On the De Branges Theorem

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Recently, Todorov and Wilf independently realized that de Branges' original proof of the Bieberbach and Milin conjectures and the proof that was later given by Weinstein deal with the same special function system that de Branges had introduced in his work.

In this article, we present an elementary proof of this statement based on the defining differential equations system rather than the closed representation of de Branges' function system. Our proof does neither use special functions (like Wilf's) nor the residue theorem (like Todorov's) nor the closed representation (like both), but is purely algebraic.

On the other hand, by a similar algebraic treatment, the closed representation of de Branges' function system is derived.

Our whole contribution can be looked at as the study of properties of the Koebe function. Therefore, in a very elementary manner it is shown that the known proofs of the Bieberbach and Milin conjectures can be understood as a consequence of the Löwner differential equation, plus properties of the Koebe function.

Keywords: de Branges theorem; Bieberbach conjecture; Milin conjecture; Chebyshev polynomials; Jacobi polynomials; Gegenbauer polynomials; de Branges functions; Weinstein functions; Löwner theory; Löwner differential equation; hypergeometric representations; generating functions; computer algebra.

1 INTRODUCTION

Let S denote the family of analytic and univalent functions $f(z) = z + a_2 z^2 + \ldots$ of the unit disk $\mathbb{D}$. S is compact with respect to the topology of locally uniform convergence so that $k_n := \max_{f \in S} |a_n(f)|$ exists. In 1916 Bieberbach [3] proved that $k_2 = 2$, with equality if and only if f is a rotation of the Koebe function

$$K(z) := \frac{z}{(1-z)^2} = \frac{1}{4}\left(\left(\frac{1+z}{1-z}\right)^2 - 1\right) = \sum_{n=1}^{\infty} n z^n, \tag{1}$$

and in a footnote he mentioned "Vielleicht ist überhaupt $k_n = n$". This statement is known as the *Bieberbach conjecture*.

In 1923 Löwner [14] proved the Bieberbach conjecture for $n = 3$. His method was to embed a univalent function $f(z)$ into a *Löwner chain*, i.e., a family $\{f(z,t) | t \geq 0\}$ of univalent functions of the form

$$f(z,t) = e^t z + \sum_{n=2}^{\infty} a_n(t) z^n, \quad (z \in \mathbb{D}, t \geq 0, a_n(t) \in \mathbb{C}(n \geq 2))$$

which start with f

$$f(z,0) = f(z),$$

and for which the relation

$$\text{Re } p(z,t) = \text{Re } \left(\frac{\dot{f}(z,t)}{z f'(z,t)} \right) > 0 \quad (z \in \mathbb{D}) \tag{2}$$

is satisfied. Here $'$ and $\dot{}$ denote the partial derivatives with respect to z and t, respectively. Equation (2) is referred to as the *Löwner differential equation*, and geometrically it states that the image domains of f_t expand as t increases.

The history of the Bieberbach conjecture showed that it was easier to obtain results about the *logarithmic coefficients* of a univalent function f, i.e., the coefficients d_n of the expansion

$$\varphi(z) = \ln \frac{f(z)}{z} =: \sum_{n=1}^{\infty} d_n z^n$$

rather than for the coefficients a_n of f itself. So Lebedev and Milin [13] in the mid sixties developed methods to exponentiate such information. They proved that if for $f \in S$ the *Milin conjecture*

$$\sum_{k=1}^{n} (n+1-k) \left(k |d_k|^2 - \frac{4}{k} \right) \leq 0$$

on its logarithmic coefficients is satisfied for some $n \in \mathbb{N}$, then the Bieberbach conjecture for the index $n+1$ follows.

In 1984 de Branges [4] verified the Milin, and therefore the Bieberbach conjecture, and in 1990, Weinstein [18] gave a different proof. A reference concerning de Branges' proof is [5], and a German language summary of the history of the Bieberbach conjecture and its proofs was given in [8].

Both proofs use special function systems, and independently, Todorov [17] and Wilf [19] showed that these are essentially the same.

In this article, we present an elementary proof of this statement. Our considerations are based on the defining differential equations system rather than the closed representation of de Branges' function system. Our proof neither uses special functions (like Wilf's) nor the residue theorem (like Todorov's) nor the closed representation (like both), but is purely algebraic. On the other hand, by a similar algebraic treatment, the closed representation of de Branges' function system is derived.

Our whole contribution can be looked at as the study of properties of the Koebe function. Therefore, in a very elementary manner it is shown that the known proofs of the Bieberbach and Milin conjectures can be understood as a consequence of the Löwner differential equation, plus properties of the Koebe function.

2 THE LÖWNER CHAIN OF THE KOEBE FUNCTION

In this section, we consider the Löwner chain

$$w(z, t) := K^{-1}(e^{-t} K(z)) \quad (z \in \mathbb{D}, t \geq 0) \tag{3}$$

of bounded univalent functions in the unit disk $\mathbb{D}$ which is defined in terms of the Koebe function (1). Since K maps the unit disk onto the entire plane slit along the negative x-axis in the interval $(-\infty, 1/4]$, $w(\mathbb{D}, t)$ is the unit disk with a radial slit increasing with t. The function $w(z, t)$ is implicitly given by the equation

$$K(z) = e^t K(w(z, t)), \tag{4}$$

and satisfies the Löwner type differential equation (we omit the arguments)

$$\dot{w} = -\frac{1 - w}{1 + w} w \tag{5}$$

(compare e.g. [15], Chapter 6) which is obtained differentiating (4) with respect to t

$$0 = e^t K(w) + e^t K'(w) \dot{w}, \tag{6}$$

hence

$$\dot{w} = -\frac{K(w)}{K'(w)} = -\frac{w}{(1 - w)^2} \frac{(1 - w)^3}{1 + w} = \frac{1 - w}{1 + w} w.$$

In this section, we deduce a closed representation of the Taylor coefficients $A_n(t)$ of

$$z(z, t) = \sum_{n=1}^{\infty} A_n(t) z^n \tag{7}$$

In particular, by the normalization of the Koebe function, we have $\left.\frac{K(z)}{z}\right|_{z=0} = 1$, hence by (4)

$$\frac{K(z)}{z} = e^t \frac{K(w(z,t))}{z} = e^t \frac{K(w(z,t))}{w(z,t)} \frac{w(z,t)}{z}$$

and letting $z \to 0$ therefore gives $A_1(t) = e^{-t}$. To deduce the general result in an elementary way (for a shorter deduction using Gegenbauer polynomials, see Section 3), we begin with some lemmas. The first lemma states a linear partial differential equation (different from the nonlinear Löwner differential equation (5)) for $w(z,t)$:

LEMMA 1 (Partial differential equation).   *The function $w(z,t)$ satisfies the linear partial differential equation*

$$(z - 1)zw'(z,t) = (z + 1)\dot{w}(z,t) \tag{8}$$

with the initial function

$$w(z,0) = z.$$

Proof   Differentiating (4) with respect to both z and t yields the equations

$$K'(z) = e^t K'(w(z,t))w'(z,t)$$

and (6), from which we deduce

$$\frac{zw'(z,t)}{\dot{w}(z,t)} = -\frac{zK'(z)}{e^t K'(w(z,t))} \frac{K'(w(z,t))}{K(w(z,t))} = -\frac{zK'(z)}{e^t K(w(z,t))}$$

$$= \frac{zK'(z)}{K(z)} = -\frac{1 + z}{1 - z}$$

where we used (4) once again, and (1). The initial function is determined trivially.   ∎

As a consequence we have for the coefficients $A_n(t)$ of $w(z,t)$:

LEMMA 2 (Differential equations system for coefficient functions).   *The coefficients $A_n(t)$ satisfy the system of linear differential equations*

$$(n - 1)A_{n-1}(t) - nA_n(t) = \dot{A}_{n-1}(t) + \dot{A}_n(t), \quad A_n(0) = 0 \quad (n \geq 2) \tag{9}$$

and

$$-A_1(t) = \dot{A}_1(t), \quad A_1(0) = 1. \tag{10}$$

Proof This follows directly by summing (8) for $n = 0, \ldots, \infty$, and equating coefficients. ∎

Starting with the solution $A_1(t) = e^{-t}$ of (10), by induction we see that $A_n(t)$ is a polynomial of degree n in e^{-t}. Therefore we may introduce the variable $y := e^{-t}$, and define the polynomials $B_n(y)$ by

$$A_n(t) = B_n(y) = B_n(e^{-t}) = \sum_{j=1}^{n} a_j^{(n)} e^{-jt} = \sum_{j=1}^{n} a_j^{(n)} y^j, \qquad (11)$$

so that in terms of $B_n(y)$, Lemma 2 reads as follows:

LEMMA 3 (Differential equations system for coefficient functions). *The functions $B_n(y)$ satisfy the system of linear differential equations*

$$y(B_n'(y) + B_{n-1}'(y)) = nB_n(y) - (n-1)B_{n-1}(y), \quad B_n(1) = 0 \quad (n \geq 2). \qquad (12)$$

For the numbers $a_j^{(n)}$, we deduce

LEMMA 4 (Recurrence equations). *For the numbers $a_j^{(n)}$ defined by (11), the simple recurrence equation*

$$(n-j)a_j^{(n)} = (n-1+j)a_j^{(n-1)} \quad (1 \leq j \leq n-1, n \geq 2) \qquad (13)$$

is valid. Therefore, we have

$$a_j^{(n)} = \binom{n+j-1}{n-j} a_j^{(j)} \quad (1 \leq j \leq n, n \geq 2) \qquad (14)$$

and the initial value

$$a_1^{(n)} = n.$$

Proof For $j = n$, Equation (14) is trivial. Therefore assume $1 \leq j \leq n-1, n \geq 2$. Substituting (11) into (12), and equating coefficients of y^j ($1 \leq j \leq n-1$) results in (13). From (13), we get the telescoping product

$$a_j^{(n)} = \frac{n-1+j}{n-j} a_j^{(n-1)} = \frac{(n-1+j)(n-2+j)\cdots(2j)}{(n-j)(n-j-1)\cdots 1} a_j^{(j)}$$

$$= \binom{n-1+j}{n-j} a_j^{(j)},$$

and hence (14).

Using $A_1(t) = e^{-t}$, we get $a_1^{(1)} = 1$, so that by (14), we finally have

$$a_1^{(n)} = \binom{n}{n-1} a_1^{(1)} = n,$$

and we are done.   ∎

Our next step is to derive an ordinary differential equation valid for $B_n(y)$:

LEMMA 5 (Ordinary differential equation for coefficient functions).   *The function* $B_n(y)$ $(n \geq 1)$ *satisfies the ordinary differential equation*

$$y^2(1-y)B_n''(y) + y(1-y)B_n'(y) + (n^2 y - 1)B_n(y) = 0. \tag{15}$$

Proof   For $n = 1$, the statement is true, so assume $n \geq 2$. We consider the function

$$\Delta_n(y) := y^2(1-y)B_n''(y) + y(1-y)B_n'(y) + (n^2 y - 1)B_n(y), \tag{16}$$

and show in a first step that $\Delta_n(y)$ satisfies the relation

$$y(\Delta_n'(y) + \Delta_{n-1}'(y)) - n\Delta_n(y) + (n-1)\Delta_{n-1}(y) = 0. \tag{17}$$

(In other words, we show that $\Delta_n(y)$ satisfies the same system of differential equations (12) as $B_n(y)$.)

To prove (17), we first solve (12) for $B_{n-1}'(y)$:

$$B_{n-1}'(y) = \frac{n}{y}B_n(y) - \frac{n-1}{y}B_{n-1}(y) - B_n'(y).$$

We take this equation and the first two derivatives thereof as replacement rules for any occurrence of $B_{n-1}'(y)$, $B_{n-1}''(y)$, and $B_{n-1}'''(y)$ in the left hand side of (17). The resulting term reduces to zero. This procedure can be easily done with the aid of a computer algebra system, and we leave these elementary algebraic transformations to the reader.

Therefore, by (15)–(16) we have $\Delta_1(y) \equiv 0$ and further $\Delta_n(1) = 0$ for all $n \in \mathbb{N}$. From the induction hypothesis $\Delta_{n-1}(y) \equiv 0$, we get the initial value problem

$$y\Delta_n'(y) - n\Delta_n(y) = 0, \quad \text{and} \quad \Delta_n(1) = 0,$$

and by integration the unique solution $\Delta_n(y) \equiv 0$ is deduced.   ∎

Obviously there is a corresponding ordinary differential equation for $A_n(t)$, namely

$$(1 - e^t)\ddot{A}_n(t) + (e^t - n^2)A_n(t) = 0,$$

which is simpler than (15) in the sense that it does not contain the first derivative explicitly, but which does not have polynomial coefficients since e^t-terms occur.

As a consequence of the preceding lemmas, we find the following closed form representation of a_i'.

THEOREM 1 (Coefficient representation of Löwner chain of Koebe function). *For the numbers $a_j^{(n)}$ defined by (11), we have the closed form representation*

$$a_j^{(n)} = 2(-1)^{j+1} \binom{n+j-1}{n-j} \frac{(2j-1)!}{(j-1)!(j+1)!} \quad (1 \le j \le n, n \ge 1). \quad (18)$$

Therefore, by (11), we have further

$$A_n(t) = \sum_{j=1}^{n} 2(-1)^{j+1} \binom{n+j-1}{n-j} \frac{(2j-1)!}{(j-1)!(j+1)!} e^{-jt} \quad (n \ge 1), \quad (19)$$

and finally by (7)

$$w(z,t) = \sum_{n=1}^{\infty} \sum_{j=1}^{n} 2(-1)^{j+1} \binom{n+j-1}{n-j} \frac{(2j-1)!}{(j-1)!(j+1)!} e^{-jt} z^n. \quad (20)$$

Proof By (14), it remains to prove that

$$a_j^{(j)} = 2(-1)^{j+1} \frac{(2j-1)!}{(j-1)!(j+1)!}. \quad (21)$$

Substituting (11) into the ordinary differential equation (15), and equating coefficients gives

$$(n+1-j)(n-1+j)a_{j-1}^{(n)} + (j-1)(j+1)a_j^{(n)} = 0,$$

so that in particular for $n = j$

$$a_j^{(j)} = -\frac{2j-1}{(j-1)(j+1)} a_{j-1}^{(j)} = 2\frac{2j-1}{j+1} a_{j-1}^{(j-1)} \quad (1 \le j \le n-1, n \ge 2),$$

by an application of (13), and therefore (21) follows from $a_2^{(2)} = -2$. It is easily checked that (18) remains true for $n = 1$. ∎

3 CONNECTION WITH THE GEGENBAUER POLYNOMIALS

In this section, we again deduce the closed form representation for $A_n(t)$, this time utilizing an explicit representation of $w(z, t)$ in terms of Gegenbauer polynomials. Observe that this section is not necessary for our development, but it shows some interesting connections for the reader who is familiar with orthogonal polynomials and generating functions. Solving $w = K(z) = \frac{z}{(1-z)^2}$ for z leads to the representation

$$K \cdots = \frac{1 + 2w - \sqrt{1 + 4w}}{2w}$$

for the inverse of the Koebe function. Therefore, substituting $e^{-t} K(z)$, we obtain the representation

$$w(z, t) = \frac{-1 + (1 + x)z - z^2 + (1 - z)\sqrt{1 + z^2 - 2xz}}{z(x - 1)} \tag{22}$$

where we simplified the result changing variable according to $e^{-t} = \frac{1-x}{2}$.

Since $\sqrt{1 + z^2 - 2xz}$ is the generating function of the Gegenbauer polynomials $C_n^{(-1/2)}(x)$ (see e.g. [1], (22.9.3)), (22) implies for $n \geq 2$

$$A_n(t) = \frac{1}{x - 1} \left(C_{n+1}^{(-1/2)}(x) - C_n^{(-1/2)}(x) \right). \tag{23}$$

On the other hand, it is well-known that $C_n^{(-1/2)}(x)$ has the (hypergeometric) representation

$$C_n^{(-1/2)}(x) = 2 \sum_{j=0}^{n-1} \frac{(1 - n)_j (n)_j}{j!(2)_j} \left(\frac{1 - x}{2} \right)^{j+1} \tag{24}$$

when expanded at $x = 1$, which can be obtained from [1], (22.5.46), as limiting case. Here, $(a)_j := a(a + 1) \cdots (a + j - 1)$ as usual denotes the *Pochhammer symbol* (or *shifted factorial*). Therefore, we obtain for the difference (23)

$$A_n(t) = \frac{1}{x - 1} \left(C_{n+1}^{(-1/2)}(x) - C_n^{(-1/2)}(x) \right)$$

$$= -\sum_{j=0}^{n} \frac{(-n)_j (n + 1)_j}{j!(2)_j} \left(\frac{1 - x}{2} \right)^j + \sum_{j=0}^{n-1} \frac{(1 - n)_j (n)_j}{j!(2)_j} \left(\frac{1 - x}{2} \right)^j$$

$$= \sum_{j=1}^{n} \frac{(1 - n)_{j-1} (n + 1)_{j-1}}{2(j - 1)!(2)_j} \left(\frac{1 - x}{2} \right)^j$$

$$= n \frac{1 - x}{2} \sum_{j=0}^{n-1} \frac{(1 - n)_j (1 + n)_j}{j!(3)_j} \left(\frac{1 - x}{2} \right)^j$$

$$= n e^{-t} \sum_{j=0}^{n-1} \frac{(1 - n)_j (1 + n)_j}{j!(3)_j} e^{-jt}$$

$$= \sum_{j=0}^{n} 2(-1)^{j+1} \binom{n + j - 1}{j} \frac{(2j - 1)!}{} e^{-jt}$$

We note in passing that the method presented in [9]–[11] finds the ordinary differential equation for $B_n(y)$ of Lemma 5, and furthermore a pure recurrence equation with respect to n, automatically. Actually, this differential equation generated by our MATHEMATICA [20] implementation [10] was an essential tool to discover the short proof of Theorem 1. Moreover, the same implementation discovers the power series representation (24) automatically.

4 THE DE BRANGES AND WEINSTEIN FUNCTIONS

In [4] de Branges showed that the Milin conjecture is valid if for all $n \geq 2$ the *de Branges functions* $\tau_k^n : \mathbb{R}^+ \to \mathbb{R}$ $(k = 1, \ldots, n+1)$ defined by the system of differential equations

$$\tau_{k+1}^n(t) - \tau_k^n(t) = \frac{\tau_k^n(t)}{k} + \frac{\dot{\tau}_{k+1}^n(t)}{k+1} \quad (k = 1, \ldots, n) \tag{25}$$

$$\tau_{n+1}^n \equiv 0 \tag{26}$$

with the initial values

$$\tau_k^n(0) = n + 1 - k \tag{27}$$

have the properties

$$\lim_{t \to \infty} \tau_k^n(t) = 0, \tag{28}$$

and

$$\dot{\tau}_k^n(t) \leq 0 \quad (t \in \mathbb{R}^+) \tag{29}$$

The relation (28) is easily checked using standard methods for ordinary differential equations, whereas (29) is a deep result.

L. de Branges gave an explicit representation of the function system $\tau_k^n(t)$ ([4, 7, 16]) (that we don't use, though, see Section 5, however), with which the proof of the de Branges theorem was completed as soon as de Branges realized that (29) was a theorem previously proved by Askey and Gasper [2].

Note that the derivatives $\dot{\tau}_k^n(t)$ are characterized by the same system of differential equations (25), the equation

$$\dot{\tau}_n^n(t) = -n e^{-nt} \tag{30}$$

and the initial values

$$\dot{\tau}_k^n(0) = \begin{cases} -k & \text{if } n - k \text{ even} \\ 0 & \text{if } n - k \text{ odd} \end{cases} \tag{31}$$

as replacements for (26) and (27) (see e.g. [5], p. 685).

On the other hand, Weinstein [18] uses the Löwner chain (3), and shows the validity of Milin's conjecture if for all $n \geq 2$ the *Weinstein functions* $\Lambda_k^n : \mathbb{R}^+ \to \mathbb{R}$ $(k = 1, \ldots, n+1)$ defined by

$$\frac{e^t w(z, t)^{k+1}}{1 - w^2(z, t)} =: \sum_{n=k}^{\infty} \Lambda_k^n(t) z^{n+1} = W_k(z, t) \tag{32}$$

satisfy the relations

$$\Lambda_k^n(t) \geq 0 \quad (t \in \mathbb{R}^+, k, n \in \mathbb{N}). \tag{33}$$

Weinstein did not identify the functions $\Lambda_k^n(t)$, but was able to prove (33) without an explicit representation.

Independently, both Todorov [17] and Wilf [19] proved — using the explicit representation of the de Branges functions — the following

THEOREM 2 (Connection between de Branges and Weinstein functions)   *For all* $n \in \mathbb{N}$, $k = 1, \ldots, n$, *one has the identity*

$$\dot{\tau}_k^n(t) = -k\Lambda_k^n(t), \tag{34}$$

i.e., the de Branges and the Weinstein functions essentially are the same, and the main inequalities (29) and (33) are identical.   ∎

In this section, we give a very elementary proof of this result, which in view of (32) can be looked at as a property of the Koebe function.

Firstly, we realize that (again, we omit the arguments of $w(z, t)$)

$$W_{k+1}(z, t) = \frac{e^t w^{k+2}}{1 - w^2} = w W_k(z, t), \tag{35}$$

and that further

$$W_k(z, t) = \frac{e^t w^{k+1}}{1 - w^2} = \frac{e^t w}{(1 - w)^2} \frac{1 - w}{1 + w} w^k = K(z) \frac{1 - w}{1 + w} w^k,$$

so that with (5) in particular

$$W_1(z, t) = K(z) \frac{1 - w}{1 + w} w = -K(z) w \tag{36}$$

Moreover, we get the relation

$$W_k(z, t) + W_{k+1}(z, t) = (1 + w)W_k(z, t) - K(z)(1 - w)w^k$$

$$= K(z)w^k - K(z)w^{k+1}.$$

Taking derivative with respect to t, this identity implies

$$\dot{W}_k(z, t) + \dot{W}_{k+1}(z, t) = K(z) \cdot kw^{k-1}\dot{w} - K(z) \cdot (k + 1)w^k\dot{w}$$

$$= -k \cdot K(z)\frac{1 - w}{1 + w}w^k + (k + 1) \cdot K(z)\frac{1 - w}{1 + w}w^{k+1}$$

$$= (k + 1)W_{k+1}(z, t) - kW_k(z, t) \qquad (37)$$

where again, we utilized the Löwner differential equation (5) for $w(z, t)$.

Equating coefficients it follows that the same system of differential equations is valid for $\Lambda_k^n(t)$, and therefore for $y_k^n(t) := -k\Lambda_k^n(t)$ we get the differential equations system

$$y_{k+1}^n(t) - y_k^n(t) = \frac{\dot{y}_k^n(t)}{k} + \frac{\dot{y}_{k+1}^n(t)}{k + 1}$$

of de Branges (25). From

$$\Lambda_n^n(t) = \lim_{z \to 0} \frac{W_n(z, t)}{z^n} = \lim_{z \to 0} \frac{e^t}{1 - w^2(z, t)} \left(\frac{w(z, t)}{z}\right)^{n+1}$$

$$= e^t \left(e^{-(n+1)t}\right) = e^{-nt}$$

which follows from (32), we realize that

$$y_n^n(t) = -ne^{-nt}$$

so that (30) is satisfied.

To show (34), it therefore remains to prove (31) which can be read off from

$$W_k(z, 0) = \sum_{n=k}^{\infty} \Lambda_k^n(0)z^{n+1} = \frac{z^k}{1 - z^2} = \sum_{j=0}^{\infty} z^{2j+k}.$$

This finishes the proof of Theorem 2.

5 CLOSED FORM REPRESENTATION OF WEINSTEIN FUNCTIONS

In this section, we show how — in a similar manner as we derived the closed form representation of the coefficients of Koebe's Löwner chain $w(z, t)$ in Section 2 — the closed form representation of $\dot{\tau}_k^n(t) = -k\Lambda_k^n(t)$ that was given by de Branges, can be deduced in an elementary way, only utilizing the properties of $w(z, t)$ that we developed in Section 2. In particular, the known proofs of the Bieberbach and Milin conjectures may be regarded as a consequence of the Löwner differential equation, plus properties of the Koebe function.

Since $W_1(z, t)$ is given by (36) in terms of $w(z, t)$, and $W_k(z, t)$ satisfies the recurrence (35), from the representation (20) of $w(z, t)$ we deduce by induction that the coefficients $\Lambda_k^n(t)$ of $W_k(z, t)$ have a representation

$$\Lambda_k^n(t) = \sum_{i=k}^{n} a_j^{(n,k)} e^{-it} \quad (n \geq k). \tag{38}$$

Substituting $\Lambda_k^n(t)$ according to (32) in (37), and equating coefficients, we obtain for $n \geq k + 1 \geq 2$

$$\dot{\Lambda}_k^n(t) + \dot{\Lambda}_{k+1}^n(t) = (k + 1)\Lambda_{k+1}^n(t) - k\Lambda_k^n(t). \tag{39}$$

If we substitute now (38) in (39), and equate coefficients, again, then we get the simple recurrence equation ($n \geq j \geq k \geq 2$)

$$a_j^{(n,k)} = -\frac{j - k + 1}{j + k} a_j^{(n,k-1)}$$

for the coefficients $a_j^{(n,k)}$ which (by telescoping) generates

$$a_j^{(n,k)} = (-1)^{k-1} \frac{(j - 1)!(j + 1)!}{(j - k)!(j + k)!} a_j^{(n,1)} \quad (n \geq j \geq k \geq 2). \tag{40}$$

Therefore, to get a closed form representation of $a_j^{(n,k)}$, we need only one for $a_j^{(n,1)}$, and we are done. To obtain this result, we observe that

$$\Lambda_1^n(t) = -\sum_{l=1}^{n} (n + 1 - l)\dot{A}_l(t)$$

following from (7) and (36). Using the definition $a_j^{(n,k)} := 0$ for $j < k$, and the representation (19) of $A_l(t)$ that we gave in Theorem 1. we deduce in a straightforward manner

$$\sum_{j-1}^{n} a_j^{(n,1)} e^{-jt} = \Lambda_1^n(t) = -\sum_{l-1}^{n} (n+1-l)\dot{A}_l(t)$$

$$= \sum_{l=1}^{n} (n+1-l) \sum_{j=1}^{n} 2j(-1)^{j+1} \binom{l+j-1}{l-j} \frac{(2j-1)!}{(j-1)!(j+1)!} e^{-jt}$$

$$= \sum_{j=1}^{n} 2j(-1)^{j+1} \frac{(2j-1)!}{(j-1)!(j+1)!} \left(\sum_{l=1}^{n} (n+1-l) \binom{l+j-1}{l-j} \right) e^{-jt}$$

where we changed the order of summation. Equating coefficients, we therefore see that

$$a_j^{(n,1)} = 2j(-1)^{j+1} \frac{(2j-1)!}{(j-1)!(j+1)!} \left(\sum_{l=j}^{n} (n+1-l) \binom{l+j-1}{l-j} \right).$$

Since for $b_l := (n+1-l) \binom{l+j-1}{l-j}$, one has

$$b_l = s_l - s_{l-1} \quad \text{with} \quad s_l := \frac{(j+l)(n+1+j+2jn-2jl)}{2j(2j+1)} \binom{l+j-1}{l-j},$$

(i.e., s_l is an antidifference of b_l which is found by Gosper's algorithm, see [6], [12]) which can easily be checked, and since $s_{j-1} \equiv 0$ it turns out that

$$\sum_{l=j}^{n} b_l = \sum_{l=j}^{n} (n+1-l) \binom{l+j-1}{l-j}$$

$$= \sum_{l=j}^{n} (s_l - s_{l-1}) = s_n - s_{j-1} = \frac{(j+n)(n+1+j)}{2j(2j+1)} \binom{n+j-1}{n-j}.$$

Therefore, using (40), we finally have

$$a_j^{(n,k)} = (-1)^{k-1} \frac{(j-1)!(j+1)!}{(j-k)!(j+k)!} a_j^{(n,1)}$$

$$= (-1)^{k-1} \frac{(j-1)!(j+1)!}{(j-k)!(j+k)!} 2j(-1)^{j+1} \frac{(2j-1)!}{(j-1)!(j+1)!} \times$$

$$\left(\sum_{l=j}^{n} (n+1-l) \binom{l+j-1}{l-j} \right)$$

$$= (-1)^{k+j} \frac{(2j-1)!}{(j-k)!(j+k)!} \frac{(j+n)(n+1+j)}{(2j+1)} \binom{n+j-1}{n-j}$$

$$= (-1)^{k+j} \binom{2j}{k} \binom{n+j+1}{ } \tag{41}$$

and hence

$$\Lambda_k^n(t) = \sum_{j=k}^n a_j^{(n,k)} e^{-jt} = \sum_{j=k}^n (-1)^{k+j} \binom{2j}{j-k} \binom{n+j+1}{n-j} e^{-jt}. \qquad (42)$$

This, by (34), gives de Branges' closed representation for $\dot{\tau}_k^n(t)$.

Note that in our presentation no knowledge about hypergeometric functions is needed. On the other hand, from representation (41) one can read off

$$\frac{a_{j+1}^{(n,k)}}{a_j^{(n,k)}} = \frac{(j+1/2)(j+n+2)(j-n)}{(j+3/2)(j+k+1)(j-k+1)},$$

and since k is an integer, we may substitute $j \mapsto j + k$ (i.e., shift the summation variable) which leads to

$$\frac{a_{j+k+1}^{(n,k)}}{a_{j+k}^{(n,k)}} = \frac{(j+k+1/2)(j+n+k+2)(j-n+k)}{(j+k+3/2)(j+2k+1)(j+1)}$$

and to the initial value

$$e^{-kt} a_k^{(n,k)} = e^{-kt} \binom{n+k+1}{n-k}$$

so that (42) reads

$$\Lambda_k^n(t) = e^{-kt} \binom{n+k+1}{n-k} {}_3F_2 \left(\begin{matrix} k+1/2, n+k+2, -n+k \\ k+3/2, 2k+1 \end{matrix} \middle| e^{-t} \right).$$

Note, that similarly from (19) one gets the hypergeometric representations

$$A_n(t) = n e^{-t} {}_2F_1 \left(\begin{matrix} 1-n, n+1 \\ 3 \end{matrix} \middle| e^{-t} \right),$$

and for the Gegenbauer polynomials $C_n^{(-1/2)}(x)$ by (24)

$$C_n^{(-1/2)}(x) = (1-x) \, {}_2F_1 \left(\begin{matrix} 1-n, n \\ 2 \end{matrix} \middle| \frac{1-x}{2} \right).$$

6 GENERATING FUNCTION OF THE de BRANGES FUNCTIONS

In this final section, we give a very simple representation of the generating function $B_k(z, t)$ of the de Branges functions

$$B_k(z, t) = \sum_{n=k}^{\infty} \tau_k^n(t) \, z^{n+1}$$

from which one can directly deduce de Branges main inequality $\tau_k^n(t) \geq 0$ (see [4], [5]) without utilizing the inequality for the derivatives $\dot{\tau}_k^n(t) \leq 0$.

Whereas de Branges considered the Milin conjecture for *fixed* $n \in \mathbb{N}$, and therefore introduced $\tau_k^n(t)$ ($k = 1, \ldots, n+1$), we take a fixed $k \in \mathbb{N}$ and the generating function of $\tau_k^n(t)$ with respect to n, hence all $n \geq k$ are considered *at the same time*

THEOREM 3   *The generating function of the de Branges functions has the representation*

$$B_k(z, t) = \sum_{n=k}^{\infty} \tau_k^n(t) \, z^{n+1} = K(z) \, w(z, t)^k$$

$$= K(z) \left(\frac{4e^{-t}z}{\left(1 - z + \sqrt{1 - 2xz + z^2}\right)^2} \right)^k, \tag{43}$$

$(x = 1 - 2e^{-t})$. *Moreover one has the hypergeometric representation*

$$B_k(z, t) = K(z)^{k+1} \, e^{-kt} \, {}_2F_1\left(\begin{array}{c} k, k+1/2 \\ 2k+1 \end{array} \right| \left. -4K(z)e^{-t} \right)$$

$$= \sum_{j=k}^{\infty} (-1)^{j+k} \, \frac{2k}{j+k} \binom{2j-1}{j-k} K(z)^{j+1} \, e^{-jt} \tag{44}$$

being a Taylor series representation with respect to $y = e^{-t} = \frac{1-x}{2}$.

Proof   Define $B_k(z, t)$ by

$$B_k(z, t) := K(z) \, w(z, t)^k ; \tag{45}$$

then

$$\dot{B}_k(z, t) = K(z) \, k \, w(z, t)^{k-1} \, \dot{w}(z, t) .$$

Using (5), we therefore get

$$\frac{\dot{B}_{k+1}(z,t)}{k+1} + \frac{\dot{B}_k(z,t)}{k} = K(z)\,w(z,t)^{k-1}\,\dot{w}(z,t)\,(1+w(z,t))$$

$$= -K(z)\,w(z,t)^k\,(1-w(z,t))$$

$$= K(z)\,w(z,t)^{k+1} \quad K(z)\,w(z,t)^k$$

$$= B_{k+1}(z,t) - B_k(z,t)\,. \tag{46}$$

By the definition (45) of $B_k(z,t)$ its Taylor series for $z=0$ starts with a z^{k+1} term, hence we may write

$$B_k(z,t) = \sum_{n=k}^{\infty} \tau_k^n(t)\,z^{n+1} \tag{47}$$

and we can assume $\tau_{n+1}^n(t) \equiv 0$, hence (26). Substituting (47) in (46) yields furthermore (25) by equating coefficients of z^{n+1}. The $(n+1)^{st}$ Taylor coefficient of

$$B_k(z,0) = K(z)\,w(z,0)^k = \frac{z^{k+1}}{(1-z)^2}$$

equals $n+1-k$, hence the initial values (27) are satisfied, and therefore $\tau_k^n(t)$ form the de Branges functions.

Starting with (22), a calculation shows that $w(z,t)$ has the explicit representation

$$w(z,t) = \frac{4e^{-t}z}{\left(1-z+\sqrt{1-2xz+z^2}\right)^2}\,,$$

hence the right hand representation of (43) follows.

In a similar manner as we derived the closed form representation of the coefficients of Koebe's Löwner chain $w(z,t)$ in Section 2 (or by the method presented in [9–11]), one deduces (44). ∎

To deduce the inequalities $\tau_k^n(t) \geq 0$ as announced, we remark that the Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$ have the generating function

$$\sum_{n=0}^{\infty} P_n^{(\alpha,\beta)}(x)\,z^n = \frac{2^{\alpha+\beta}}{\sqrt{1-2xz+z^2}}\,\frac{1}{\left(1-z+\sqrt{1-2xz+z^2}\right)^\alpha}$$

$$\times \frac{1}{\left(1 \cdots \sqrt{}\right)^\beta}$$

(see e.g. [1], (22.9.1)), hence

$$B_k(z,t) = K(z) \left(\frac{4e^{-t}z}{\left(1 - z + \sqrt{1 - 2xz + z^2}\right)^2} \right)^k$$

$$= z^{k+1} e^{-kt} \frac{1}{1-z} \frac{2^{2k}}{\sqrt{1-2xz+z^2}} \frac{1}{\left(1 - z + \sqrt{1-2xz+z^2}\right)^{2k}}$$

$$\cdot \frac{\sqrt{1-2xz+z^2}}{1-z}$$

$$= z^{k+1} e^{-kt} \cdot \sum_{n=0}^{\infty} \sum_{j=0}^{n} P_j^{(2k,0)}(x) z^n \cdot \sum_{n=0}^{\infty} \sum_{j=0}^{n} C_j^{(-1/2)}(x) z^n ,$$

and the result follows from the positivity of the Jacobi polynomial sums

$$\sum_{j=0}^{n} P_j^{(2k,0)}(x) \geq 0 \tag{48}$$

(see [2], Theorem 3) and the positivity of the Taylor coefficients (with respect to z) of the function

$$\frac{\sqrt{1-2xz+z^2}}{1-z}$$

(see [2], Theorem D). Hence, again, as in de Branges' original proof, the Askey-Gasper result (48) does the main job.

References

[1] Abramowitz, M. and Stegun, I.A. (1964). *Handbook of Mathematical Functions*. Dover Publ., New York.
[2] Askey, R. and Gasper, G. (1976). Positive Jacobi polynomial sums II. *Amer. J. Math.*, **98**, 709–737.
[3] Bieberbach, L. (1916). Über die Koeffizienten derjenigen Potenzreihen, welche eine schlichte Abbildung des Einheitskreises vermitteln. *S.-B. Preuss. Akad. Wiss.*, **38**, 940–955.
[4] de Branges, L. (1985). A proof of the Bieberbach conjecture. *Acta Math.*, **154**, 137–152.
[5] FitzGerald, C.H. and Pommerenke, Ch. (1985). The de Branges Theorem on univalent functions. *Trans. Amer. Math. Soc.*, **290**, 683–690.
[6] Gosper Jr., R.W. (1978). Decision procedure for indefinite hypergeometric summation. *Proc. Natl. Acad. Sci. USA*, **75**, 40–42.
[7] Henrici, P. (1986). *Applied and Computational Complex Analysis, Vol. 3: Discrete Fourier Analysis — Cauchy Integrals — Construction of Conformal Maps — Univalent Functions*. John Wiley & Sons, New York.
[8] Koepf, W. (1994). Von der Bieberbachschen Vermutung zum Satz von de Branges sowie der Beweisvariante von Weinstein. In *Jahrbuch Überblicke Mathematik 1994*, Vieweg-Verlag, Braunschweig-Wiesbaden,
[9] Koepf, W. (1992). Power Series in Computer Algebra.

[10] Koepf, W. (1994). A package on formal power series. *Mathematica Journal*, **4**, 62–69.

[11] Koepf, W. (1994). Algorithmic work with orthogonal polynomials and special functions. *Konrad-Zuse-Zentrum Berlin (ZIB)*, Preprint SC 94–5.

[12] Koepf, W (1995) Algorithms for m-fold hypergeometric summation *J Symbolic Computation*, **20**, 399–417.

[13] Lebedev, N.A. and Milin, I.M. (1965). An inequality. *Vestnik Leningrad Univ.*, **20**, 157–158 (Russian).

[14] Löwner, K. (1923). Untersuchungen über schlichte konforme Abbildungen des Einheitskreises I *Math Ann.*, **89**, 103–121.

[15] Pommerenke, Chr. (1973). *Univalent functions*. Vandenhoeck & Ruprecht, Göttingen-Zürich.

[16] Schmersau, D. (1990). Untersuchungen zur Rekursion von L. de Branges. *Complex Variables*, **15**, 115–124

[17] Todorov, P. (1992). A simple proof of the Bieberbach conjecture. *Bull. Cl. Sci., VI. Sér., Acad. R. Belg. 3*, **12**, 335–356.

[18] Weinstein, L. (1991). The Bieberbach conjecture. *International Mathematics Research Notice*. **5**. 61–64.

[19] Wilf, H. (1994). A footnote on two proofs of the Bieberbach de Branges Theorem. *Bull. London Math. Soc.*, **26**, 61–63.

[20] Wolfram, St. (1988). MATHEMATICA. *A system for doing mathematics by Computer*. Addison-Wesley Publ. Comp., Redwood City.