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On linearization and connection coefficients for generalized Hermite polynomials. (English summary)
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This paper deals with the connection and linearization problems for generalized Hermite polynomials $\mathcal{H}_n^\mu(x)$, which are orthogonal on $\mathbb{R}$ with respect to the weight $|x|^{2\mu}e^{-x^2}$, for $\mu > -1/2$.

For any two different sets of generalized Hermite polynomials, $\mathcal{H}_n^{\mu_1}(x)$ and $\mathcal{H}_n^{\mu_2}(x)$, the authors find the connection formula

$$\frac{\mathcal{H}_n^{\mu_2}(x)}{n!} = \sum_{m=0}^{[n/2]} C_{n-2m}(n)\frac{\mathcal{H}_{n-2m}^{\mu_1}(x)}{(n-2m)!}$$

and give both an explicit formula for the connection coefficients C_{n-2m} and a linear recurrence relation between C_{n-2m} and C_{n-2m-2}. In particular, the sign properties of the coefficients C_{n-2m} are deduced: for instance, if $\mu_2 > \mu_1$ they alternate in sign as m varies.

The standard linearization problem (or Clebsch-Gordan type problem) asks one to write the products $\mathcal{H}_i^\mu(x)\mathcal{H}_j^\mu(x)$ as linear combinations of the same family of polynomials $\mathcal{H}_k^\mu(x)$. The authors obtain

$$\mathcal{H}_i^\mu(x)\mathcal{H}_j^\mu(x) = \sum_{k=0}^{\min(i,j)} L_{ij}(i+j-2k)\mathcal{H}_{i+j-2k}^\mu(x)$$

and give a linear recurrence relation involving $L_{ij}(i+j-2k)$, $L_{ij}(i+j-2k-2)$, $L_{ij}(i+j-2k-4)$, and $L_{ij}(i+j-2k-6)$ (depending on i and j only three consecutive coefficients are needed). The sign behaviour of the linearization coefficients is studied, also.

In both the connection and the linearization problems, some of the computations are carried out with the computer algebra system Maple.

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References

1. R. Askey, Orthogonal Polynomials and Special Functions, in: CBMS-NSF Regional Conference Series in Appl. Math., vol. 21, SIAM, Philadelphia, Pennsylvania, 1975. MR0481145 (58 #1288)
2. R. Szwarc, Orthogonal polynomials and a discrete boundary value problem II, SIAM J. Math. Anal. 23 (1992) 959–964. MR1166568 (93i:33007)
3. R. Álvarez-Nodarse, R.J. Yá nez, J.S. Dehesa, Modified Clebsch-Gordan-type expansions for products of discrete hypergeometric polynomials, J. Comput. Appl. Math. 89 (1998) 171–197. MR1625947 (99h:33058)
4. S. Lewanowicz, The hypergeometric function approach to the connection problem for the

classical orthogonal polynomials, Tech. report, Inst. Computer Sci, Univ. of Wroclaw, Feb. 1998.

5. A. Ronveaux, M.N. Hounkonnou, S. Belmehdi, Generalized linearization problems, J. Phys. A: Math. Gen. 28 (1995) 4423–4430. MR1351939 (96i:33020)
6. A. Ronveaux, S. Belmehdi, E. Godoy, A. Zarzo, Recurrence relation approach for connection coefficients. Application to discrete orthogonal polynomials, in: Proc. Inter. Workshop on Symmetries and Integrability of Difference Equations, in: CRM Proc. and Lecture Notes Series, vol. 9, AMS, Providence R.I, 1996, pp. 321–337.
7. A. Ronveaux, Orthogonal polynomials: connection and linearization coefficients, in: M. Alfaro et al. (Ed.), Proc. Inter. Workshop on Orthogonal Polynomials in Mathematical Physics, Leganés, Madrid, Spain, 1996. MR1466775
8. A. Ronveaux, A. Zarzo, E. Godoy, Recurrence relations for connection coefficients between two families of orthogonal polynomials, J. Comput. Appl. Math. 62 (1995) 67–73. MR1361274
9. S. Lewanowicz, Second-order recurrence relations for the linearization coefficients of the classical orthogonal polynomials, J. Comput. Appl. Math. 69 (1996) 159–160. MR1391617 (97k:33010)
10. W. Koepf, D. Schmersau, Representations of orthogonal polynomials, J. Comput. Appl. Math. 90 (1998) 57–94. MR1627168 (2000d:33005)
11. Y. Ben Cheikh, H. Chaggara, Connection coefficients via lowering operators, J. Comput. Appl. Math. 178 (2005) 45–61. MR2127869 (2006f:33004)
12. Y. Ben Cheikh, H. Chaggara, Connection coefficients between Boas-Buck polynomial sets, J. Math. Anal. Appl. 319 (2006) 665–689. MR2227931 (2007c:33008)
13. H. Chaggara, I. Lamiri, Linearization coefficients for Boas-Buck polynomial sets, Appl. Math. Comput. 189 (2007) 1533–1549. MR2332108 (2008b:33018)
14. M.W. Wilson, Nonnegative expansions of polynomials, Proc. Amer. Math. Soc. 24 (1970) 100–102. MR0287244 (44 #4451)
15. R. Lasser, Linearization of the product of associated Legendre polynomials, SIAM J. Math. Anal. 14 (1983) 403–408. MR0688586 (84e:33013)
16. R. Askey, G. Gasper, Jacobi polynomial expansions of Jacobi polynomials with non-negative coefficients, Proc. Camb. Phil. Soc. 70 (1971) 243–255. MR0296369 (45 #5430)
17. D.K. Dimitrov, Connection coefficients and zeros of orthogonal polynomials, J. Comput. Appl. Math. 133 (2001) 331–340. MR1858291 (2002k:33007)
18. H. Chaggara, W. Koepf, Duplication coefficients for discrete polynomial sets via generating functions, Complex Var. Elliptic Equ. 52 (2007) 237–249. MR2326190 (2008f:33010)
19. W. Koepf, Hypergeometric Summation, Vieweg, Braunschweig, Wiesbaden, 1998. MR1644447 (2000c:33002)
20. T. Sprenger, Algorithmen für mehrfache Summen, Diploma Thesis at the University of Kassel, 2004, pp. 1–85.
21. G. Polya, G. Szegö, Problems and Theorems in Analysis, vol. 1, Springer-Verlag, New York, Heidelberg and Berlin, 1972. MR0344042 (49 #8782)
22. T.S. Chihara, Generalized Hermite polynomials, Ph.D. Thesis, Purdue, 1955. MR2612324
23. T.S. Shao, T.C. Chen, R.M. Frank, Table of zeros and gaussien weights of certain associated

Laguerre polynomials and generalized Hermite polynomials, Math. Comp. 18 (1964) 598–616. MR0166397 (29 #3674)
24. M. Rosenblum, Generalized Hermite polynomials and the Bose-like oscillator calculus, Oper. Theory Adv. Appl. 73 (1994) 369–396. MR1320555 (96b:33005)
25. Y. Ben Cheikh, M. Gaied, Dunkl-Appell d-orthogonal polynomials, Integral Transforms Spec. Funct. 18 (2007) 581–597. MR2348603 (2008k:42074)
26. H. Dette, Characterizations of generalized Hermite and sieved ultraspherical polynomials, Trans. Amer. Math. Soc. 348 (1996) 691–711. MR1311912 (96g:33010)
27. H. Chaggara, Operational rules and a generalized Hermite polynomials, J. Math. Anal. Appl. 332 (2007) 11–21. MR2319641 (2008h:33029)
28. Y. Ben Cheikh, Some results on quasi-monomiality, Appl. Math. Comput. 141 (2003) 63–76. MR1984228 (2004h:33030)
29. H.M. Srivastava, H.L. Manocha, A Treatise on Generating Functions, John Wiley and Sons, New York, Chichester, Brisbane, Toronto, 1984. MR0750112 (85m:33016)
30. L. Carlitz, Products of Appell polynomials, Collect. Math. 112 (1963) 133–138.
31. K. Wegschaider, Computer generated proofs of binomial multi-sum identities, Diploma Thesis at the J. Kepler University of Linz, 1997, pp. 1–99.
32. M. Van Hoeij, Finite singularities and hypergeometric solutions of linear recurrence equations, J. Pure Appl. Algebra 139 (1998) 109–131. MR1700540 (2001h:39023)
33. E.D. Rainville, Special Functions, The Macmillan Company, New York, 1960. MR0107725 (21 #6447)
34. G.N. Watson, A note on the polynomials of Hermite and Laguerre, J. Lond. Math. Soc. 13 (1938) 29–32. MR1574524

Note: This list reflects references listed in the original paper as accurately as possible with no attempt to correct errors.