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CLOSE-TO-CONVEX FUNCTIONS AND LINEAR-INVARIANT FAMILIES

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1. Introduction. Landau showed in 1925 [6] that in the class S of normalized schlicht functions on the unit disk we can get a distortion theorem for the n-th derivative if we have ensured the first n Bieberbach coefficient estimates to be correct.

We shall modify this result for linear-invariant families. Families of close-to-convex functions and of functions of bounded boundary rotation will be showed to be linear-invariant.

Because of the coefficient estimate for close-to-convex functions and functions of bounded boundary rotation derived by Aharonov and Friedland [1], it is possible to get the distortion theorem for the n-th derivative for all n, but here we obtain the same conclusion more elementarily (and without using the linear-invariance), just because the coefficient estimate is given for all n.

All functions f considered here are analytic functions on the unit disk with normalization $f(0)=0, f'(0)=1$, and they are locally schlicht, i.e., $\{z|f'(z)=0\}=\emptyset$. Let N be the class of such functions.

Pommerenke defined a linear-invariant family in [9] and showed some properties of such families. A subset F of N is called linear-invariant if it is closed under the re-normalized composition with a schlicht automorphism of the unit disk. If the modulus of the second Taylor coefficient is bounded in F, we define the order α of the linear-invariant family to be

$$\alpha := \tfrac{1}{2} \sup_{f \in F} |f''(0)|. \tag{1}$$

An example of a linear-invariant family of order 2 is the class S of normalized schlicht functions on the unit disk.

Pommerenke [9] (pp. 115—116) generalized the well-known Bieberbach distortion theorems [2] (see [12] p. 178) for S to the concept of linear-invariant families and showed for a linear-invariant family F of order α the relations

$$\begin{cases} |f(z)| \leqq \dfrac{1}{2\alpha}\left(\left(\dfrac{1+|z|}{1-|z|}\right)^{\alpha}-1\right), \\ |f'(z)| \leqq \dfrac{(1+|z|)^{\alpha-1}}{(1-|z|)^{\alpha+1}}. \end{cases} \tag{2}$$

We want to give further examples of linear-invariant families. Let V_k be the class of functions of bounded boundary rotation $k\pi$ (see Lehto [7])

$$V_k := \left\{ f\in N \;\middle|\; \forall r\in[0, 1[\left[\int_0^{2\pi} \left|\mathrm{Re}\left(1+z\frac{f''}{f'}\right)\right| d\vartheta \leqq k\pi \right],\; z = re^{i\vartheta} \right\}$$

for $k\in[2, \infty[$. Let further C_β be the class of close-to-convex functions of order β defined by Reade [11] and Pommerenke [10],

$$C_\beta := \left\{ f\in N \;\middle|\; \exists\varphi \text{ schlicht with convex range } \left[\left|\arg\frac{f'}{\varphi'}\right| \leqq \beta\frac{\pi}{2} \right] \right\},$$

for $\beta\in[0, \infty[$.

Properties of these classes are given in the book of Schober [12] (Chapter 2).

As special cases we have $V_2=C_0$, the well-known class of normalized convex functions, and C_1, the class of close-to-convex functions defined by Kaplan [5]. The classes V_k and C_β are increasing in k and β, respectively, and until $k=4$ and $\beta=1$ they contain only schlicht functions.

Aharonov and Friedland [1] showed that the Taylor coefficients of functions in C_β as well as in V_k are dominated in modulus by the corresponding coefficients of the function h_α defined by

$$h_\alpha(z) := \frac{1}{2\alpha}\left(\left(\frac{1+z}{1-z}\right)^\alpha - 1\right)$$

with $\alpha:=k/2$ resp. $\alpha:=\beta+1$. That means: For $f\in V_{2\alpha}$ or $f\in C_{\alpha-1}$ we have

$$|f^{(n)}(0)| \leqq h_\alpha^{(n)}(0). \tag{3}$$

In the proof of this inequality they used the inclusion

$$V_{2\alpha} \subset C_{\alpha-1}. \tag{4}$$

As closed normal families all classes V_k and C_β are compact with respect to the topology of locally uniform convergence.

Now we prove the linear-invariance of these classes.

2. Lemma. *For every $\beta\in[0, \infty[$ the family C_β is linear-invariant of order $\beta+1$. For every $k\in[2, \infty[$ the family V_k is linear-invariant of order $k/2$.*

Proof. Reade [11] and Pommerenke [10] showed the desired property for C_β if $\beta\in[0, 1]$. In this case the functions are all schlicht and so this property follows from a geometrical description of the classes.

We now take an arbitrary $\beta\in[0, \infty[$. Let $f\in C_\beta$ with convex φ such that

$$\left|\arg\frac{f'}{\varphi'}\right| \leqq \beta\frac{\pi}{2}.$$

Our first step will be to show that C_β has the rotation-invariance property, which means

$$f\in C_\beta \Rightarrow f_x\in C_\beta$$

whenever $|x|=1$ and

$$f_x(z) := \frac{f(xz)}{x}.$$

The function φ_1 defined by $\varphi_1(z):=\varphi(xz)/x$ has convex range and obeys the inequality

$$\left|\arg\frac{f_x'}{\varphi_1'}\right| \leqq \beta\frac{\pi}{2}.$$

So C_β inherits this property from C_0. We show now that C_β inherits the linear-invariance property, too. Therefore it is enough to show that for

$$l(z) = \frac{z+r}{1+rz} \quad \text{with} \quad r\in[0, 1[$$

and for $f\in C_\beta$ also the function

$$g := \frac{f\circ l - f\circ l(0)}{(f\circ l)'(0)}$$

is in C_β. Now we have to find a convex φ_2 with

$$\left|\arg\frac{g'(z)}{\varphi_2'(z)}\right| \leqq \beta\frac{\pi}{2}.$$

We get

$$\left|\arg\frac{g'(z)}{\varphi_2'(z)}\right| = \left|\arg\frac{f'(l)\,l'(z)}{f'(r)\,\varphi_2'(z)}\right| = \left|\arg\frac{f'(l)}{\varphi_2'(l)} + \arg\frac{\varphi_2'(l)\,l'(z)}{\varphi_2'(z)f'(r)}\right|.$$

Since f is in C_β, this expression will be less than or equal to $\beta\pi/2$ if we take

$$\varphi_2 := \frac{\varphi\circ l}{f'(r)}.$$

One sees from the geometric definition that the convexity of φ implies the convexity of φ_2.

The order is given by the coefficient domination theorem (3).

In the case of the families V_k the same argumentation gives the order. The linear-invariance property is a consequence of the geometrical interpretation of the definition. Because the ranges of f and g are similar, the limit boundary rotation of the two functions coincide,

$$\lim_{r\to 1}\int_0^{2\pi}\left|\operatorname{Re}\left(1+\frac{zf''}{f'}\right)\right| d\vartheta = \lim_{r\to 1}\int_0^{2\pi}\left|\operatorname{Re}\left(1+\frac{zg''}{g'}\right)\right| d\vartheta,$$

since the integrals are monotone in r (see [8], p. 12) and the suprema are equal. Lehto [7] (p. 12) already used the linear-invariance of V_k. □

Now we come to our main result.

3. Theorem. *Let* $\alpha\in[1,\infty[$, *let* F *be a linear-invariant family of order* α *and* $n\geqq 2$. *If for all* $f\in F$ *and all* m, $2\leqq m\leqq n$,

$$|f^{(m)}(0)|\leqq h_\alpha^{(m)}(0),$$

then the corresponding distortion theorems

$$|f^{(m)}(re^{i\vartheta})|\leqq h_\alpha^{(m)}(r)$$

hold for all $r\in[0,1[$ *and all* $\vartheta\in\mathbf{R}$.

In particular we get for all linear-invariant families of order α

$$|f''(re^{i\vartheta})|\leqq 2(\alpha+r)\frac{(1+r)^{\alpha-2}}{(1-r)^{\alpha+2}}=h_\alpha''(r).$$

Proof. We generalize a result due to Landau [6] (see [12], p. 179).

We want to transform the information about $|f^{(m)}(0)|$ from the origin to an arbitrary point. Every linear-invariant family is of course rotation-invariant, and so we only need to consider a positive real point r.

Let be $f\in F$ and l the Möbius-transform with

$$l(z)=\frac{z+r}{1+rz}$$

and let g be the composition

$$g=f\circ l.$$

If g has the expansion

$$g(z)=\sum_{m=0}^{\infty}c_m z^m,$$

we get for f

$$f(z)=g\circ l^{-1}(z)=g\left(\frac{z-r}{1-rz}\right)=\sum_{m=0}^{\infty}c_m\left(\frac{z-r}{1-rz}\right)^m.$$

Because of the generalized product rule

$$f=uv\Rightarrow f^{(n)}=\sum_{k=0}^{n}\binom{n}{k}u^{(k)}v^{(n-k)}$$

and the formula

$$[(z-r)^m]^{(n)}|_{z=r}=n!\,\delta_{nm}$$

we get

$$f^{(n)}(r)=\sum_{m=0}^{n}c_m\binom{n}{m}m!\,[(1-rz)^{-m}]^{(n-m)}\Big|_{z=r}$$

and further

$$f^{(n)}(r)=n!\sum_{m=1}^{n}c_m r^{n-m}\binom{n-1}{m-1}(1-r^2)^{-n}. \tag{5}$$

Because of the linear-invariance property it follows from $f \in F$ that

$$\frac{g-g(0)}{g'(0)} \in F$$

and so the given coefficient estimate shows

$$|c_m| \leqq \frac{h_\alpha^{(m)}(0)}{m!}|g'(0)| \quad \text{for all} \quad m \leqq n.$$

If we take (5) with $n:=1$ we get

$$|c_1| = |g'(0)| = (1-r^2)|f'(r)|.$$

At that stage we utilize the linear-invariance property for the second time, using the distortion theorem (2) for the first derivative. So we get

$$|c_m| \leqq \frac{h_\alpha^{(m)}(0)}{m!}\left(\frac{1+r}{1-r}\right)^\alpha \quad \text{for all} \quad m \leqq n$$

and

$$|f^{(n)}(r)| \leqq n! \sum_{m=1}^{n} \frac{h_\alpha^{(m)}(0)}{m!}\left(\frac{1+r}{1-r}\right) r^{n-m}\binom{n-1}{m-1}(1-r^2)^{-n} \tag{6}$$

(as all terms here are positive). We shall show that the right-hand term equals $h_\alpha^{(n)}(r)$. With $f:=h_\alpha$ we get

$$h_\alpha \circ l(z) = \frac{1}{2\alpha}\left(\left(\frac{1+l(z)}{1-l(z)}\right)^\alpha - 1\right) = \frac{1}{2\alpha}\left(\left(\frac{1+r}{1-r}\right)^\alpha\left(\frac{1+z}{1-z}\right)^\alpha - 1\right)$$

and we write

$$h_\alpha \circ l(z) = Ah_\alpha(z) + B$$

with

$$A = \left(\frac{1+r}{1-r}\right)^\alpha,$$

$$B = h_\alpha(r).$$

So we have

$$h_\alpha \circ l^{(m)}(0) = \left(\frac{1+r}{1-r}\right)^\alpha h_\alpha^{(m)}(0),$$

and the right-hand side of (6) gets the form

$$n! \sum_{m=1}^{n} \frac{h_\alpha \circ l^{(m)}(0)}{m!} r^{n-m}\binom{n-1}{m-1}(1-r^2)^{-n}.$$

Looking back to formula (5) we see that this is an expression for $h_\alpha^{(n)}(r)$. So we get our conclusion for the index $m:=n$. For $m<n$ the proof coincides with the given one and our result follows.

In the special case $n:=2$ we get the distortion theorem because of the definition of the order. (Bieberbach was the first who proved this distortion theorem in the class S [3].) □

4. Corollary. *Let* $\alpha\in[1,\infty[$ *and* $n\in N_0$. *Then the following equality holds:*

$$\max_{f\in C_{\alpha-1}}\max_{\vartheta\in R}|f^{(n)}(re^{i\vartheta})| = \max_{f\in V_{2\alpha}}\max_{\vartheta\in R}|f^{(n)}(re^{i\vartheta})| = h_\alpha^{(n)}(r).$$

Proof. Because of the compactness of the classes the maximum exists. Formulae (2) for $n\in\{0,1\}$ and our theorem for $n\geqq 2$ show what maximum we can hope to get.

The well-known results

$$h_\alpha\in V_{2\alpha} \quad\text{and}\quad h_\alpha\in C_{\alpha-1}$$

make the results sharp. □

5. Remark. The theorem we proved shows that the linear-invariance property helps us to obtain successive distortion theorems for the n-th derivative in an arbitrary linear-invariant family from the corresponding coefficient estimates.

But if we have — as in the cases C_β and V_k — the coefficient estimates for all n, we can get the distortion theorems more elementarily and without using the linear-invariance property from the following Lemma.

The Lemma arises from a note of Doppel and Volkmann [4], who used it solving a similar problem for another class.

6. Lemma. *Let in the unit disk*

$$f(z)=\sum_{n=0}^{\infty}a_nz^n$$

and

$$g(z)=\sum_{n=0}^{\infty}b_nz^n$$

with $b_n\in[0,\infty[$ *for all* n. *If*

$$|a_n|\leqq b_n$$

holds for all n, *we get*

$$|f^n(z)|\leqq g^{(n)}(|z|)$$

for all z *in the unit disk.*

Proof. The identity

$$f^{(n)}(z)=n!\sum_{k=n}^{\infty}\binom{k}{n}a_kz^{k-n}$$

ane the corresponding one for g imply

$$|f^n(z)|=n!\left|\sum_{k=n}^{\infty}\binom{k}{n}a_kz^{k-n}\right|\leqq n!\sum_{k=n}^{\infty}\left|\binom{k}{n}a_kz^{k-n}\right|=n!\sum_{k=n}^{\infty}\binom{k}{n}|a_k|\,|z|^{k-n}$$

$$\leqq n!\sum_{k=n}^{\infty}\binom{k}{n}b_k|z|^{k-n}=g^{(n)}(|z|).\quad\square$$

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