On a connection between formulas about q–gamma functions

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In this short communication, we want to pay attention to a few wrong formulas which are unfortunately cited and used in a dozen papers afterwards. We prove that the provided relations and asymptotic expansion about the q-gamma function are not correct. This is illustrated by numerous concrete counterexamples. The error came from the wrong assumption about the existence of a parameter which does not depend on anything. Here, we apply a similar procedure and derive a correct formula for the q-gamma function.

Keywords: q–Gamma function; asymptotic expansion; boundary functions.

2000 Mathematics Subject Classification: 33D05, 11A67

1. Introduction

Since J. Thomae (1869) and F. H. Jackson (1904) defined the q-gamma function, it plays an important role in the theory of the basic hypergeometric series [4] and its applications [7]. Its properties and different representations were discussed in numerous papers, such as in [3], [11] and [10]. A few successful algorithms for its numerical evaluation are introduced in [6] and [5] and [1]. An asymptotic expansion of the q-gamma function was provided in [2].

Here, we will make observations on the asymptotic expansions given in [8, 9].

Let $q \in [0,\ 1)$. A q-number $[a]_q$ is

$$[a]_q := \frac{1-q^a}{1-q}, \quad a \in \mathbb{R}.$$

The factorial of a positive integer number $[n]_q$ is given by

$$[0]_q! := 1, \quad [n]_q! := [n]_q[n-1]_q \cdots [1]_q, \qquad (n \in \mathbb{N}).$$

An important role in q–calculus plays the q-Pochhammer symbol defined by

$$(a;q)_0 = 1, \qquad (a;q)_n = \prod_{i=0}^{n-1}(1-aq^i) \quad \big(n \in \mathbb{N} \cup \{+\infty\}\big),$$

and

$$(a;q)_\lambda = \frac{(a;q)_\infty}{(aq^\lambda;q)_\infty} \qquad (|q|<1,\ \lambda \in \mathbb{C})\ .$$

The q-gamma function

$$\Gamma_q(z) = (q;q)_{z-1}\,(1-q)^{1-z} = \frac{(q;q)_\infty}{(q^z;q)_\infty}\,(1-q)^{1-z} \qquad (0<q<1,\ z \notin \mathbb{Z}^-) \tag{1.1}$$

has the following properties:

$$\Gamma_q(z+1) = [z]_q\,\Gamma_q(z) \quad (z \in \mathbb{C}), \quad \Gamma_q(n+1) = [n]_q! \quad (n \in \mathbb{N}_0).$$

In particular,

$$\lim_{q\to 1-} \Gamma_q(z) = \Gamma(z).$$

The exact q–Gauss multiplication formula can be found in [4] or [3]:

$$\Gamma_q(nx)\prod_{k=1}^{n-1}\Gamma_{q^n}\left(\frac{k}{n}\right) = [n]_q^{nx-1}\prod_{k=0}^{n-1}\Gamma_{q^n}\left(x+\frac{k}{n}\right) \quad (x>0;\ n \in \mathbb{N}).$$

Equivalently, substituting $z = nx$, it can be written in the form

$$\Gamma_q(z)\prod_{k=1}^{n-1}\Gamma_{q^n}\left(\frac{k}{n}\right) = [n]_q^{z-1}\prod_{k=0}^{n-1}\Gamma_{q^n}\left(\frac{z+k}{n}\right) \quad (z>0;\ n \in \mathbb{N}). \tag{1.2}$$

2. Our corrections to the paper [8]

Starting from the definition

$$\Gamma_q(x) = (q;q)_\infty\,(1-q)^{1-x}(q^x;q)_\infty^{-1},$$

we can write

$$\Gamma_q(x) = (q;q)_\infty\,(1-q)^{1/2}(1-q)^{1/2-x}e^{-\log(q^x;q)_\infty}.$$

Hence the function $\Gamma_q(x)$ can be written in the form

$$\Gamma_q(x) = a(q)\cdot(1-q)^{1/2-x}\,e^{\mu(x)} \quad (a(q) \in \mathbb{R}), \tag{2.1}$$

where

$$0 < a(q) = (q;q)_\infty\,(1-q)^{1/2} < 1, \qquad \mu(x,q) = -\log(q^x;q)_\infty. \tag{2.2}$$

Let

$$\psi(x,q) = \frac{q^x}{(1-q)(1-q^x)}.$$

From the estimate

$$0<\mu(x,q)<\psi(x,q)\qquad(0<q<1,\ x>0),$$

it exists $\theta(x,q)\in(0,1)$ such that

$$\mu(x,q)=\theta(x,q)\cdot\psi(x,q).$$

Therefore, relation (2.1) becomes

$$\Gamma_q(x)=a(q)\cdot(1-q)^{1/2-x}\ e^{\theta(x,q)\cdot\psi(x,q)}. \tag{2.3}$$

On the other hand, formula (1.2) can be written in the form

$$a_p(q)\Gamma_q(x)=[p]_q^x\prod_{k=0}^{n-1}\Gamma_{q^p}\left(\frac{x+k}{p}\right)\quad(x>0;\ p\in\mathbb{N}), \tag{2.4}$$

where

$$a_p(q)=[p]_q\,\Gamma_{q^p}\Big(\frac{1}{p}\Big)\Gamma_{q^p}\Big(\frac{2}{p}\Big)\cdots\Gamma_{q^p}\Big(\frac{p}{p}\Big).$$

Substituting $q\to q^p$ and $x\to k/p$ into the definition (1.1) of the q–gamma function, we have

$$\Gamma_{q^p}\Big(\frac{k}{p}\Big)=\frac{(q^p;q^p)_\infty}{(q^k;q^p)_\infty}\,(1-q^p)^{1-k/p}=(1-q^p)^{1-k/p}\lim_{n\to\infty}\frac{(q^p;q^p)_n}{(q^k;q^p)_n}.$$

Using moreover

$$\prod_{k=1}^{p}(1-q^p)^{1-k/p}=(1-q^p)^{\frac{p-1}{2}},$$

the following holds:

$$\begin{aligned}a_p(q)&=[p]_q\prod_{k=1}^{p}\Gamma_{q^p}\Big(\frac{k}{p}\Big)=[p]_q\prod_{k=1}^{p}(1-q^p)^{1-k/p}\lim_{n\to\infty}\frac{(q^p;q^p)_n}{(q^k;q^p)_n}\\&=[p]_q\prod_{k=1}^{p}(1-q^p)^{1-k/p}\lim_{n\to\infty}\frac{(q^p;q^p)_n^p}{\prod_{k=1}^{p}(q^k;q^p)_n}\\&=[p]_q(1-q^p)^{\frac{p-1}{2}}\lim_{n\to\infty}\frac{(q^p;q^p)_n^p}{\prod_{k=1}^{p}(q^k;q^p)_n}.\end{aligned}$$

The following identity is valid

$$\prod_{k=1}^{p}(q^k;q^p)_n=(q;q)_{np}.$$

Using estimate (2.3), we get

$$\Gamma_{q^p}(n+1)=a(q^p)\cdot(1-q^p)^{-n-1/2}\cdot e^{\theta(n+1,q^p)\cdot\psi(n+1,q^p)}$$

Since

$$\frac{(q^p;q^p)_n^p}{(1-q^p)^{np}}=\Gamma_{q^p}^p(n+1)=a^p(q^p)\cdot(1-q^p)^{p(-1/2-n)}\cdot e^{p\cdot\theta(n+1,q^p)\cdot\psi(n+1,q^p)},$$

and

$$\frac{\prod_{k=1}^{p}(q^k;q^p)_n}{(1-q)^{np}} = \frac{(q;q)_{np}}{(1-q)^{np}} = \Gamma_q(np+1) = a(q)\cdot(1-q)^{-1/2-np}\cdot e^{\theta(np+1,q)\cdot\psi(np+1,q)},$$

we have

$$a_p(q) = \frac{a^p(q^p)}{a(q)}[p]_q^{1/2}\lim_{n\to\infty}\frac{e^{p\cdot\theta(n+1,q^p)\cdot\psi(n+1,q^p)}}{e^{\theta(np+1,q)\cdot\psi(np+1,q)}}.$$

From

$$\lim_{n\to\infty}\psi(n+1,q^p) = \lim_{n\to\infty}\psi(np+1,q) = 0 \qquad (0<q<1;\ p\in\mathbb{N}),$$

we find

$$a_p(q) = [p]_q^{1/2}\frac{a^p(q^p)}{a(q)}.$$

In that manner, the parameter $a_p(q)$ from formula (2.4) is expressed via the parameter $a(q)$ from formula (2.3).

3. Faults in paper [8]

In the very beginning, the author has supposed that $\Gamma_q(x)$ for $0<q<1;\ x>0$, can be written in the form

$$\Gamma_q(x) = a\cdot(1-q)^{1/2-x}\ e^{\mu(x)} \quad (a\in\mathbb{R}),$$

where

$$\mu(x,q) = -\log(q^x;q)_\infty > 0.$$

His efforts in looking for $\mu(x)$ we shortened a lot by starting from the definition of $\Gamma_q(x)$. From the fact that

$$0<\mu(x)<\frac{q^x}{(1-q)(1-q^x)}\ ,$$

and

$$(1-q)(1-q^x) = 1-q-q^x+q^{x+1} > 1-q-q^x,$$

the author in [8] concluded wrongly that

$$0<\mu(x)<\frac{q^x}{(1-q)-q^x}\ .$$

But, expression $1-q-q^x$ is not positive for all $q\in(0,1)$ and $x>0$. Indeed,

$$1-q-q^x\le 0 \ \Leftrightarrow\ 1-q\le q^x \ \Leftrightarrow\ x\cdot\log q\ge\log(1-q) \ \Leftrightarrow\ x\le\frac{\log(1-q)}{\log q}\ .$$

Example 3.1. We examined the sign changes of the function $h_q(x)\equiv 1-q-q^x$ for different q and x. Notice that $x\to+\infty$ if $q\to 1^-$.

Table 1. Unique real zero of the function $h_q(x)$ and the sign changes for random values of q and x

q	$x:\ 1-q-q^x=0$
0.1	0.045758
0.3	0.296248
0.5	1.0000
0.7	3.37555
0.9	21.8543

x	q	$1-q-q^x$
1.10500	0.592727	−0.15378
2.27287	0.752038	−0.275286
6.47584	0.816692	−0.0861563
43.2362	0.946066	−0.0370453
60.1635	0.954814	−0.0167368

This estimate should be written in the from

$$0 < \mu(x) < \frac{q^x}{(1-q)-q^x} \qquad \left(0<q<1;\ x > \frac{\log(1-q)}{\log q}\right).$$

Furthermore, from the estimate

$$0 < \mu(x) < \frac{q^x}{(1-q)-q^x}\ ,$$

the author in [8] concluded wrongly that

$$\mu(x) = \frac{\theta q^x}{(1-q)-q^x}\ ,$$

where θ is a number independent of x between 0 and 1.

Example 3.2. We find counterexamples which show that θ depends on x and q. At the first table, we fixed $q = 0.9$ and take a few random values for x. In another we changed the rule of variables.

Table 2. The dependence of parameter θ from x and q

x	q	θ
3.78377	0.9	−7.27980
13.2554	0.9	−1.58344
20.6473	0.9	−0.139893
25.7471	0.9	0.342512
32.2948	0.9	0.673069
43.8850	0.9	0.904181

x	q	θ
10.5	0.063920	1.00000
10.5	0.234682	1.00000
10.5	0.494904	0.99898
10.5	0.618621	0.98504
10.5	0.806515	0.473541
10.5	0.915828	−4.19862

In continuation, the author in [8] got the wrong formulas (2.21)-(2.27). He concluded that

$$a_p = \sqrt{[2]_q}\Gamma_{q^2}(1/2),$$

and

$$\Gamma_q(x) = \sqrt{[2]_q}\Gamma_{q^2}(1/2)(1-q)^{1/2-x}\, e^{\theta \frac{q^x}{(1-q)-q^x}} \quad (0<\theta<1).$$

The following wrong version of the q–Gauss multiplication formula was provided

$$[n]_q^{1/2-x}[2]_q^{(n-1)/2}\Gamma_{q^2}^{n-1}(1/2)\Gamma_q(x) = \prod_{k=0}^{n-1}\Gamma_{q^n}\left(\frac{x+k}{n}\right) \quad (x>0;\ n\in\mathbb{N}).$$

In a special case, for $n = 2$, it agrees with the exact q–Legendre relation. Also, when $q \to 1$, it reduces to well-known formulas for gamma-function.

4. Bounds of the q-gamma function

Let

$$g(x) = \ln \Gamma_q(x)$$

Since

$$g(x+1) = \ln \Gamma_q(x+1) = \ln([x]_q \Gamma_q(x)) = \ln[x]_q + g(x), \tag{4.1}$$

by induction, we get

$$g(x+n) = \sum_{k=0}^{n-1} \ln[x+k]_q + g(x) \qquad (n \in \mathbb{N}).$$

It is known that $g(x)$ is a convex function.

Lemma 4.1. *If $x \in (0,1)$ and $n \in \mathbb{N}$, then*

$$g(n) + x \ln[x+n-1]_q \leq g(x+n) \leq (1-x)g(n) + xg(n+1)$$

Proof. Since

$$x + n = (1-x)n + x(n+1),$$

we can write

$$g(x+n) = g((1-x)n + x(n+1)) \leq (1-x)g(n) + xg(n+1).$$

Let us find a lower bound for $\Gamma_q(x)$. Since

$$n = (1-x)(x+n) + x(x+n-1),$$

and because of the convexity of the function $g(x)$, we have

$$g(n) \leq (1-x)g(x+n) + xg(x+n-1).$$

Applying (4.1), for $x \to x+n-1$, we can write

$$g(x+n) = \ln[x+n-1]_q + g(x+n-1),$$

wherefrom

$$g(n) \leq (1-x)g(x+n) + x(g(x+n) - \ln[x+n-1]_q) = g(x+n) - x\ln[x+n-1]_q,$$

i.e.,

$$g(n) + x\ln[x+n-1]_q \leq g(x+n). \square$$

Theorem 4.1. *The following bounds are valid:*

$$[n-1]_q!\ [n-1+x]_q^x \leq \Gamma_q(n+x) \leq [n-1]_q!\ [n]_q^x\,, \qquad (n \in \mathbb{N}_0;\ 0 \leq x < 1).$$

Proof. According to the upper bound for $g(x)$, we get e. g.

$$\ln\Gamma_q(x+n) \le (1-x)\ln\Gamma_q(n) + x\ln\Gamma_q(n+1).$$

Hence

$$\Gamma_q(x+n) \le ([n-1]_q!)^{1-x}([n]_q!)^x,$$

wherefrom

$$\Gamma_q(x+n) \le [n-1]_q!\ [n]_q^x.$$

According to the lower bound for $g(x)$, we get

$$\ln\Gamma_q(n) + x\ln[x+n-1]_q \le \ln\Gamma_q(x+n),$$

i.e.,

$$\Gamma_q(n)\ [n+x-1]_q^x \le \Gamma_q(n+x).\ \Box$$

Theorem 4.2.

$$[n-(1-x)]_q \le \left(\frac{\Gamma_q(n+x)}{[n-1]_q!}\right)^{1/x} \le [n]_q\,, \qquad (n\in\mathbb{N}_0;\ 0\le x<1).$$

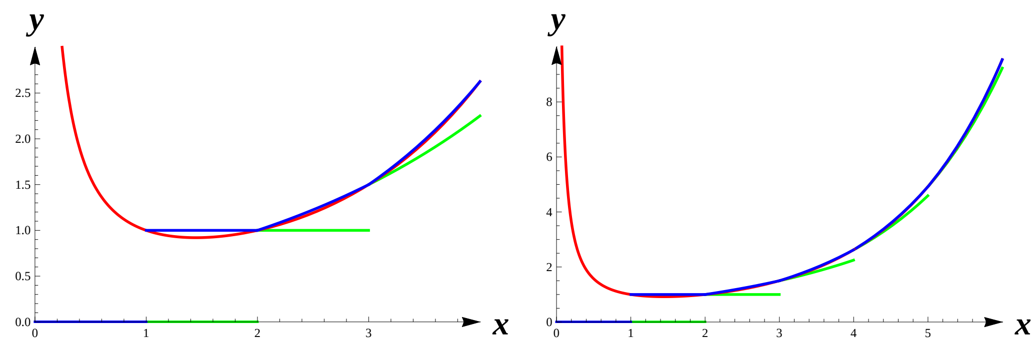

Fig. 1. $\Gamma_q(x)$ and its boundary functions for $q=0.5$.

Theorem 4.3. *For any $n\in\mathbb{N}$ and $x\in(0,1)$ there exists $\theta=\theta(n,x,q)\in(0,1)$ such that*

$$\Gamma_q(n+x) = [n-1]_q!\ [n-\theta(1-x)]_q^x.$$

Introducing $y=n+x$ $(n\in\mathbb{N}_0;\ 0\le x<1)$ and denoting $n=\lfloor y\rfloor$, we can write

$$[\lfloor y\rfloor-1]_q!\ [y-1]_q^{y-\lfloor y\rfloor} \le \Gamma_q(y) \le [\lfloor y\rfloor-1]_q!\ [\lfloor y\rfloor]_q^{y-\lfloor y\rfloor} \qquad (y>1).$$

Theorem 4.4. *For any $y\in(1,+\infty)\setminus\mathbb{N}$, it exists $\theta=\theta(y,q)\in(0,1)]$ such that*

$$\Gamma_q(y) = [\lfloor y\rfloor-1]_q!\ [\lfloor y\rfloor-\theta(1-(y-\lfloor y\rfloor))]_q^{y-\lfloor y\rfloor}.$$

Acknowledgement. This paper is supported by the Ministry of Science and Technological Development of the Republic Serbia, projects No 174011.

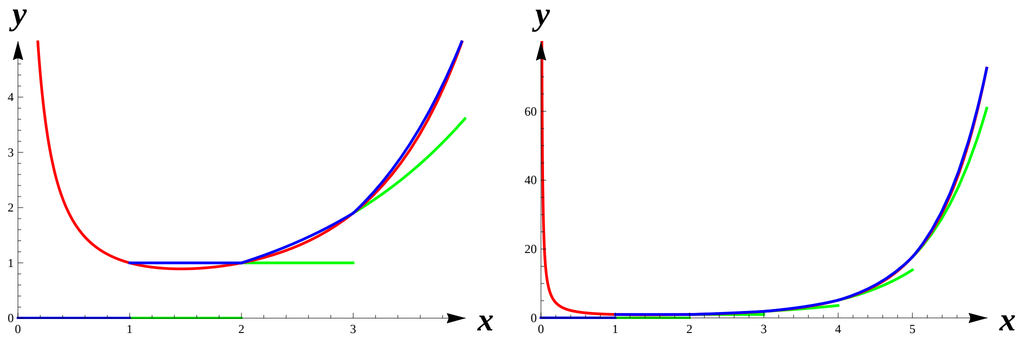

Fig. 2. $\Gamma_q(x)$ and its bounds for $q = 0.9$.

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