ON A NEW CLASS OF LAPLACE TRANSFORMS OF GENERALIZED HYPERGEOMETRIC FUNCTIONS

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Abstract. Very recently Masjed-Jamei and Koepf established several interesting and useful generalizations of classical summation theorems for the series ${}_2F_1$, ${}_3F_2$, ${}_4F_3$, ${}_5F_4$ and ${}_6F_5$. The main objective of this paper is to provide a new class of Laplace transforms of generalized hypergeometric functions by employing these summation theorems. Several new and known special cases have also been considered.

1. Introduction

The generalized hypergeometric function with p numerator and q denominator parametres is defined [1, 2, 17] as

$$ {}_pF_q\left[\begin{matrix} a_1, & \cdots, & a_p \\ b_1, & \cdots, & b_q \end{matrix}; z\right] = \sum_{n=0}^{\infty} \frac{(a_1)_n \cdots (a_p)_n}{(b_1)_n \cdots (b_q)_n} \frac{z^n}{n!}, \tag{1} $$

where $(a)_n$ is the well known Pochhammer symbol [10] for any complex number a defined as

$$ \begin{aligned} (a)_n &= \frac{\Gamma(a+n)}{\Gamma(a)} \\ &= \begin{cases} 1, & (n = 0,\, a \in \mathbb{C} \setminus \{0\}) \\ a(a+1)\cdots(a+n-1), & (n \in \mathbb{N},\, a \in \mathbb{C}), \end{cases} \end{aligned} \tag{2} $$

where $\Gamma(z)$ is the well known gamma function defined by

$$ \Gamma(z) = \int_0^{\infty} e^{-x} x^{z-1} dx \tag{3} $$

for $\mathrm{Re}(z) > 0$.

For details about the convergence conditions of (1) and other properties, we refer to [17].

It is not out of place to mention here that whenever a generalized hypergeometric function reduces to gamma function, the results are very important from the application point of view. Here, we shall mention the following classical summation theorems [1, 2] so that the paper may be self contained. These are

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- Gauss Theorem for $\mathrm{Re}(c-a-b)>0$

$$ {}_2F_1\left[\begin{matrix} a, & b \\ & c \end{matrix};1\right]=\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \tag{4}$$

- Kummer's Theorem

$$ {}_2F_1\left[\begin{matrix} a, & b \\ & 1+a-b \end{matrix};-1\right]=\frac{\Gamma(1+a-b)\Gamma(1+\frac{1}{2}a)}{\Gamma(1-b+\frac{1}{2}a)\Gamma(1+a)} \tag{5}$$

- Second Gauss Theorem

$$ {}_2F_1\left[\begin{matrix} a, & b \\ & \frac{1}{2}(a+b+1) \end{matrix};\frac{1}{2}\right]=\frac{\sqrt{\pi}\,\Gamma(\frac{1}{2}(a+b+1))}{\Gamma(\frac{1}{2}(a+1))\Gamma(\frac{1}{2}(b+1))} \tag{6}$$

- Bailey's Theorem

$$ {}_2F_1\left[\begin{matrix} a, & 1-a \\ & b \end{matrix};\frac{1}{2}\right]=\frac{\Gamma(\frac{1}{2}b)\Gamma(\frac{1}{2}(b+1))}{\Gamma(\frac{1}{2}(a+b))\Gamma(\frac{1}{2}(b-a+1))} \tag{7}$$

- Dixon's Theorem

$$ {}_3F_2\left[\begin{matrix} a, & b, & c \\ & 1+a-b, & 1+a-c \end{matrix};1\right] = \frac{\Gamma(1+\frac{1}{2}a)\Gamma(1+a-b)\Gamma(1+a-c)\Gamma(1-b-c+\frac{1}{2}a)}{\Gamma(1+a)\Gamma(1-b+\frac{1}{2}a)\Gamma(1-c+\frac{1}{2}a)\Gamma(1+a-b-c)} \tag{8}$$

- Watson's Theorem

$$ {}_3F_2\left[\begin{matrix} a, & b, & c \\ & \frac{1}{2}(a+b+1), & 2c \end{matrix};1\right] = \frac{\sqrt{\pi}\,\Gamma(c+\frac{1}{2})\Gamma(\frac{1}{2}(a+b+1))\Gamma(c-\frac{1}{2}(a+b-1))}{\Gamma(\frac{1}{2}(a+1))\Gamma(\frac{1}{2}(b+1))\Gamma(c-\frac{1}{2}(a-1))\Gamma(c-\frac{1}{2}(b-1))} \tag{9}$$

- Whipple's Theorem

$$ {}_3F_2\left[\begin{matrix} a, & 1-a, & b \\ & c, & 2b-c+1 \end{matrix};1\right] = \frac{\pi\,2^{1-2b}\,\Gamma(c)\Gamma(2b-c+1)}{\Gamma(\frac{1}{2}(a+c))\Gamma(b+\frac{1}{2}(a-c+1))\Gamma(\frac{1}{2}(1-a+c))\Gamma(b+1-\frac{1}{2}(a+c))} \tag{10}$$

- Pfaff-Saalschütz Theorem

$$ {}_3F_2\left[\begin{matrix} a, & b, & -n \\ & c, & 1+a+b-c-n \end{matrix};1\right]=\frac{(c-a)_n(c-b)_n}{(c)_n(c-a-b)_n} \tag{11}$$

- Second Whipple's Theorem

$$ {}_4F_3\left[\begin{matrix} a, & 1+\frac{1}{2}a, & b, & c \\ & \frac{1}{2}a, & a-b+1, & a-c+1 \end{matrix};-1\right]=\frac{\Gamma(a-b+1)\Gamma(a-c+1)}{\Gamma(a+1)\Gamma(a-b-c+1)} \tag{12}$$

- Dougall's Theorem

$$ {}_5F_4\left[\begin{matrix} a, & 1+\frac{1}{2}a, & c, & d, & e \\ & \frac{1}{2}a, & a-c+1, & a-d+1, & a-e+1 \end{matrix};1\right] = \frac{\Gamma(a-c+1)\Gamma(a-d+1)\Gamma(a-e+1)\Gamma(a-c-d-e+1)}{\Gamma(a+1)\Gamma(a-d-e+1)\Gamma(a-c-e+1)\Gamma(a-c-d+1)} \tag{13}$$

- Second Dougall's Theorem

$$(14)\quad {}_7F_6\left[\begin{matrix} a, & 1+\frac{1}{2}a, & b, & c, & d, & 1+2a-b-c-d+n, & -n \\ \frac{1}{2}a, & a-b+1, & a-c+1, & a-d+1, & b+c+d-a-n, & a+1+n \end{matrix};1\right]$$
$$= \frac{(a+1)_n(a-b-c+1)_n(a-b-d+1)_n(a-c-d+1)_n}{(a+1-b)_n(a+1-c)_n(a+1-d)_n(a+1-b-c-d)_n}$$

For finite sums of hypergeometric series, we will use the following symbol

$${}_p\overset{(m)}{F}_q\left[\begin{matrix} a_1, & \cdots, & a_p \\ b_1, & \cdots, & b_q \end{matrix};z\right] = \sum_{n=0}^{m} \frac{\prod_{i=1}^{p}(a_i)_n}{\prod_{i=1}^{q}(b_i)_n}\frac{z^n}{n!},$$

where for instance

$${}_p\overset{(-1)}{F}_q(z) = 0, \quad {}_p\overset{(0)}{F}_q(z) = 1, \quad {}_p\overset{(1)}{F}_q(z) = 1 + \frac{a_1\cdots a_p}{b_1\cdots b_q}z.$$

By using the following relation [16],

$$(15)\quad {}_pF_q\left[\begin{matrix} a_1,\cdots, & a_{p-1}, & 1 \\ b_1,\cdots, & b_{q-1}, & m \end{matrix};z\right]$$
$$= \frac{\Gamma(b_1)\cdots\Gamma(b_{q-1})}{\Gamma(a_1)\cdots\Gamma(a_{p-1})}\frac{\Gamma(a_1-m+1)\cdots\Gamma(a_{p-1}-m+1)}{\Gamma(b_1-m+1)\cdots\Gamma(b_{q-1}-m+1)}\frac{(m-1)!}{z^{m-1}}$$
$$\times\left\{{}_{p-1}F_{q-1}\left[\begin{matrix} a_1-m+1,\cdots, & a_{p-1}-m+1 \\ b_1-m+1,\cdots, & b_{q-1}-m+1 \end{matrix};z\right]\right.$$
$$\left.-\,{}_{p-1}\overset{(m-2)}{F}_{q-1}\left[\begin{matrix} a_1-m+1,\cdots, & a_{p-1}-m+1 \\ b_1-m+1,\cdots, & b_{q-1}-m+1 \end{matrix};z\right]\right\},$$

very recently Masjed-Jamei and Koepf [14] have established generalizations of the classical summation theorems (4) to (14) in the following form:

$$(16)$$
$${}_3F_2\left[\begin{matrix} a, & b, & 1 \\ & c, & m \end{matrix};1\right]$$
$$= \frac{\Gamma(m)\Gamma(c)\Gamma(a-m+1)\Gamma(b-m+1)}{\Gamma(a)\Gamma(b)\Gamma(c-m+1)}$$
$$\times\left\{\frac{\Gamma(c-m+1)\Gamma(c-a-b+m-1)}{\Gamma(c-a)\Gamma(c-b)} - {}_2\overset{(m-2)}{F}_1\left[\begin{matrix} a-m+1, & b-m+1 \\ & c-m+1 \end{matrix};1\right]\right\}$$
$$= \Omega_1$$

(17)

$$
\begin{aligned}
&{}_3F_2\left[\begin{matrix} a, \quad b, \quad 1 \\ a-b+m,\ m \end{matrix};-1\right] \\
&= (-1)^{m-1}\frac{\Gamma(m)\Gamma(a-b+m)\Gamma(a-m+1)\Gamma(b-m+1)}{\Gamma(a)\Gamma(b)\Gamma(a-b+1)} \\
&\times\left\{\frac{\Gamma(a-b+1)\Gamma(1+\frac{1}{2}(a-m+1))}{\Gamma(2+a-m)\Gamma(m-b+\frac{1}{2}(a-m+1))} - {}^{(m-2)}_{\quad 2}F_1\left[\begin{matrix} a-m+1,\ b-m+1 \\ a-b+1 \end{matrix};-1\right]\right\} \\
&= \Omega_2
\end{aligned}
$$

(18)

$$
\begin{aligned}
&{}_3F_2\left[\begin{matrix} a, \quad b, \quad 1 \\ \frac{1}{2}(a+b+1),\ m \end{matrix};\frac{1}{2}\right] \\
&= 2^{m-1}\frac{\Gamma(m)\Gamma(\frac{1}{2}(a+b+1))\Gamma(a-m+1)\Gamma(b-m+1)}{\Gamma(a)\Gamma(b)\Gamma(-m+1+\frac{1}{2}(a+b+1))} \\
&\times\left\{\frac{\sqrt{\pi}\,\Gamma(-m+1+\frac{1}{2}(a+b+1))}{\Gamma(1+\frac{1}{2}(a-m))\Gamma(1+\frac{1}{2}(b-m))} - {}^{(m-2)}_{\quad 2}F_1\left[\begin{matrix} a-m+1,\ b-m+1 \\ -m+1+\frac{1}{2}(a+b+1) \end{matrix};\frac{1}{2}\right]\right\} \\
&= \Omega_3
\end{aligned}
$$

(19)

$$
\begin{aligned}
&{}_3F_2\left[\begin{matrix} a,\ 2m-a-1,\ 1 \\ b, \quad m \end{matrix};\frac{1}{2}\right] \\
&= 2^{m-1}\frac{\Gamma(m)\Gamma(b)\Gamma(a-m+1)\Gamma(m-a)}{\Gamma(a)\Gamma(2m-a-1)\Gamma(b-m+1)} \\
&\times\left\{\frac{\Gamma(\frac{1}{2}(b-m+1))\Gamma(\frac{1}{2}(b-m+2))}{\Gamma(-m+1+\frac{1}{2}(a+b))\Gamma(\frac{1}{2}(b-a+1))} - {}^{(m-2)}_{\quad 2}F_1\left[\begin{matrix} a-m+1,\ m-a \\ b-m+1 \end{matrix};\frac{1}{2}\right]\right\} \\
&= \Omega_4
\end{aligned}
$$

(20)

$$
\begin{aligned}
&{}_4F_3\left[\begin{matrix} a, \quad b, \quad c, \quad 1 \\ a-b+m,\ a-c+m,\ m \end{matrix};1\right] \\
&= \frac{\Gamma(m)\Gamma(a-b+m)\Gamma(a-c+m)\Gamma(a+1-m)\Gamma(b+1-m)\Gamma(c+1-m)}{\Gamma(a)\Gamma(b)\Gamma(c)\Gamma(a-b+1)\Gamma(a-c+1)} \\
&\times\left\{\frac{\Gamma(\frac{1}{2}(a+3-m))\Gamma(a-b+1)\Gamma(a-c+1)\Gamma(-b-c+\frac{1}{2}(a+3m-1))}{\Gamma(a+2-m)\Gamma(-b+\frac{1}{2}(a+m+1))\Gamma(-c+\frac{1}{2}(a+m+1))\Gamma(a-b-c+m)}\right. \\
&\quad\left. - {}^{(m-2)}_{\quad 3}F_2\left[\begin{matrix} a-m+1,\ b-m+1,\ c-m+1 \\ a-b+1, \quad a-c+1 \end{matrix};1\right]\right\} \\
&= \Omega_5
\end{aligned}
$$

(21)

$$
{}_4F_3\left[\begin{matrix} a, & b, & c, & 1 \\ \frac{1}{2}(a+b+1), & 2c+1-m, & m \end{matrix};1\right]
= \frac{\Gamma(m)\Gamma(\frac{1}{2}(a+b+1))\Gamma(2c+1-m)\Gamma(a+1-m)\Gamma(b+1-m)\Gamma(c+1-m)}{\Gamma(a)\Gamma(b)\Gamma(c)\Gamma(-m+\frac{1}{2}(a+b+3))\Gamma(2c-2m+2)}
$$

$$
\times\left\{\frac{\sqrt{\pi}\,\Gamma(c-m+\frac{3}{2})\Gamma(-m+\frac{1}{2}(a+b+3))\Gamma(c-\frac{1}{2}(a+b-1))}{\Gamma(1+\frac{1}{2}(a-m))\Gamma(1+\frac{1}{2}(b-m))\Gamma(c+1-\frac{1}{2}(a+m))\Gamma(c+1-\frac{1}{2}(b+m))}
- {}^{(m-2)}_{3}F_2\left[\begin{matrix} a-m+1, & b-m+1, & c-m+1 \\ -m+1+\frac{1}{2}(a+b+1), & 2c-2m+2 \end{matrix};1\right]\right\}
= \Omega_6
$$

(22)

$$
{}_4F_3\left[\begin{matrix} a, & 2m-1-a, & b, & 1 \\ c, & 2b-c+1, & m \end{matrix};1\right]
= \frac{\Gamma(m)\Gamma(c)\Gamma(2b-c+1)\Gamma(m-a)\Gamma(a+1-m)\Gamma(b+1-m)}{\Gamma(a)\Gamma(b)\Gamma(2m-1-a)\Gamma(c+1-m)\Gamma(2b-c-m+2)}
$$

$$
\times\left\{\frac{\pi\, 2^{2m-2b-1}\,\Gamma(c-m+1)}{\Gamma(-m+1+\frac{1}{2}(a+c))\Gamma(-m+1+b+\frac{1}{2}(a-c+1))\Gamma(\frac{1}{2}(1-a+c))}
\times\frac{\Gamma(2b-c-m+2)}{\Gamma(b+1-\frac{1}{2}(a+c))} - {}^{(m-2)}_{3}F_2\left[\begin{matrix} a-m+1, & b-m+1, & m-a \\ c-m+1, & 2b-c-m+2 \end{matrix};1\right]\right\}
= \Omega_7
$$

(23)

$$
{}_4F_3\left[\begin{matrix} a, & b, & -n+m-1, & 1 \\ c, & 1+a+b-c-n, & m \end{matrix};1\right] = \frac{(m-1)!\,(1-c)_{m-1}}{(1-a)_{m-1}(1-b)_{m-1}}
\times\frac{(c-a-b+n)_{m-1}}{(n+2-m)_{m-1}}\times\left\{\frac{(c-a)_n(c-b)_n}{(c+1-m)_n(c-a-b+m-1)_n}
- {}^{(m-2)}_{3}F_2\left[\begin{matrix} a-m+1, & b-m+1, & -n \\ c-m+1, & 2+a+b-c-m-n \end{matrix};1\right]\right\}
= \Omega_8
$$

(24)

$$
{}_5F_4\left[\begin{matrix} a, & \frac{1}{2}(a+m+1), & b, & c, & 1 \\ \frac{1}{2}(a+m-1), & a-b+m, & a-c+m, & m \end{matrix};-1\right] = (-1)^{m-1}\Gamma(m)
\times\frac{\Gamma(\frac{1}{2}(a+m-1))\Gamma(a-b+m)\Gamma(a-c+m)\Gamma(\frac{1}{2}(a-m+3))\Gamma(a-m+1)}{\Gamma(a)\Gamma(b)\Gamma(c)\Gamma(\frac{1}{2}(a+m+1))\Gamma(\frac{1}{2}(a-m+1))}
$$

$$
\times\frac{\Gamma(b+1-m)\Gamma(c+1-m)}{\Gamma(a-b+1)\Gamma(a-c+1)}\times\left\{\frac{\Gamma(1+a-b)\Gamma(1+a-c)}{\Gamma(2-m+a)\Gamma(m+a-b-c)}
- {}^{(m-2)}_{4}F_3\left[\begin{matrix} a-m+1, & b-m+1, & \frac{1}{2}(a-m+3), & c-m+1 \\ \frac{1}{2}(a-m+1), & a-b+1, & a-c+1 \end{matrix};-1\right]\right\}
= \Omega_9
$$

(25)

$$
\begin{aligned}
&{}_6F_5\left[\begin{matrix} a, & \frac{1}{2}(a+m+1), & c, & d, & e, & 1 \\ \frac{1}{2}(a+m-1), & a-c+m, & a-d+m, & a-e+m, & m \end{matrix};1\right] \\
&= \frac{\Gamma(m)\Gamma(\frac{1}{2}(a+m-1))\Gamma(a-c+m)\Gamma(a-d+m)\Gamma(a-e+m)}{\Gamma(a-c+1)\Gamma(a-d+1)\Gamma(a-e+1)} \\
&\times \frac{\Gamma(a-m+1)\Gamma(\frac{1}{2}(a-m+3))\Gamma(c+1-m)\Gamma(d+1-m)\Gamma(e+1-m)}{\Gamma(a)\Gamma(c)\Gamma(d)\Gamma(e)\Gamma(\frac{1}{2}(a+m+1))\Gamma(\frac{1}{2}(a-m+1))} \\
&\times \left\{ \frac{\Gamma(a-c+1)\Gamma(a-d+1)\Gamma(a-e+1)\Gamma(a-c-d-e+2m-1)}{\Gamma(2-m+a)\Gamma(a-c-e+m)\Gamma(a-d-e+m)\Gamma(a-c-d+m)} \right. \\
&\left. - {}_5F_4^{(m-2)}\left[\begin{matrix} a-m+1, & c-m+1, & \frac{1}{2}(a-m+3), & d-m+1, & e-m+1 \\ \frac{1}{2}(a-m+1), & a-c+1, & a-d+1, & a-e+1 \end{matrix};1\right] \right\} \\
&= \Omega_{10}
\end{aligned}
$$

(26)

$$
\begin{aligned}
&{}_8F_7\left[\begin{matrix} a, & \frac{1}{2}(a+m+1), & b, & c, & d, & 2a-b-c-d+2m-1+n, & m-n-1, & 1 \\ \frac{1}{2}(a+m-1), & a-b+m, & a-c+m, & a-d+m, & b+c+d-a+1-m-n, & a+n+1, & m \end{matrix};1\right] \\
&= (-1)^{m-1}(m-1)! \times \frac{(\frac{1}{2}(3-a-m))_{m-1}(1-a+b-m)_{m-1}}{(\frac{1}{2}(1-a-m))_{m-1}(1-a)_{m-1}} \\
&\times \frac{(1-a+c-m)_{m-1}(1-a+d-m)_{m-1}(m+n+a-b-c-d)_{m-1}(-a-n)_{m-1}}{(1-b)_{m-1}(1-c)_{m-1}(1-d)_{m-1}(b+c+d-2a+2-2m-n)_{m-1}(n+2-m)_{m-1}} \\
&\times \left\{ \frac{(a-m+2)_n(a-b-c+m)_n(a-b-d+m)_n(a-c-d+m)_n}{(a-b+1)_n(a-c+1)_n(a-d+1)_n(a-b-c-d+2m-1)_n} \right. \\
&\left. - {}_7F_6^{(m-2)}\left[\begin{matrix} a-m+1, & \frac{1}{2}(a-m+3), & b-m+1, & c-m+1, & d-m+1, & 2a-b-c-d+m+n, & -n \\ \frac{1}{2}(a-m+1), & a-b+1, & a-c+1, & a-d+1, & b+c+d+-a+2-2m-n, & a-m+n+2 \end{matrix};1\right] \right\} \\
&= \Omega_{11}
\end{aligned}
$$

It is interesting to mention here that for $m = 1$, the results (16) to (26) reduce to the results (4) to (14), respectively. For other generalizations and extensions of the results (5) to (10), we refer to [7, 11, 12, 13, 18].

On the other hand, we define the (direct) Laplace transform of a function $f(t)$ of a real variable t as the integral $g(s)$ over a range of the complex parameter s as

$$
g(s) = \mathcal{L}\{f(t); s\} = \int_0^\infty e^{-st} f(t)dt \tag{27}
$$

provided the integral exists in the Lebesgue sense. For more details, see for instance [3] or [4]. It is interesting to mention here that in view of the formula

$$
\int_0^\infty e^{-st} t^{\alpha-1} dt = \Gamma(\alpha) s^{-\alpha} \tag{28}
$$

provided $\mathrm{Re}(s) > 0$ and $\mathrm{Re}(\alpha) > 0$, by utilizing (1), with $p \leq q$, it is not difficult to show that the Laplace transform of a generalized hypergeometric function ${}_pF_q$ is obtained as [5, 15, 19]:

$$\int_0^\infty e^{-st}t^{\nu-1}{}_pF_q\left[\begin{matrix}a_1, & \cdots, & a_p\\ b_1, & \cdots, & b_q\end{matrix}; wt\right]dt \tag{29}$$
$$= \Gamma(\nu)s^{-\nu}\;{}_{p+1}F_q\left[\begin{matrix}\nu, \ a_1, & \cdots, & a_p\\ b_1, & \cdots, & b_q\end{matrix}; \frac{w}{s}\right]$$

provided that when $p < q$, $\mathrm{Re}(\nu) > 0$, $\mathrm{Re}(s) > 0$ for w arbitrary, or $p = q > 0$, $\mathrm{Re}(\nu) > 0$ and $\mathrm{Re}(s) > \mathrm{Re}(w)$.

Here, the interchange of order of integration and summation when integrating the left-hand side of (29) with respect to t is easily seen to be justified by the uniform convergence of the series (1).

The aim of this paper is to provide a new class of Laplace transforms of generalized hypergeometric functions by employing the summation theorems (16) to (26). Several new and known special cases have also been considered.

2. Laplace transforms of generalized hypergeometric functions

In this section, we shall establish several new, interesting and elementary Laplace transforms of generalized hypergeometric functions asserted in the following theorems that follow directly from (29) and (16) - (26).

Theorem 2.1. *For $m \in \mathbb{N}$, Re$(s) > 0$, Re$(a) > 0$ and Re$(c - a - b + m) > 1$, the following result holds true.*

$$\int_0^\infty e^{-st}\,t^{a-1}\,{}_2F_2\left[\begin{matrix}b, & 1\\ c, & m\end{matrix}; st\right]dt = \Gamma(a)\,s^{-a}\,\Omega_1, \tag{30}$$

where Ω_1 is the same as given in (16).

Theorem 2.2. *For $m \in \mathbb{N}$, Re$(s) > 0$ and Re$(c - a - b + m) > 1$, the following result holds true.*

$$\int_0^\infty e^{-st}\,{}_2F_2\left[\begin{matrix}a, & b\\ c, & m\end{matrix}; st\right]dt = s^{-1}\Omega_1, \tag{31}$$

where Ω_1 is the same as given in (16).

Theorem 2.3. *For $m \in \mathbb{N}$, Re$(s) > 0$ and Re$(a) > 0$, the following result holds true.*

$$\int_0^\infty e^{-st}\,t^{a-1}\,{}_2F_2\left[\begin{matrix}b, & 1\\ a-b+m, & m\end{matrix}; -st\right]dt = \Gamma(a)\,s^{-a}\,\Omega_2, \tag{32}$$

where Ω_2 is the same as given in (17).

Theorem 2.4. *For $m \in \mathbb{N}$, Re$(s) > 0$ and Re$(b) > 0$, the following result holds true.*

$$\int_0^\infty e^{-st}\,t^{b-1}\,{}_2F_2\left[\begin{matrix}a, & 1\\ a-b+m, & m\end{matrix}; -st\right]dt = \Gamma(b)\,s^{-b}\,\Omega_2, \tag{33}$$

where Ω_2 is the same as given in (17).

Theorem 2.5. *For $m \in \mathbb{N}$ and $Re(s) > 0$, the following result holds true.*

$$\int_0^\infty e^{-st}\, {}_2F_2\left[\begin{matrix} a, & b \\ a-b+m, & m\end{matrix}; -st\right] dt = s^{-1}\Omega_2, \tag{34}$$

where Ω_2 is the same as given in (17).

Theorem 2.6. *For $m \in \mathbb{N}$, $Re(s) > 0$ and $Re(a) > 0$, the following result holds true.*

$$\int_0^\infty e^{-st}\, t^{a-1}\, {}_2F_2\left[\begin{matrix} b, & 1 \\ \frac{1}{2}(a+b+1), & m\end{matrix}; \frac{1}{2}st\right] dt = \Gamma(a)\, s^{-a}\, \Omega_3, \tag{35}$$

where Ω_3 is the same as given in (18).

Theorem 2.7. *For $m \in \mathbb{N}$ and $Re(s) > 0$, the following result holds true.*

$$\int_0^\infty e^{-st}\, {}_2F_2\left[\begin{matrix} a, & b \\ \frac{1}{2}(a+b+1), & m\end{matrix}; \frac{1}{2}st\right] dt = s^{-1}\Omega_3, \tag{36}$$

where Ω_3 is the same as given in (18).

Theorem 2.8. *For $m \in \mathbb{N}$, $Re(s) > 0$ and $Re(a) > 0$, the following result holds true.*

$$\int_0^\infty e^{-st}\, t^{a-1}\, {}_2F_2\left[\begin{matrix} 2m-a-1, & 1 \\ b, & m\end{matrix}; \frac{1}{2}st\right] dt = \Gamma(a)\, s^{-a}\, \Omega_4, \tag{37}$$

where Ω_4 is the same as given in (19).

Theorem 2.9. *For $m \in \mathbb{N}$, $Re(s) > 0$ and $Re(2m-a-1) > 0$, the following result holds true.*

$$\int_0^\infty e^{-st}\, t^{2m-a-2}\, {}_2F_2\left[\begin{matrix} a, & 1 \\ b, & m\end{matrix}; \frac{1}{2}st\right] dt = \Gamma(2m-a-1)\, s^{a+1-2m}\, \Omega_4, \tag{38}$$

where Ω_4 is the same as given in (19).

Theorem 2.10. *For $m \in \mathbb{N}$ and $Re(s) > 0$, the following result holds true.*

$$\int_0^\infty e^{-st}\, {}_2F_2\left[\begin{matrix} a, & 2m-a-1 \\ b, & m\end{matrix}; \frac{1}{2}st\right] dt = s^{-1}\Omega_4, \tag{39}$$

where Ω_4 is the same as given in (19).

Theorem 2.11. *For $m \in \mathbb{N}$, $Re(s) > 0$, $Re(a) > 0$ and $Re(a-2b-2c+3m) > 1$, the following result holds true.*

$$\int_0^\infty e^{-st}\, t^{a-1}\, {}_3F_3\left[\begin{matrix} b, & c, & 1 \\ a-b+m, & a-c+m, & m\end{matrix}; st\right] dt = \Gamma(a)\, s^{-a}\, \Omega_5, \tag{40}$$

where Ω_5 is the same as given in (20).

Theorem 2.12. *For $m \in \mathbb{N}$, $Re(s) > 0$, $Re(b) > 0$ and $Re(a - 2b - 2c + 3m) > 1$, the following result holds true.*

$$\int_0^\infty e^{-st}\, t^{b-1}\, {}_3F_3\left[\begin{matrix} a, & c, & 1 \\ a-b+m, & a-c+m, & m \end{matrix}; st\right] dt = \Gamma(b)\, s^{-b}\, \Omega_5, \tag{41}$$

where Ω_5 is the same as given in (20).

Theorem 2.13. *For $m \in \mathbb{N}$, $Re(s) > 0$ and $Re(a-2b-2c+3m) > 1$, the following result holds true.*

$$\int_0^\infty e^{-st}\, {}_3F_3\left[\begin{matrix} a, & b, & c \\ a-b+m, & a-c+m, & m \end{matrix}; st\right] dt = s^{-1}\Omega_5, \tag{42}$$

where Ω_5 is the same as given in (20).

Theorem 2.14. *For $m \in \mathbb{N}$, $Re(s) > 0$, $Re(a) > 0$ and $Re(2c - a - b) > -1$, the following result holds true.*

$$\int_0^\infty e^{-st}\, t^{a-1}\, {}_3F_3\left[\begin{matrix} b, & c, & 1 \\ \frac{1}{2}(a+b+1), & 2c+1-m, & m \end{matrix}; st\right] dt = \Gamma(a)\, s^{-a}\, \Omega_6, \tag{43}$$

where Ω_6 is the same as given in (21).

Theorem 2.15. *For $m \in \mathbb{N}$, $Re(s) > 0$, $Re(c) > 0$ and $Re(2c - a - b) > -1$, the following result holds true.*

$$\int_0^\infty e^{-st}\, t^{c-1}\, {}_3F_3\left[\begin{matrix} a, & b, & 1 \\ \frac{1}{2}(a+b+1), & 2c+1-m, & m \end{matrix}; st\right] dt = \Gamma(c)\, s^{-c}\, \Omega_6, \tag{44}$$

where Ω_6 is the same as given in (21).

Theorem 2.16. *For $m \in \mathbb{N}$, $Re(s) > 0$ and $Re(2c - a - b) > -1$, the following result holds true.*

$$\int_0^\infty e^{-st}\, {}_3F_3\left[\begin{matrix} a, & b, & c \\ \frac{1}{2}(a+b+1), & 2c+1-m, & m \end{matrix}; st\right] dt = s^{-1}\Omega_6, \tag{45}$$

where Ω_6 is the same as given in (21).

Theorem 2.17. *For $m \in \mathbb{N}$, $Re(s) > 0$, $Re(a) > 0$ and $Re(b - m + 1) > 0$, the following result holds true.*

$$\int_0^\infty e^{-st}\, t^{a-1}\, {}_3F_3\left[\begin{matrix} 2m-a-1, & b, & 1 \\ c, & 2b-c+1, & m \end{matrix}; st\right] dt = \Gamma(a)\, s^{-a}\, \Omega_7, \tag{46}$$

where Ω_7 is the same as given in (22).

Theorem 2.18. *For $m \in \mathbb{N}$, $Re(s) > 0$, $Re(b) > 0$ and $Re(b - m + 1) > 0$, the following result holds true.*

$$\int_0^\infty e^{-st}\, t^{b-1}\, {}_3F_3\left[\begin{matrix} a, & 2m-a-1, & 1 \\ c, & 2b-c+1, & m \end{matrix}; st\right] dt = \Gamma(b)\, s^{-b}\, \Omega_7, \tag{47}$$

where Ω_7 is the same as given in (22).

Theorem 2.19. *For $m \in \mathbb{N}$, $Re(s) > 0$, $Re(2m-a-1) > 0$ and $Re(b-m+1) > 0$, the following result holds true.*

$$\int_0^\infty e^{-st}\, t^{2m-a-2}\, {}_3F_3\left[\begin{matrix} a, & c, & 1 \\ c, & 2b-c+1, & m \end{matrix}; st\right] dt = \Gamma(2m-a-1)\, s^{a+1-2m}\, \Omega_7, \tag{48}$$

where Ω_7 is the same as given in (22).

Theorem 2.20. *For $m \in \mathbb{N}$, $Re(s) > 0$ and $Re(b-m+1) > 0$, the following result holds true.*

$$\int_0^\infty e^{-st}\, {}_3F_3\left[\begin{matrix} a, & 2m-a-1, & b \\ c, & 2b-c+1, & m \end{matrix}; st\right] dt = s^{-1}\Omega_7, \tag{49}$$

where Ω_7 is the same as given in (22).

Theorem 2.21. *For $m \in \mathbb{N}$, $n \in \mathbb{N}_0$, $Re(s) > 0$ and $Re(a) > 0$, the following result holds true.*

$$\int_0^\infty e^{-st}\, t^{a-1}\, {}_3F_3\left[\begin{matrix} b, & -n+m-1, & 1 \\ c, & 1+a+b-c-n, & m \end{matrix}; st\right] dt = \Gamma(a)\, s^{-a}\, \Omega_8, \tag{50}$$

where Ω_8 is the same as given in (23).

Theorem 2.22. *For $m \in \mathbb{N}$, $n \in \mathbb{N}_0$ and $Re(s) > 0$, the following result holds true.*

$$\int_0^\infty e^{-st}\, {}_3F_3\left[\begin{matrix} a, & b, & -n+m-1 \\ c, & 1+a+b-c-n, & m \end{matrix}; st\right] dt = s^{-1}\Omega_8, \tag{51}$$

where Ω_8 is the same as given in (23).

Theorem 2.23. *For $m \in \mathbb{N}$, $Re(s) > 0$, $Re(a) > 0$ and $Re(a-2b-2c+3m) > 2$, the following result holds true.*

$$\int_0^\infty e^{-st}\, t^{a-1}\, {}_4F_4\left[\begin{matrix} \frac{1}{2}(a+m+1), & b, & c, & 1 \\ \frac{1}{2}(a+m-1), & a-b+m, & a-c+m, & m \end{matrix}; -st\right] dt = \Gamma(a)\, s^{-a}\, \Omega_9, \tag{52}$$

where Ω_9 is the same as given in (24).

Theorem 2.24. *For $m \in \mathbb{N}$, $Re(s) > 0$, $Re(c) > 0$ and $Re(a-2b-2c+3m) > 2$, the following result holds true.*

$$\int_0^\infty e^{-st}\, t^{c-1}\, {}_4F_4\left[\begin{matrix} a, & \frac{1}{2}(a+m+1), & b, & 1 \\ \frac{1}{2}(a+m-1), & a-b+m, & a-c+m, & m \end{matrix}; -st\right] dt = \Gamma(c)\, s^{-c}\, \Omega_9, \tag{53}$$

where Ω_9 is the same as given in (24).

Theorem 2.25. *For $m \in \mathbb{N}$, $Re(s) > 0$, $Re(a+m+1) > 0$ and $Re(a-2b-2c+3m) > 2$, the following result holds true.*

$$\begin{aligned} &\int_0^\infty e^{-st}\, t^{\frac{1}{2}(a+m-1)}\, {}_4F_4\left[\begin{matrix} a, & b, & c, & 1 \\ \frac{1}{2}(a+m-1), & a-b+m, & a-c+m, & m \end{matrix}; -st\right] dt \\ &\quad = \Gamma(\frac{1}{2}(a+m+1))\, s^{-\frac{1}{2}(a+m+1)}\, \Omega_9, \end{aligned} \tag{54}$$

where Ω_9 is the same as given in (24).

Theorem 2.26. *For $m \in \mathbb{N}$, $Re(s) > 0$ and $Re(a-2b-2c+3m) > 2$, the following result holds true.*

$$\int_0^\infty e^{-st}\, {}_4F_4\left[\begin{matrix} a, & \frac{1}{2}(a+m+1), & b, & c \\ \frac{1}{2}(a+m-1), & a-b+m, & a-c+m, & m \end{matrix}; -st\right] dt = s^{-1}\Omega_9, \tag{55}$$

where Ω_9 is the same as given in (24).

Theorem 2.27. *For $m \in \mathbb{N}$, $Re(s) > 0$, $Re(a) > 0$ and $Re(2a-2c-2d-2e+3m) > 2$, the following result holds true.*

(56)
$$\int_0^\infty e^{-st}\, t^{a-1}\, {}_5F_5\left[\begin{matrix} \frac{1}{2}(a+m+1), & c, & d, & e, & 1 \\ \frac{1}{2}(a+m-1), & a-c+m, & a-d+m, & a-e+m, & m \end{matrix}; st\right] dt$$
$$= \Gamma(a)\, s^{-a}\, \Omega_{10},$$

where Ω_{10} is the same as given in (25).

Theorem 2.28. *For $m \in \mathbb{N}$, $Re(s) > 0$, $Re(c) > 0$ and $Re(2a-2c-2d-2e+3m) > 2$, the following result holds true.*

(57)
$$\int_0^\infty e^{-st}\, t^{c-1}\, {}_5F_5\left[\begin{matrix} a, & \frac{1}{2}(a+m+1), & d, & e, & 1 \\ \frac{1}{2}(a+m-1), & a-c+m, & a-d+m, & a-e+m, & m \end{matrix}; st\right] dt$$
$$= \Gamma(c)\, s^{-c}\, \Omega_{10},$$

where Ω_{10} is the same as given in (25).

Theorem 2.29. *For $m \in \mathbb{N}$, $Re(s) > 0$, $Re(a+m+1) > 0$ and $Re(2a-2c-2d-2e+3m) > 2$, the following result holds true.*

(58)
$$\int_0^\infty e^{-st}\, t^{\frac{1}{2}(a+m-1)}\, {}_5F_5\left[\begin{matrix} a, & c, & d, & e, & 1 \\ \frac{1}{2}(a+m-1), & a-c+m, & a-d+m, & a-e+m, & m \end{matrix}; st\right] dt$$
$$= \Gamma(\frac{1}{2}(a+m+1))\, s^{-\frac{1}{2}(a+m+1)}\, \Omega_{10},$$

where Ω_{10} is the same as given in (25).

Theorem 2.30. *For $m \in \mathbb{N}$, $Re(s) > 0$ and $Re(2a-2c-2d-2e+3m) > 2$, the following result holds true.*

(59)
$$\int_0^\infty e^{-st}\, {}_5F_5\left[\begin{matrix} a, & \frac{1}{2}(a+m-1), & c, & d, & e \\ \frac{1}{2}(a+m-1), & a-c+m, & a-d+m, & a-e+m, & m \end{matrix}; st\right] dt$$
$$= s^{-1}\Omega_{10},$$

where Ω_{10} is the same as given in (25).

Theorem 2.31. *For $m \in \mathbb{N}$, $Re(s) > 0$ and $Re(a) > 0$, the following result holds true.*

(60)
$$\int_0^\infty e^{-st}\, t^{a-1} \times$$
$${}_7F_7\left[\begin{matrix} \frac{1}{2}(a+m+1), & b, & c, & d, & 2a-b-c-d+2m-1+n, & m-n-1, & 1 \\ \frac{1}{2}(a+m-1), & a-b+m, & a-c+m, & a-d+m, & b+c+d-a+1-m-n, & a+n+1, & m \end{matrix}; st\right] dt$$
$$= \Gamma(a)\, s^{-a}\, \Omega_{11},$$

where Ω_{11} is the same as given in (26).

Theorem 2.32. *For $m \in \mathbb{N}$, $Re(s) > 0$ and $Re(a+m+1) > 0$, the following result holds true.*

(61)

$$\int_0^\infty e^{-st}\, t^{\frac{1}{2}(a+m-1)} \times$$
$${}_7F_7\left[\begin{matrix} a, & b, & c, & d, & 2a-b-c-d+2m-1+n, & m-n-1, & 1 \\ \frac{1}{2}(a+m-1), & a-b+m, & a-c+m, & a-d+m, & b+c+d-a+1-m-n, & a+n+1, & m \end{matrix}; st\right] dt$$
$$= \Gamma(\frac{1}{2}(a+m+1))\, s^{-\frac{1}{2}(a+m+1)}\, \Omega_{11},$$

where Ω_{11} is the same as given in (26).

Theorem 2.33. *For $m \in \mathbb{N}$, $Re(s) > 0$ and $Re(b) > 0$, the following result holds true.*

(62)

$$\int_0^\infty e^{-st}\, t^{b-1} \times$$
$${}_7F_7\left[\begin{matrix} \frac{1}{2}(a+m+1), & b, & c, & d, & 2a-b-c-d+2m-1+n, & m-n-1, & 1 \\ \frac{1}{2}(a+m-1), & a-b+m, & a-c+m, & a-d+m, & b+c+d-a+1-m-n, & a+n+1, & m \end{matrix}; st\right] dt$$
$$= \Gamma(b)\, s^{-b}\, \Omega_{11},$$

where Ω_{11} is the same as given in (26).

Theorem 2.34. *For $m \in \mathbb{N}$, $Re(s) > 0$ and $Re(2a-b-c-d+2m-1+n) > 0$, the following result holds true.*

(63)

$$\int_0^\infty e^{-st}\, t^{2a-b-c-d+2m+n-2} \times$$
$${}_7F_7\left[\begin{matrix} \frac{1}{2}(a+m+1), & a, & b, & c, & d, & m-n-1, & 1 \\ \frac{1}{2}(a+m-1), & a-b+m, & a-c+m, & a-d+m, & b+c+d-a+1-m-n, & a+n+1, & m \end{matrix}; st\right] dt$$
$$= \Gamma(2a-b-c-d+2m-1+n)\, s^{-(2a-b-c-d+2m-1+n)}\, \Omega_{11},$$

where Ω_{11} is the same as given in (26).

Theorem 2.35. *For $m \in \mathbb{N}$ and $Re(s) > 0$, the following result holds true.*

(64)

$$\int_0^\infty e^{-st}\, {}_7F_7\left[\begin{matrix} \frac{1}{2}(a+m+1), & a, & b, & c, & d, & m-n-1, & 1 \\ \frac{1}{2}(a+m-1), & a-b+m, & a-c+m, & a-d+m, & b+c+d-a+1-m-n, & a+n+1, & m \end{matrix}; st\right] dt$$
$$= s^{-1}\Omega_{11},$$

where Ω_{11} is the same as given in (26).

Proof. In order to establish the result (30) asserted in the theorem 2.1, we proceed as follows. In (29), if we take $p = q = 2$, $\nu = a$, $a_1 = b$, $a_2 = 1$, $b_1 = c$, $b_2 = m$ and $w = s$, we get

$$\int_0^\infty e^{-st}\, t^{a-1}\, {}_2F_2\left[\begin{matrix} b, & 1 \\ c, & m \end{matrix}; st\right] dx = s^{-a}\Gamma(a)\, {}_3F_2\left[\begin{matrix} a, & b, & 1 \\ & c, & m \end{matrix}; 1\right]. \tag{65}$$

We now observe that the ${}_3F_2$ appearing on the right-hand side of (65) can be evaluated with the help of the result (16) and we easily arrive at the right-hand side of (30). This completes the proof of (30) asserted in the theorem 2.1.

In exactly the same manner, the results (31) to (64) asserted in the theorem 2.2 to 2.35 can be evaluated. We however omit the details.

□

3. Corollaries

In this section, we shall mention some of the very interesting special cases of our main findings.

(a) In Theorem 2.1, if we take $m = 1, 2, 3$, we get the following results.

$$\int_0^\infty e^{-st}\, t^{a-1}\, {}_1F_1\left[\begin{matrix} b \\ c \end{matrix}; st\right] dt = \frac{\Gamma(a)\Gamma(c)\Gamma(c-a-b)}{s^a\,\Gamma(c-a)\Gamma(c-b)}, \tag{66}$$

$$\int_0^\infty e^{-st}\, t^{a-1}\, {}_2F_2\left[\begin{matrix} b, & 1 \\ c, & 2 \end{matrix}; st\right] dt = \frac{(c-1)\Gamma(a-1)}{s^a\,(b-1)} \left\{\frac{\Gamma(c-1)\Gamma(c-a-b+1)}{\Gamma(c-a)\Gamma(c-b)} - 1\right\} \tag{67}$$

and

$$\int_0^\infty e^{-st}\, t^{a-1}\, {}_2F_2\left[\begin{matrix} b, & 1 \\ c, & 3 \end{matrix}; st\right] dt = \frac{2\Gamma(a)(c-2)_2}{s^a\,(a-2)_2(b-2)_2} \left\{\frac{\Gamma(c-2)\Gamma(c-a-b+2)}{\Gamma(c-a)\Gamma(c-b)} - \frac{ab+c-2a-2b+2}{c-2}\right\}. \tag{68}$$

(b) In Theorem 2.4, if we take $m = 1, 2, 3$, we get the following results.

$$\int_0^\infty e^{-st}\, t^{b-1}\, {}_1F_1\left[\begin{matrix} a \\ 1+a-b \end{matrix}; -st\right] dt = \frac{2^{-a}\Gamma(b)\Gamma(\frac{1}{2})\Gamma(1+a-b)}{s^b\,\Gamma(\frac{1}{2}a+\frac{1}{2})\Gamma(1+\frac{1}{2}a-b)}, \tag{69}$$

$$\int_0^\infty e^{-st}\, t^{b-1}\, {}_2F_2\left[\begin{matrix} a, & 1 \\ 2+a-b, & 2 \end{matrix}; -st\right] dt = \frac{(a-b+1)\Gamma(b)}{s^b\,(a-1)(b-1)} \left\{1 - \frac{\Gamma(1+a-b)\Gamma(\frac{1}{2}a+\frac{1}{2})}{\Gamma(a)\Gamma(\frac{1}{2}a-b+\frac{3}{2})}\right\} \tag{70}$$

and

$$\int_0^\infty e^{-st}\, t^{a-1}\, {}_2F_2\left[\begin{matrix} b, & 1 \\ 3+a-b, & 3 \end{matrix}; -st\right] dt = \frac{2(a-b+1)_2\,\Gamma(a)}{s^b\,(a-2)_2(b-2)_2} \left\{\frac{\Gamma(\frac{1}{2}a)\Gamma(1+a-b)}{\Gamma(a-1)\Gamma(\frac{1}{2}a-b+2)} - \frac{3a+b-ab-3}{1+a-b}\right\}. \tag{71}$$

(c) In Theorem 2.7, if we take $m = 1, 2, 3$, we get the following results.

$$\int_0^\infty e^{-st}\, t^{a-1}\, {}_1F_1\left[\begin{matrix} b \\ \frac{1}{2}(a+b+1)\end{matrix}; \frac{1}{2}st\right] dt = \frac{\sqrt{\pi}\Gamma(a)\Gamma(\frac{1}{2}(a+b+1))}{s^a\,\Gamma(\frac{1}{2}(a+1))\Gamma(\frac{1}{2}(b+1))}, \tag{72}$$

$$\begin{aligned}&\int_0^\infty e^{-st}\, t^{a-1}\, {}_2F_2\left[\begin{matrix} b, \quad 1 \\ \frac{1}{2}(a+b+1), 2\end{matrix}; \frac{1}{2}st\right] dt \\ &= \frac{(a+b-1)\Gamma(a-1)}{s^a\,(b-1)}\left\{\frac{\sqrt{\pi}\,\Gamma(\frac{1}{2}(a+b-1))}{\Gamma(\frac{1}{2}a)\Gamma(\frac{1}{2}b)} - 1\right\}\end{aligned} \tag{73}$$

and

$$\begin{aligned}&\int_0^\infty e^{-st}\, t^{a-1}\, {}_2F_2\left[\begin{matrix} b, \quad 1 \\ \frac{1}{2}(a+b+1), 3\end{matrix}; \frac{1}{2}st\right] dt = \frac{2\Gamma(a)(a+b-1)(a+b-3)}{s^a\,(a-2)_2(b-2)_2} \\ &\times\left\{\frac{\sqrt{\pi}\,\Gamma(\frac{1}{2}(a+b-3))}{\Gamma(\frac{1}{2}(a-1))\Gamma(\frac{1}{2}(b-1))} - \frac{ab-a-b+1}{a+b-3}\right\}.\end{aligned} \tag{74}$$

(d) In Theorem 2.10, if we take $m = 1, 2, 3$, we get the following results.

$$\int_0^\infty e^{-st}\, t^{a-1}\, {}_1F_1\left[\begin{matrix} 1-a \\ b\end{matrix}; \frac{1}{2}st\right] dt = \frac{\Gamma(a)\Gamma(\frac{1}{2}b)\Gamma(\frac{1}{2}(b+1))}{s^a\,\Gamma(\frac{1}{2}(a+b))\Gamma(\frac{1}{2}(b-a+1))}, \tag{75}$$

$$\begin{aligned}&\int_0^\infty e^{-st}\, t^{a-1}\, {}_2F_2\left[\begin{matrix} 3-a, \quad 1 \\ b, \quad 2\end{matrix}; \frac{1}{2}st\right] dt \\ &= \frac{2(1-b)\Gamma(a)}{s^a\,(1-a)_2}\left\{\frac{\Gamma(\frac{1}{2}(b-1))\Gamma(\frac{1}{2}b)}{\Gamma(\frac{1}{2}(a+b)-1)\Gamma(\frac{1}{2}(b-a+1))} - 1\right\}\end{aligned} \tag{76}$$

and

$$\begin{aligned}&\int_0^\infty e^{-st}\, t^{a-1}\, {}_2F_2\left[\begin{matrix} 5-a, \quad 1 \\ b, \quad 3\end{matrix}; \frac{1}{2}st\right] dt = \frac{8(b-2)_2\Gamma(a)}{s^a\,(a-4)_4} \\ &\times\left\{\frac{\Gamma(\frac{1}{2}(b-1))\Gamma(\frac{1}{2}(b-2))}{\Gamma(\frac{1}{2}(a+b)-2)\Gamma(\frac{1}{2}(b-a+1))} - \frac{5a-a^2+2b-10}{2(b-2)}\right\}.\end{aligned} \tag{77}$$

(e) In Theorem 2.11, if we take $m = 1, 2, 3$, we get the following results.

$$\begin{aligned}&\int_0^\infty e^{-st}\, t^{a-1}\, {}_2F_2\left[\begin{matrix} b, \quad c \\ 1+a-b, \ 1+a-c\end{matrix}; st\right] dt \\ &= \frac{\Gamma(\frac{1}{2}a)\Gamma(1+a-b)\Gamma(1+a-c)\Gamma(1+\frac{1}{2}a-b-c)}{2\,s^a\,\Gamma(1+\frac{1}{2}a-b)\Gamma(1+\frac{1}{2}a-c)\Gamma(1+a-b-c)},\end{aligned} \tag{78}$$

$$\begin{aligned}&\int_0^\infty e^{-st}\, t^{a-1}\, {}_3F_3\left[\begin{matrix} b, \quad c, \quad 1 \\ a-b+2, \ a-c+2, \ 2\end{matrix}; st\right] dt \\ &= \frac{\Gamma(a-1)(1+a-b)(1+a-c)}{s^a\,(b-1)(c-1)} \\ &\quad\times\left\{\frac{\Gamma(\frac{1}{2}(a+1))\Gamma(1+a-b)\Gamma(1+a-c)\Gamma(\frac{1}{2}a-b-c+\frac{5}{2})}{\Gamma(a)\Gamma(\frac{1}{2}a-b+\frac{3}{2})\Gamma(\frac{1}{2}a-c+\frac{3}{2})\Gamma(2+a-b-c)} - 1\right\}\end{aligned} \tag{79}$$

and

$$\int_0^\infty e^{-st}\, t^{a-1}\, {}_3F_3\left[\begin{matrix} b, & c, & 1 \\ a-b+3, & a-c+3, & 3 \end{matrix}; st\right] dt \tag{80}$$
$$= \frac{2(a-b+1)_2(a-c+1)_2\,\Gamma(a)}{s^a\,(a-2)_2(b-2)_2(c-2)_2}$$
$$\times\left\{\frac{\Gamma(\frac{1}{2}a)\Gamma(1+a-b)\Gamma(1+a-c)\Gamma(\frac{1}{2}a-b-c+4)}{\Gamma(a-1)\Gamma(\frac{1}{2}a-b+2)\Gamma(\frac{1}{2}a-c+2)\Gamma(3+a-b-c)}\right.$$
$$\left. -\frac{(a-2)(b-2)(c-2)}{(a-b+1)(a-c+1)} - 1\right\}.$$

(f) In Theorem 2.14, if we take $m = 1, 2, 3$, we get the following results.

$$\int_0^\infty e^{-st}\, t^{a-1}\, {}_2F_2\left[\begin{matrix} b, & c \\ \frac{1}{2}(a+b+1), & 2c \end{matrix}; st\right] dt \tag{81}$$
$$= \frac{\sqrt{\pi}\Gamma(a)\Gamma(c+\frac{1}{2})\Gamma(\frac{1}{2}(a+b+1))\Gamma(c-\frac{1}{2}(a+b-1))}{s^a\,\Gamma(\frac{1}{2}(a+1)\Gamma(\frac{1}{2}(b+1))\Gamma(c-\frac{1}{2}(a-1))\Gamma(c-\frac{1}{2}(b-1))},$$

(82)

$$\int_0^\infty e^{-st}\, t^{a-1}\, {}_3F_3\left[\begin{matrix} b, & c, & 1 \\ \frac{1}{2}(a+b+1), & 2c-1, & 2 \end{matrix}; st\right] dt$$
$$= \frac{(a+b-1)\Gamma(a-1)}{s^a\,(b-1)}\left\{\frac{\sqrt{\pi}\,\Gamma(c-\frac{1}{2})\Gamma(\frac{1}{2}(a+b-1))\Gamma(c-\frac{1}{2}(a+b-1))}{\Gamma(\frac{1}{2}a)\Gamma(\frac{1}{2}b)\Gamma(c-\frac{1}{2}a)\Gamma(c-\frac{1}{2}b)} - 1\right\}$$

and

$$\int_0^\infty e^{-st}\, t^{a-1}\, {}_3F_3\left[\begin{matrix} b, & c, & 1 \\ \frac{1}{2}(a+b+1), & 2c-2, & 3 \end{matrix}; st\right] dt \tag{83}$$
$$= \frac{(2c-3)(a+b-1)(a+b-3)\,\Gamma(a)}{s^a\,(c-1)(a-2)_2(b-2)_2}$$
$$\times\left\{\frac{\sqrt{\pi}\,\Gamma(c-\frac{3}{2})\Gamma(\frac{1}{2}(a+b-3))\Gamma(c-\frac{1}{2}(a+b-1))}{\Gamma(\frac{1}{2}(a-1))\Gamma(\frac{1}{2}(b-1))\Gamma(c-\frac{1}{2}(a+1))\Gamma(c-\frac{1}{2}(b+1))}\right.$$
$$\left. -\frac{(a-2)(b-2)}{a+b-3} - 1\right\}.$$

(g) In Theorem 2.17, if we take $m = 1, 2, 3$, we get the following results.

$$\int_0^\infty e^{-st}\, t^{a-1}\, {}_2F_2\left[\begin{matrix} 1-a, & b \\ c, & 2b-c+1 \end{matrix}; st\right] dt \tag{84}$$
$$= \frac{\pi\, 2^{1-2b}\,\Gamma(a)\Gamma(c)\Gamma(2b-c+1)}{s^a\,\Gamma(\frac{1}{2}(a+c))\Gamma(b+\frac{1}{2}(a-c+1))\Gamma(\frac{1}{2}(1-a+c))\Gamma(b+1-\frac{1}{2}(a+c))},$$

(85)

$$\int_0^\infty e^{-st}\, t^{a-1}\, {}_3F_3\left[\begin{matrix} 3-a, & b, & 1 \\ c, & 2b-c+1, & 2 \end{matrix}; st\right] dt = \frac{(c-1)(c-2b)\Gamma(a)}{s^a\,(a-2)_2(b-1)}$$
$$\times\left\{\frac{\pi 2^{3-2b}\Gamma(c-1)\Gamma(2b-c)}{\Gamma(\frac{1}{2}(a+c)-1)\Gamma(b+\frac{1}{2}(a-c-1))\Gamma(\frac{1}{2}(1-a+c))\Gamma(b+1-\frac{1}{2}(a+c))} - 1\right\}$$

and

(86)

$$\int_0^\infty e^{-st}\, t^{a-1}\, {}_3F_3\left[\begin{matrix} 5-a, & b, & 1 \\ c, & 2b-c+1, & 3 \end{matrix}; st\right] dt = \frac{2(c-2)_2(2b-c-1)_2\Gamma(a)}{s^a\,(a-4)_4(b-2)_2}$$
$$\times\left\{\frac{\pi\, 2^{5-2b}\,\Gamma(c-2)\Gamma(2b-c+1)}{\Gamma(\frac{1}{2}(a+c)-2)\Gamma(b+\frac{1}{2}(a-c-3))\Gamma(\frac{1}{2}(1-a+c))\Gamma(b+1-\frac{1}{2}(a+c))}\right.$$
$$\left.-\frac{(a-2)(3-a)(b-2)}{(c-2)(2b-c-1)} - 1\right\}.$$

(h) In Theorem 2.21, if we take $m = 1, 2, 3$, we get the following results.

$$\int_0^\infty e^{-st}\, t^{a-1}\, {}_2F_2\left[\begin{matrix} -n, & b \\ 1+a+b-c-n, & c \end{matrix}; st\right] dt = \frac{\Gamma(a)(c-a)_n(c-b)_n}{s^a\,(c)_n(c-a-b)_n}, \tag{87}$$

$$\int_0^\infty e^{-st}\, t^{a-1} {}_3F_3\left[\begin{matrix} -n+1, & b, & 1 \\ 1+a+b-c-n, & c, & 2 \end{matrix}; st\right] dt \tag{88}$$
$$= \frac{(1-c)(c-a-b+n)\Gamma(a-1)}{n(1-b)\,s^a}\left\{\frac{(c-a)_n(c-b)_n}{(c)_n(c-a-b+1)_n} - 1\right\}$$

and

$$\int_0^\infty e^{-st}\, t^{a-1} {}_3F_3\left[\begin{matrix} -n+2, & b, & 1 \\ 1+a+b-c-n, & c, & 3 \end{matrix}; st\right] dt \tag{89}$$
$$= \frac{2(1-c)_2(c-a-b+n)_2\,\Gamma(a)}{s^a\,(1-a)_2(1-b)_2}$$
$$\times\left\{\frac{(c-a)_n(c-b)_n}{(c-2)_n(c-a-b+2)_n} + \frac{n(a-2)(b-2)}{(c-2)(a+b-c-n-1)} - 1\right\}.$$

(i) In Theorem 2.24, if we take $m = 1, 2, 3$, we get the following results.

$$\int_0^\infty e^{-st}\, t^{c-1}\, {}_3F_3\left[\begin{matrix} a, & \frac{1}{2}(a+2), & b \\ \frac{1}{2}a, & a-b+1, & a-c+1 \end{matrix}; -st\right] dt \tag{90}$$
$$= \frac{\Gamma(1+a-b)\Gamma(1+a-c)\Gamma(c)}{s^c\,\Gamma(1+a)\Gamma(1+a-b-c)},$$

$$\int_0^\infty e^{-st}\, t^{c-1}\, {}_4F_4\left[\begin{matrix} a, & \frac{1}{2}(a+3), & b, & 1 \\ \frac{1}{2}(a+1), & a-b+2, & a-c+2, & 2 \end{matrix}; -st\right] dt$$
$$= \frac{(1+a-b)(1+a-c)\Gamma(c)}{s^c\,(a+1)(b-1)(c-1)}\left\{1 - \frac{\Gamma(1+a-b)\Gamma(1+a-c)}{\Gamma(a)\Gamma(2+a-b-c)}\right\} \tag{91}$$

and

$$\int_0^\infty e^{-st}\, t^{c-1}\, {}_4F_4\left[\begin{matrix} a, & \frac{1}{2}(a+4), & b, & 1 \\ \frac{1}{2}(a+3), & a-b+3, & a-c+3, & 3 \end{matrix}; -st\right] dt$$
$$= \frac{2(1+a-b)_2(1+a-c)_2\,\Gamma(c)}{s^c\,(a+2)(a-1)(b-2)_2(c-2)_2}$$
$$\times\left\{\frac{\Gamma(1+a-b)\Gamma(1+a-c)}{\Gamma(a-1)\Gamma(3+a-b-c)} + \frac{a(b-2)(c-2)}{(1+a-b)(1+a-c)} - 1\right\}. \tag{92}$$

(j) In Theorem 2.27, if we take $m = 1, 2, 3$, we get the following results.

$$\int_0^\infty e^{-st}\, t^{a-1}\, {}_4F_4\left[\begin{matrix} c, & \frac{1}{2}(a+2), & d, & e \\ \frac{1}{2}a, & a-c+1, & a-d+1, & a-e+1 \end{matrix}; st\right] dt$$
$$= \frac{\Gamma(1+a-c)\Gamma(1+a-d)\Gamma(1+a-e)\Gamma(1+a-c-d-e)}{s^a\,\Gamma(1+a-d-e)\Gamma(1+a-c-e)\Gamma(1+a-c-d)}, \tag{93}$$

$$\int_0^\infty e^{-st}\, t^{a-1}\, {}_5F_5\left[\begin{matrix} c, & \frac{1}{2}(a+3), & d, & e, & 1 \\ \frac{1}{2}(a+1), & a-c+2, & a-d+2, & a-e+2, & 2 \end{matrix}; st\right] dt$$
$$= \frac{(1+a-c)(1+a-d)(1+a-e)\,\Gamma(a)}{s^a\,(1+a)(c-1)(d-1)(e-1)}$$
$$\times\left\{\frac{\Gamma(1+a-c)\Gamma(1+a-d)\Gamma(1+a-e)\Gamma(3+a-c-d-e)}{\Gamma(a)\Gamma(2+a-d-e)\Gamma(2+a-c-e)\Gamma(2+a-c-d)} - 1\right\} \tag{94}$$

and

$$\int_0^\infty e^{-st}\, t^{a-1}\, {}_5F_5\left[\begin{matrix} c, & \frac{1}{2}(a+4), & d, & e, & 1 \\ \frac{1}{2}(a+2), & a-c+3, & a-d+3, & a-e+3, & 3 \end{matrix}; st\right] dt$$
$$= \frac{2(1+a-c)_2(1+a-d)_2(1+a-e)_2\,\Gamma(a)}{s^a\,(a-1)(a+2)(c-2)_2(d-2)_2(e-2)_2}$$
$$\times\left\{\frac{\Gamma(1+a-c)\Gamma(1+a-d)\Gamma(1+a-e)\Gamma(5+a-c-d-e)}{\Gamma(a-1)\Gamma(3+a-d-e)\Gamma(3+a-c-e)\Gamma(3+a-c-d)}\right.$$
$$\left. - \frac{a(c-2)(d-2)(e-2)}{(1+a-c)(1+a-d)(1+a-e)}\right\}. \tag{95}$$

(k) In Theorem 2.31, if we take $m = 1, 2, 3$, we get the following results.

(96)

$$\begin{aligned}
&\int_0^\infty e^{-st}\, t^{a-1} \\
&\times {}_6F_6\left[\begin{matrix} b, & \frac{1}{2}(a+2), & c, & d, & 2a-b-c-d+n+1, & -n \\ \frac{1}{2}a, & 1+a-b, & 1+a-c, & 1+a-d, & b+c+d-a-n, & a+n+1 \end{matrix}; st\right] dt \\
&= \frac{\Gamma(a)(1+a)_n(a-b-c+1)_n(a-b-d+1)_n(a-c-d+1)_n}{s^a\,(1+a-b)_n(1+a-c)_n(1+a-d)_n(1+a-b-c-d)_n},
\end{aligned}$$

(97)

$$\begin{aligned}
&\int_0^\infty e^{-st}\, t^{a-1} \\
&\times {}_7F_7\left[\begin{matrix} b, & \frac{1}{2}(a+3), & c, & d, & 2a-b-c-d+n+3, & 1-n, & 1 \\ \frac{1}{2}(a+1), & 1+a-b, & 1+a-c, & 1+a-d, & b+c+d-a-n-1, & a+n+1, & 2 \end{matrix}; st\right] dt \\
&= \frac{(b-a-1)(c-a-1)(d-a-1)(n+2+a-b-c-d)(a+n)\Gamma(a)}{n\,s^a\,(1+a)(1-b)(1-c)(1-d)(b+c+d-2a-2-n)} \\
&\times \left\{1 - \frac{(a)_n(a-b-c+2)_n(a-b-d+2)_n(a-c-d+2)_n}{(1+a-b)_n(1+a-c)_n(1+a-d)_n(3+a-b-c-d)_n}\right\}
\end{aligned}$$

and

(98)

$$\begin{aligned}
&\int_0^\infty e^{-st}\, t^{a-1} \\
&\times {}_7F_7\left[\begin{matrix} b, & \frac{1}{2}(a+4), & c, & d, & 2a-b-c-d+n+5, & 2-n, & 1 \\ \frac{1}{2}(a+2), & 3+a-b, & 3+a-c, & 3+a-d, & b+c+d-a-n-2, & a+n+1, & 3 \end{matrix}; st\right] dt \\
&= \frac{(a-2)(b-a-2)_2(c-a-2)_2(d-a-2)_2}{s^a\,(a+2)(1-a)_2(1-b)_2(1-c)_2(1-d)_2} \\
&\times \frac{(-a-n)_n(3+n+a-b-c-d)_2\,\Gamma(a)}{(n-1)_2(b+c+d-2a-4-n)_2} \\
&\times \left\{\frac{(a-1)_n(a-b-c+3)_n(a-b-d+3)_n(a-c-d+3)_n}{(a-b+1)_n(a-c+1)_n(a-d+1)_n(a-b-c-d+5)_n}\right. \\
&\left.+\frac{na(b-2)(c-2)(d-2)(2a-b-c+n+3)}{(a-b+1)(a-c+1)(a-d+1)(b+c+d-a-n-4)(n+a-1)} - 1\right\}.
\end{aligned}$$

(l) The results (69), (72) and (75) are recorded in [8] and also in [19].

(m) In Theorem 2.12, if we take $m = 1$, we get a known result obtained recently by Kim *et al.* [9].

(n) In Theorem 2.15, if we take $m = 1$, we get a known result obtained recently by Kim *et al.* [9].

(o) In Theorem 2.18, if we take $m = 1$, we get a known result obtained recently by Kim *et al.* [9].

Similarly other results can be obtained.

Remark. For evaluation of Eulerian's type integrals involving generalized hypergeometric functions by employing the summation theorems, (16) to (26), we refer an interesting paper by Jun *et al.* [6].

Conclusion Remark

In this paper, several Laplace transforms involving generalized hypergeometric functions have been evaluated in terms of gamma function by employing very recently obtained summation theorems by Masjed-Jamei and Koepf. A few new, interesting and elementary Laplace transforms have also been given as special cases of our main findings.

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Authors' contributions

All authors contributed equally to writing of this paper. All authors read and approved the final manuscript.

Competing interest

The authors declare that they have no competing interests.

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