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On Close-to-convex Functions and Linearly Accessible Domains

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Lewandowski ([9] and [10]) proved that a function is close-to-convex if and only if its image domain is linearly accessible [2]. The proof of both implications was astonishingly complicated, so Lewandowski himself looked for some simpler argument. He and Bielecki [1] finally gave an elegant proof that a close-to-convex function is linearly accessible using a Löwner type argument, but the problem of giving a simpler proof for the other implication remained open.

Pommerenke [11] introduced the notion of close-to-convex functions of order β and gave a geometrical description of their image domains without proof.

We give here an elementary proof of Lewandowski's and Pommerenke's results using the Carathéodory kernel theorem, Schwarz–Christoffel mappings and a certain approximation argument for functions with positive real part, which seems to be of some interest by its own.

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1. UNIVALENT FUNCTIONS

We consider functions that are analytic in the unit disk

$$\mathbb{D} := \{z \in \mathbb{C} \mid |z| < 1\}$$

A function is called *univalent* (or *schlicht*) if it is one-to-one. The Riemann mapping theorem guarantees the existence of a univalent map $f\colon \mathbb{D} \to G$ for each simply connected domain $G \subsetneqq \mathbb{C}$. Moreover f is uniquely determined except of the composition with rotations $z \mapsto e^{i\alpha} z$ of $\mathbb{D}$.

If (G_n) is a sequence of simply connected domains with $a \in G_n$, $n \in \mathbb{N}$, then the largest domain G containing a and having the property that each compact subset of G lies in all but a finite number of the domains G_n is called the *kernel* of (G_n). If no such domain exists then the kernel is $\{a\}$. A sequence (G_n) is said to converge to G, if each subsequence has the kernel G. We write $G_n \to G$. The *Carathéodory kernel theorem* states that a sequence (f_n) of univalent functions with $f_n(0) = a$ and $f'_n(0) > 0$ converges locally uniformly to f, if and only if $f_n(\mathbb{D})$ converges to $f(\mathbb{D})$.

If we speak about convergence of a sequence (f_n) of analytic functions, we mean locally uniform convergence and write $f_n \to f$. The family A of analytic functions of $\mathbb{D}$ together with this topology is a Fréchet space, i.e. a locally convex complete metrizable linear space.

A sequence of univalent functions not converging locally uniformly to ∞ is normal, and there is a convergent subsequence. The limit function is univalent or constant. (See [3], [4], [12].)

2. FUNCTIONS WITH POSITIVE REAL PART

Let P denote the subset of A of functions p with positive real part that are normalized by $p(0) = 1$.

A function of the form

$$p(z) = \int_{\partial\mathbb{D}} \frac{1 + xz}{1 - xz}\, d\mu(x), \tag{1}$$

where μ denotes a Borel probability measure on $\partial\mathbb{D}$, clearly has positive real part, because the kernel functions have this property. The famous *Herglotz representation theorem* states that the converse is also true. This is equivalent to the fact that the *extreme points* of P (i.e. the points which have no proper convex representation within the convex set P) are the kernel functions of representation (1), which map $\mathbb{D}$ univalently onto the right halfplane $\{w \in \mathbb{C} \mid \operatorname{Re} w > 0\}$ (see e.g. [13], [5]); we write $\mathrm{E}P = \left\{ \frac{1 + xz}{1 - xz} \,\middle|\, x \in \partial\mathbb{D} \right\}$. By the Krein–Milman theorem their *closed convex hull* $\overline{\mathrm{co}}(\mathrm{E}P)$ is all of P and so their *convex hull* $\mathrm{co}(\mathrm{E}P)$ lies dense in P with respect to the topology of locally uniform convergence (which makes P compact). Thus each function $p \in P$ can be locally uniformly approximated by functions p_n of the form

$$p_n(z) = \sum_{k=1}^{n} \mu_k \frac{1 + x_k z}{1 - x_k z}, \qquad |x_k| = 1, \quad \mu_k > 0, \quad k = 1, \ldots, n,$$

$$\sum_{k=1}^{n} \mu_k = 1, \qquad n \in \mathbb{N}. \tag{2}$$

The functions of representation (2) form the so-called *Carathéodory boundary* of P.

A function f is called *subordinate* to g, if $f = g \circ w$ for some function $w \in A$ with $w(0) = 0$ and $w(\mathbb{D}) \subset \mathbb{D}$; we write $f \prec g$. If g is univalent then $f \prec g$ if and only if $f(0) = g(0)$ and $f(\mathbb{D}) \subset g(\mathbb{D})$, and so $p \in P$ iff $p \prec \frac{1 + z}{1 - z}$.

A similar compact family of some interest is the class $\tilde{P}$ of functions p normalized by $p(0) = 1$ for which there is some $\alpha \in \mathbb{R}$ such that the real part of $e^{i\alpha} p$ is positive. One sees that $p \in \tilde{P}$ iff $p \prec \frac{1 + yz}{1 - z}$, where $y = e^{-2i\alpha}$, and a slight modification of

Herglotz's theorem gives that each function $p \in \tilde{P}$ can be approximated by functions of the form

$$p_n(z) = \sum_{k=1}^{n} \mu_k \frac{1 + y x_k z}{1 - x_k z}, \qquad |y| = |x_k| = 1, \quad \mu_k > 0, \quad k = 1, \dots, n,$$

$$\sum_{k=1}^{n} \mu_k = 1, \qquad n \in \mathbb{N}. \tag{3}$$

Hence the following lemma holds

LEMMA 2.1 *The functions of the form* (3) *form a dense subset of* $\tilde{P}$.

(For details see e.g. [5, Chapter 3], and [7].)

LEMMA 2.2 *Each function of the form* (3) *has a representation*

$$p_n(z) = \prod_{k=1}^{n} \frac{1 - y_k z}{1 - x_k z}, \tag{4}$$

where

$$|x_k| = |y_k| = 1, \qquad k = 1, \dots, n \tag{5}$$

and

$$\arg x_1 < \arg y_1 < \arg x_2 < \arg y_2 < \cdots < \arg x_n < \arg y_n < \arg x_1 + 2\pi. \tag{6}$$

Proof The function p_n given by (3) is rational in $\hat{\mathbb{C}}$ of degree n with exactly n poles at the points x_k, and $p_n(0) = 1$, so that (4) holds. As a convex combination of functions subordinate to $\frac{1 + yz}{1 - z}$ also $p_n \prec \frac{1 + yz}{1 - z}$, and so $p_n(\mathbb{D})$ lies in some halfplane H whose boundary contains the origin, and in particular p_n is nonvanishing in $\mathbb{D}$. From this it follows that $|y_k| \leqq 1$, $k = 1, \dots, n$. On the other hand

$$p_n(\infty) = -y \sum_{k=1}^{n} \mu_k = -y = \prod_{k=1}^{n} \frac{y_k}{x_k},$$

so that $\prod_{k=1}^{n} |y_k| = 1$, which leads to (5). From (4) it follows with the aid of the identity

$$\arg(1 + x) = \tfrac{1}{2} \arg x, \qquad |x| = 1, \quad x \neq -1, \tag{7}$$

that for $e^{i\theta} \neq \bar{x}_k, \bar{y}_k$, $k = 1, \dots, n$

$$\arg(p_n(e^{i\theta})) = \tfrac{1}{2} \arg\left(\prod_{k=1}^{n} \frac{y_k}{x_k} \right) \qquad (\text{mod } \pi), \tag{8}$$

so that the curve $\{p_n(e^{i\theta})\}$ lies on a line l through the origin, and $p_n(\mathbb{D}) \subset H$ then implies that $p_n(\mathbb{D}) = H$ where H denotes that halfplane with $l = \partial H$ and $1 \in H$. In particular, $p_n(e^{i\theta})$ does not contain a turning point θ_0 where $p_n'(e^{i\theta_0}) = 0$. Suppose now that (6) does not hold. Then there exist two zeros of $p_n(e^{i\theta})$, θ_1 and θ_2, say, without pole between them (on $\partial\mathbb{D}$), so that $p_n(e^{i\theta})$ must change its direction on l for some $\theta_0 \in]\theta_1, \theta_2[$. Here $p_n'(e^{i\theta_0}) = 0$, and we have a contradiction. ∎

On the other hand, functions of the form (4)–(6) are elements of $\tilde{P}$ as the following lemma shows.

LEMMA 2.3 *The functions of the form* (4)–(6) *form a dense subset of* $\tilde{P}$.

Proof By Lemma 2.1 the functions of form (3) are dense in $\tilde{P}$, and by Lemma 2.2 they have a representation of the form (4)–(6). Now we show that functions of the form (4)–(6) lie in $\tilde{P}$, which gives the result.

As above we get (8), and the curve $\{p_n(e^{i\theta})\}$ lies on a line l through the origin. Now we shall show that $p_n'(z) \neq 0$ for $z \in \partial\mathbb{D}$, and from this it follows that $p_n(\mathbb{D})$ must lie on one side of l, because $p_n(e^{i\theta})$ does not change its direction by moving on l while θ varies from 0 to 2π, and this gives that $p \in \tilde{P}$.

The zeros $\overline{y_k}$ and the poles $\overline{x_k}$ of p_n are pairwise different by (6), so that they have order one and $p_n'(\overline{x_k}), p_n'(\overline{y_k}) \neq 0$, $k = 1, \ldots, n$. It remains to show that

$$z\frac{p_n'}{p_n}(z) \neq 0 \qquad \text{for} \quad z \in \partial\mathbb{D}, \quad z \neq \bar{x}_k, \bar{y}_k, \quad k = 1, \ldots, n.$$

From representation (4) it follows for $z \in \partial\mathbb{D}$ that

$$z\frac{p_n'}{p_n}(z) = \sum_{k=1}^{n}\left(\frac{1}{1-y_k z} - \frac{1}{1-x_k z}\right).$$

The real part of this sum equals zero because the same is true for each summand. On the other hand, we get for $z = e^{i\theta}$

$$\operatorname{Im}\left(z\frac{p_n'}{p_n}(z)\right) = \frac{1}{2}\sum_{k=1}^{n}\left(\cot\frac{\varphi_k - \theta}{2} - \cot\frac{\psi_k - \theta}{2}\right),$$

if we write

$$\varphi_k := \arg \bar{x}_k; \qquad \psi_k := \arg \bar{y}_k.$$

Now let $\theta \in [0, 2\pi]$ be given. Then rearrange the values of φ_k and ψ_k modulo 2π, such that

$$\begin{aligned} \theta < \varphi_N < \psi_N < \varphi_{N+1} < \psi_{N+1} < \cdots < \varphi_n < \psi_n < \varphi_1 < \psi_1 < \cdots \\ < \varphi_{N-1} < \psi_{N-1} < \theta + 2\pi \end{aligned} \tag{9}$$

or

$$\begin{aligned} \theta < \psi_N < \varphi_N < \psi_{N+1} < \varphi_{N+1} < \cdots < \psi_n < \varphi_n < \psi_1 < \varphi_1 < \cdots \\ < \psi_{N-1} < \varphi_{N-1} < \theta + 2\pi \end{aligned} \tag{10}$$

holds which is possible by (6) if $\theta \neq \varphi_k, \psi_k$, $k = 1, \ldots, n$. Write

$$a_k := \cot\frac{\psi_k - \theta}{2}; \qquad b_k := \cot\frac{\varphi_k - \theta}{2},$$

then

$$\operatorname{Im}\left(z\frac{p_n'}{p_n}(z)\right) = \frac{1}{2}\sum_{k=1}^{n}(b_k - a_k).$$

Suppose now. (9) holds, then $b_k - a_k > 0$, $k = 1, \ldots, n$, because the function cot is strictly decreasing in $]0, \pi[$, so that $\operatorname{Im}(zp_n'/p_n(z)) > 0$. If (10) holds it follows similarly that $\operatorname{Im}(zp_n'/p_n)) < 0$. This finishes the proof that zp_n'/p_n has no zero on $\partial\mathbb{D}$. ■

3. POLYGONS AND SCHWARZ–CHRISTOFFEL MAPPINGS

Let $f \in A$ be continuous in $\bar{\mathbb{D}}$ and have a Riemann surface F as image domain whose boundary consists of a finite number of linear arcs, such that the boundary correspondence $\partial\mathbb{D} \to \partial F$ is one-to-one. Then F is called a *polygon*. Let F have n vertices of interior angles $\alpha_k\pi$, $k = 1, \ldots, n$. We do not suppose f to be univalent, so that $\alpha_k > 2$ is possible, whereas for univalent polygons

$$\alpha_k \leqq 2, \qquad k = 1, \ldots, n. \tag{11}$$

If we have a bounded vertex then

$$\alpha_k > 0. \tag{12}$$

If a vertex lies at infinity we measure the angle on the Riemann sphere and have

$$\alpha_k \geqq 0, \tag{13}$$

where $\alpha_k = 0$ is a zero angle which corresponds to two parallel rays of ∂F.

Let now x_k be the *prevertices*, i.e. the preimages under f of the vertices $f(x_k)$. Then the *Schwarz–Christoffel formula* is the representation

$$\frac{f''}{f'}(z) = -2 \sum_{k=1}^{n} \frac{\mu_k}{z - x_k}, \tag{14}$$

where

$$2\mu_k\pi := \begin{cases} (1-\alpha_k)\pi & \text{if } f(x_k) \text{ is bounded} \\ (1+\alpha_k)\pi & \text{if } f(x_k) \text{ is unbounded} \end{cases} \tag{15}$$

denotes the *outer angles*. The formula

$$\sum_{k=1}^{n} \mu_k = 1 \tag{16}$$

in the bounded (univalent) case corresponds both to the rule for the sum of angles in an n-gon and to the fact that the increment of the tangent direction is exactly 2π when surrounding the polygon on ∂F one time.

On the other hand, if f fulfills (14) and (16), such that $x_k \in \partial\mathbb{D}$ for $k = 1, \ldots, n$, then the Riemann image surface $f(\mathbb{D})$ is a polygon.

If $f(x_k)$ is bounded then relation (12) yields

$$\mu_k < \tfrac{1}{2}, \tag{17}$$

whereas for unbounded $f(x_k)$ relations (13) and (15) give

$$\mu_k \geqq \tfrac{1}{2}. \tag{18}$$

If f is univalent, then (11) leads to

$$\mu_k \geqq -\tfrac{1}{2}. \tag{19}$$

(See [8], [15].)

4. CONVEX AND STARLIKE FUNCTIONS

A function $f \in A$ is called *convex* if it maps $\mathbb{D}$ univalently onto a convex domain, and it is called *starlike* if it maps $\mathbb{D}$ univalently onto a domain which is starlike with respect to $f(0)=0$.

Clearly a polygon is convex if $\alpha_k < 1$, $k=1,\ldots,n$ or equivalently if $\mu_k > 0$, $k=1,\ldots,n$. So by (14) it follows that

$$1 + z\frac{f''}{f'}(z) = \sum_{k=1}^{n} \mu_k \frac{1+x_k z}{1-x_k z}, \tag{20}$$

if one uses (16). Thus

$$1 + z\frac{f''}{f'} \in P \tag{21}$$

On the other hand, if (21) holds, then by (2) f can be approximated by convex Schwarz–Christoffel mappings, and the Carathéodory kernel theorem shows that $f(\mathbb{D})$ is convex. So (21) is a necessary and sufficient condition for f to be convex.

It is well known that a function f is starlike if and only if

$$z\frac{f'}{f} \in P \tag{22}$$

(see e.g. [12]).

By (2) and Lemmas 2.1 and 2.2 the function zf'/f can be approximated by functions of the form

$$z\frac{f_n'}{f_n}(z) = p_n(z) = \sum_{k=1}^{n} \mu_k \frac{1+x_k z}{1-x_k z} = \prod_{k=1}^{n} \frac{1-y_k z}{1-x_k z},$$

$$|x_k| = |y_k| = 1, \quad \mu_k > 0, \quad k=1,\ldots,n, \quad \sum_{k=1}^{n} \mu_k = 1, \quad n \in \mathbb{N}$$

with the property (6), so that

$$\frac{f_n''}{f_n'} = \frac{p_n(z)-1}{z} + \frac{p_n'}{p_n}(z) = -2\sum_{k=1}^{n} \frac{-1/2}{z-y_k} - 2\sum_{k=1}^{n} \frac{\mu_k + 1/2}{z-x_k},$$

from which we can see that f_n is a Schwarz–Christoffel mapping with n finite vertices of interior angle 2π, and alternating n vertices at ∞. This is a special case of linear accessibility which will be considered now.

5. LINEARLY ACCESSIBLE DOMAINS AND CLOSE-TO-CONVEX FUNCTIONS

A domain F is called *(angularly) accessible* of order β, $\beta \in [0, 1]$, if it is the complement of the union of rays that are pairwise disjoint except that the origin of one ray may lie on another one of the rays, and such that every ray is the bisector of a sector of angle $(1-\beta)\pi$ which wholly lies in the complement of F. If $\beta = 1$ then F is called *(strictly) linearly accessible* (see [2], [14], [11]). A function f is called close-to-convex of order β, $\beta \in [0, 1]$, (for reasons which shall be seen later) if $f(\mathbb{D})$ is accessible of order β. We shall give an analytical characterization for f to be close-to-convex of order β, which is for $\beta = 1$ originally due to Lewandowski [9]–[10] and for $\beta < 1$ to Pommerenke [11] (who did not give a proof for his statement) and has been the original definition of close-to-convexity due to Kaplan [6]. Therefore we use Lemma 2.3.

THEOREM 5.1 *Let f be univalent and $f(\mathbb{D})$ accessible of order β. Then there exist a convex function g and a function $p \in \tilde{P}$ such that the representation*

$$f' = g' \; p^{\beta} \tag{23}$$

holds.

Proof Suppose firstly, $\beta = 1$. Then by the geometrical definition we have $f(\mathbb{D}) = \mathbb{C} \setminus \bigcup_{t \in T} \gamma_t$, where γ_t are rays that are pairwise disjoint except that the origin of one ray may lie on another one of the rays, and T is a suitably chosen parameter set, which is separable (e.g. $T \subset \mathbb{R}^3$). Choose a dense subset $\{t_n \in T \mid n \in \mathbb{N}\}$ of T and define f_n by

$$f_n(\mathbb{D}) := \mathbb{C} \setminus \bigcup_{k=1}^{n} \gamma_{t_k}. \tag{24}$$

There is no loss of generality to assume that (γ_{t_k}) are pairwise disjoint, because if some of the chosen rays have their origins lying on another ray, we shorten them by $1/n$ and get the same conclusion. Obviously $f_n \to f$, because $f_n(\mathbb{D}) \to f(\mathbb{D})$ in the sense of Carathéodory kernel convergence. This shows that it suffices to show the conclusion for functions f_n satisfying (24), because $\{f_n\}$ is a normal family and the functions f with representation (23) form a closed subset of A.

Observe that f_n is a certain Schwarz–Christoffel mapping with n finite vertices at the points $w_k =: f_n(y_k)$, say. The interior angle at each of those hairpin vertices is 2π. The other n vertices alternate with w_k and lie at $\infty =: f_n(x_k)$, say. The interior angles $\alpha_k \pi$ at those vertices satisfy $\alpha_k \geqq 0$, and their sum fulfills $\sum_{k=1}^{n} \alpha_k \pi = 2\pi$, because f_n is univalent (in other words: the rays are traversed at ∞ systematically with increasing argument when surrounding the polygon), so that by (14) and (15)

$$\frac{f_n''}{f_n'}(z) = -2 \sum_{k=1}^{n} \frac{-1/2}{z-y_k} - 2 \sum_{k=1}^{n} \frac{(1+\alpha_k)/2}{z-x_k} = \sum_{k=1}^{n} \left(\frac{1}{z-y_k} - \frac{1}{z-x_k} \right) - 2 \sum_{k=1}^{n} \frac{\alpha_k/2}{z-x_k}. \tag{25}$$

The choise (4) gives a function $p \in \tilde{P}$ as Lemma 2.3 shows because (5) and (6) are fulfilled, and

$$\frac{g''}{g'}(z) := -2 \sum_{k=1}^{n} \frac{\alpha_k/2}{z - x_k}, \qquad g'(0) := f_n'(0)$$

gives a convex polygon. Then from (25) it follows that

$$\frac{f_n''}{f_n'} = \frac{p'}{p} + \frac{g''}{g'}, \qquad f_n'(0) = g'(0)$$

which is equivalent to $f_n' = g' \cdot p$.

Now suppose $0 < \beta < 1$. Then for each γ_t, $t \in T$, the sector S_t of angle $(1 - \beta)\pi$ which is symmetric with respect to γ_t lies in $\mathbb{C} \setminus f(\mathbb{D})$. Define here

$$f_n(\mathbb{D}) := \mathbb{C} \setminus \bigcup_{k=1}^{n} S_{t_k} \tag{26}$$

for a certain dense subset $\{t_n \in T \mid n \in \mathbb{N}\}$ of T. Then $f_n \to f$, and it suffices to show the conclusion for functions f_n satisfying (26).

Observe that $f_n(\mathbb{D})$ is a polygon with $2n$ vertices, n of them of interior angle $(1 + \beta)\pi$ at the origins of S_{t_k}, $k = 1, \ldots, n$. Let the sectors S_{t_k} be ordered in the same way as their origins—which are vertices of $f_n(\mathbb{D})$—when traversing $\partial\mathbb{D}$ in positive sense. Now the polygon $f_n(\mathbb{D})$ has a finite vertex between the origins S_{t_k} and $S_{t_{k+1}}$ when surrounding $f_n(\mathbb{D})$ if they intersect, and has a vertex at ∞ if they do not. Let α_k be the angle between the directions of γ_{t_k} and $\gamma_{t_{k+1}}$. Then in either case the outer angle is seen to be $2\mu_k\pi = (\alpha_k + \beta)\pi$, so that

$$\frac{f_n''}{f_n'}(z) = \beta \sum_{k=1}^{n} \left(\frac{1}{z - y_k} - \frac{1}{z - x_k}\right) - 2 \sum_{k=1}^{n} \frac{\alpha_k/2}{z - x_k}.$$

Because $\sum_{k=1}^{n} \alpha_k\pi = 2\pi$, this gives the result as above. ■

It is decisive that the converse is also true. For this reason the functions are called close-to-convex.

THEOREM 5.2 *Let $\beta \in]0, 1]$ and let f have a representation of the form* (23) *for some convex function g and some $p \in \tilde{P}$. Then f is univalent and $f(\mathbb{D})$ is accessible of order β.*

Proof The function $h = f \circ g^{-1}$ is defined in the convex domain $g(\mathbb{D})$ and fulfils for $z_1, z_2 \in g(\mathbb{D})$

$$h(z_2) - h(z_1) = \int_{z_1}^{z_2} h'(z)\, dz = (z_2 - z_1) \int_0^1 h'(tz_2 + (1 - t)z_1)\, dt \neq 0,$$

since $\mathrm{Re}(e^{i\alpha}h') = \mathrm{Re}(e^{i\alpha}f'/g') > 0$ for some $\alpha \in \mathbb{R}$, so that h and therefore f is univalent.

We prove the rest of the result also by an approximation argument. Therefore we need to know that the family of domains that are accessible of order β is closed with respect to Carathéodory kernel convergence, i.e. a convergent sequence of domains which are accessible of order β that does not converge to a singleton converges to a domain accessible of order β.

Suppose G_n are accessible of order β and $G_n \to G$. Each boundary point $w \in \partial G$ is the limit point of a sequence w_n of boundary points of G_n. Each w_n is the vertex of a sector S_n which lies in $\mathbb{C} \setminus G_n$. Let γ_n denote the bisector of S_n. Then one chooses a subsequence such that there is a limit direction of the directions of γ_n and thus a limit ray γ. Let S be the corresponding symmetric sector of angle $(1-\beta)\pi$. Carathéodory kernel convergence shows that $S \subset \mathbb{C} \setminus G$. Furthermore a simple argument also shows that the rays which correspond to different boundary points of G are pairwise disjoint. For the details see [2, Lemma 3].

Suppose now, f has a representation (23). Then

$$\frac{f''}{f'} = \frac{g''}{g'} + \beta \frac{p'}{p}. \tag{27}$$

Each function of this form can be approximated by functions f_n of the same form where g is a convex Schwarz–Christoffel mapping and p has a representation (4)–(6) as Lemma 2.3 shows. So we get for the approximants

$$\frac{f_n''}{f_n'}(z) = -2 \sum_{k=1}^{n} \frac{\beta/2}{z - x_k} - 2 \sum_{k=1}^{n} \frac{-\beta/2}{z - y_k} - 2 \sum_{k=1}^{m} \frac{\mu_k}{z - w_k} \tag{28}$$

where the numbers x_k, y_k alternate with each other on $\partial\mathbb{D}$, $\mu_k > 0$, $|w_k| = 1$, $k = 1, \ldots, m$, $\sum_{k=1}^{m} \mu_k = 1$, and $n, m \in \mathbb{N}$. Without loss of generality we can assume that g is bounded (i.e. $\mu_k < \frac{1}{2}$, $k = 1, \ldots, m$) because otherwise we approximate g by bounded convex polygons. On similar reasons we suppose that the numbers w_k are pairwise different from x_k and y_k.

From (28) one sees that $f_n(\mathbb{D})$ is a polygon and because it has the form (27) it is *a priori* close-to-convex and hence univalent.

Now suppose first, $\beta = 1$. Then there are n vertices at ∞ of angle zero, and alternatively n finite hairpin vertices of angle 2π. Furthermore there are m finite convex vertices.

At first we prove that the complement E of $F := f_n(\mathbb{D})$ contains the n rays γ_k, $k = 1, \ldots, n$, which come from the hairpin vertices O_k. Clearly a segment σ of γ_k containing O_k lies in E. Suppose now that there is a point $Q \in \gamma_k$ which lies in F. Then there is a curve Γ which connects O_k with Q within F because O_k is an accessible boundary point. The segment of γ_k from O_k to Q and Γ encloses a bounded region. It contains points of ∂F other than those of σ, and without loss of generality we can assume that there are those which come after O_k when traversing on $\partial\mathbb{D}$ in positive sense. Call the corresponding part of ∂F from O_k to the next vertex at infinity δ. Now because δ is unbounded it must cross γ_k between σ and Q. But this contradicts the fact that all vertices of δ are convex. Thus $\gamma_k \subset E$.

The rays γ_k, $k = 1, \ldots, n$, are pairwise disjoint because of the univalence. Let them be ordered in the same way as their origins O_k when traversing $\partial\mathbb{D}$ in positive sense.

When traversing from O_k to O_{k+1} along ∂F there is exactly one vertex at ∞ (of angle zero) between O_k and O_{k+1}, because the numbers x_k, y_k are alternating on $\partial\mathbb{D}$. So the rays γ_k are separated by half parallel strips and lie in components G_k of E which are pairwise disjoint.

Furthermore $E = \bigcup_{k=1}^{n} G_k$, because in a neighborhood of infinity E has exactly n components (f has exactly n poles on $\partial\mathbb{D}$), so that an additional component would be bounded contradicting the simple connectivity of F.

So, for to fill E with rays that are pairwise disjoint, it is enough to do this for the components G_k. But this is easily done.

Take the parallels of γ_k from O_k to the next vertex P_1 with origins on ∂F. Because all vertices before the next vertex at ∞ are convex, we may choose from P_1 on as direction of a new family of parallel rays the boundary direction of F before P_1, and fill the remaining sector arbitrarily. Note that in this case the origin of some ray lies on another one of the rays. Continue the procedure from P_1 to the next vertex P_2 and so on until $P_j = \infty$. Finally apply the same process from O_k to the last vertex at ∞ before O_k. This gives a suitable representation of G_k as union of rays that are pairwise disjoint and finishes the proof for $\beta = 1$.

Now suppose, $0 < \beta < 1$. Because $\mu_k < \frac{1}{2}$, $k = 1, \ldots, m$, f is bounded, i.e. all vertices are finite. There are exactly n vertices of angle $(1-\beta)\pi$, alternately n vertices of angle $(1+\beta)\pi$, and finally m convex vertices. The vertices O_k, $k = 1, \ldots, n$, of interior angle $(1+\beta)\pi$ define sectors S_k, $k = 1, \ldots, n$, of angle $(1-\beta)\pi$ which lie in $E := \mathbb{C} \setminus f(\mathbb{D})$. Let γ_k denote the bisector of S_k, $k = 1, \ldots, n$. Because all other $n+m$ vertices are bounded and convex, E can be filled with rays γ_t, $t \in T$, that are pairwise disjoint such that for each γ_t the symmetric sector S_t of angle $(1-\beta)\pi$ lies in E, if we choose γ_t to be parallel to γ_k in a neighborhood of O_k, $k = 1, \ldots, n$. This finishes the proof. ■

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