On Incomplete Symmetric Orthogonal Polynomials of Jacobi Type

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Abstract. In this paper, by using the extended Sturm-Liouville theorem for symmetric functions, we introduce the following differential equation

$$x^2(1-x^{2m})\,\Phi_n''(x)-2x\big((a+mb+1)x^{2m}-a+m-1\big)\Phi_n'(x)+(\alpha_n x^{2m}+\beta+\frac{1-(-1)^n}{2}\gamma)\,\Phi_n(x)=0\,,$$

in which $\beta=-2s(2s+2a-2m+1)$; $\gamma=2s(2s+2a-2m+1)-2(2r+1)(r+a-m+1)$ and $\alpha_n=\big(mn+2s+(r-s+(m-1)/2)(1-(-1)^n)\big)\big(mn+2s+2a+1+2mb+(r-s+(m-1)/2)(1-(-1)^n)\big)$ and show that one of its basic solutions is a class of incomplete symmetric polynomials orthogonal with respect to the weight function $|x|^{2a}(1-x^{2m})^b$ on [-1,1]. We also obtain the norm square value of this orthogonal class.

Keywords. Special functions, Extended Sturm-Liouville problems for symmetric functions, Incomplete symmetric orthogonal polynomials of Jacobi type, weight function, generalized ultraspherical polynomials.

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1. Introduction

In the classical case, systems of orthogonal polynomials are set up such that the *n*th polynomial $\Phi_n(x)$ has exact degree *n*. Such systems are most often complete and in particular form a basis of the space of polynomials. In this work we will consider incomplete sets of orthogonal polynomials such that the system ($\Phi_n(x)$) does not contain polynomials of every degree. Although such systems do not have all properties as in the classical case, they can nevertheless directly be applied to functions approximation since we will compute their norm square values.

It is well known when partial differential equations are solved by the method of separation of variables, the problem reduces to the solution of ordinary differential equations [1]. The solutions of these ordinary equations can, in many interesting problems of mathematical physics, be expressed in terms of special functions. In order to obtain such solutions of the partial differential equations in specific cases, we have to impose some additional conditions on the problem solutions to finally have a unique solution. These conditions eventually transform the problem to a boundary value problem [1]. Hence, we first focus our discussion on the solutions of boundary value problems obtained by the method of separation of variables. It has been shown [1] that the method of separation of variables can extensively be applied for solving partial differential equations of the form

$$\rho(x,y,z)\left(A^*(t)\frac{\partial^2 u}{\partial t^2}+B^*(t)\frac{\partial u}{\partial t}\right)=Lu\,, \tag{1}$$

where

$$Lu=\operatorname{div}\left(k(x,y,z)\operatorname{grad}u\right)-q(x,y,z)u\ , \tag{2}$$

$$\operatorname{div}\vec{\mathrm{F}}=\text{Divergence of vector F}\ , \tag{3}$$

$$\operatorname{grad}\mathrm{F}=\frac{\partial F}{\partial x}\vec{i}+\frac{\partial F}{\partial y}\vec{j}+\frac{\partial F}{\partial z}\vec{k}\ . \tag{4}$$

The equation (1) describes the propagation of a vibration such as electromagnetic or acoustic waves if $A^*(t)=1$ and $B^*(t)=0$ and describes transfer processes such as heat transfer or the diffusion of particles in a medium when $A^*(t)=0$ and $B^*(t)=1$ and finally describes the corresponding time-independent processes if $A^*(t)=0$ and $B^*(t)=0$ [1].
To have a unique solution of equation (1), which corresponds to an actual physical problem, some supplementary conditions must be imposed as we said. The most typical conditions are initial or boundary conditions [1]. The initial conditions for equation (1) are usually the values of $u(x,y,z,t)$ and $\partial u(x,y,z,t)/\partial t$, while the simplest boundary condition is in the form

$$\left.\left[\alpha(x,y,z)u+\beta(x,y,z)\frac{\partial u}{\partial\eta}\right]\right|_S=0\,, \tag{5}$$

where $\alpha(x,y,z)$ and $\beta(x,y,z)$ are given functions, S is the surface bounding the domain where (5) is to be solved and $\partial u/\partial\eta$ is the derivative in the direction of the outward normal to S. Particular solutions of (1) under the boundary conditions (5) will be found if one looks for a solution of the form

$$u(x,y,z,t)=T(t)\,v(x,y,z)\,. \tag{6}$$

By substituting (6) into the main equation (1) one respectively gets

$$A^*(t)T''+B^*(t)T'+\lambda T=0\,, \tag{7}$$

$$Lv+\lambda\rho v=0\,, \tag{8}$$

where λ is a constant. Clearly (7) can be solved for typical problems in mathematical physics. However, to solve (8) we should use a boundary condition that follows from (5), namely

$$\left[\alpha(x,y,z)v+\beta(x,y,z)\frac{\partial v}{\partial \eta}\right]\Bigg|_S=0\,, \tag{9}$$

The described problem is known as multidimensional boundary value problem. Nevertheless, it can be simplified to a one-dimensional problem if (8) is reduced to an equation of the form

$$L\,y+\lambda\;\rho(x)\,y=0, \tag{10}$$

where

$$L\,y=\frac{d}{dx}\left(k(x)\frac{dy}{dx}\right)-q(x)y,\qquad k(x)>0,\quad \rho(x)>0. \tag{11}$$

The equation (10) should be considered on an open interval, say (a,b), with boundary conditions in the form

$$\begin{aligned}\alpha_1 y(a)+\beta_1 y'(a)=0\ ,\\ \alpha_2 y(b)+\beta_2 y'(b)=0\ ,\end{aligned} \tag{12}$$

where α_1, α_2 and β_1, β_2 are the given constants and $k(x),k'(x),q(x)$ and $\rho(x)$ in (10) and (11) are to be assumed continuous for $x\in[a,b]$.
The simplified boundary value problem (10)-(12) is called a regular Sturm-Liouville problem [1]. Moreover, if one of the boundary points a and b is singular (i.e. $k(a)=0$ or $k(b)=0$), the problem will be transformed to a singular Sturm-Liouville problem. In this case, one can ignore boundary conditions (12) and directly obtain the orthogonality relation.
By noting this, now suppose $y_n(x)$ and $y_m(x)$ are two solutions (eigenfunctions) of equation (10). According to Sturm-Liouville theory [1], they should be orthogonal with respect to the weight function $\rho(x)$ on (a,b) under the conditions (12), i.e.

$$\int_a^b \rho(x)y_n(x)y_m(x)\,dx=\left(\int_a^b \rho(x)y_n^2(x)\,dx\right)\delta_{n,m}\quad\text{if}\quad \delta_{n,m}=\begin{cases}0 & (n\neq m),\\ 1 & (n=m).\end{cases} \tag{13}$$

Many important special functions in theoretical and mathematical physics are the solutions of regular or singular Sturm-Liouville problems that satisfy the orthogonality relation (13). For instance, the associated Legendre functions [1], Bessel functions [1], trigonometric sequences related to Fourier analysis [2,5], ultraspherical functions [2,5], Hermite functions [1] and so on [3,4] are particular solutions of some Sturm-Liouville problems.
Fortunately, most of these mentioned functions are symmetric and satisfy the symmetry property $\Phi_n(-x)=(-1)^n\Phi_n(x)$. They have found various applications in mathematical physics and engineering [1,5]. Now, if we can extend the above-mentioned examples symmetrically and preserve their orthogonality property, it seems that we will be able to find some new applications in physics and engineering which extend the previous applications. In this paper, we extend one of the classical symmetric orthogonal sequences and obtain its orthogonality property directly.

For this purpose, we should first refer to a key theorem in [3] in which a symmetric generalization of usual Sturm-Liouville problems with symmetric solutions is presented.

1.1. Theorem [3]. *Let* $\Phi_n(x) = (-1)^n \Phi_n(-x)$ *be a sequence of independent symmetric functions that satisfies the differential equation*

$$A(x)\Phi_n''(x) + B(x)\Phi_n'(x) + \left(\lambda_n C(x) + D(x) + (1-(-1)^n)E(x)/2\right)\Phi_n(x) = 0\,, \tag{14}$$

where $A(x)$, $B(x)$, $C(x)$, $D(x)$ *and* $E(x)$ *are real functions and* $\{\lambda_n\}$ *is a sequence of constants. If* $A(x)$, $(C(x) > 0)$, $D(x)$ *and* $E(x)$ *are even functions and* $B(x)$ *is odd then*

$$\int_{-v}^{v} W^*(x)\Phi_n(x)\Phi_m(x)\,dx = \left(\int_{-v}^{v} W^*(x)\Phi_n^2(x)\,dx\right)\delta_{n,m}\,, \tag{15}$$

where $W^*(x)$ *denotes the corresponding weight function as*

$$W^*(x) = C(x)\exp\left(\int \frac{B(x)-A'(x)}{A(x)}dx\right) = \frac{C(x)}{A(x)}\exp\left(\int \frac{B(x)}{A(x)}dx\right). \tag{16}$$

Of course, the weight function defined in (16) must be positive and even on $[-v,v]$ *and* $x = v$ *must be a root of the function*

$$A(x)K(x) = A(x)\exp\left(\int \frac{B(x)-A'(x)}{A(x)}dx\right) = \exp\left(\int \frac{B(x)}{A(x)}dx\right), \tag{17}$$

i.e. $A(v)K(v) = 0$. *Since* $K(x) = W^*(x)/C(x)$ *is an even function it follows that* $A(-v)K(-v) = 0$ *automatically.*

As we said, by using this theorem, many symmetric orthogonal functions can be generalized [3,4]. Here we introduce incomplete symmetric orthogonal polynomials of Jacobi type as the solutions of a generalized Sturm-Liouville equation of type (14). For this purpose, let us consider the shifted Jacobi polynomials on [0,1] as:

$$P_{n,+}^{(\alpha,\beta)}(x) = \sum_{k=0}^{n} (-1)^k \binom{n+\alpha+\beta+k}{k}\binom{n+\alpha}{n-k} x^k\,, \tag{18}$$

that satisfy the differential equation [2,5]

$$x(1-x)y'' - ((\alpha+\beta+2)x - (\alpha+1))\,y' + n(n+\alpha+\beta+1)y = 0\,;\quad y = P_{n,+}^{(\alpha,\beta)}(x)\,, \tag{19}$$

and orthogonality relation [5]

$$\int_0^1 x^{\alpha}(1-x)^{\beta}\, P_{n,+}^{(\alpha,\beta)}(x)P_{m,+}^{(\alpha,\beta)}(x)dx = \frac{\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{(2n+\alpha+\beta+1)\,\Gamma(n+1)\,\Gamma(n+\alpha+\beta+1)}\,\delta_{n,m}. \quad (20)$$

By referring to the main Theorem 1.1 and noting the Jacobi differential equation (19), we can construct a differential equation of type (14) whose solutions are orthogonal with respect to an even weight function, say $|x|^{2a}(1-x^{2m})^{b}$, on the symmetric interval [-1,1]. Hence, let us first substitute

$$g(x) = x^{\lambda} P_{n,+}^{(\alpha,\beta)}(x^{\theta})\ ;\ \lambda,\theta \in \mathbf{R}, \quad (21)$$

into equation (19) to obtain the differential equation of $g(x)$ as

$$\begin{aligned} &x^2(1-x^{\theta})g'' + x\big((2\lambda-(\alpha+\beta+1)\theta-1)x^{\theta} - 2\lambda+\alpha\theta+1\big)g' \\ &\qquad + \big((\theta^2 n(n+\alpha+\beta+1)+(\alpha+\beta+1)\theta\lambda-\lambda^2)x^{\theta}+\lambda^2-\alpha\theta\lambda\big)g = 0. \end{aligned} \quad (22)$$

If we write for convenience

$$\begin{cases} 2\lambda-(\alpha+\beta+1)\theta-1 = p, \\ -2\lambda+\alpha\theta+1 = q, \end{cases} \text{ or equivalently } \alpha = \frac{q+2\lambda-1}{\theta} \text{ and } \beta = -\frac{p+q}{\theta}-1, \quad (23)$$

then the differential equation (22) is changed to

$$\begin{aligned} &x^2(1-x^{\theta})g'' + x\big(p\,x^{\theta}+q\big)g' + \big((\theta n+\lambda)(\theta n+\lambda-p-1)\,x^{\theta} - \lambda(\lambda+q-1)\big)g = 0 \\ &\qquad \Leftrightarrow\ g = x^{\lambda} P_{n,+}^{(\frac{q+2\lambda-1}{\theta},\,-\frac{p+q}{\theta}-1)}(x^{\theta}). \end{aligned} \quad (24)$$

Now, by noting Theorem 1.1, let us define the following odd and even *polynomial* sequences

$$\begin{cases} \Phi_{2n}(x) = x^{2s} P_{n,+}^{(\frac{q+4s-1}{2m},\,-\frac{p+q}{2m}-1)}(x^{2m})\ ;\ \ \lambda = 2s,\ s\in \mathbf{Z}^{+} \text{ and } \theta = 2m,\ m\in\mathbf{N}, \\ \Phi_{2n+1}(x) = x^{2r+1} P_{n,+}^{(\frac{q+4r+1}{2m},\,-\frac{p+q}{2m}-1)}(x^{2m})\ ;\ \ \lambda = 2r+1,\ r\in \mathbf{Z}^{+} \text{ and } \theta = 2m,\ m\in\mathbf{N}, \end{cases} \quad (25)$$

and assume from now that $\sigma_n = \frac{1-(-1)^n}{2} = \begin{cases} 0 & \text{if } n = 2k \\ 1 & \text{if } n = 2k+1 \end{cases}$; $k\in\mathbf{Z}$. It is clear that we generally have

$$u+(w-u)\sigma_n = \frac{u+w}{2} + (-1)^n\,\frac{u-w}{2} = \begin{cases} u & \text{if } n = 2k, \\ w & \text{if } n = 2k+1. \end{cases} \quad (26)$$

Therefore, the polynomial sequence $\Phi_n(x)$ defined in (25) can be written in a unique form as

$$\Phi_n(x) = (x^{2s} + (x^{2r+1} - x^{2s})\sigma_n)\, P_{[n/2],+}^{(\frac{q+2s+2r+(-1)^n(2s-2r-1)}{2m}, -\frac{p+q}{2m}-1)}(x^{2m}). \tag{27}$$

According to definitions (25) and differential equation (24), $\Phi_{2n}(x)$ should satisfy the equation

$$\begin{aligned} &x^2(1-x^{2m})\Phi''_{2n}(x) + x\left(px^{2m}+q\right)\Phi'_{2n}(x) \\ &\qquad + \left((2mn+2s)(2mn+2s-p-1)x^{2m} - 2s(2s+q-1)\right)\Phi_{2n}(x) = 0, \end{aligned} \tag{28}$$

and $\Phi_{2n+1}(x)$ should satisfy

$$\begin{aligned} &x^2(1-x^{2m})\Phi''_{2n+1}(x) + x\left(px^{2m}+q\right)\Phi'_{2n+1}(x) \\ &\qquad + \left((2mn+2r+1)(2mn+2r-p)x^{2m} - (2r+1)(2r+q)\right)\Phi_{2n+1}(x) = 0. \end{aligned} \tag{29}$$

Hence, combining these two equations finally gives

$$\begin{aligned} &x^2(1-x^{2m})\Phi''_n(x) + x\left(px^{2m}+q\right)\Phi'_n(x) + \\ &\{(mn+2s+(2r+1-m-2s)\sigma_n)(mn+2s-p-1+(2r+1-m-2s)\sigma_n)x^{2m} \\ &\qquad -2s(2s+q-1) - \left((2r+1)(2r+q) - 2s(2s+q-1)\right)\sigma_n\}\ \Phi_n(x) = 0, \end{aligned} \tag{30}$$

which is a special case of the generalized Sturm-Liouville equation (14). Also, the weight function corresponding to (30) takes the form

$$W(x) = x^{2m}\exp(\int \frac{x(px^{2m}+q) - (2x-(2m+2)x^{2m+1})}{x^2(1-x^{2m})}dx) = K\, x^{2m+q-2}(1-x^{2m})^{-\frac{p+q}{2m}-1}. \tag{31}$$

Note that we can, without loss of generality, suppose that $K=1$ and since $W(x)$ must be positive, the weight function (31) can be considered as $|x|^{2a}(1-x^{2m})^b$ for $2a = 2m+q-2$ and $b = -1-(p+q)/2m$.

1.2. Corollary. Suppose in the generic equation (14) that

$$\begin{aligned} &A(x) = x^2(1-x^{2m}) && \text{an even function,} \\ &B(x) = -2x\left((a+mb+1)x^{2m} - a+m-1\right) && \text{an odd function,} \\ &C(x) = x^{2m} > 0 && \text{an even function,} \\ &D(x) = -2s(2s+2a-2m+1) && \text{an even function,} \\ &E(x) = 2s(2s+2a-2m+1) - 2(2r+1)(r+a-m+1) && \text{an even function,} \\ &\lambda_n = (mn+2s+(2r+1-m-2s)\sigma_n)(mn+2s+2a+1+2mb+(2r+1-m-2s)\sigma_n). \end{aligned} \tag{32}$$

Then, the differential equation corresponding to options (32) has a polynomial solution as

$$\Phi_n^{(r,s)}(x;a,b,m) = (x^{2s} + (x^{2r+1} - x^{2s})\sigma_n)\, P_{[n/2],+}^{(\frac{a+1-m+s+r}{m} + (-1)^n \frac{2s-2r-1}{2m},\, b)}(x^{2m}), \tag{33}$$

which satisfies the orthogonality relation (34)

$$\int_{-1}^{1} x^{2a}(1-x^{2m})^b \Phi_n^{(r,s)}(x;a,b,m)\Phi_k^{(r,s)}(x;a,b,m)\,dx = \left(\int_{-1}^{1} x^{2a}(1-x^{2m})^b \left(\Phi_n^{(r,s)}(x;a,b,m)\right)^2 dx\right)\delta_{n,k}.$$

To compute the norm square value of (34) we can directly use the orthogonality relation (20) so that for $n = 2j$ we have

$$N_{2j} = \int_{-1}^{1} x^{2a}(1-x^{2m})^b \left(\Phi_{2j}^{(r,s)}(x;a,b,m)\right)^2 dx = \int_{-1}^{1} x^{2a+4s}(1-x^{2m})^b \left(P_{j,+}^{(\frac{2a+4s+1-2m}{2m},\, b)}(x^{2m})\right)^2 dx$$

$$= \frac{1}{m}\int_0^1 t^{\frac{2a+4s+1-2m}{2m}}(1-t)^b \left(P_{j,+}^{(\frac{2a+4s+1-2m}{2m},\, b)}(t)\right)^2 dx = \frac{\Gamma(j + \frac{2a+4s+1}{2m})\Gamma(j+b+1)}{(2mj + a + 2s + \frac{1}{2} + mb)\Gamma(j+1)\Gamma(j + \frac{2a+4s+1}{2m} + b)}, \tag{35}$$

and for $n = 2j+1$ the norm square is

$$N_{2j+1} = \int_{-1}^{1} x^{2a}(1-x^{2m})^b \left(\Phi_{2j+1}^{(r,s)}(x;a,b,m)\right)^2 dx = \int_{-1}^{1} x^{2a+4r+2}(1-x^{2m})^b \left(P_{j,+}^{(\frac{2a+4r+3-2m}{2m},\, b)}(x^{2m})\right)^2 dx =$$

$$\frac{1}{m}\int_0^1 t^{\frac{2a+4r+3-2m}{2m}}(1-t)^b \left(P_{j,+}^{(\frac{2a+4r+3-2m}{2m},\, b)}(t)\right)^2 dx = \frac{\Gamma(j + \frac{2a+4r+3}{2m})\Gamma(j+1+b)}{(2mj + a + 2r + \frac{3}{2} + mb)\Gamma(j+1)\Gamma(j + \frac{2a+4r+3}{2m} + b)}. \tag{36}$$

Therefore, combining both relations (35) and (36) gives

$$N_n = \frac{\Gamma(\frac{n-\sigma_n}{2} + \frac{2a+4s+1}{2m} + \frac{2r+1-2s}{m}\sigma_n)\,\Gamma(\frac{n-\sigma_n}{2} + b + 1)}{(m(n-\sigma_n) + a + 2s + \frac{1}{2} + mb + (2r+1-2s)\sigma_n)\,\Gamma(\frac{n-\sigma_n}{2} + 1)\,\Gamma(\frac{n-\sigma_n}{2} + \frac{2a+4s+1}{2m} + b + \frac{2r+1-2s}{m}\sigma_n)}. \tag{37}$$

This value shows that the orthogonality (34) is valid if and only if $b > -1$; $2a + 4s + 1 > 0$; $2a + 4s + 1 + 2mb > 0$; $m \in \mathbf{N}$; $2a + 4r + 3 > 0$; $2a + 4r + 3 + 2mb > 0$ and finally $(-1)^{2a} = 1$ because the weight function must be even. Now, it is a good position to present some practical examples.

Example 1. Find the standard properties of incomplete symmetric polynomials orthogonal with respect to the weight function $x^4\sqrt{1-x^4}$ on [-1,1].
To solve the problem, it is sufficient in (33) to choose $m = a = 2$ and $b = 1/2$ to get the polynomials

$$\Phi_n^{(r,s)}(x;2,\frac{1}{2},2)=(x^{2s}+(x^{2r+1}-x^{2s})\sigma_n)\,P_{[n/2],+}^{(\frac{1+s+r}{2}+(-1)^n\frac{2s-2r-1}{4},\frac{1}{2})}(x^4);\;r,s\in\mathbf{Z}^+, \tag{38}$$

that satisfy the differential equation

$$\begin{aligned}&x^2(1-x^4)\Phi_n''(x)+2x\left(1-4x^4\right)\Phi_n'(x)+\\&\{(2n+2s+(2r-1-2s)\sigma_n)(2n+2s+7+(2r-1-2s)\sigma_n)x^4\\&\qquad -2s(2s+1)+\left(2s(2s+1)-2(2r+1)(r+1)\right)\sigma_n\}\,\Phi_n(x)=0,\end{aligned} \tag{39}$$

and orthogonality relation

$$\begin{aligned}&\int_{-1}^{1}x^4\sqrt{1-x^4}\,\Phi_n^{(r,s)}(x;2,\frac{1}{2},2)\Phi_k^{(r,s)}(x;2,\frac{1}{2},2)\,dx\\&=\frac{\Gamma(\frac{n-\sigma_n}{2}+s+\frac{5}{4}+\frac{2r-2s+1}{2}\sigma_n)\Gamma(\frac{n-\sigma_n+3}{2})}{(2(n-\sigma_n)+\frac{7}{2}+2s+(2r-2s+1)\sigma_n)\Gamma(\frac{n-\sigma_n}{2}+1)\Gamma(\frac{n-\sigma_n}{2}+s+\frac{7}{4}+\frac{2r-2s+1}{2}\sigma_n)}\delta_{n,k}.\end{aligned} \tag{40}$$

As (38) shows, $\Phi_n^{(r,s)}(x;2,1/2,2)$ are incomplete symmetric polynomials with the degrees respectively $\{2s, 2r+1, 2s+4, 2r+5, 2s+8, 2r+9, ...\}$. For instance we have

$$\text{degrees of }\Phi_n^{(1,0)}(x;2,\frac{1}{2},2)=\{0,3,4,7,8,11,...\}, \text{degrees of }\Phi_n^{(2,0)}(x;2,\frac{1}{2},2)=\{0,5,4,9,8,13,...\}$$
$$\text{and degrees of }\Phi_n^{(1,1)}(x;2,\frac{1}{2},2)=\{2,3,6,7,10,11,...\}.$$

Example 2. A generalization of generalized ultraspherical polynomials (GUP)

It is known that the generalized ultraspherical polynomials [4,5] are orthogonal with respect to the weight function $x^{2a}(1-x^2)^b$ on [-1,1]. Now if $m=1$ in (33), a generalization of GUP as

$$\Phi_n^{(r,s)}(x;a,b,1)=(x^{2s}+(x^{2r+1}-x^{2s})\sigma_n)\,P_{[n/2],+}^{(a+s+r+(-1)^n(s-r-\frac{1}{2}),b)}(x^2), \tag{41}$$

is derived for $r=s=0$ that satisfies the differential equation

$$\begin{aligned}&x^2(1-x^2)\Phi_n''(x)-2x\left((a+b+1)x^2-a\right)\Phi_n'(x)+\\&\{(n+2s+(2r-2s)\sigma_n)(n+2s+2a+1+2b+(2r-2s)\sigma_n)x^2\\&\qquad -2s(2s+2a-1)+\left(2s(2s+2a-1)-2(2r+1)(r+a)\right)\sigma_n\}\,\Phi_n(x)=0,\end{aligned} \tag{42}$$

and orthogonality relation

$$\int_{-1}^{1} x^{2a}(1-x^2)^b\,\Phi_n^{(r,s)}(x;a,b,1)\,\Phi_k^{(r,s)}(x;a,b,1)\,dx=\delta_{n,k}\times$$

$$\frac{\Gamma(\frac{n-\sigma_n+2a+4s+1}{2}+(2r+1-2s)\sigma_n)\,\Gamma(\frac{n-\sigma_n}{2}+b+1)}{(n-\sigma_n+a+2s+\frac{1}{2}+b+(2r+1-2s)\sigma_n)\,\Gamma(\frac{n-\sigma_n}{2}+1)\,\Gamma(\frac{n-\sigma_n+2a+4s+1}{2}+b+(2r+1-2s)\sigma_n)}. \quad (43)$$

Again, as (41) shows, $\Phi_n^{(r,s)}(x;a,b,1)$ are incomplete symmetric polynomials with the degrees respectively $D^{(r,s)}$ = {2s, 2r +1, 2s + 2, 2r + 3, 2s + 4, 2r + 5, ...} though it is *complete* for $r=s=0$, because in this case we have $D^{(0,0)}$ = {0,1,2,3,...} = $\mathbf{Z}^+$.

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