)}, \tag{17}$$

but this can be found by using a symmetry argument.

Table 5. Bounds for the extreme zeros of $p_n(x;a,b,c,d)$, $\operatorname{Im}(a) \leqq \operatorname{Im}(b)$, $c = \bar{a}, d = \bar{b}$, for different values of a, b and $n.B_n(0,2)$, $B_n(0,1)$, $B_n(1,0)$ and $B_n(2,0)$ are the bounds in (16), (15), (14) and (17) respectively.

n	a	b	$x_{n,1}$	$B_n(0,2)$	$B_n(0,1)$	$B_n(1,0)$	$B_n(2,0)$	$x_{n,n}$
5	$1-20i$	$3-20i$	17.94	20	20	20	20	22.06
5	$1-20i$	$1+155i$	-140.16	-139.09	-125.83	-9.17	4.09	5.16
5	$1-15i$	$1+15i$	-12.55	-12.27	-10.0 0	10.00	12.27	12.55
5	$10+i$	$1+15i$	-15.46	-14.63	-14.07	-10.33	-9.80	-7.86
15	$1-20i$	$1+i$	-2.01	-0.69	0.31	18.69	19.69	21.01
15	$1-20i$	$1-10i$	7.46	10.15	10.63	19.38	19.85	22.54

4. Inner bounds for the extreme zeros of Continuous Dual Hahn polynomials

The parameter shifted Continuous Dual Hahn polynomials $\tilde{S}_n(x^2; a+k, b+l, c+m)$, are orthogonal in the interval $(0;\infty)$ with respect to $\sigma_{k,l,m}(x^2)w(x;a,b,c)$, where

$$\sigma_{k,l,m}(x^2) = |(a+ix)_k(b+ix)_l(c+ix)_m|^2 \tag{18}$$

and satisfy the mixed three-term recurrence relations

$$\begin{aligned}(x^2 - B_n(k,l,m))\tilde{S}_{n-1}(x^2;a,b,c) \\ = a_{k+l+m-2}(x^2)\tilde{S}_n(x^2;a,b,c) - d_n\sigma_{k,l,m}(x^2)\tilde{S}_{n-2}(x^2;a+k,b+l,c+m),\end{aligned}$$

where $a_{k+l+m-2}$ is a polynomial of degree $k+l+m-2$ in x^2 when $k+l+m \in \{2,3,\dots\}$, and $a_0,\ a_1,\ d_n$ and $B_n(k,l,m)$ are constants. From Theorem 1.1 we deduce that each point $B_n(k,l,m)$ such that $k+l+m \in \{0,1,2,3,4\}$ will be an upper (lower) bound for the smallest (largest) zero of $\tilde{S}_n(x^2;a,b,c)$.

The mixed three-term recurrence relations that provide us with relatively good upper bounds for the lowest zeros are those that involve a total parameter shift of four units, i.e., when we shift

(1) one parameter by four units;
(2) two of the parameters by two units each;
(3) one parameter by three units and another one by one unit;
(4) two parameters by one unit and the third one by two units.

Neither the weight function, nor the zeros of the polynomial $\tilde{S}_n(x^2; a, b, c)$ depend on the order in which the parameters a, b and c occur and shifting a by four units leads to exactly the same bound as shifting b or c by four units, provided that the parameter with the smallest numerical value is shifted. In order to illustrate the method used, we provide the mixed recurrence relations obtained when a is shifted by two units.

$$\begin{aligned}\tilde{S}_n(x^2; a, b, c) &= \frac{B_n(2,0,0) - x^2}{(a+b)(a+c) + (2a+1)(n-1)} \tilde{S}_{n-1}(x^2; a, b, c) \\ &\quad - \frac{(n-1)\,(b+c+n-2)\,\sigma_{2,0,0}(x^2)}{(a+c)_2\,(a+b)_2\,((a+b)(a+c)+(2a+1)(n-1))} \tilde{S}_{n-2}(x^2; a+2, b, c),\end{aligned} \tag{19}$$

where, from (18),

$$\sigma_{j,0,0}(x^2) = \prod_{k=0}^{j-1} (a+k)^2 + x^2$$

and the inner bound obtained in this case is

$$B_n(2,0,0) = ab + ac + bc + (2a+1)(n-1). \tag{20}$$

Shifting both a and b by two units leads to the mixed three-term recurrence relation

$$\begin{aligned}\frac{a_2(x^2)}{a+b+2n} \tilde{S}_n(x^2; a, b, c) &= -\big(x^2 - B_n(2,2,0)\big)(a+b+1)\tilde{S}_{n-1}(x^2; a, b, c) \\ &\quad - \frac{(n-1)(a+b+n)_2\,\sigma_{2,2,0}(x^2)}{(a+b+2n)(a+b)_4(a+c)_2} \tilde{S}_{n-2}(x^2; a+2, b+2, c),\end{aligned}$$

where a_2 is a polynomial of degree two in x^2 :

$$\begin{aligned}a_2(x^2) &= 4\,a^2bc + 4\,a^3bc + 6\,a^2b^2c + 3\,a^2bc^2 + 4\,ab^3c + 3\,ab^2c^2 + 6\,abc^2 + 4\,ab^2c - 6\,abc \\ &\quad + a + b - a^3 + (a+b+1)\,(2\,ab + a + b)\,n^2 + 3\,a^2b^3 - 3\,a^2b + a^4b + a^2b^2 + 3\,a^3b^2 \\ &\quad + 3\,b^2c^2 + 2\,ac^2 + 2\,bc^2 + b^3c^2 - 2\,b^2c + b^4c + 2\,b^3c + ab^4 - 3\,ab^2 + a^3c^2 + 3\,a^2c^2 \\ &\quad + 2\,a^3c + a^4c - 2\,a^2c + ab - 3\,ac - 3\,bc - b^3 \\ &\quad + (3\,a^3b + a^3c + 5\,a^2b^2 + 5\,a^2bc + 3\,ab^3 + 5\,ab^2c + b^3c + a^3 + 3\,a^2b + 4\,a^2c \\ &\quad + 3\,ab^2 + 10\,abc + b^3 + 4\,b^2c - a^2 - 5\,ab + 3\,ac - b^2 + 3\,bc - 2\,a - 2\,b)\,n \\ &\quad - (n-1)\,(2\,a(a+b+c+n) + 2\,b(b+c+n) + 2\,c + 2\,n - 1)\,x^2 - (n-1)\,x^4\end{aligned}$$

and

$$B_n(2,2,0) = \frac{(a+b)(ac+bc+2c+n-1)}{a+b+2n} + ab. \tag{21}$$

The relation obtained when we shift a by four units can be downloaded from `http://www.mathematik.uni-kassel.de/~koepf/Publikationen`. The bound obtained in this case is included in our examples and is given by

$$\begin{aligned} B_n(4,0,0) = \Big(&2\,(2\,a+3)\,(a^2+3\,a+1)\,n^3 + (6\,a^4+6\,a^3\,b+6\,a^3\,c+10\,a^2\,b\,c+24\,a^3 \\ &+ 22\,a^2\,b+22\,a^2\,c+30\,a\,b\,c+22\,a^2+18\,a\,b+18\,a\,c+18\,b\,c)\,n^2 \\ &+ (2\,a^5+6\,a^4\,b+6\,a^4\,c+4\,a^3\,b^2+16\,a^3\,b\,c+4\,a^3\,c^2+10\,a^2\,b^2\,c+10\,a^2\,b\,c^2 \\ &+ 6\,a\,b^2\,c^2+3\,a^4+18\,a^3\,b+18\,a^3\,c+13\,a^2\,b^2+42\,a^2\,b\,c+13\,a^2\,c^2 \\ &+ 24\,a\,b^2\,c+24\,a\,b\,c^2+9\,b^2\,c^2-16\,a^3+5\,a^2\,b+5\,a^2\,c+9\,a\,b^2+12\,a\,b\,c \\ &+ 9\,a\,c^2+9\,b^2\,c+9\,b\,c^2-35\,a^2-9\,a\,b-9\,a\,c-9\,b\,c-22\,a-6)n \\ &+ (a+c)_2\,(a+b)_2\,(a\,b+a\,c+b\,c-2\,a+2\,b+2\,c-5)\Big) \\ &\Big/ \Big((4\,a+6)\,n^3+(6\,a^2+6\,a\,b+6\,a\,c+2\,b\,c+6\,a+8\,b+8\,c-4)\,n^2 \\ &+ (4\,a^3+6\,a^2\,b+6\,a^2\,c+2\,a\,b^2+8\,a\,b\,c+2\,a\,c^2+2\,b^2\,c+2\,b\,c^2 \\ &+ 6\,a^2+6\,a\,b+6\,a\,c+2\,b^2+6\,b\,c+2\,c^2+2\,a-2\,b-2\,c+2)n \\ &+ (a+c)_2\,(a+b)_2\Big). \end{aligned} \tag{22}$$

In Tables 6 and 7 we provide some examples that illustrate the quality of these bounds. The point

$$B_n = (a+b+n-1)(a+c+n-1) + (n-1)(b+c+n-2) - a^2, \tag{23}$$

obtained from the three-term recurrence relation satisfied by the Continuous Dual Hahn polynomials [1, Equation 9.3.4], is in each case the best lower bound for the largest zero of these polynomials. It is clear that B_n is an increasing function of n and therefore a more accurate lower bound for the largest zero of $\tilde{S}_n(x^2;a,b,c)$ than the bound in (2). In Table 6 the best upper bound for the smallest zero obtained in each case is printed in bold. Furthermore, when only one parameter is shifted, we shift the smallest one and where two parameters are shifted, we shift the smallest parameters.

Table 6. Bounds for the extreme zeros of $\tilde{S}_n(x^2;a,b,c)$ for different values of a,b,c and n. $B_n(4,0,0)$, $B_n(2,2,0)$, $B_n(2,0,0)$ and B_n are the bounds in (22), (21), (20) and (23) respectively.

n	$\{a,b,c\}$	$x^2_{n,1}$	$B_n(4,0,0)$	$B_n(2,2,0)$	$B_n(2,0,0)$	Bound in (2)	B_n	$x^2_{n,n}$
6	{7,7,7}	63.91	120.68	**112.00**	222.00	267.83	402	581.83
6	{7,8,9}	85.53	148.14	**143.79**	266.00	326.83	476	690.30
6	{1,19,40}	389.85	**504.83**	572.13	834.00	1134.83	1464	2147.23
6	{7,8,40}	312.91	440.85	**436.56**	731.00	946.83	1251	1828.50
6	{7,39,40}	1204.09	**1614.00**	1799.72	2188.00	2558.83	3018	4285.71
31	{7,8,9}	29.34	98.629	**91.65**	641.00	1506	3401	5829.19
31	{1,19,40}	114.82	**157.85**	240.95	909.00	3214	6189	10788.25

Table 7. Best inner bounds for the extreme zeros of $\tilde{S}_n(x^2;a,b,c)$ obtained by this method (taken from Table 6) for different values of a,b,c and n.

n	$\{a,b,c\}$	$x_{n,1}$	Upper Bound for $x_{n,1}$	$\sqrt{B_n}$	$x_{n,n}$
6	{7,7,7}	7.99	10.58	20.05	24.12
6	{7,8,9}	9.25	11.99	21.82	26.27
6	{1,19,40}	19.74	22.47	38.26	46.34
6	{7,8,40}	17.69	20.89	35.37	42.76
6	{7,39,40}	34.70	40.17	54.94	65.47
31	{7,8,9}	5.42	9.93	58.32	76.35
31	{1,19,40}	10.72	12.56	78.67	103.87

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