PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 101, Number 1, September 1987

ON THE FEKETE-SZEGÖ PROBLEM FOR CLOSE-TO-CONVEX FUNCTIONS

WOLFRAM KOEPF

ABSTRACT. Let S be the familiar class of normalized univalent functions in the unit disk. Fekete and Szegö proved the well-known result

$$\max_{f\in S}|a_3 - \lambda a_2^2| = 1 + 2e^{-2\lambda/(1-\lambda)}$$

for $\lambda \in [0,1]$. We consider the corresponding problem for the family C of close-to-convex functions and get

$$\max_{f\in C}|a_3 - \lambda a_2^2| = \begin{cases} 3-4\lambda & \text{if } \lambda\in[0,1/3], \\ 1/3+4/(9\lambda) & \text{if } \lambda\in[1/3,2/3], \\ 1 & \text{if } \lambda\in[2/3,1]. \end{cases}$$

As an application it is shown that $||a_3|-|a_2|| \le 1$ for close-to-convex functions, in contrast to the result in S

$$\max_{f\in S}||a_3|-|a_2|| = 1.029\dots.$$

1. Introduction. Let S denote the family of univalent functions f of the unit disk, normalized by

$$f(z) = z + a_2 z^2 + a_3 z^3 + \cdots. \tag{1}$$

Let St denote the subset of starlike functions, i.e. functions that have a starlike range with respect to the origin. Further let C denote the family of close-to-convex functions, which have been introduced by Kaplan [**4**]. A function f, normalized by (1), is called close-to-convex if there exist a starlike function g and a real number α, such that

$$\operatorname{Re}(e^{i\alpha} z f'(z)/g(z)) > 0, \qquad \alpha \in]-\pi/2, \pi/2[.$$

It turns out that a function is close-to-convex if and only if it maps the unit disk univalently onto a domain whose complement is the union of half-lines, which are pairwise disjoint up to possibly equal tips (see [**6**–**7**, **1**]).

A well-known function of this kind is the Koebe function k with

$$k(z) = \sum_{n=1}^{\infty} n z^n = \frac{z}{(1-z)^2} = \frac{1}{4}\left(\left(\frac{1+z}{1-z}\right)^2 - 1\right),$$

which maps the unit disk onto the complement of the half-line $]-\infty, -1/4]$, as the last representation shows.

Many extremal problems within the class S are solved by the Koebe function. On the other hand, the Koebe function satisfies

$$|a_3 - \lambda a_2^2| = |3 - 4\lambda|,$$

Received by the editors November 19, 1985 and, in revised form, May 30, 1986.
1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 30C45, 30C50.
Key words and phrases. Close-to-convex functions, univalent functions.

whereas Fekete and Szegö showed [**3**]

$$\max_{f\in S}|a_3-\lambda a_2^2| = 1+2e^{-2\lambda/(1-\lambda)}$$

for $\lambda\in[0,1]$,

For $\lambda=0,1$ the Koebe function gives the maximum, but there is no $\lambda_0\in]0,1[$ such that the functional $|a_3-\lambda_0 a_2^2|$ is maximized by k. We shall show that

$$\max_{f\in C}|a_3-\lambda a_2^2| = 3-4\lambda$$

for $\lambda\in[0,1/3]$, so that for close-to-convex functions the situation is quite different. This result implies furthermore that

$$\max_{f\in C}|\,|a_3|-|a_2|\,| = 1,$$

in contrast to the known estimate in S,

$$\max_{f\in S}|\,|a_3|-|a_2|| = 1.029\ldots$$

(see e.g. [**2**, Theorem 3.11]). Moreover we show that

$$\max_{f\in C}|a_3-\lambda a_2^2| = \begin{cases} 1/3+4/(9\lambda) & \text{if } \lambda\in[1/3,2/3],\\ 1 & \text{if } \lambda\in[2/3,1].\end{cases}$$

2. Preliminary results. Here we give some lemmas which will be used in the next section to solve the main problem.

Recall that a function f is called close-to-convex of order β if there exist a starlike function g and a real number α, such that

$$|\arg(e^{i\alpha}zf'(z)/g(z))| < \beta\pi/2.$$

LEMMA 1 (see [**5**, Lemma 1]). *Let $f\in C$. Then the function h, defined by*

$$h'(z) = (f'(z^2))^{1/2}, \qquad h(0)=0, \tag{2}$$

is an odd close-to-convex function of order $1/2$.

LEMMA 2 (see [**8**, p. 166, formula (10)]). *Let $p(z)=1+p_1z+p_2z^2+\cdots$ and* $\operatorname{Re}p>0$. *Then*

$$|p_2-p_1^2/2|\le 2-|p_1|^2/2.$$

LEMMA 3. *Let $g(z)=z+b_2z^2+b_3z^3+\cdots\in St$. Then*

$$|b_3-\lambda b_2^2|\le\max\{1,|3-4\lambda|\}$$

which is sharp for the Koebe function k if $|\lambda-3/4|\ge 1/4$ and for $(k(z^2))^{1/2}=z/(1-z^2)$ if $|\lambda-3/4|\le 1/4$.

PROOF. Because $g\in St$, the function

$$zg'(z)/g(z) = 1+b_2z+(2b_3-b_2^2)z^2+\cdots = 1+p_1z(3)+p_2z^2+\cdots$$

has positive real part, so that $|p_2-\frac12 p_1^2|\le 2-|p_1|^2/2$ by Lemma 2. Let now $\lambda\in\mathbf{C}$. Then by (3) we have

$$\begin{aligned}|b_3-\lambda b_2^2| &= \tfrac12|p_2+(1-2\lambda)p_1^2| = \tfrac12\left|p_2-\tfrac12p_1^2+\left(\tfrac32-2\lambda\right)p_1^2\right|\\ &\le \tfrac12\left(2-\tfrac12|p_1|^2+\left|\tfrac32-2\lambda\right||p_1|^2\right)^2.\end{aligned}$$

If now $|\lambda - 3/4| \leq \frac{1}{4}$, then

$$|b_3 - \lambda b_2^2| \leq \tfrac{1}{2}\left(2 - \tfrac{1}{2}|p_1|^2 + \tfrac{1}{2}|p_1|^2\right) = 1.$$

On the other hand, if $|\lambda - 3/4| \geq \frac{1}{4}$, then we use $|p_1| \leq 2$ (see e.g. [**8**, Corollary 2.3]), and get

$$\begin{aligned} |b_3 - \lambda b_2^2| &\leq 1 + \tfrac{1}{2}\left(\left|\tfrac{3}{2} - 2\lambda\right| - \tfrac{1}{2}\right)|p_1|^2 \\ &\leq 1 + |3 - 4\lambda| - 1 = |3 - 4\lambda|. \quad \square \end{aligned}$$

3. Main results. The first step of the solution of the Fekete-Szegö problem for close-to-convex functions is the special case $\lambda = 1/3$.

THEOREM 1. *Let $f(z) = z + a_2 z^2 + a_3 z^3 + \cdots \in C$. Then $|a_3 - \frac{1}{3}a_2^2| \leq \frac{5}{3}$.*

PROOF. Let $f \in C$. Then by Lemma 1 the function h, defined by (2), is an odd close-to-convex function of order 1/2.

For such functions, the author gave sharp bounds on the coefficients (see [**5**, Theorem 1]), in particular, the fifth coefficient of h is bounded in modulus by 1/2. On the other hand the fifth coefficient of h is given by $\frac{3}{10}(a_3 - \frac{1}{3}a_2^2)$, which implies the result. $\square$

The next corollary follows easily from the theorem using $|a_2| \leq 2$ (see e.g. [**2**, Theorem 2.2]).

COROLLARY 1. *Let $\lambda \in [0, 1/3]$. Then*

$$\max_{f \in C} |a_3 - \lambda a_2^2| = 3 - 4\lambda.$$

The maximum is attained by the Koebe function.

Another consequence of the theorem is the following result about successive coefficients of close-to-convex functions.

COROLLARY 2. *Let $f \in C$. Then $\left||a_3| - |a_2|\right| \leq 1$.*

PROOF. It is well known that $|a_2| - |a_3| \leq 1$ for all $f \in S$ (see e.g. [**2**, Theorem 3.11]). Moreover, if $|a_2| \leq 1$, then also $|a_3| - |a_2| \leq 1$ (see e.g. [**2**, proof of Theorem 3.11]). Now let $f \in C$ and $|a_2| \in [1, 2]$. Then Theorem 1 implies that

$$\begin{aligned} |a_3| - |a_2| &\leq |a_3 - \tfrac{1}{3}a_2^2| + \tfrac{1}{3}|a_2|^2 - |a_2| \\ &\leq \tfrac{5}{3} + \tfrac{1}{3}|a_2|^2 - |a_2| \leq 1, \end{aligned}$$

as $|a_2|$ is in the above range. $\square$

The following notation will be used throughout the paper. For $f(z) = z + a_2 z^2 + a_3 z^3 + \cdots \in C$ there is a representation of the form

$$f'(z) = \frac{g(z)}{z} \cdot \tilde{p}(z) \tag{4}$$

with some function $g(z) = z + b_2 z^2 + b_3 z^3 + \cdots \in St$ and some function $\tilde{p}(z) = 1 + \tilde{p}_1 z + \tilde{p}_2 z^2 + \cdots$ such that $\mathrm{Re}(e^{i\alpha}\tilde{p}(z)) > 0$, $\alpha \in]-\pi/2, \pi/2[$. Then the function $p(z) = 1 + p_1 z + p_2 z^2 + \cdots$, defined by

$$\tilde{p}_n = \cos\alpha \cdot e^{-i\alpha} \cdot p_n, \qquad n \in \mathbf{N}, \tag{5}$$

has positive real part. Comparing coefficients in (4) we get

$$3a_3 = b_3 + \tilde{p}_1 b_2 + \tilde{p}_2, \qquad 2a_2 = b_2 + \tilde{p}_1,$$

so that

$$a_3 - \lambda a_2^2 = \tfrac{1}{3}(b_3 - \tfrac{3}{4}\lambda b_2^2) + \tfrac{1}{3}(\tilde{p}_2 - \tfrac{3}{4}\lambda \tilde{p}_1^2) + \tilde{p}_1 b_2(\tfrac{1}{3} - \lambda/2). \tag{6}$$

Now we consider the case $\lambda = 2/3$.

THEOREM 2. *Let* $f(z) = z = a_2 z^2 + a_3 z^3 + \cdots \in C$. *Then* $|a_3 - \frac{2}{3}a_2^2| \le 1$.

PROOF. From (6) it follows that

$$|a_3 - \tfrac{2}{3}a_2^2| \le \tfrac{1}{3}|b_3 - \tfrac{1}{2}b_2^2| + \tfrac{1}{3}|\tilde{p}_2 - \tfrac{1}{2}\tilde{p}_1^2|.$$

From (5) we get

$$\begin{aligned}\tilde{p}_2 - \tfrac{1}{2}\tilde{p}_1^2 &= \cos\alpha \cdot e^{-i\alpha}(p_2 - \tfrac{1}{2}\cos\alpha \cdot e^{-i\alpha}p_1^2)\\ &= \cos\alpha \cdot e^{-i\alpha}(p_2 - \tfrac{1}{2}p_1^2 + \mu p_1^2),\end{aligned}$$

where $|2\mu|^2 = |1 - \cos\alpha \cdot e^{-i\alpha}|^2 = \sin^2\alpha$. Now we get with the aid of Lemmas 2 and 3 that

$$\begin{aligned}\left|a_3 - \frac{2}{3}a_2^2\right| &\le \frac{1}{3} + \frac{1}{3}\cos\alpha\left(2 - \frac{|p_1|^2}{2}\right) + \frac{1}{3}\cos\alpha|\sin\alpha|\frac{|p_1|^2}{2}\\ &\le 1 - \cos\alpha\frac{|p_1|^2}{6}(1 - |\sin\alpha|) \le 1. \quad \square\end{aligned}$$

An easy consequence using $|a_3 - a_2^2| \le 1$ is

COROLLARY 3. *Let* $\lambda \in [2/3, 1]$. *Then*

$$\max_{f \in C}|a_3 - \lambda a_2^2| = 1.$$

The maximum is attained by the function $(k(z^2))^{1/2}$. $\square$

We remark that Theorem 2 provides a direct proof of $|a_3| - |a_2| \le 1$ for $|a_2| \le 3/2$ (compare with the proof of Corollary 2), namely

$$\begin{aligned}|a_3| - |a_2| &\le |a_3 - \tfrac{2}{3}a_2^2| + \tfrac{2}{3}|a_2|^2 - |a_2|\\ &\le 1 + \tfrac{2}{3}|a_2|^2 - |a_2| \le 1\end{aligned}$$

if $|a_2| \in [0, 3/2]$.

It remains to consider the case $\lambda \in]1/3, 2/3[$.

THEOREM 3. *Let* $\lambda \in]1/3, 2/3[$. *Then*

$$\max_{f \in C}|a_3 - \lambda a_2^2| = \frac{1}{3} + \frac{4}{9\lambda}.$$

The maximum is attained by the function f, *which is defined by*

$$f'(z) = \frac{1}{(1-z)^2} \cdot \left(t\frac{1+z}{1-z} + (1-t)\frac{1+z^2}{1-z^2}\right), \qquad f(0) = 0,$$

where $t = 2/(3\lambda) - 1$.

PROOF. Consider equation (6). We use the estimate $|b_3 - \frac{3}{4}\lambda b_2^2| \le 3(1-\lambda)$, which comes from Lemma 3, further equations (5) and $|b_2| \le 2$, getting

$$|a_3 - \lambda a_2^2| \le 1 - \lambda + \frac{\cos\alpha}{3}\left|p_2 - \frac{3}{4}\lambda\cos\alpha \cdot e^{-i\alpha}p_1^2\right| + \cos\alpha\left(\frac{2}{3} - \lambda\right)|p_1|.$$

Writing $\frac{3}{4}\lambda\cos\alpha\cdot e^{-i\alpha}=\frac{1}{2}-\mu$, we have

$$|2\mu|^2=|1-\tfrac{3}{2}\lambda\cos\alpha\cdot e^{-i\alpha}|^2=1-(3\lambda-\tfrac{9}{4}\lambda^2)\cos^2\alpha,$$

which implies with the aid of Lemma 2 that

$$\left|p_2-\frac{3}{4}\lambda\cos\alpha\cdot e^{-i\alpha}p_1^2\right|\le 2+\frac{|p_1|^2}{2}\left(\sqrt{1-\left(3\lambda-\frac{9}{4}\lambda^2\right)\cos^2\alpha}-1\right),$$

so that—using the notations $y:=\cos\alpha$ and $p:=|p_1|$—it follows that

$$\begin{aligned}|a_3-\lambda a_2^2|&\le 1-\lambda+y\left(\frac{2}{3}+\frac{p^2}{6}\left(\sqrt{1-\left(3\lambda-\frac{9}{4}\lambda^2\right)y^2}-1\right)+p\left(\frac{2}{3}-\lambda\right)\right)\\&=:F_\lambda(p,y).\end{aligned}$$

For further simplification we shall use the notation $\gamma:=2-3\lambda$.

Now we shall show that F_λ attains its maximum value for $(p,y)\in[0,2]\times[0,1]$ at the point $(4/(3\lambda)-2,1)$. Observe that

$$F_\lambda\left(\frac{4}{3\lambda}-2,1\right)=\frac{1}{3}+\frac{4}{9\lambda}. \tag{7}$$

Suppose now that F_λ attains its maximum value at an interior point $(p_0,y_0)\in\,]0,2[\times]0,1[$. Then the partial derivates $\partial F_\lambda/\partial p$ and $\partial F_\lambda/\partial y$ must vanish at (p_0,y_0). The equality $(\partial F_\lambda/\partial p)(p_0,y_0)=0$ gives the relation

$$\sqrt{1-\left(3\lambda-\frac{9}{4}\lambda^2\right)y_0^2}-1=-\frac{\gamma}{p_0}, \tag{8}$$

so that

$$\left(3\lambda-\frac{9}{4}\lambda^2\right)y_0^2=\frac{2\gamma}{p_0}-\frac{\gamma^2}{p_0^2}.$$

Now, $(\partial F_\lambda/\partial y)(p_0,y_0)=0$ implies

$$\frac{2}{3}+\frac{\gamma p_0}{6}=\frac{p_0^2(2\gamma/p_0-\gamma^2/p_0^2)}{6(1-\gamma/p_0)},$$

so that, by solving the quadratic equation for p_0, we get

$$\gamma p_0=2\left(1-\sqrt{1-\gamma^2}\right). \tag{9}$$

Therefore, at (p_0,y_0) the value of F_λ becomes, using (8) and (9),

$$\begin{aligned}F_\lambda(p_0,y_0)&=1-\lambda+y\left(\frac{2}{3}+\frac{1}{3}\left(1-\sqrt{1-\gamma^2}\right)\right)\\&\le\frac{4+\gamma-\sqrt{1-\gamma^2}}{3},\end{aligned} \tag{10}$$

because $y\le 1$.

Since $\lambda\in]1/3,2/3[$, the number γ lies between 0 and 1 so that there is some $\delta\in\,]0,\pi/2[$ with $\gamma=\cos\delta$ and $\sqrt{1-\gamma^2}=\sin\delta$. The evident inequality $1<\cos\delta+\sin\delta$

implies

$$\begin{aligned} 2 - \cos\delta &< 1 + \sin\delta \\ &\Rightarrow (2-\cos\delta)(1-\sin\delta) < 1 - \sin^2\delta = \cos^2\delta \\ &\Rightarrow (2-\gamma)\left(1-\sqrt{1-\gamma^2}\right) < \gamma^2 \\ &\Rightarrow (2-\gamma)\left(4+\gamma-\sqrt{1-\gamma^2}\right) < 6-\gamma \\ &\Rightarrow \frac{4+\gamma-\sqrt{1-\gamma^2}}{3} < \frac{1}{3} + \frac{4}{3(2-\gamma)} = \frac{1}{3} + \frac{4}{9\lambda}. \end{aligned}$$

Thus, using (7) and (10), we get a contradiction to our assumption that F_λ attains its maximum value at (p_0, y_0), so that the maximum must be attained at a boundary point.

In both cases $y = 0$ and $p = 0$ an easy computation shows that the maximal value (7) is not attained. If $y = 1$ we have

$$F_\lambda(p,1) =: G_\lambda(p) = \frac{5}{3} - \lambda + \left(\frac{2}{3} - \lambda\right) p - \frac{\lambda}{4} p^2.$$

Because $G_\lambda(2) = 3 - 4\lambda$ is not maximal, the local maximum at $p = 4/(3\lambda) - 2$—given by $dG_\lambda(p)/dp = 0$—is global. This leads to the maximal value (7).

Now it remains to prove that

$$F_\lambda(p,y) \le \frac{1}{3} + \frac{4}{9\lambda}$$

for $p = 2,\ y \in]0,1[$. This statement is equivalent to

$$H_\gamma(y) := 2y\left(\sqrt{1-\left(1-\frac{\gamma^2}{4}\right)y^2} + \gamma\right) \le \frac{4}{2-\gamma} - \gamma \tag{11}$$

for $\gamma = 2 - 3\lambda \in]0,1[$. Because we already know that $H_\gamma(y) \le 4/(2-\gamma) - \gamma$ when $y \in \{0,1\}$, it suffices to show (11) for points with $dH_\gamma(y)/dy = 0$. This leads to

$$\left(1-\frac{\gamma^2}{4}\right) y^2 = \frac{4-\gamma^2+\gamma\sqrt{8+\gamma^2}}{8}. \tag{12}$$

Observe that $0 \le y \le 1$ when (12) is satisfied. Squaring inequality (11) and substituting (12) gives the following inequality:

$$\begin{aligned} &4\left(\frac{4-\gamma^2+\gamma\sqrt{8+\gamma^2}}{8}\right)\left(\frac{\sqrt{8+\gamma^2}-\gamma}{4} + \gamma\right)^2 \\ &\qquad \le \left(1-\frac{\gamma^2}{4}\right)\left(\frac{4}{2-\gamma} - \gamma\right)^2. \end{aligned} \tag{13}$$

It remains to prove (13). A lengthy calculation gives—after multiplying with the number $(2-\gamma)$, which is positive—the equivalent version

$$\begin{aligned} \gamma(2-\gamma)(8+\gamma^2)^{3/2} &\le (4+2\gamma)(4-2\gamma+\gamma^2)^2 - (2-\gamma)(8+20\gamma^2-\gamma^4) \\ &= 48 - 24\gamma - 24\gamma^2 + 28\gamma^3 - 2\gamma^4 + \gamma^5. \end{aligned}$$

The right-hand side turns out to be positive:

$$48 - 24\gamma - 24\gamma^2 + 28\gamma^3 - 2\gamma^4 + \gamma^5 > 28\gamma^3 - 2\gamma^4 + \gamma^5 = \gamma^3(28 - 2\gamma + \gamma^2) > \gamma^3(26 + \gamma^2) \geq 0,$$

so that equivalently, squaring again

$$\gamma^2(2-\gamma)^2(8+\gamma^2)^3 \leq (48 - 24\gamma - 24\gamma^2 + 28\gamma^3 - 2\gamma^4 + \gamma^5)^2.$$

A further lengthy computation gives the equivalent reformulation

$$\begin{aligned}\gamma^8 - 2\gamma^7 + 17\gamma^6 - 12\gamma^5 - 70\gamma^4 + 184\gamma^3 - 118\gamma^2 - 72\gamma + 72 \\ = (1-\gamma)^2(\gamma^6 + 16\gamma^4 + 20\gamma^3 - 46\gamma^2 + 72\gamma + 72) \\ = (1-\gamma)^2(\gamma^6 + 16\gamma^4 + 20\gamma^3 + 26\gamma^2 + 72\gamma(1-\gamma) + 72) \geq 0,\end{aligned}$$

which is trivially true. This finishes the proof of the inequality

$$|a_3 - \lambda a_2^2| \leq \frac{1}{3} + \frac{4}{9\lambda}.$$

From our considerations it follows that equality holds if $b_2 = 2$ and $b_3 = 3$ (so that g is a rotation of k), and if $\alpha = 0$, $p_2 = 2$, and $p_1 = 4/(3\lambda) - 2$; the function

$$\tilde{p}(z) = t\left(\frac{1+z}{1-z}\right) + (1-t)\left(\frac{1+z^2}{1-z^2}\right), \qquad t = \frac{2}{3\lambda} - 1,$$

satisfies these conditions, which makes the result sharp. □

References

1. A. Bielecki and Z. Lewandowski, *Sur un théorème concernant les fonctions univalentes linéairement accessibles de M. Biernacki*, Ann. Polon. Math. **12** (1962), 61–63.
2. P. L. Duren, *Univalent functions*, Springer-Verlag, New York-Berlin-Heidelberg-Tokyo, 1983.
3. M. Fekete and G. Szegö, *Eine Bemerkung über ungerade schlichte Functionen*, J. London Math. Soc. **8** (1933), 85–89.
4. W. Kaplan, *Close-to-convex schlicht functions*, Michigan Math. J. **1** (1952), 169–185.
5. W. Koepf, *Coefficients of symmetric functions of bounded boundary rotation* (to appear).
6. Z. Lewandowski, *Sur l'identité de certaines classes de fonctions univalentes*. I, Ann. Univ. Mariae Curie-Skłodowska **12** (1958), 131–146.
7. ——, *Sur l'identité de certaines classes de fonctions univalentes*. II, Ann. Univ. Mariae Curie-Skłodowska **14** (1960), 19–46.
8. Ch. Pommerenke, *Univalent functions*, Vandenhoeck & Ruprecht, Göttingen, 1975.

FREIE UNIVERSITÄT BERLIN, FACHBEREICH MATHEMATIK, ARNIMALLEE 3, 1000 BERLIN 33, FEDERAL REPUBLIC OF GERMANY