A uniqueness theorem for functions of positive real part

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Abstract:

Functions of positive real part have been studied extensively, especially many extremal problems. Usually functions of the so-called Carathéodory boundary are solutions of extremal problems. In many cases, however, it is more difficult to study the case of equality than the extremal problem itself.

In this paper we show a general uniqueness theorem for functions of positive real part, and give some applications.

1 Introduction

We consider functions that are analytic in the unit disk

$$\mathbb{D} := \{z \in \mathbb{C} \mid |z| < 1\} \ .$$

A function is called *univalent* if it is one-to-one. The Riemann mapping theorem guarantees the existence of a univalent map $f : \mathbb{D} \to G$ for each simply connected domain $G \subsetneqq \mathbb{C}$. Moreover f is uniquely determined except of the composition with rotations $z \mapsto e^{i\alpha}z$ of $\mathbb{D}$.

If we speak about convergence of a sequence (f_n) of analytic functions, we mean locally uniform convergence and write $f_n \to f$. The family A of *analytic functions of* $\mathbb{D}$ together with this topology is a Fréchet space, i.e. a locally convex complete metrizable linear space.

The family S of univalent functions that are normalized by $f(0) = 0$, $f'(0) = 1$, i.e.

$$f(z) = z + a_2 z^2 + a_3 z^3 + \cdots , \tag{1}$$

forms a compact subset of A.

A function $f \in A$ is called *starlike* if it maps $\mathbb{D}$ univalently onto a domain which is starlike with respect to $f(0) = 0$. It is well-known that a function f is starlike if and only if

$$z\frac{f'}{f} \in P$$

where P denotes the subset of A of functions p with positive real part that are normalized by $p(0) = 1$ (see e.g. [9]). Let St denote the family of starlike functions that are normalized by (1).

Many extremal problems in St have been studied. As a particularly nice example we mention the result of Leung ([7], see e.g. [3], § 5.10) that for $f(z) = z + a_2\, z^2 + a_3\, z^3 + \ldots \in St$ the inequality

$$\Big| |a_{n+1}| - |a_n| \Big| \leqq 1 \qquad (n \in \mathbb{N}) \tag{2}$$

holds with equality for some $n \in \mathbb{N}$ if and only if

$$f(z) = \frac{z}{(1-xz)(1-yz)} \tag{3}$$

with $|x| = |y| = 1$ and $x/y = \zeta$ where $\zeta^k = 1$ for some k in $\mathbb{N}$.

We evidently have equality for all n in $\mathbb{N}$ if $\zeta = 1$, i.e. $x = y$. So now assume, possibly after a rotation, that $x = e^{i\theta}$, $y = e^{-i\theta}$, where $0 < |\theta| < \pi$. Then

$$\frac{z}{(1-ze^{i\theta})(1-ze^{-i\theta})} = \frac{z}{1-2z\cos\theta + z^2} = \sum_{k=1}^{\infty} \frac{\sin(k\theta)}{\sin\theta} z^k ,$$

and

$$\Big||a_{n+1}| - |a_n|\Big| = \left|\frac{|\sin((n+1)\theta)| - |\sin(n\theta)|}{\sin\theta}\right| < 1 \tag{4}$$

unless $\sin(n\theta) = 0$ or $\sin((n+1)\theta) = 0$.

In fact, (4) is a special case of the triangle inequality

$$|\sin(A+B)| \leqq |\sin A| + |\sin B| \tag{5}$$

for real A, B with equality if and only if $\sin A = 0$ or $\sin B = 0$. In fact

$$|\sin(A+B)| \leqq |\sin A||\cos B| + |\cos A||\sin B| \leqq |\sin A| + |\sin B| ,$$

with strict inequality if $\sin A \sin B \neq 0$, so that $|\cos A| < 1$ and $|\cos B| < 1$.

We deduce from (5) that

$$|\sin A| = |\sin(A+B-B)| \leqq |\sin(A+B)| + |\sin B| ,$$

so that

$$\Big||\sin(A+B)| - |\sin A|\Big| < |\sin B| ,$$

unless one of $\sin A$, $\sin B$ and $\sin(A+B)$ is zero. Writing $A = n\theta$, $B = \theta$, we obtain (4).

To derive inequality (2) Leung makes clever use of the Lebedev-Milin inequalities (see e.g. [3], § 5.1). To prove the case of equality, however, he uses deep variational methods. (Therefore in the book of Duren ([3], § 5.10) the case of equality is omitted.)

In this paper we deduce a general uniqueness theorem for functions in P. As an application we point out that our general uniqueness theorem covers the case of equality in Leung's result, too.

2 Functions with positive real part

A function of the form

$$p(z) = \int_{\partial\mathbb{D}} \frac{1+xz}{1-xz} d\mu(x) , \tag{6}$$

where μ denotes a Borel probability measure on $\partial\mathbb{D}$, clearly has positive real part, because the kernel functions have this property. The famous *Herglotz representation theorem* states that the converse is also true. This is equivalent to the fact that the *extreme points* of P (i.e. the points which have no proper convex representation within the convex set P) are the kernel functions of representation (6), which map $\mathbb{D}$ univalently onto the right halfplane $\{w \in \mathbb{C} \mid \operatorname{Re} w > 0\}$ (see e.g. [10], [5]); we write $\mathrm{E}\,(P) = \{\frac{1+xz}{1-xz} \mid x \in \partial\mathbb{D}\}$. By the Krein-Milman theorem their *closed convex hull* $\overline{\mathrm{co}}\,(\mathrm{E}\,P)$ is all of P and so their *convex hull* $\mathrm{co}\,(\mathrm{E}\,P)$ lies dense in P with respect to the topology of locally uniform convergence (which makes P compact), so that each function $p \in P$ can be locally uniformly approximated by functions p_n of the form

$$p_n(z) = \sum_{k=1}^{n} \mu_k \frac{1+x_k z}{1-x_k z}, \quad |x_k| = 1,\ \mu_k > 0 \quad (k = 1, \ldots, n), \quad \sum_{k=1}^{n} \mu_k = 1, \quad n \in \mathbb{N}. \tag{7}$$

The functions of the form (7) give the so-called *Carathéodory boundary* of P.

A function f is called *subordinate* to g, if $f = g \circ \omega$ for some function $\omega \in A$ with $\omega(0) = 0$ and $\omega(\mathbb{D}) \subset \mathbb{D}$; we write $f \prec g$. The *subordination principle* states that if g is univalent then $f \prec g$ if and only if $f(0) = g(0)$ and $f(\mathbb{D}) \subset g(\mathbb{D})$, and so $p \in P$ if and only if $p \prec \frac{1+z}{1-z}$. If $f \prec g$ then by Schwarz's Lemma $f(\mathbb{D}_r) \subset g(\mathbb{D}_r)$ for all r in $]0, 1[$ where $\mathbb{D}_r := \{z \in \mathbb{C} \mid |z| < r\}$. By B we denote the family of functions $\omega \in A$ with $\omega(0) = 0$ and $\omega(\mathbb{D}) \subset \mathbb{D}$.

A compact family which is similar to P is the class $\widetilde{P}$ of functions p normalized by $p(0) = 1$ for which there is some $\alpha \in \mathbb{R}$ such that the real part of $e^{i\alpha}p$ is positive. One sees that $p \in \widetilde{P}$ if and only if $p \prec \frac{1+yz}{1-z}$, where $y = e^{-2i\alpha}$ and $|\alpha| < \pi/2$. A slight modification of Herglotz's theorem gives that each function $p \in \widetilde{P}$ can be approximated by functions of the form

$$p_n(z) = \sum_{k=1}^{n} \mu_k \frac{1+yx_k z}{1-x_k z}, \quad |y| = |x_k| = 1,\ \mu_k > 0 \quad (k = 1, \ldots, n), \quad \sum_{k=1}^{n} \mu_k = 1, \quad n \in \mathbb{N}\,.$$

From this property in ([6], Lemma 2.3) I deduced the following

Lemma 1 *The functions p_n $(n \in \mathbb{N})$ with a representation of the form*

$$p_n(z) = \prod_{k=1}^{n} \frac{1-y_k z}{1-x_k z}\ ,$$

where

$$|x_k| = |y_k| = 1 \qquad (k = 1, \ldots, n),$$

and

$$\arg \overline{x_1} < \arg \overline{y_1} < \arg \overline{x_2} < \arg \overline{y_2} < \cdots < \arg \overline{x_n} < \arg \overline{y_n} < \arg \overline{x_1} + 2\pi$$

form a dense subset of $\widetilde{P}$.

3 Uniqueness statements for functions with positive real part

It is an easy consequence of Schwarz' Lemma that $p \in P$ implies $|p_1| \leqq 2$ with equality if and only if $p(z) = \frac{1+xz}{1-xz}$ $(x \in \partial\mathbb{D})$. This includes the uniqueness statement that

$$p \in P,\ p_1 = 2x \quad (x \in \partial\mathbb{D}) \quad \Longrightarrow \quad p(z) = \frac{1+xz}{1-xz}\ .$$

We shall now give a generalization of this statement the proof of which is a consequence of the Carathéodory-Toeplitz-Fejér theory on positive harmonic functions, in particular of the following

Theorem due to Carathéodory, see [1], or [2], Theorem VI:

Theorem I *Suppose that* $p(z) = 1 + p_1 z + p_2 z^2 + \cdots \in P$, $n \in \mathbb{N}$, *and*

$$D_n := \begin{vmatrix} 2 & p_1 & p_2 & \cdots & p_n \\ \overline{p_1} & 2 & p_1 & \cdots & p_{n-1} \\ \overline{p_2} & \overline{p_1} & 2 & \cdots & p_{n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \overline{p_n} & \overline{p_{n-1}} & \overline{p_{n-2}} & \cdots & 2 \end{vmatrix} = 0\,.$$

Then $D_j = 0$ *for all* $j > n$, *and* p *is the only function in* P *with the coefficients* $p_1, p_2, \ldots, p_n$, *i.e. it is uniquely determined by those first* n *coefficients.*

As a consequence we get the following uniqueness statement for functions with positive real part.

Theorem 1 *Suppose that* $p(z) = 1 + p_1 z + p_2 z^2 + \cdots \in P$, $n \in \mathbb{N}$, *and that for all* j, $1 \leqq j \leqq n$ *we have*

$$p_j = 2\sum_{k=1}^{n} t_k x_k^j, \quad \sum_{k=1}^{n} t_k = 1, \quad t_k > 0,\ x_k \in \partial\mathbb{D} \quad (k = 1, \ldots, n), \tag{8}$$

then (8) *holds for all* j *in* $\mathbb{N}$, *i.e.*

$$p(z) = \sum_{k=1}^{n} t_k \left(\frac{1 + x_k z}{1 - x_k z} \right) .$$

Proof: Observe that in our case

$$D_n = \begin{vmatrix} 2\sum\limits_{k=1}^{n} t_k & 2\sum\limits_{k=1}^{n} t_k x_k & 2\sum\limits_{k=1}^{n} t_k x_k^2 & \cdots & 2\sum\limits_{k=1}^{n} t_k x_k^n \\ 2\sum\limits_{k=1}^{n} t_k \overline{x_k} & 2\sum\limits_{k=1}^{n} t_k & 2\sum\limits_{k=1}^{n} t_k x_k & \cdots & 2\sum\limits_{k=1}^{n} t_k x_k^{n-1} \\ 2\sum\limits_{k=1}^{n} t_k \overline{x_k^2} & 2\sum\limits_{k=1}^{n} t_k \overline{x_k} & 2\sum\limits_{k=1}^{n} t_k & \cdots & 2\sum\limits_{k=1}^{n} t_k x_k^{n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 2\sum\limits_{k=1}^{n} t_k \overline{x_k^n} & 2\sum\limits_{k=1}^{n} t_k \overline{x_k^{n-1}} & 2\sum\limits_{k=1}^{n} t_k \overline{x_k^{n-2}} & \cdots & 2\sum\limits_{k=1}^{n} t_k \end{vmatrix}$$

$$
= 2^n \begin{vmatrix} \sum_{k=1}^n t_k & \sum_{k=1}^n t_k x_k & \sum_{k=1}^n t_k x_k^2 & \cdots & \sum_{k=1}^n t_k x_k^n \\ \sum_{k=1}^n t_k x_k^{-1} & \sum_{k=1}^n t_k & \sum_{k=1}^n t_k x_k & \cdots & \sum_{k=1}^n t_k x_k^{n-1} \\ \sum_{k=1}^n t_k x_k^{-2} & \sum_{k=1}^n t_k x_k^{-1} & \sum_{k=1}^n t_k & \cdots & \sum_{k=1}^n t_k x_k^{n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \sum_{k=1}^n t_k x_k^{-n} & \sum_{k=1}^n t_k x_k^{-(n-1)} & \sum_{k=1}^n t_k x_k^{-(n-2)} & \cdots & \sum_{k=1}^n t_k \end{vmatrix} \tag{9}
$$

$$
= 2^n \begin{vmatrix} 0 & 1 & 1 & \cdots & 1 \\ 0 & x_1^{-1} & x_2^{-1} & \cdots & x_n^{-1} \\ 0 & x_1^{-2} & x_2^{-2} & \cdots & x_n^{-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & x_1^{-n} & x_2^{-n} & \cdots & x_n^{-n} \end{vmatrix} \cdot \begin{vmatrix} 0 & 0 & 0 & \cdots & 0 \\ t_1 & t_1 x_1 & t_1 x_1^2 & \cdots & t_1 x_1^n \\ t_2 & t_2 x_2 & t_2 x_2^2 & \cdots & t_2 x_2^n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ t_n & t_n x_n & t_n x_n^2 & \cdots & t_n x_n^n \end{vmatrix} = 0
$$

as the determinants of both factor matrices trivially vanish. So by Theorem I it follows that $D_j = 0$ for $j > n$, and the uniqueness of p, establishing the result. □

Note that we only used $x_k \overline{x_k} = 1$ $(k = 1, \ldots, n)$ but not the facts that t_k is positive, and $\sum_{k=1}^n t_k = 1$. In particular, the determinant (9) vanishes for arbitrary t_k, x_k $(k = 1, \ldots, n)$.

As a consequence of the theorem we get

Corollary 1 *Suppose that* $p(z) = 1 + p_1 z + p_2 z^2 + \cdots \in P$ *where* $p_1/2 \in \mathbb{D}$, *and further that* $x \in \partial\mathbb{D}$ *and* $y \in \partial\mathbb{D}$, $0 < t < 1$, *and* $p_1 = 2\Big(tx + (1-t)y\Big)$. *(Such a representation exists for* $(2 - |p_1|)/4 \leqq t \leqq (2 + |p_1|)/4$*).*
If now $p_2 = 2\Big(tx^2 + (1-t)y^2\Big)$, *then* p *is uniquely determined and*

$$
p(z) = t\left(\frac{1+xz}{1-xz}\right) + (1-t)\left(\frac{1+yz}{1-yz}\right) . \tag{10}
$$

The functions of form (10) are the extremals also for the next problem. This is a typical example of an extremal problem, which occurs frequently (see e.g. [9], p. 166, formula (10)) but for which the case of equality has not been explicitly studied.

Lemma 2 *Suppose that* $p(z) = 1 + p_1 z + p_2 z^2 + \cdots \in P$. *Then*

$$
\left| p_2 - \frac{1}{2} p_1^2 \right| \leqq 2 - \frac{1}{2} |p_1|^2 \tag{11}
$$

with equality if and only if p *is of form* (10) *with* x, y *in* $\partial\mathbb{D}$ *and* t *in* $[0, 1]$.

Proof: For $p(z) = 1 + p_1 z + p_2 z^2 + \cdots \in P$ we have $\omega(z) := \frac{1}{z} \cdot \frac{p(z)-1}{p(z)+1} = \omega_0 + \omega_1 z + \omega_2 z^2 + \cdots \in B$, and it follows that $|\omega_1| \leqq 1 - |\omega_0|^2$ with equality if and only if $\omega(z) = w^2 \dfrac{z+a}{1+\overline{a}z}$ for some $w \in \partial\mathbb{D}$

and $a \in \mathbb{D}$ (see e.g. [4], Kapitel VIII, Satz 2). This inequality is equivalent to (11). If equality occurs in (11), then

$$p(z) = \frac{1+z\omega(z)}{1-z\omega(z)} = \frac{1+w^2 z\frac{z+a}{1+\overline{a}z}}{1-w^2 z\frac{z+a}{1+\overline{a}z}} = \frac{1+z(\overline{a}+w^2 a)+w^2 z^2}{1+z(\overline{a}-w^2 a)-w^2 z^2}$$

$$= \frac{1+wz(2\,\mathrm{Re}\,aw)+w^2 z^2}{1-wz(2i\,\mathrm{Im}\,aw)-w^2 z^2},$$

and so writing $b = aw$ we deduce that

$$p(\overline{w}z) = \frac{1+z(b+\overline{b})+z^2}{1-z(b-\overline{b})-z^2}. \tag{12}$$

It is therefore sufficient to establish the representation (10) for the functions $p(\overline{w}z)$ given by (12).

We write $b = \cos\theta - i\sin\psi$, and note that, since $|b| < 1$, we have

$$|\cos\theta| < |\cos\psi|.$$

Now

$$p(\overline{w}z) = \frac{(1+ze^{i\theta})(1+ze^{-i\theta})}{(1+ze^{i\psi})(1-ze^{-i\psi})} = t\frac{1-ze^{i\psi}}{1+ze^{i\psi}} + (1-t)\frac{1+ze^{-i\psi}}{1-ze^{-i\psi}},$$

where $t = \frac{1}{2}\left(1-\frac{\cos\theta}{\cos\psi}\right)$, so that $0 < t < 1$. This proves (10).

On the other hand a calculation shows that the functions of form (10) with $x, y \in \partial\mathbb{D}$ and $t \in [0,1]$ give actually equality in (11). □

We remark that Corollary 1 also follows from Lemma 2.

4 Successive coefficients of starlike functions

In this section we derive the extremal functions (3) for the successive coefficient result (2) for functions $f \in St$ by our method. We follow the lines of the proof given in ([3], Theorem 5.12). Assume, equality holds in (2) for some $n \in \mathbb{N}$ $(n \geq 2)$. Let

$$p(z) = \sum_{k=0}^{\infty} p_k z^k := \frac{zf'(z)}{f(z)}$$

denote the function of positive real part associated with f. In Leung's proof the third Lebedev-Milin inequality is applied to the function

$$\ln\left((1-yz)\,\frac{f(z)}{z}\right) = \sum_{k=1}^{\infty} \alpha_k z^k = \sum_{k=1}^{\infty} \frac{p_k - y^k}{k} z^k \tag{13}$$

for some $y \in \mathbb{C}$ with $|y| = 1$, so that by the equality statement for the third Lebedev-Milin inequality (see e.g. [3], § 5.1) it follows that

$$\alpha_k = \frac{x^k}{k} \ \ (k = 1, \ldots, n), \quad |x| = 1,$$

By (13) we have

$$p_k = x^k + y^k \ \ (k = 1, \ldots, n), \quad |x| = |y| = 1 \ ,$$

and an application of Corollary 1 yields (10) with $t = 1/2$, and so

$$p(z) = \frac{1}{2}\left(\frac{1+xz}{1-xz} + \frac{1+yz}{1-yz}\right) \ .$$

Finally an integration yields (3).

5 Coefficients of the logarithmic derivative and an application

In §2 we presented a dense subset of $\widetilde{P}$. As an application of the solution of the coefficient problem for the logarithmic derivative of functions in $\widetilde{P}$ we derive a family of inequalities for sets of consecutive points on the unit circle.

Theorem 2 *Suppose that* $p \in \widetilde{P}$ *and that* $z\frac{p'}{p}(z) = \sum_{j=1}^{\infty} \gamma_j z^j$. *Then, if* $m \in \mathbb{N}$, *we have* $|\gamma_m| \leqq 2m$, *and this is sharp as is shown by* $p(z) = \frac{1+xz^m}{1-xz^m}$ *where* $x \in \partial\mathbb{D}$.

Proof: Since $p \in \widetilde{P}$, there is a number $x \in \partial\mathbb{D}$ such that $p \prec \frac{1+xz}{1-z}$, so that

$$\ln p \prec \ln(1 + xz) - \ln(1 - z) \ .$$

The last function on the right hand side has the expansion

$$G(z) := -\ln(1-z) = \sum_{k=1}^{\infty} \frac{z^k}{k} \ ,$$

so that, if $m \in \mathbb{N}$ and $g \prec G$

$$|a_m(g)| \leqq a_1(G) = 1 \ ,$$

as the coefficients $a_m(G)$ form a decreasing and convex sequence of positive real numbers (see e.g. [8], Theorem 216). For $f(z) \prec F(z) := \ln(1 + xz)$ it also follows that

$$|a_m(f)| \leqq a_1(G) = 1$$

as $F(z) = -G(-xz)$ and so we have (with some $\omega \in B$)

$$|a_m(\ln p)| = |a_m(F \circ \omega) + a_m(G \circ \omega)| \leqq |a_m(f)| + |a_m(g)| \leqq 2 \ ,$$

implying the result. For the function $p(z) = \frac{1+xz^m}{1-xz^m}$ equality holds as is easily verified, which finishes the proof. □

Applying the theorem to the dense subset of $\widetilde{P}$ of Lemma 1 leads to

Corollary 2 *Suppose that* $m \in \mathbb{N}$, $n \in \mathbb{N}$, *that* $x_k \in \partial\mathbb{D}$ *and* $y_k \in \partial\mathbb{D}$ *for* $1 \leqq k \leqq n$ *and that*

$$\arg x_1 < \arg y_1 < \arg x_2 < \arg y_2 < \cdots < \arg x_n < \arg y_n < \arg x_1 + 2\pi \ .$$

Then

$$\left| \sum_{k=1}^{n} (x_k^m - y_k^m) \right| \leqq 2m \ .$$

For a fixed m *equality occurs if* $n = m$, $x_k = e^{2\pi ik/m} x_0$, *and* $y_k = e^{\pi i/m} x_k$ $(k = 1, \ldots, m)$ *for some* $x_0 \in \partial\mathbb{D}$.

We remark that for $m = 1$ the Corollary is a statement about the sum of the lengths of the vectors $x_k - y_k$, which can be proven also by geometrical means. In this sense Corollary 2 is a geometrical statement.

Acknowledgement I thank the reviewer of this paper very much for his important remarks that made several parts of the paper much easier to read.

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