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Algorithmic approach for formal Fourier series. **(English summary)**

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The study of trigonometric series started at the beginning of the nineteenth century. Joseph Fourier made the important observation that almost every function on a closed interval can be decomposed into the sum of sine and cosine functions. This technique to develop a function into a trigonometric series was published for the first time in 1822 by Fourier. The resulting series is nowadays called Fourier series. Since Fourier's time, many different approaches to the concept of Fourier series have been discovered, each of which emphasizes different aspects of the topic. Some of the more powerful and elegant approaches are based on mathematical ideas and tools that were not available at the time Fourier completed his original work.

Despite the importance of Fourier series, the method used up to the present time to compute them via computer algebra systems (CAS) was essentially based on the same principle as in Fourier's time, i.e. by the evaluation of certain integrals. Unfortunately this technique is not completely successful for many functions. Although numeric values of the Fourier coefficients might be available, symbolic values are often not accessible. As a matter of fact, modern CAS like *Maple* or *Mathematica* can compute such integrals in many cases for a given $n\in\mathbb{Z}$; however, if one is interested in the Fourier coefficients for all $n\in\mathbb{Z}$, then n is considered a given symbolic variable and such integrals can be computed only in a few cases. More precisely, the symbolic computation of the Fourier coefficients of an integrable function $f\colon[a,b]\to\mathbb{R}$ using the standard definition of (real or complex) Fourier coefficients may turn out to be quite involved because of the integer parameter n that appears within their expressions [see, e.g., I. N. Bronshteĭn et al., *Taschenbuch der Mathematik*, English translation, 7th revised and enlarged edition, Verl. Harri Dtsch., Frankfurt am Main, 2008; and general references therein].

Consider, for example, the function given as

$$f(t)=\cos 5t\ln(2+\cos 5t).$$

With the use of the algorithms presented in the paper, its Fourier series (polynomial) $\mathcal{F}(f)(t)$ in the interval $I=[0,2\pi/5]$ is given by

$$\mathcal{F}(f)(t)=2(2-\sqrt{3})+\big(2\sqrt{3}-7/2+\ln(2+\sqrt{3})-\ln 2\big)\cos 5t+\sum_{n\geq 2}\frac{2(-2+\sqrt{3})^n(\sqrt{3}+2n)}{(n+1)(n-1)}\cos 5nt.$$

Note that the algorithmic approach is applicable to the rich family of trigonometric holonomic functions, whereas the Fourier coefficients of the above function f cannot be successfully computed by current CAS using the standard Fourier formulas.

In this article, an algorithmic approach is introduced to compute those Fourier coefficients, involving differential equations of a particular form, as well as recurrence equations. This approach extrapolates the computation of the Fourier series for functions for which the computation of Fourier coefficients via the usual definition is out of reach for current CAS.

A holonomic recurrence equation for a_n, i.e. a recurrence equation which is linear, homogeneous and has polynomial coefficients, can be written in operator notation as $L(a_n) = 0$. Following S. Lewanowicz et al. [Numer. Algorithms **23** (2000), no. 1, 31–50; MR1767103], the operator L can be interpreted as a non-commutative polynomial via the commutator rule $Sn - nS = S$, where S denotes the shift operator $Sa_n = a_{n+1}$. The last section shows how this algorithm can be used to factorize such recurrence operators in certain cases.

To the best of our knowledge the algorithms designed in the paper under review are new and were first introduced in the second author's Ph.D. thesis [*Algorithmic computation of formal Fourier series*, Univ. Kassel, 2010]. *Christian Lavault*