On Incomplete Symmetric Orthogonal Polynomials of Laguerre Type

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Abstract. By using the extended Sturm-Liouville theorem for symmetric functions, we introduce the following differential equation

$$x^2\Phi_n''(x)-2x\left(m\,x^{2m}-a+m-1\right)\Phi_n'(x)+(\alpha_n x^{2m}+\beta+\frac{1-(-1)^n}{2}\gamma)\,\Phi_n(x)=0\,,$$

in which $\beta=-2s(2s+2a-2m+1)$; $\gamma=2s(2s+2a-2m+1)-2(2r+1)(r+a-m+1)$ and $\alpha_n=2m\left(mn+2s+(r-s+(m-1)/2)(1-(-1)^n)\right)$ and show that one of its basis solutions is a class of incomplete symmetric polynomials orthogonal with respect to the weight function $\left|x\right|^{2a}\exp(-x^{2m})$ on $(-\infty,\infty)$. We also obtain the norm square value of this orthogonal class.

Keywords. Special functions, extended Sturm-Liouville problems for symmetric functions, incomplete symmetric orthogonal polynomials of Laguerre type, weight function, generalized Hermite polynomials.

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1. Introduction

In the classical case, systems of orthogonal polynomials are set up such that the *n*th polynomial $\Phi_n(x)$ has exact degree *n*. Such systems form a basis of the space of polynomials and are most often complete. In this work we will consider incomplete sets of polynomials orthogonal with respect to the weight function $\left|x\right|^{2a}\exp(-x^{2m})$ on $(-\infty,\infty)$ such that the system ($\Phi_n(x)$) does not contain polynomials of every degree. Although such systems do not have all properties as in the classical case, they can nevertheless directly be applied to function approximation since we will explicitly compute their norm square values.
We start our discussion with the equation

$$L\,y + \lambda\;\rho(x)\,y = 0\,, \tag{1}$$

in which

$$L\,y = \frac{d}{dx}\left(k(x)\frac{dy}{dx}\right) - q(x)y\,, \qquad k(x) > 0\,, \quad \rho(x) > 0\,, \tag{2}$$

and

$$\begin{aligned} \alpha_1 y(a) + \beta_1 y'(a) = 0\,, \\ \alpha_2 y(b) + \beta_2 y'(b) = 0\,, \end{aligned} \tag{3}$$

where α_1, α_2 and β_1, β_2 are given constants and $k(x), k'(x), q(x)$ and $\rho(x)$ in (1) and (2) are to be assumed continuous for $x \in [a,b]$.

The boundary value problem (1)-(3) is called a regular Sturm-Liouville problem [1]. In this sense, if one of the boundary points a and b is singular (i.e. $k(a) = 0$ or $k(b) = 0$), the problem will be transformed to a singular Sturm-Liouville problem and one can then ignore the boundary conditions (3) and obtain the orthogonality relation directly.

Now suppose $y_n(x)$ and $y_m(x)$ are two solutions (eigenfunctions) of equation (1). According to Sturm-Liouville theory [1], they should be orthogonal with respect to the weight function $\rho(x)$ on (a,b) under the conditions (3), i.e.

$$\int_a^b \rho(x)\,y_n(x)\,y_m(x)\,dx = \left(\int_a^b \rho(x)\,y_n^2(x)\,dx\right)\delta_{n,m} \quad \text{where} \quad \delta_{n,m} = \begin{cases} 0 & (n \neq m), \\ 1 & (n = m). \end{cases} \tag{4}$$

Many important special functions in theoretical and mathematical physics are solutions of regular or singular Sturm-Liouville problems that satisfy the orthogonality relation (4). For instance, the associated Legendre functions [1], Bessel functions [1], trigonometric sequences related to Fourier analysis [2,5], ultraspherical functions [2,5], Hermite functions [1] and so on [3] are particular solutions of some Sturm-Liouville problems.

Fortunately, most of these mentioned functions are symmetric and satisfy the symmetry property $\Phi_n(-x) = (-1)^n\,\Phi_n(x)$. Hence, they have found various applications in mathematical physics and engineering [1,5]. Now, if we can extend the above-mentioned examples symmetrically and preserve their orthogonality property, it seems that we will be able to find some new applications in physics and engineering which might extend the previous applications. In this paper, we extend one of the classical symmetric orthogonal sequences and obtain its orthogonality property directly.

For this purpose, we should first refer to a key theorem in [3] in which a symmetric generalization of usual Sturm-Liouville problems with symmetric solutions is presented.

1.1. Theorem [3]. *Let $\Phi_n(x) = (-1)^n\,\Phi_n(-x)$ be a sequence of independent symmetric functions that satisfies the differential equation*

$$A(x)\,\Phi_n''(x) + B(x)\,\Phi_n'(x) + \big(\lambda_n\,C(x) + D(x) + (1-(-1)^n)E(x)/2\big)\Phi_n(x) = 0\,, \tag{5}$$

where $A(x)$, $B(x)$, $C(x)$, $D(x)$ *and* $E(x)$ *are real functions and* $\{\lambda_n\}$ *is a sequence of constants. If* $A(x)$, $(C(x)>0)$, $D(x)$ *and* $E(x)$ *are even functions and* $B(x)$ *is odd then*

$$\int_{-v}^{v} W^*(x)\Phi_n(x)\Phi_m(x)\,dx = \left(\int_{-v}^{v} W^*(x)\Phi_n^2(x)\,dx\right)\delta_{n,m}, \tag{6}$$

where $W^*(x)$ *denotes the corresponding weight function as*

$$W^*(x) = C(x)\exp\left(\int \frac{B(x)-A'(x)}{A(x)}dx\right) = \frac{C(x)}{A(x)}\exp\left(\int \frac{B(x)}{A(x)}dx\right). \tag{7}$$

Of course, the weight function defined in (16) must be positive and even on $[-v,v]$ *and* $x=v$ *must be a root of the function*

$$A(x)K(x) = A(x)\exp\left(\int \frac{B(x)-A'(x)}{A(x)}dx\right) = \exp\left(\int \frac{B(x)}{A(x)}dx\right), \tag{8}$$

i.e. $A(v)K(v)=0$. *In this sense, since* $K(x)=W^*(x)/C(x)$ *is an even function so* $A(-v)K(-v)=0$ *automatically.*

As mentioned, using this theorem many symmetric orthogonal functions can be generalized [3]. Here we intend to introduce incomplete symmetric orthogonal polynomials of Laguerre type as solutions of a generalized Sturm-Liouville equation of type (5). For this purpose, we first consider the generalized Laguerre polynomials on $[0,\infty)$ as:

$$L_n^{(\alpha)}(x) = \sum_{k=0}^{n} \frac{(-1)^k}{k!}\binom{n+\alpha}{n-k}x^k, \tag{9}$$

that satisfy the differential equation [2,5]

$$x\,y'' + (\alpha+1-x)\,y' + n\,y = 0\,; \quad y = L_n^{(\alpha)}(x), \tag{10}$$

and the orthogonality relation [5]

$$\int_0^\infty x^\alpha e^{-x} L_n^{(\alpha)}(x)L_m^{(\alpha)}(x)\,dx = \frac{\Gamma(n+\alpha+1)}{n!}\,\delta_{n,m}. \tag{11}$$

By referring to Theorem 1.1 and the Laguerre differential equation (10), we can now construct a differential equation of type (5) whose solutions are orthogonal with respect to an even weight function, namely $|x|^{2a}\exp(-x^{2m})$, on the symmetric interval $(-\infty,\infty)$. Hence, we first substitute

$$g(x) = x^\lambda L_n^{(\alpha)}(x^\theta);\ \lambda,\theta \in \mathbf{R}, \tag{12}$$

into equation (10) to obtain the differential equation of $g(x)$ as

$$x^2 g'' + x\left(-2\lambda + \alpha\theta + 1 - \theta x^\theta\right)g' + \left(\theta(n\theta + \lambda)x^\theta + \lambda(\lambda - \alpha\theta)\right)g = 0. \tag{13}$$

If for convenience we take

$$-2\lambda + \alpha\theta + 1 = q \quad \text{or equivalently } \alpha = \frac{q + 2\lambda - 1}{\theta}, \tag{14}$$

then the differential equation (13) is transformed to

$$x^2 g'' + x\left(q - \theta x^\theta\right)g' + \left(\theta(\theta n + \lambda)x^\theta - \lambda(\lambda + q - 1)\right)g = 0 \Leftrightarrow g = x^\lambda L_n^{(\frac{q+2\lambda-1}{\theta})}(x^\theta). \tag{15}$$

By referring to Theorem 1.1, let us now define the following odd and even *polynomial* sequences

$$\begin{cases} \Phi_{2n}(x) = x^{2s} L_n^{(\frac{q+4s-1}{2m})}(x^{2m})\,; \quad \lambda = 2s,\ s \in \mathbf{Z}^+ \text{ and } \theta = 2m,\ m \in \mathbf{N}, \\ \Phi_{2n+1}(x) = x^{2r+1} L_{n,+}^{(\frac{q+4r+1}{2m})}(x^{2m})\,; \quad \lambda = 2r+1,\ r \in \mathbf{Z}^+ \text{ and } \theta = 2m,\ m \in \mathbf{N}, \end{cases} \tag{16}$$

and assume from now that $\sigma_n = \dfrac{1-(-1)^n}{2} = \begin{cases} 0 & \text{if } n = 2k \\ 1 & \text{if } n = 2k+1 \end{cases}$; $k \in \mathbf{Z}$. It is clear that we generally have

$$u + (w - u)\sigma_n = \frac{u+w}{2} + (-1)^n \frac{u-w}{2} = \begin{cases} u & \text{if } n = 2k, \\ w & \text{if } n = 2k+1. \end{cases} \tag{17}$$

Therefore, the polynomial sequence $\Phi_n(x)$ defined in (16) can be written in a unique form as

$$\Phi_n(x) = (x^{2s} + (x^{2r+1} - x^{2s})\sigma_n) L_{[n/2]}^{(\frac{q+2s+2r+(-1)^n(2s-2r-1)}{2m})}(x^{2m}). \tag{18}$$

According to definitions (16) and equation (15), $\Phi_{2n}(x)$ should satisfy the equation

$$x^2 \Phi''_{2n}(x) - x\left(2m x^{2m} - q\right)\Phi'_{2n}(x) + \left(4m(mn + s)x^{2m} - 2s(2s + q - 1)\right)\Phi_{2n}(x) = 0, \tag{19}$$

and $\Phi_{2n+1}(x)$ should satisfy

$$x^2 \Phi''_{2n+1}(x) - x\left(2m x^{2m} - q\right)\Phi'_{2n+1}(x) + \left(2m(2mn + 2r + 1)x^{2m} - (2r+1)(2r+q)\right)\Phi_{2n+1}(x) = 0. \tag{20}$$

Hence, combining these two equations finally gives

$$x^2\Phi''_n(x) - x\left(2m\,x^{2m} - q\right)\Phi'_n(x) + \{2m\,(mn + 2s + (2r+1-m-2s)\,\sigma_n)\,x^{2m} \\ - 2s(2s+q-1) - \left((2r+1)(2r+q) - 2s(2s+q-1)\right)\sigma_n\}\ \Phi_n(x) = 0\,, \tag{21}$$

which is a special case of the generalized Sturm-Liouville equation (5). In this way, the weight function corresponding to equation (21) takes the form

$$W(x) = x^{2m}\exp\left(\int \frac{-x(2m\,x^{2m} - q) - 2x}{x^2}\,dx\right) = K\,x^{2m+q-2}\exp(-x^{2m})\,. \tag{22}$$

Note that we can, without loss of generality, suppose that $K=1$ and since $W(x)$ must be positive, and therefore the weight function (31) can be considered as $|x|^{2a}\exp(-x^{2m})$ for $2a = 2m + q - 2$.

1.2. Corollary. Suppose in the generic equation (5) that

$$\begin{array}{ll} A(x) = x^2 & \text{an even function,} \\ B(x) = -2x\left(m\,x^{2m} - a + m - 1\right) & \text{an odd function,} \\ C(x) = x^{2m} > 0 & \text{an even function,} \\ D(x) = -2s(2s + 2a - 2m + 1) & \text{an even function,} \\ E(x) = 2s(2s+2a-2m+1) - 2(2r+1)(r+a-m+1) & \text{an even function,} \\ \lambda_n = 2m\,(mn + 2s + (2r+1-m-2s)\,\sigma_n)\,. & \end{array} \tag{23}$$

Then, the differential equation corresponding to options (23) has a polynomial solution as

$$\Phi_n^{(r,s)}(x;a,m) = (x^{2s} + (x^{2r+1} - x^{2s})\,\sigma_n)\,L_{[n/2]}^{(\frac{a+1-m+s+r}{m} + (-1)^n\frac{2s-2r-1}{2m})}(x^{2m})\,, \tag{24}$$

which satisfies the orthogonality relation (25)

$$\int_{-\infty}^{\infty} x^{2a}\exp(-x^{2m})\Phi_n^{(r,s)}(x;a,m)\Phi_k^{(r,s)}(x;a,m)\,dx = \left(\int_{-\infty}^{\infty} x^{2a}\exp(-x^{2m})\left(\Phi_n^{(r,s)}(x;a,m)\right)^2 dx\right)\delta_{n,k}\,.$$

To compute the norm square value of (25) we can directly use the orthogonality relation (11) so that for $n = 2j$ we have

$$N_{2j} = \int_{-\infty}^{\infty} x^{2a}\exp(-x^{2m})\left(\Phi_{2j}^{(r,s)}(x;a,m)\right)^2 dx = \int_{-\infty}^{\infty} x^{2a+4s}\exp(-x^{2m})\left(L_j^{(\frac{2a+4s+1-2m}{2m})}(x^{2m})\right)^2 dx \\ = \frac{1}{m}\int_0^{\infty} t^{\frac{2a+4s+1-2m}{2m}}\,e^{-t}\left(L_j^{(\frac{2a+4s+1-2m}{2m})}(t)\right)^2 dx = \frac{1}{m\,j!}\Gamma\left(j + \frac{2a+4s+1}{2m}\right), \tag{26}$$

and for $n=2j+1$ the norm square is

$$N_{2j+1}=\int_{-\infty}^{\infty}x^{2a}\exp(-x^{2m})\left(\Phi_{2j+1}^{(r,s)}(x;a,m)\right)^2 dx=\int_{-\infty}^{\infty}x^{2a+4r+2}\exp(-x^{2m})\left(L_j^{(\frac{2a+4r+3-2m}{2m})}(x^{2m})\right)^2 dx$$

$$=\frac{1}{m}\int_0^{\infty}t^{\frac{2a+4r+3-2m}{2m}}e^{-t}\left(L_j^{(\frac{2a+4r+3-2m}{2m})}(t)\right)^2 dx=\frac{1}{m\,j!}\Gamma(j+\frac{2a+4r+3}{2m}). \tag{27}$$

Therefore, combining both relations (35) and (36) gives

$$N_n=\frac{1}{m((n-\sigma_n)/2)!}\Gamma\left(\frac{n-\sigma_n}{2}+\frac{2a+4s+1}{2m}+\frac{2r+1-2s}{m}\sigma_n\right). \tag{28}$$

This value shows that the orthogonality (25) is valid if and only if $2a+4s+1>0$; $2a+4r+3>0$ and finally $(-1)^{2a}=1$ because the weight function must be even.

Now, it is time to present some practical examples.

Example 1. Find the standard properties of incomplete symmetric polynomials orthogonal with respect to the weight function $x^4\exp(-x^4)$ on $(-\infty,\infty)$.
To solve the problem, it is sufficient in (24) to choose $m=a=2$ to get the polynomials

$$\Phi_n^{(r,s)}(x;2,2)=(x^{2s}+(x^{2r+1}-x^{2s})\sigma_n)\,L_{[n/2]}^{(\frac{1+s+r}{2}+(-1)^n\frac{2s-2r-1}{4})}(x^4);\ r,s\in\mathbf{Z}^+, \tag{29}$$

that satisfy the differential equation

$$\begin{aligned}x^2\Phi_n''(x)-2x\left(2x^4-1\right)\Phi_n'(x)+\{4(2n+2s+(2r-1-2s)\sigma_n)x^4\\ -2s(2s+1)+\left(2s(2s+1)-2(2r+1)(r+1)\right)\sigma_n\}\ \Phi_n(x)=0,\end{aligned} \tag{30}$$

and the orthogonality relation

$$\int_{-\infty}^{\infty}x^4\exp(-x^4)\,\Phi_n^{(r,s)}(x;2,2)\,\Phi_k^{(r,s)}(x;2,2)\,dx=\frac{1}{2((n-\sigma_n)/2)!}\Gamma\left(\frac{n-\sigma_n}{2}+s+\frac{5}{4}+\frac{2r-2s+1}{2}\sigma_n\right)\delta_{n,k}. \tag{31}$$

As (29) shows, $\Phi_n^{(r,s)}(x;2,2)$ are incomplete symmetric polynomials with the degrees respectively $\{2s,\ 2r+1,\ 2s+4,\ 2r+5,\ 2s+8,\ 2r+9,\ ...\}$. For instance we have

degrees of $\Phi_n^{(1,0)}(x;2,2)=\{0,3,4,7,8,11,...\}$, degrees of $\Phi_n^{(2,0)}(x;2,2)=\{0,5,4,9,8,13,...\}$ and degrees of $\Phi_n^{(1,1)}(x;2,2)=\{2,3,6,7,10,11,...\}$.

Example 2. A generalization of generalized Hermite polynomials (GHP)

It is known that the generalized Hermite polynomials [5] are orthogonal with respect to the weight function $|x|^{2a}\exp(-x^2)$ on $(-\infty,\infty)$. Now if $m=1$ in (24), a generalization of GHP as

$$\Phi_n^{(r,s)}(x;a,1)=(x^{2s}+(x^{2r+1}-x^{2s})\sigma_n)\,L_{[n/2]}^{(a+s+r+(-1)^n(s-r-\frac{1}{2}))}(x^2), \tag{32}$$

is derived for $r=s=0$ that satisfies the differential equation

$$\begin{aligned}x^2\Phi_n''(x)-2x\left(x^2-a\right)\Phi_n'(x)+\{2\,(n+2s+(2r-2s)\sigma_n)\,x^2-2s(2s+2a-1)\\ +\left(2s(2s+2a-1)-2(2r+1)(r+a)\right)\sigma_n\}\ \Phi_n(x)=0,\end{aligned} \tag{33}$$

and the orthogonality relation

$$\int_{-\infty}^{\infty}x^{2a}\exp(-x^2)\Gamma\,\Phi_n^{(r,s)}(x;a,1)\,\Phi_k^{(r,s)}(x;a,1)\,dx=\frac{\Gamma(\frac{n-\sigma_n+2a+4s+1}{2}+(2r+1-2s)\sigma_n)}{((n-\sigma_n)/2)!}\delta_{n,k}\,. \tag{34}$$

Again, as (32) shows, $\Phi_n^{(r,s)}(x;a,1)$ are incomplete symmetric polynomials with the degrees respectively $D^{(r,s)}=\{2s,\,2r+1,\,2s+2,\,2r+3,\,2s+4,\,2r+5,\,...\}$ though they are *complete* for $r=s=0$, because in this case we have $D^{(0,0)}=\{0,1,2,3,...\}=\mathbf{Z}^+$. Finally we add that there is also an incomplete symmetric class of Jacobi polynomials [4] that are orthogonal with respect to the weight function $|x|^{2a}(1-x^{2m})^b$ on $[-1,1]$.

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