On Moments of Classical Orthogonal Polynomials

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Abstract

In this work, using the inversion coefficients and some connection coefficients between some polynomial sets, we give explicit representations of the moments of all the orthogonal polynomials belonging to the Askey-Wilson scheme. Generating functions for some of these moments are also provided.

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1. Introduction

A sequence of polynomials $\{p_n(x)\}$, where $p_n(x)$ is of exact degree n in x, is said to be orthogonal with respect to a Lebesgue-Stieltjes measure $d\alpha(x)$ if

$$\int_{-\infty}^{\infty} p_m(x)p_n(x)d\alpha(x) = 0, \quad m \neq n. \tag{1}$$

Implicit in this definition is the assumption that the moments

$$\mu_n = \int_{-\infty}^{\infty} x^n d\alpha(x), \quad n = 0, 1, 2, \ldots, \tag{2}$$

are finite. If the nondecreasing, real-valued, bounded function $\alpha(x)$ also happens to be absolutely continuous with $d\alpha(x) = \rho(x)dx$, $\rho(x) \geq 0$, then (1) and (2) reduce to

$$\int_{-\infty}^{\infty} p_m(x)p_n(x)\rho(x)dx = 0, \quad m \neq n, \tag{3}$$

and

$$\mu_n = \int_{-\infty}^{\infty} \rho(x)x^n dx, \quad n = 0, 1, 2, \ldots, \tag{4}$$

respectively, and the sequence $\{p_n(x)\}$ is said to be orthogonal with respect to the weight function $\rho(x)$. If on the other hand, $\alpha(x)$ is a step-function with jumps ρ_j at $x = x_j$, $j = 0, 1, 2, \ldots$, then (1) and (2) take the form of a sum:

$$\sum_{j=0}^{\infty} p_m(x_j)p_n(x_j)\rho_j = 0, \quad m \neq n \tag{5}$$

and

$$\mu_n = \sum_{j=0}^{\infty} x_j^n \rho_j, \quad n = 0, 1, 2, \ldots. \tag{6}$$

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The polynomials p_n, in monic form, are given explicitly in terms of the moments by [37]

$$p_n(x) = \frac{K_n}{d_{n-1}} \begin{vmatrix} \mu_0 & \mu_1 & \cdots & \mu_n \\ \mu_1 & \mu_2 & \cdots & \mu_{n+1} \\ \vdots & \vdots & \ddots & \vdots \\ \mu_{n-1} & \mu_{n+1} & \cdots & \mu_{2n-1} \\ 1 & x & \cdots & x^n \end{vmatrix},$$

where

$$d_n = \begin{vmatrix} \mu_0 & \mu_1 & \cdots & \mu_n \\ \mu_1 & \mu_2 & \cdots & \mu_{n+1} \\ \vdots & \vdots & \ddots & \vdots \\ \mu_{n-1} & \mu_n & \cdots & \mu_{2n-1} \\ \mu_n & \mu_{n+1} & \cdots & \mu_{2n} \end{vmatrix} \quad \text{under the condition} \quad d_n \neq 0,\ n \geq 0.$$

The previous representation shows that the moments characterize fully the orthogonal family $\{p_n(x)\}$. In [1, P. 295, Theorem 6.3.3] for example, the authors used the moments of the Jacobi polynomials to give the hypergeometric representation of these polynomials.

Note that the classical continuous and discrete orthogonal polynomial families are very much related to probability theory [34] (see also [24]). In the continuous case, the measures of the Hermite, Laguerre and Jacobi polynomials are the normal, the Gamma and the Beta distributions, respectively. In the discrete case, the measures of the Charlier, the Meixner, the Krawtchouk and the Hahn polynomials are the Poisson, the Pascal, the binomial and the hypergeometric distributions. Of course moments play an important role in probability theory and statistics (see [24]).

Despite the important role that the moments play in various topics of orthogonal polynomials and applications to other domains such as statistics and probability theory, no exhaustive repository of moments for the well-known classical orthogonal polynomials can be found in the literature. The book by Koekoek, Lesky and Swarttouw [26] which is one of the best and most famous documents containing almost all kinds of formulas and relations for the Askey-Wilson scheme does not provide information about the moments. In addition, despite the fact that almost all the moments of the classical orthogonal polynomials of the continuous, the discrete and q-discrete have been previously published in the litterature (see for example [3, 9, 10]) , it is not the case for the classical orthogonal polynomials of the quadratic and the q-quadratic variable. It becomes therefore a very interesting task to investigate this topic in order not only to make also available in the litterature the moments of the classical orthogonal polynomials of a quadratic and q-quadratic variable but also to provide and exhaustive reposipory of the moments of all classical orthogonal polynomials.

The paper is organized as follows:

- In Section 2, we present some basic definitions and give some important properties that will be used throughout the paper;
- in Section 3, some useful Taylor formulas for polynomials and applications are given and used to find connection coefficients between suitable polynomial bases;
- in Section 4, we use the results given in Section 3 to deduce explicit representations of the (canonical)

moments of all the orthogonal families listed in [26]. Some generating functions for these moments are also provided.

The results of this paper appeared in the Ph.D. thesis [31] of the first author.

2. Definitions and miscellaneous properties

In this section we recall basic definitions and introduce some difference operators that will be useful along this paper.

Definition 1. [26, P. 4] *The Pochhammer symbol or shifted factorial is defined by*

$$(a)_0 := 1 \quad \textit{and} \quad (a)_n = a(a+1)(a+2)\cdots(a+n-1), \quad a \neq 0 \quad n = 1, 2, 3, \dots .$$

The following notation (falling factorial) will also be used:

$$a^{\underline{0}} := 1 \quad \text{and} \quad a^{\underline{n}} = a(a-1)(a-2)\cdots(a-n+1), \quad n = 1, 2, 3, \dots .$$

It should be noted that the Pochhammer symbol and the falling factorial are linked as follows:

$$(-a)_n = (-1)^n a^{\underline{n}}.$$

Definition 2. [26, P. 5] *The hypergeometric series ${}_rF_s$ is defined by*

$${}_rF_s\left(\begin{array}{c} a_1, \cdots, a_r \\ b_1, \cdots, b_s \end{array} \middle| z\right) := \sum_{n=0}^{\infty} \frac{(a_1, \cdots, a_r)_n}{(b_1, \cdots, b_s)_n} \frac{z^n}{n!},$$

where

$$(a_1, \dots, a_r)_n = (a_1)_n \cdots (a_r)_n.$$

Definition 3. [26, P. 11] *The q-variant of the shifted factorial, also called q-Pochhammer symbol, is defined by*

$$\begin{aligned} (a;q)_0 &= 1, \\ (a;q)_n &= (1-a)(1-aq)\cdots(1-aq^{n-1}), \quad n = 1, 2, \dots . \end{aligned}$$

For $n = \infty$ we set

$$(a;q)_\infty = \prod_{n=0}^{\infty}(1-aq^n), \ |q| < 1.$$

In order to deal with some families of orthogonal polynomials and other basic hypergeometric functions, the following more general notation (see [25])

$$(x \ominus y)_q^n = (x-y)(x-qy)\cdots(x-q^{n-1}y), \tag{7}$$

which is the so-called q-power basis, will be used.

Definition 4. [26, P. 15] *The q-hypergeometric function denoted by ${}_r\phi_s$ is defined by*

$${}_r\phi_s\left(\begin{array}{c} a_1, a_2, \cdots, a_r \\ b_1, b_2, \cdots, b_s \end{array} \middle| q; z\right) = \sum_{n=0}^{\infty} \frac{(a_1, \cdots, a_r; q)_n}{(b_1, \cdots, b_s; q)_n} \left[(-1)^n q^{\binom{n}{2}}\right]^{1+s-r} \frac{z^n}{(q,q)_n},$$

where

$$(a_1, a_2, \cdots, a_m; q)_n = (a_1; q)_n (a_2, q)_n \cdots (a_m; q)_n.$$

We will also use the following common notations

$$[a]_q = \frac{1-q^a}{1-q}, \quad a \in \mathbb{C}, \quad q \neq 1, \tag{8}$$

$$\left[\begin{array}{c} n \\ m \end{array} \right]_q = \frac{(q;q)_n}{(q;q)_m (q;q)_{n-m}}, \quad 0 \leq m \leq n, \tag{9}$$

called the q-bracket and the q-binomial coefficient, respectively.

The following difference and divided difference operators will also be frequently used.

Definition 5. *Let f be a function of the variable x.*

1. *The forward and the backward difference operators Δ and ∇ are, respectively, defined by:*

$$\Delta f(x) = f(x+1) - f(x), \qquad \nabla f(x) = f(x) - f(x-1).$$

For $m \in \mathbb{N}_{\geq 0} = \{1, 2, 3, \dots\}$, one sets

$$\Delta^{m+1} f(x) = \Delta(\Delta^m f(x)), \quad \textit{and} \quad \Delta^0 f(x) = f(x).$$

2. *The q-difference operator D_q is defined as:*

$$D_q f(x) := \frac{f(x) - f(qx)}{(1-q)x} \quad \textit{if } x \neq 0 \ ,$$

and $D_q f(0) = f'(0)$ provided that f is differentiable at $x = 0$.

If m is a nonnegative integer, we have

$$D_q^{m+1} f(x) = D_q \left(D_q^m f(x) \right); \qquad D_q^0 f(x) = f(x).$$

3. *The difference operator $\mathcal{D}$ is defined as follows:*

$$\mathcal{D} f(x) = f\left(x + \frac{i}{2}\right) - f\left(x - \frac{i}{2}\right), \quad \textit{where} \quad i^2 = -1.$$

4. *The divided difference operator $\mathbf{D}$ is defined as follows:*

$$\mathbf{D} f(x^2) = \frac{f((x+\frac{i}{2})^2) - f((x-\frac{i}{2})^2)}{2ix}, \quad i^2 = -1.$$

3. Taylor formulas, power derivatives and connection formulas

In this section, we give some tools for the computations of the moments given in the next section. Some Taylor formulas are proved, the power derivatives of some operators are given. As applications, we compute some connection coefficients between suitable polynomial sets that appear in the computation of the moments.

Proposition 6 (See e.g. [2, 23])**.** *Let $f(x)$ be a polynomial of degree n in the variable x. The following expansion formula holds*

$$f(x) = \sum_{m=0}^{n} \frac{[D_q^m f](y)}{[m]_q!} (x \ominus y)_q^m. \tag{10}$$

Proof. We assume f is a polynomial of degree n, and we write

$$f(x) = \sum_{m=0}^{n} c_m (x \ominus y)_q^m.$$

Next we apply the operator D_q k times to both sides of this relation and get

$$D_q^k f(x) = \sum_{m=k}^{n} c_m \frac{[m]_q!}{[m-k]_q!} (x \ominus y)_q^{m-k}.$$

Taking $x = y$, it follows that $[D_q^k f](y) = c_k [k]_q!$ and the proposition follows. $\square$

Corollary 7. *We have the following connection formula between the q-power and the power bases*

$$x^n = \sum_{m=0}^{n} y^{n-m} \begin{bmatrix} n \\ m \end{bmatrix}_q (x \ominus y)_q^m. \tag{11}$$

Remark 8. *This corollary will be useful for the computations of the canonical moments of the Al Salam Carlitz I polynomials.*

Theorem 9. *Define the polynomial basis $\theta_n(a,x)$ by*

$$\theta_n(a,x) = (a - ix)_n (a + ix)_n = \prod_{k=0}^{n-1} (x^2 + (a+k)^2), \quad \theta_0(a,x) = 1.$$

If f is a polynomial of degree n in x^2, then

$$f(x) = \sum_{k=0}^{n} f_k \theta_k(a,x),$$

where

$$f_k = \frac{\mathbf{D}^k f(i(a + \frac{k}{2}))}{k!}.$$

Proof. First remark that $\theta_k(a, ai) = 0$ for all $k > 0$. Hence

$$\mathbf{D}^j f(x) = \sum_{k=j}^{n} f_k \frac{k!}{(k-j)!} \theta_{k-j}(a + \frac{j}{2}, x) = f_j j! + \sum_{k=j+1}^{n} f_k \frac{k!}{(k-j)!} \theta_{k-j}(a + \frac{j}{2}, x)$$

and for $x = i\left(a + \frac{j}{2}\right)$, we get

$$\mathbf{D}^j f\left(i\left(a + \frac{j}{2}\right)\right) = f_j j!.$$

$\square$

Theorem 10 (see [11])**.** *Let k be a nonnegative integer. Then*

$$\mathbf{D}^k f(x) = \sum_{l=0}^{k} \frac{(-k)_l}{l!} \frac{(2ix - k - 2l)}{(2ix - k + l)_{k+1}} f\left(x + \frac{k-2l}{2} i\right). \tag{12}$$

Corollary 11. *The following result is valid*

$$\mathbf{D}^k x^{2n} = \sum_{l=0}^{k} \frac{(-k)_l}{l!} \frac{(2ix - k + 2l)}{(2ix - k + l)_{k+1}} \left(x + \frac{k-2l}{2} i\right)^{2n}. \tag{13}$$

Proof. Take $f(x) = x^{2n}$ in (12) to get the result. $\square$

Corollary 12. *The following connection formula is valid*

$$x^{2n} = (-1)^n \sum_{k=0}^{n} \frac{1}{k!} \sum_{l=0}^{k} \frac{(-k)_l}{l!} \frac{(-2a - 2k + 2l)}{(-2a - 2k + l)_{k+1}} (a + k - l)^{2n} \theta_k(a,x). \tag{14}$$

Proof. The proof follows from Theorem 9 and Theorem 10 with $f(x) = x^{2n}$. $\square$

Remark 13. *This connection is useful for the computation of the moments of the Wilson and the Continuous Dual Hahn polynomials.*

Theorem 14. *Define the polynomial basis $\eta_n(a,x) = (a+ix)_n$. If f is a polynomial of degree n in x, then*

$$f(x) = \sum_{k=0}^{n} f_k \eta_k(a,x),$$

where

$$f_k = \frac{(-1)^k}{k!}\mathcal{D}^k f\left(i\left(a+\frac{k}{2}\right)\right).$$

Proof. First remark that $\eta_k(a,ai) = 0$ for all $k>0$. Hence

$$\mathcal{D}^j f(x) = \sum_{k=j}^{n}(-1)^k f_k \frac{k!}{(k-j)!}\eta_{k-j}(a+\frac{j}{2},x) = (-1)^j f_j j! + \sum_{k=j+1}^{n} f_k \frac{k!}{(k-j)!}\eta_{k-j}(a+\frac{j}{2},x)$$

and for $x = i\left(a+\frac{j}{2}\right)$, we get

$$\mathcal{D}^j f\left(i\left(a+\frac{j}{2}\right)\right) = (-1)^j j! f_j.$$

This proves the theorem. $\square$

Remark 15. *Note that in Theorem 14, there is a need to have an explicit representation of $\mathcal{D}^k f(x)$ in order to have a better expression of the Taylor formula. The following proposition gives the required expression.*

Proposition 16 (Power of $\mathcal{D}$). *Let k be a nonnegative integer, then the following relation holds*

$$\mathcal{D}^k f(x) = \sum_{l=0}^{k}(-1)^l\binom{k}{l} f\left(x+\frac{k-2l}{2}i\right). \tag{15}$$

Proof. The proof is done by induction. The relation is obvious for $k=1$. Assume it is true for a fixed integer $k>0$. Then, we have

$$\begin{aligned}
\mathcal{D}^{k+1} f(x) &= \mathcal{D}(\mathcal{D}^k f(x)) \\
&= \sum_{l=0}^{k}(-1)^l\binom{k}{l}\mathcal{D}f\left(x+\frac{k-2l}{2}i\right) \\
&= \sum_{l=0}^{k}(-1)^l\binom{k}{l}\left(f\left(x+\frac{k-2l+1}{2}i\right) - f\left(x+\frac{k-2l-1}{2}i\right)\right) \\
&= \sum_{l=0}^{k}(-1)^l\binom{k}{l} f\left(x+\frac{k-2l+1}{2}i\right) + \sum_{l=1}^{k+1}(-1)^l\binom{k}{l-1} f\left(x+\frac{k-2l+1}{2}i\right) \\
&= \sum_{l=0}^{k+1}(-1)^l\binom{k+1}{l} f\left(x+\frac{k-2l+1}{2}i\right).
\end{aligned}$$

$\square$

Corollary 17. *The following connection formula is valid.*

$$x^n = \sum_{k=0}^{n}\frac{1}{k!}\sum_{l=0}^{k}(-1)^l\binom{k}{l}((a+l)i)^n \eta_k(a,x). \tag{16}$$

Proof. First, we apply Theorem 14 with $f(x) = x^n$ to get

$$x^n = \sum_{k=0}^{n} \left(\frac{(-1)^k}{k!} \mathcal{D}^k x^n_{\big|x=i\left(a+\frac{k}{2}\right)} \right) \eta_k(a, x).$$

Next, using Proposition 16, we have

$$\mathcal{D}^k x^n = \sum_{l=0}^{k} (-1)^l \binom{k}{l} \left(x + \frac{k-2l}{2} i \right)^n .$$

Then, we have

$$\begin{aligned} \mathcal{D}^k x^n_{\big|i\left(a+\frac{k}{2}\right)} &= \sum_{l=0}^{k} (-1)^l \binom{k}{l} \left((a+k-l)i\right)^n \\ &= (-1)^k \sum_{l=0}^{k} (-1)^l \binom{k}{l} \left((a+l)i\right)^n . \end{aligned}$$

This completes the proof. □

Remark 18. *This connection is useful for the computation of the moments of the Continuous Hahn and the Meixner-Pollaczek polynomials.*

Proposition 19 (see [11])**.** *The following q-derivative rule is valid.*

$$(\mathcal{D}_q^n f)(x) = \frac{2^n q^{\frac{n(1-n)}{4}}}{(q^{1/2} - q^{-1/2})^n} \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q \frac{q^{k(n-k)} z^{2k-n} \breve{f}(q^{(n-2k)/2} z^2)}{(q^{1+n-2k} z^2; q)_k (q^{2k-n+1} z^{-2}; q)_{n-k}} \tag{17}$$

where $\breve{f}(z) = f((z + 1/z)/2)$, $z = e^{i\theta}$, $x = \cos\theta$.

Proposition 20 (see [21])**.** *If $f(x)$ is a polynomial in $x = \cos\theta$ of degree n, then*

$$f(x) = \sum_{k=0}^{n} f_k (ae^{i\theta}, ae^{-i\theta}; q)_k \tag{18}$$

where

$$f_k = \frac{(q-1)^k}{(2a)^k (q;q)_k} q^{-\frac{k(k-1)}{4}} (\mathcal{D}_q^k f)(x_k),$$

with

$$x_k = \frac{1}{2}(aq^{k/2} + q^{-k/2}/a).$$

Corollary 21. *If $f(x)$ is a polynomial of degree n in $x = \cos\theta$, then*

$$f(x) = \sum_{k=0}^{n} f_k (ae^{i\theta}, ae^{-i\theta}; q)_k,$$

with

$$f_k = q^k \sum_{j=0}^{k} \frac{q^{-(k-j)^2} a^{2(j-k)} \breve{f}(aq^{k-j})}{(q, q^{1+2(k-j)} a^2; q)_j (q, q^{-1-2(k-j)} a^{-2}; q)_{k-j}}. \tag{19}$$

Remark 22. *Note that, by a change of variable $j := k - j$, the p_k's in Corollary 21 can be written as*

$$f_k = q^k \sum_{j=0}^{k} \frac{q^{-j^2} a^{-2j} \breve{f}(aq^j)}{(q, q^{1+2j} a^2; q)_{k-j} (q, q^{-1-2j} a^{-2}; q)_j}.$$

Corollary 23. *The following connection formula is valid.*

$$x^n = \sum_{k=0}^{n} q^k \sum_{j=0}^{k} \frac{q^{-j^2} a^{-2j} (aq^j + a^{-1} q^{-j})^k}{(q, q^{1+2j} a^2; q)_{k-j} (q, q^{-1-2j} a^{-2}; q)_j} (ae^{i\theta}, ae^{-i\theta}; q)_k, \quad x = \cos\theta. \tag{20}$$

Remark 24. *This connection is useful for the computation of the moments of orthogonal polynomials on q-quadratic lattices.*

4. Moments of classical orthogonal polynomials

4.1. Canonical moments and generalized moments

Definition 25. *Let $\{p_n(x)\}$ be a polynomial set, i.e. $\deg(p_n(x)) = n$, orthogonal with respect to a Lebesgue-Stieltjes measure $d\alpha(x)$. Let $\{\theta_n(x)\}$ be a polynomial set. The numbers*

$$\mu_n(\theta_k(x)) = \int_{-\infty}^{\infty} \theta_n(x) d\alpha(x), \quad n = 0, 1, 2, \dots \tag{21}$$

are the moments with respect to $\theta_n(x)$ of the family $\{p_n(x)\}$, they are called generalized moments.

Remark 26. *Note that in the previous definition, if $\theta_n(x) = x^n$, then the generalized moments are the cononical moments.*

Theorem 27. *Let $\{\theta_n(x)\}$ be a polynomial set. Assume that one can find explicit representations of the coefficients $C_m(n)$ in the expansion*

$$x^n = \sum_{m=0}^{n} C_m(n)\theta_m(x). \tag{22}$$

Then, the canonical moments μ_n can be computed from the generalized moments $\mu_n(\theta_k(x))$ using the relation

$$\mu_n = \sum_{m=0}^{n} C_m(n)\mu_m(\theta_k(x)). \tag{23}$$

Proof. Assume the conditions of the theorem are satisfied. We have

$$\begin{aligned} \mu_n &= \int_{-\infty}^{\infty} x^n d\alpha(x) = \int_{-\infty}^{\infty} \left(\sum_{m=0}^{n} C_m(n)\theta_m(x) \right) d\alpha(x) \\ &= \sum_{m=0}^{n} C_m(n) \int_{-\infty}^{\infty} \theta_m(x) d\alpha(x) = \sum_{m=0}^{n} C_m(n)\mu_m(\theta_k(x)). \end{aligned}$$

□

Theorem 28. *Assume that the coefficients $I_m(n)$ (called inversion coefficients) in the expansion*

$$\theta_n(x) = \sum_{m=0}^{n} I_m(n)P_m(x) \tag{24}$$

are given. Then, for all $n \in \mathbb{N}$, the generalized moments of the family $(P_n)_n$ with respect to the basis $\theta_n(x)$ can be computed by the formula

$$\mu_n(\theta_k(x)) = I_0(n)P_0\mu_0. \tag{25}$$

Proof. Using the expansion (24), we have

$$\mu_n(\theta_k(x)) = \frac{1}{P_0}(\theta_n(x), P_0) = \frac{1}{P_0}\sum_{k=0}^{n} I_k(n)(P_n, P_0) = \frac{1}{P_0} I_0(n)(P_0, P_0) = I_0(n)P_0\mu_0,$$

where (f, g) is the inner product defined by

$$(f, g) = \int_{-\infty}^{\infty} f(x)g(x)d\alpha(x).$$

□

Note that this result was announced in [20].

Corollary 29. *Using the notations of Theorems 27 and 28, the canonical moments of the orthogonal family $\{p_n\}$ can be computed for all $n \in \mathbb{N}$ by the formula*

$$\mu_n = \mu_0 p_0 \sum_{m=0}^{n} C_m(n) I_0(m). \tag{26}$$

4.2. Continuous orthogonal polynomials

Note that by $P_n^{(\alpha,\beta)}(x),\ C_n^{(\lambda)}(x),\ T_n(x),\ U_n(x),\ P_n(x),\ L_n^{(\alpha)}(x),\ H_n(x),\ B_n^{(\alpha)}(x)$, we denote, respectively, the Jacobi, Gegenbauer (ultraspherical), Chebyshev of first kind, Chebyshev of second kind, Legendre, Laguerre, Hermite and Bessel polynomials. They have the following hypergeometric representations (see [26])

$$\begin{aligned}
P_n^{(\alpha,\beta)}(x) &= \frac{(\alpha+1)_n}{n!}\,{}_2F_1\left(\begin{array}{c} -n, n+\alpha+\beta+1 \\ \alpha+1 \end{array}\middle|\,\frac{1-x}{2}\right),\ \alpha>-1,\ \beta>-1 \quad (J1)\\
&= (-1)^n\frac{(\beta+1)_n}{n!}\,{}_2F_1\left(\begin{array}{c} -n, n+\alpha+\beta+1 \\ \beta+1 \end{array}\middle|\,\frac{1+x}{2}\right), \quad (J2)\\
C_n^{(\lambda)}(x) &= \frac{(2\lambda)_n}{\left(\lambda+\frac{1}{2}\right)_n}P_n^{\left(\lambda-\frac{1}{2},\lambda-\frac{1}{2}\right)}(x)\\
&= \frac{(2\lambda)_n}{n!}\,{}_2F_1\left(\begin{array}{c} -n, n+2\lambda \\ \lambda+\frac{1}{2} \end{array}\middle|\,\frac{1-x}{2}\right), \quad \lambda\neq 0.
\end{aligned}$$

$$\begin{aligned}
T_n(x) &= \frac{P_n^{\left(-\frac{1}{2},-\frac{1}{2}\right)}(x)}{P_n^{\left(-\frac{1}{2},-\frac{1}{2}\right)}(1)} = {}_2F_1\left(\begin{array}{c} -n, n \\ \frac{1}{2} \end{array}\middle|\,\frac{1-x}{2}\right),\\
U_n(x) &= (n+1)\frac{P_n^{\left(\frac{1}{2},\frac{1}{2}\right)}(x)}{P_n^{\left(\frac{1}{2},\frac{1}{2}\right)}(1)} = (n+1){}_2F_1\left(\begin{array}{c} -n, n+2 \\ \frac{3}{2} \end{array}\middle|\,\frac{1-x}{2}\right),\\
P_n(x) &= {}_2F_1\left(\begin{array}{c} -n, n+1 \\ 1 \end{array}\middle|\,\frac{1-x}{2}\right)\\
L_n^{(\alpha)}(x) &= \frac{(\alpha+1)_n}{n!}\,{}_1F_1\left(\begin{array}{c} -n \\ \alpha+1 \end{array}\middle|\,x\right),\ \alpha>-1,\\
H_n(x) &= (2x)^n{}_2F_0\left(\begin{array}{c} -\frac{n}{2}, -\frac{n-1}{2} \\ - \end{array}\middle|\,-\frac{1}{x^2}\right),\\
B_n^{(\alpha)}(x) &= {}_2F_0\left(\begin{array}{c} -n, n+\alpha+1 \\ - \end{array}\middle|\,-\frac{x}{2}\right),\ n=0,1,\ldots,N,\ \alpha<-2N-1.
\end{aligned}$$

In the classical continuous case the computation of the moments is rather straightforward. For example, for the Laguerre polynomials, by their definition the moments are given as values of the Gamma function. Nevertheless, since the inversion formulas can be used in principle for their computation, for the sake of completeness we state here the inversion coefficients for the classical continuous orthogonal polynomials which are given in the literature (see [6], [17], [28], [32], [33]).

Theorem 30. *The following inversion formulas are valid*

$$
\begin{aligned}
(1-x)^n &= \sum_{m=0}^{n}\binom{n}{m}\frac{(-1)^m 2^n m!(\alpha+m+1)_{n-m}}{(\alpha+\beta+m+1)_m(\alpha+\beta+2m+2)_{n-m}}P_m^{(\alpha,\beta)}(x), && (27)\\
(1+x)^n &= \sum_{m=0}^{n}\binom{n}{m}\frac{2^n m!(\beta+m+1)_{n-m}}{(\alpha+\beta+m+1)_m(\alpha+\beta+2m+2)_{n-m}}P_m^{(\alpha,\beta)}(x), && (28)\\
x^n &= (1+\alpha)_n\sum_{m=0}^{n}\frac{(-n)_m}{(1+\alpha)_m}L_m^{(\alpha)}(x), && (29)\\
x^n &= \frac{n!}{2^n}\sum_{k=0}^{\lfloor n/2\rfloor}\frac{1}{k!(n-2k)!}H_{n-2k}(x), && (30)\\
x^n &= \sum_{m=0}^{n}\binom{n}{m}\frac{(-1)^m(-2)^n}{(\alpha+m+1)_m(\alpha+2m+2)_{n-m}}B_m^{(\alpha)}(x). && (31)
\end{aligned}
$$

Remark 31. *Representations* (29) *and* (30) *were already known (see for example [28, 40]).*

Next, we provide several representations for the Jacobi polynomial moments (compare [13]).

Theorem 32. *The canonical moments of the Jacobi polynomials have the following representations:*

$$
\begin{aligned}
\mu_n &= \frac{\Gamma(\alpha+1)n!}{\Gamma(\alpha+n+2)}\,{}_2F_1\left(\begin{array}{c}-\beta, n+1\\ \alpha+n+2\end{array}\middle|\,-1\right)+(-1)^n\frac{\Gamma(\beta+1)n!}{\Gamma(\beta+n+2)}\,{}_2F_1\left(\begin{array}{c}-\alpha, n+1\\ \beta+n+2\end{array}\middle|\,-1\right)\\
&= 2^{\alpha+\beta+1}\frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{\Gamma(\alpha+\beta+2)}\,{}_2F_1\left(\begin{array}{c}-n, \alpha+1\\ \alpha+\beta+2\end{array}\middle|\,2\right) && (32)\\
&= (-1)^n 2^{\alpha+\beta+1}\frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{\Gamma(\alpha+\beta+2)}\,{}_2F_1\left(\begin{array}{c}-n, \beta+1\\ \alpha+\beta+2\end{array}\middle|\,2\right) && (33)
\end{aligned}
$$

Proof. Let us prove the first representation. We rewrite

$$\mu_n=\int_0^1 x^n(1-x)^\alpha(1+x)^\beta dx+(-1)^n\int_0^1 x^n(1+x)^\alpha(1-x)^\beta dx.$$

Next, the use of the integral representation for the Gauss hypergeometric function [26, P. 8]

$${}_2F_1\left(\begin{array}{c}a, b\\ c\end{array}\middle|\,z\right)=\frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\int_0^1 x^{b-1}(1-x)^{c-b-1}(1-zx)^{-a}dx$$

with $z=-1$ gives the desired result. In fact, for the first integral $\int_0^1 x^n(1-x)^\alpha(1+x)^\beta dx$, using the integral representation of the Gauss hypergeometric function with $b=n+1$, $c=\alpha+n+2$ and $a=-\beta$, it follows that

$$\int_0^1 x^n(1-x)^\alpha(1+x)^\beta dx=\frac{\Gamma(\alpha+1)\Gamma(n+1)}{\Gamma(\alpha+n+2)}\,{}_2F_1\left(\begin{array}{c}-\beta, n+1\\ \alpha+\beta+2\end{array}\middle|\,-1\right).$$

The second integral is computed in the same manner.

Next, we develop the second representation. For $\alpha>-1$ and $\beta>-1$, the Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$ are orthogonal in the interval $(-1;1)$ and fulfil the orthogonality relation [26, P. 217]

$$\int_{-1}^1(1-x)^\alpha(1+x)^\beta P_n^{(\alpha,\beta)}(x)P_m^{(\alpha,\beta)}(x)dx=\frac{2^{\alpha+\beta+1}}{2n+\alpha+\beta+1}\frac{\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{\Gamma(n+\alpha+\beta+1)n!}\delta_{mn}.$$

It follows that

$$\mu_0 = \int_{-1}^{1} (1-x)^{\alpha}(1+x)^{\alpha} dx = 2^{\alpha+\beta+1}\frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{\Gamma(\alpha+\beta+2)}.$$

We first prove relation (32). From the inversion formula (27) the zeroth inversion coefficient is

$$I_0(n) = 2^n \frac{\Gamma(\alpha+1+n)\Gamma(\alpha+\beta+2)}{\Gamma(\alpha+\beta+n+2)\Gamma(\alpha+1)} = 2^n \frac{(\alpha+1)_n}{(\alpha+\beta+2)_n}.$$

Hence, the generalized Jacobi moments with respect to the basis $(1-x)^n$ have the representation

$$\mu_n((1-x)^k) = 2^{n+\alpha+\beta+1}\frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{\Gamma(\alpha+\beta+2)}\frac{(\alpha+1)_n}{(\alpha+\beta+2)_n}.$$

Finally, using the binomial formula in the form

$$x^n = \sum_{m=0}^{n} (-1)^m \binom{n}{m} (1-x)^m$$

and Theorem 28, (32) follows.

In order to prove (32), we follow the same method using relation (28), and the binomial formula

$$x^n = \sum_{m=0}^{n} (-1)^{n-m} \binom{n}{m} (1+x)^m.$$

□

Using the definition and the Beta function (see e.g. [27]), one gets

Theorem 33. *The following representations for the canonical moments are valid for:*

(a) the Gegenbauer polynomials

$$\mu_n = \begin{cases} \sqrt{\pi}\frac{\Gamma\left(\lambda+\frac{1}{2}\right)}{\Gamma(\lambda+1)}\frac{(2p)!}{2^{2p}p!(\lambda+1)_p}, & \textit{if} \quad n=2p. \\ 0 & \textit{if} \quad n=2p+1; \end{cases} \tag{34}$$

(b) the Chebyshev polynomials of first kind

$$\mu_n = \begin{cases} \frac{\pi(2p)!}{2^{2p}p!^2} & \textit{if } n=2p, \\ 0 & \textit{if } n=2p+1. \end{cases} \tag{35}$$

(c) the Chebyshev polynomials of second kind

$$\mu_n = \begin{cases} \frac{\pi(2p)!}{2^{2p}p!(p+1)!} & \textit{if} \quad n=2p, \\ 0 & \textit{if} \quad n=2p+1. \end{cases} \tag{36}$$

(d) the Legendre polynomials

$$\mu_n = \begin{cases} \frac{2}{2p+1} & \textit{if} \quad n=2p \\ 0 & \textit{if} \quad n=2p+1. \end{cases} \tag{37}$$

Using the definition and the Gamma function (see e.g. [27]), one gets

Theorem 34. *The following representations for the canonical moments are valid for:*

(a) the Laguerre polynomials

$$\mu_n = \Gamma\left(n+\alpha+1\right), \quad n=0,1,2,\ldots \tag{38}$$

(b) the Hermite polynomials

$$\mu_n = \frac{1+(-1)^n}{2}\Gamma\left(\frac{n+1}{2}\right) = \begin{cases} \sqrt{\pi}\frac{(2p)!}{2^{2p}p!} & \textit{if} \quad n=2p \\ 0 & \textit{if} \quad n=2p+1 \end{cases}, \quad n=0,1,2,\ldots \tag{39}$$

4.3. Discrete orthogonal polynomials

We denote by $Q_n(x;\alpha,\beta,N)$, $M_n(x;\beta,c)$, $K_n(x;p,N)$ and $C_n(x;a)$ the Hahn, Meixner, Krawtchouk and Charlier polynomials, respectively. They have the following hypergeometric representations (see [26])

$$\begin{aligned}
Q_n(x;\alpha,\beta,N) &= {}_3F_2\left(\begin{array}{c} -n,-x,n+1+\alpha+\beta \\ \alpha+1,-N \end{array}\middle| 1\right), \\
&\quad n,x=0,1,\ldots,N,\ \alpha>-1 \text{ and } \beta>-1,\ \text{or } \alpha<-N \text{ and } \beta<-N, \\
M_n(x;\beta,c) &= {}_2F_1\left(\begin{array}{c} -n,-x \\ \beta \end{array}\middle| 1-\frac{1}{c}\right),\ \beta>0,\ 0<c<1,\ x=0,1,\ldots, \\
K_n(x;p,N) &= {}_2F_1\left(\begin{array}{c} -n,-x \\ -N \end{array}\middle| \frac{1}{p}\right),\ 0<p<1,\ n,x=0,1,\ldots,N, \\
C_n(x;a) &= {}_2F_0\left(\begin{array}{c} -n,-x \\ - \end{array}\middle| -\frac{1}{a}\right),\ a>0,\ x=0,1,\ldots.
\end{aligned}$$

In order to obtain the canonical moments for these polynomials, we need the following theorem, which can be found in in [28] and [40, Table 18] (where the polynomial systems were standardized differently).

Theorem 35. *The Hahn, the Krawtchouk, the Meixner and the Charlier polynomials fulfil the following inversion formulas*

$$x^{\underline{n}} = \sum_{m=0}^{n}\binom{n}{m}\frac{(-1)^{n-m}(\alpha+1)_n(-N)_n}{(\alpha+\beta+m+1)_m(\alpha+\beta+2m+2)_{n-m}}Q_m(x;\alpha,\beta,N), \tag{40}$$

$$\begin{aligned}
x^{\underline{n}} &= (-p)^n(-N)_n\sum_{m=0}^{n}(-1)^m\binom{n}{m}K_m(x;p,N), \\
x^{\underline{n}} &= (-1)^n(\beta)_n\left(\frac{c}{c-1}\right)^n\sum_{m=0}^{n}(-1)^m\binom{n}{m}M_m(x;\beta,c), \\
x^{\underline{n}} &= \sum_{m=0}^{n}(-1)^m\binom{n}{m}a^nC_k(x;a).
\end{aligned}$$

Theorem 36. *The following representations for the canonical moments are valid for:*

(a) the Hahn polynomials

$$\mu_n = \frac{(\alpha+\beta+1)_{N+1}}{(\alpha+\beta+1)N!}\sum_{m=0}^{n}(-1)^m\mathcal{S}_m(n)\frac{(\alpha+1)_m(-N)_m}{(\alpha+\beta+2)_m}; \tag{41}$$

(b) the Krawtchouk polynomials

$$\mu_n = \sum_{m=0}^{n}\mathcal{S}_m(n)(-N)_m(-p)^m; \tag{42}$$

(c) the Meixner polynomials

$$\mu_n = \frac{1}{(1-c)^\beta}\sum_{m=0}^{n}(-1)^m\mathcal{S}_m(n)(\beta)_m\left(\frac{c}{c-1}\right)^m; \tag{43}$$

(d) the Charlier polynomials

$$\mu_n = e^a\sum_{m=0}^{n}\mathcal{S}_m(n)a^m, \tag{44}$$

where $\mathcal{S}_m(n)$ denote the Stirling numbers of second kind defined by

$$x^n = \sum_{m=0}^{n}\mathcal{S}_m(n)x^{\underline{n}}. \tag{45}$$

Note that (44) appeared in [19] and [30] without the constant $\mu_0 = e^a$. The results of Theorem 36 also appeared in [3] and [24]. Note also that in [3] and [24], the authors used different techniques and different standardizations as compare to the current manuscript.

Proof. We give the proof for the moments of the Hahn polynomials, the other moments are obtained similarly.

The Hahn polynomials $Q_n(x;\alpha,\beta,N)$ fulfil the following orthogonality relation [26, P. 204]

$$\sum_{x=0}^{N}\binom{\alpha+x}{x}\binom{\beta+N-x}{N-x}Q_n(x;\alpha,\beta,N)Q_m(x;\alpha,\beta,N)$$
$$=\frac{(-1)^n(n+\alpha+\beta+1)_{N+1}(\beta+1)_n n!}{(2n+\alpha+\beta+1)(\alpha+1)_n(-N)_n N!}\delta_{mn}, \tag{46}$$
$$\text{for } \alpha>-1 \text{ and } \beta>-1 \text{ or } \alpha<-N \text{ and } \beta<-N.$$

With $m = n = 0$, it follows that

$$\mu_0 = \frac{(\alpha+\beta+1)_{N+1}}{(\alpha+\beta+1)N!}.$$

From the inversion formula (40), for $\theta_n(x) = x^{\underline{n}}$ in 24, we get

$$I_0(n) = (-1)^n\frac{(\alpha+1)_n(-N)_n}{(\alpha+\beta+2)_n}.$$

Therefore, the generalized Hahn moments with respect to the basis $x^{\underline{n}}$ have the representation

$$\mu_n\left(x^{\underline{k}}\right) = (-1)^n\frac{(\alpha+\beta+1)_{N+1}}{(\alpha+\beta+1)N!}\frac{(\alpha+1)_n(-N)_n}{(\alpha+\beta+2)_n}. \tag{47}$$

Using the connection (45) between the powers and the falling factorials and Corollary 29, we obtain (41). □

Whereas the canonical moments of the Krawtchouk, Meixner and Charlier polynomials are expressed in terms of the complicated Stirling numbers, they have rather simple generating functions.

Theorem 37. 1. *The canonical Krawtchouk moments have the following exponential generating function*

$$(pe^z+1-p)^N = \sum_{n=0}^{\infty}\mu_n\frac{z^n}{n!}. \tag{48}$$

2. *The canonical Meixner moments have the following exponential generating function*

$$\frac{1}{(1-ce^z)^\beta} = \sum_{n=0}^{\infty}\mu_n\frac{z^n}{n!}, \quad |ce^z|<1. \tag{49}$$

3. *The canonical Charlier moments have the following exponential generating function*

$$e^{ae^z} = \sum_{n=0}^{\infty}\mu_n\frac{z^n}{n!}. \tag{50}$$

Proof. By definition, the canonical Krawtchouk moments are given by

$$\mu_n = \sum_{k=0}^{N}k^n\binom{N}{k}p^k(1-p)^{N-k}.$$

Therefore, it follows that

$$\begin{aligned}
\sum_{n=0}^{\infty}\mu_n\frac{z^n}{n!} &= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{N}k^n\binom{N}{k}p^k(1-p)^{N-k}\right)\frac{z^n}{n!} \\
&= \sum_{k=0}^{N}\left(\binom{N}{k}p^k(1-p)^{N-k}\sum_{n=0}^{\infty}\frac{(kz)^n}{n!}\right) \\
&= \sum_{k=0}^{N}\binom{N}{k}(pe^z)^k(1-p)^{N-k} \\
&= (pe^z+1-p)^N.
\end{aligned}$$

Hence, (48) is proved.

By definition, the canonical Meixner moments are given by

$$\mu_n=\sum_{k=0}^{\infty}\frac{(\beta)_k c^k}{k!}k^n$$

It therefore follows that

$$\begin{aligned}
\sum_{n=0}^{\infty}\frac{\mu_n}{n!}z^n &= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{\infty}\frac{(\beta)_k c^k}{k!}k^n\right)\frac{z^n}{n!} \\
&= \sum_{k=0}^{\infty}\left(\frac{(\beta)_k c^k}{k!}\sum_{n=0}^{\infty}\frac{(kz)^n}{n!}\right) \\
&= \sum_{k=0}^{\infty}\frac{(\beta)_k}{k!}(ce^z)^k=\frac{1}{(1-ce^z)^\beta}.
\end{aligned}$$

This proves (49).

By definition, the canonical Charlier moments are given by

$$\mu_n=\sum_{k=0}^{\infty}\frac{a^k}{k!}k^n.$$

Therefore, we have:

$$\begin{aligned}
\sum_{n=0}^{\infty}\frac{\mu_n}{n!}z^n &= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{\infty}\frac{a^k}{k!}k^n\right)\frac{z^n}{n!} \\
&= \sum_{k=0}^{\infty}\left(\frac{a^k}{k!}\sum_{n=0}^{\infty}\frac{(kz)^n}{n!}\right) \\
&= \sum_{k=0}^{\infty}\frac{a^k e^{kz}}{k!}=\sum_{k=0}^{\infty}\frac{(ae^z)^k}{k!} \\
&= e^{ae^z}.
\end{aligned}$$

□

4.4. q-Orthogonal polynomials

These polynomials have the following q-hypergeometric representations (see e.g. [26]):

(a) The Big q-Jacobi polynomials

$$p_n(x;a,b,c;q)={}_3\phi_2\left(\begin{array}{c} q^{-n},abq^{n+1},x \\ aq,cq \end{array}\middle|\, q;q\right)$$

(b) The q-Hahn polynomials

$$Q_n(q^{-x};\alpha,\beta,N;q) = {}_3\phi_2\left(\begin{array}{c} q^{-n},\alpha\beta q^{n+1},q^{-x} \\ \alpha q, q^{-N} \end{array}\middle| q;q\right)$$

(c) The Big q-Laguerre polynomials

$$P_n(x,a,b;q) = {}_3\phi_2\left(\begin{array}{c} q^{-n},0,x \\ aq, bq \end{array}\middle| q;q\right) = \frac{1}{(b^{-1}q^{-n};q)_n}\,{}_2\phi_1\left(\begin{array}{c} q^{-n},aqx^{-1} \\ aq \end{array}\middle| q;\frac{x}{b}\right).$$

(d) The Little q-Jacobi polynomials

$$p_n(x;a,b|q) = {}_2\phi_1\left(\begin{array}{c} q^{-n},abq^{n+1} \\ aq \end{array}\middle| q;qx\right).$$

(e) The q-Meixner polynomials

$$M_n(q^{-x};b,c;q) = {}_2\phi_1\left(\begin{array}{c} q^{-n},q^{-x} \\ bq \end{array}\middle| q;-\frac{q^{n+1}}{c}\right).$$

(f) The Quantum q-Krawtchouk polynomials

$$K_n^{qtm}(q^{-x};p,N;q) = {}_2\phi_1\left(\begin{array}{c} q^{-n},q^{-x} \\ q^{-N} \end{array}\middle| q;pq^{n+1}\right).$$

(h) The q-Krawtchouk polynomials

$$K_n(q^{-x};p,N;q) \;=\; {}_3\phi_2\left(\begin{array}{c} q^{-n},q^{-x},-pq^{n} \\ q^{-N},0 \end{array}\middle| q;q\right),\quad n=0,1,2,\ldots,N.$$

(g) The Affine q-Krawtchouk polynomials

$$K_n^{\mathit{Aff}}(q^{-x};p,N;q) \;=\; {}_3\phi_2\left(\begin{array}{c} q^{-n},0,q^{-x} \\ pq, q^{-N} \end{array}\middle| q;q\right)\quad n=0,1,2,\ldots,N.$$

(i) The Little q-Laguerre polynomials

$$p_n(x,a|q) = {}_2\phi_1\left(\begin{array}{c} q^{-n},0 \\ aq \end{array}\middle| q;qx\right) = \frac{1}{(a^{-1}q^{-n};q)_n}\,{}_2\phi_0\left(\begin{array}{c} q^{-n},x^{-1} \\ 0 \end{array}\middle| q;\frac{x}{a}\right).$$

(j) The q-Laguerre polynomials

$$L_n^{(\alpha)}(x) = \frac{(q^{\alpha+1};q)_n}{(q;q)_n}\,{}_1\phi_1\left(\begin{array}{c} q^{-n} \\ q^{\alpha+1} \end{array}\middle| q;-q^{n+\alpha+1}x\right).$$

(k) The Alternative q-Charlier (also called q-Bessel) polynomials

$$K_n(x;a;q) \;=\; {}_2\phi_1\left(\begin{array}{c} q^{-n},-aq^{-n} \\ 0 \end{array}\middle| q;qx\right).$$

(l) The q-Charlier polynomials

$$C_n(q^{-x};a;q) = {}_2\phi_1\left(\begin{array}{c} q^{-n}, q^{-x} \\ 0 \end{array} \middle| q; -\frac{q^{n+1}}{a}\right).$$

(m) The Al Salam-Carlitz I polynomials

$$U_n^{(a)}(x;q) \quad = \quad (-a)^n q^{\binom{n}{2}} {}_2\phi_1\left(\begin{array}{c} q^{-n}, x^{-1} \\ 0 \end{array} \middle| q; \frac{qx}{a}\right).$$

(n) The Al Salam-Carlitz II polynomials

$$V_n^{(a)}(x;q) \quad = \quad (-a)^n q^{-\binom{n}{2}} {}_2\phi_0\left(\begin{array}{c} q^{-n}, x \\ 0 \end{array} \middle| q; \frac{q^n}{a}\right).$$

(o) The Stieltjes-Wigert polynomials

$$S_n(x;q) \quad = \quad \frac{1}{(q;q)_n} {}_1\phi_1\left(\begin{array}{c} q^{-n} \\ 0 \end{array} \middle| q; -q^{n+1}x\right).$$

(p) The Discrete q-Hermite I polynomials

$$h_n(x;q) = q^{\binom{n}{2}} \; {}_2\phi_1\left(\begin{array}{c} q^{-n}; \;\; x^{-1} \\ 0 \end{array} \middle| q; -qx\right).$$

(q) The Discrete q-Hermite II polynomials

$$\tilde{h}_n(x;q) = i^{-n} q^{-\binom{n}{2}} \; {}_2\phi_0\left(\begin{array}{c} q^{-n}; \;\; ix \\ - \end{array} \middle| q; -q^n\right).$$

These polynomials fulfill the following inversion formulas (see [4], [16], [31], [35], [3, Table 19])

Theorem 38. *The Big q-Jacobi, the q-Hahn, the Big q-Laguerre, the q-Meixner, the Quantum q-Krawtchouk, the q-Krawtchouk, the Affine q-Krawtchouk, the q-Charlier, the Al Salam-Carlitz II and* *the Discrete q-Hermite II polynomials fulfil the following inversion formulas, respectively*

$$
\begin{aligned}
(x;q)_n &= \sum_{m=0}^{n} (-1)^m \begin{bmatrix} n \\ m \end{bmatrix}_q \frac{q^{\binom{m}{2}}(aq,cq;q)_n}{(abq^{m+1};q)_m(abq^{2m+2};q)_{n-m}} P_m(x;a,b,c;q), && (51)\\
(q^{-x};q)_n &= \sum_{m=0}^{n} \begin{bmatrix} n \\ m \end{bmatrix}_q \frac{(-1)^m q^{\binom{m}{2}}(\alpha q,q^{-N};q)_n}{(\alpha\beta q^{m+1};q)_m(\alpha\beta q^{2m+2};q)_{n-m}} Q_m(q^{-x};\alpha,\beta,N|q), \\
(x;q)_n &= \sum_{m=0}^{n} (-1)^m \begin{bmatrix} n \\ m \end{bmatrix}_q q^{\binom{m}{2}}(aq,bq;q)_n P_m(x;a,b;q), \\
(q^{-x};q)_n &= \sum_{m=0}^{n} (-1)^{n-m} q^{\frac{m(5m+1)}{2}-n(m+1)} c^n \begin{bmatrix} n \\ m \end{bmatrix}_q (bq;q)_n M_m(q^{-x};b,c;q), \\
(q^{-x};q)_n &= \sum_{m=0}^{n} (-1)^m \begin{bmatrix} n \\ m \end{bmatrix}_q \frac{q^{\binom{m}{2}}}{p^{m(n+1)}}(q^{-N};q)_n (pq)^{-n} K_m^{qtm}(q^{-x};p,N|q), \\
(q^{-x};q)_n &= \sum_{m=0}^{n} \begin{bmatrix} n \\ m \end{bmatrix}_q \frac{(-1)^m q^{\binom{m}{2}}(q^{-N};q)_n}{(-pq^m;q)_m(-pq^{2m+1};q)_{n-m}} K_m(q^{-x};p,N;q), \\
(q^{-x};q)_n &= \sum_{m=0}^{n} (-1)^m \begin{bmatrix} n \\ m \end{bmatrix}_q q^{\binom{m}{2}}(pq,q^{-N};q)_n K_m^{\mathrm{Aff}}(q^{-x};p,N;q), \\
(q^{-x};q)_n &= \sum_{m=0}^{n} (-1)^{n-m} a^n \begin{bmatrix} n \\ m \end{bmatrix}_q q^{\frac{m(m+1)}{2}-n(m+1)} C_m(q^{-x};a;q), && (52)\\
(x;q)_n &= \sum_{m=0}^{n} (-1)^n \begin{bmatrix} n \\ m \end{bmatrix}_q a^{n-m} q^{m(m-n)+\binom{n}{2}} V_m^{(a)}(x;q), \\
(x;q)_n &= \sum_{m=0}^{n} (-1)^m \begin{bmatrix} n \\ m \end{bmatrix}_q q^{m(m-n)+\binom{n}{2}} \tilde{h}_m(x;q).
\end{aligned}
$$

Theorem 39. *The following representations for the canonical moments are valid:*
The Big q-Jacobi polynomials

$$\mu_n = aq\frac{(abq^2,a^{-1}c,ac^{-1}q;q)_\infty}{(aq,bq,cq,abc^{-1}q;q)_\infty}\sum_{m=0}^{n}(-1)^m \begin{bmatrix} n \\ m \end{bmatrix}_q q^{-nm+\binom{m+1}{2}}\frac{(aq,cq;q)_m}{(abq^2;q)_m}. \tag{53}$$

The q-Hahn polynomials

$$\mu_n = \frac{(\alpha\beta q^2;q)_N}{(\beta q;q)_N(\alpha q)^N}\sum_{m=0}^{n}(-1)^m \begin{bmatrix} n \\ m \end{bmatrix}_q q^{-mn+\binom{m+1}{2}}\frac{(\alpha q,q^{-N};q)_m}{(\alpha\beta q^2;q)_m}. \tag{54}$$

The Big q-Laguerre polynomials

$$\mu_n = aq\frac{(q,a^{-1}b,ab^{-1}q;q)_\infty}{(aq,bq;q)_\infty}\sum_{m=0}^{n}(-1)^m \begin{bmatrix} n \\ m \end{bmatrix}_q q^{-nm+\binom{m+1}{2}}(aq,bq;q)_m. \tag{55}$$

The q-Meixner polynomials

$$\mu_n = \frac{(-c;q)_\infty}{(-bcq;q)_\infty}\sum_{m=0}^{n} \begin{bmatrix} n \\ m \end{bmatrix}_q q^{-nm+\binom{m}{2}}c^m(bq;q)_m. \tag{56}$$

The Quantum q-Krawtchouk polynomials

$$\mu_n = \frac{p^N (q;q)_N}{(q,q;q)_N} q^{\binom{N+1}{2}} \sum_{m=0}^{n} (-1)^m \begin{bmatrix} n \\ m \end{bmatrix}_q q^{-nm+\binom{m+1}{2}} \frac{(q^{-N};q)_m}{(pq)^m} \tag{57}$$

The q-Krawtchouk polynomials

$$\mu_n = \frac{(-pq;q)_N}{p^N q^{\binom{N+1}{2}}} \frac{(-pq^{N+1};q)_n}{(-pq;q)_n} \frac{1}{q^{nN}}, \quad n = 0, 1, 2, \ldots, N. \tag{58}$$

The Affine q-Krawtchouk polynomials

$$\mu_n = (pq)^{-N} \sum_{m=0}^{n} (-1)^m \begin{bmatrix} n \\ m \end{bmatrix}_q q^{-nm+\binom{m+1}{2}} (pq, q^{-N};q)_m. \tag{59}$$

The q-Charlier polynomials

$$\mu_n = (-a;q)_\infty \left(-a^{-1}q;q\right)_n \left(\frac{a}{q}\right)^n q^{-\binom{n}{2}} \quad (\textit{compare} \;\; [10, P.\ 50]). \tag{60}$$

The Al-Salam-Carlitz II polynomials

$$\mu_n = \frac{1}{(aq;q)_\infty} \sum_{m=0}^{n} \begin{bmatrix} n \\ m \end{bmatrix}_q a^m q^{m(m-n)} \quad (\textit{see}\ [9, Eq.(10.10), P.197]). \tag{61}$$

The Discrete q-Hermite II polynomials

$$\mu_n = \frac{(q^2, -q, -q; q^2)_\infty}{(q^3, -q^2, -q^2; q^2)_\infty} \sum_{m=0}^{n} (-1)^m \begin{bmatrix} n \\ m \end{bmatrix}_q q^{m(m-n)}. \tag{62}$$

Proof. We prove the result for the Big q-Jacobi polynomials, the other results are proved similarly. The Big q-Jacobi polynomials $P_n(x;a,b,c;q)$ fulfil the following orthogonality relation [26, P. 438]

$$\begin{aligned} &\int_{cq}^{aq} \frac{(a^{-1}x, c^{-1}x;q)_\infty}{(x, bc^{-1}x;q)_\infty} P_m(x;a,b,c;q) P_n(x;a,b,c;q) d_q x \\ &\quad = aq(1-q) \frac{(abq^2, a^{-1}c, ac^{-1}q;q)_\infty}{(aq, bq, cq, abc^{-1}q;q)_\infty} \\ &\qquad \times \frac{1-abq}{1-abq^{2n+1}} \frac{(q, bq, abc^{-1}q;q)_n}{(aq, abq, cq;q)_n} (-caq^2)^n q^{\binom{n}{2}} \delta_{mn}. \end{aligned}$$

By taking $m = n = 0$ in the orthogonality relation it follows that

$$\mu_0 = aq(1-q) \frac{(abq^2, a^{-1}cac^{-1}q;q)_\infty}{(aq, bq, cq, abc^{-1}q;q)_\infty}.$$

From the inversion formula (51), we get the zeroth inversion coefficient

$$I_0(n) = \frac{(aq, cq;q)_n}{(abq^2;q)_n}.$$

Hence, the Big q-Jacobi generalized moments with respect to $(x;q)_n$ have the representation

$$\mu_n((x;q)_k) = aq(1-q) \frac{(abq^2, a^{-1}c, ac^{-1}q;q)_\infty}{(aq, bq, cq, abc^{-1}q;q)_\infty} \frac{(aq, cq;q)_n}{(abq^2;q)_n}.$$

Finally, using the connection formula (see [1], [4], [31])

$$x^n = \sum_{m=0}^{n} (-1)^m \begin{bmatrix} n \\ m \end{bmatrix}_q q^{-mn+\binom{m+1}{2}} (x;q)_m,$$

and Corollary 29, we obtain (53). □

Theorem 40 (see [3, 4, 16]). *The Little q-Jacobi, the Little q-Legendre, the Little q-Laguerre, the q-Laguerre, the Alternative q-Charlier/q-Bessel and the Stieltjes-Wigert polynomials fulfil the following inversion formulas, respectively*

$$\begin{aligned}
x^n &= \sum_{m=0}^{n} \begin{bmatrix} n \\ m \end{bmatrix}_q \frac{(-1)^m q^{\binom{m}{2}} (aq;q)_n}{(abq^{m+1};q)_m (abq^{2m+2};q)_{n-m}} p_m(x;a,b|q), \\
x^n &= \sum_{m=0}^{n} (-1)^m \begin{bmatrix} n \\ m \end{bmatrix}_q \frac{(-1)^m q^{\binom{m}{2}} (q;q)_n}{(q^{m+1};q)_m (q^{2m+2};q)_{n-m}} P_m(x|q), \\
x^n &= \sum_{m=0}^{n} (-1)^m \begin{bmatrix} n \\ m \end{bmatrix}_q q^{\binom{m}{2}} (aq;q)_n p_m(x;a|q), \\
x^n &= \sum_{m=0}^{n} (-1)^m \begin{bmatrix} n \\ m \end{bmatrix}_q q^{\frac{(m-n)(2\alpha+3m+n+1)}{2} - m(m+\alpha)} (q;q)_m (q^{m+\alpha+1};q)_{n-m} L_m^{(\alpha)}(x;q), \\
x^n &= \sum_{m=0}^{n} \begin{bmatrix} n \\ m \end{bmatrix}_q \frac{(-1)^m q^{\binom{m}{2}}}{(-aq^m;q)_m (-aq^{2m+1};q)_{n-m}} y_m(x;a|q), \\
x^n &= \sum_{m=0}^{n} (-1)^m \begin{bmatrix} n \\ m \end{bmatrix}_q q^{\frac{(m-n)(3m+n+1)}{2} - m^2} (q;q)_m S_m(x;q).
\end{aligned}$$

Theorem 41. *The following representations for the canonical moments are valid.*

(a) The Little q-Jacobi polynomials

$$\mu_n = \frac{(abq^{n+2};q)_\infty}{(aq^{n+1};q)_\infty} = \frac{(abq^2;q)_\infty}{(aq;q)_\infty} \frac{(aq;q)_n}{(abq^2;q)_n}. \tag{63}$$

(b) The Little q-Laguerre polynomials

$$\mu_n = \frac{(aq;q)_n}{(aq;q)_\infty}, \quad \textit{compare with [3, P. 91].} \tag{64}$$

(c) The q-Laguerre polynomials

$$\mu_n^{(d)} = \frac{(q, -cq^{\alpha+1}, -c^{-1}q^{-\alpha};q)_\infty}{(q^{\alpha+1}, -c, -c^{-1}q;q)_\infty} q^{-\binom{n}{2} - n(\alpha+1)} (q^{\alpha+1};q)_n, \tag{65}$$

for the discrete orthogonality, and

$$\mu_n^{(c)} = \frac{(q^{-\alpha};q)_\infty}{(q;q)_\infty} \Gamma(-\alpha)\Gamma(\alpha+1) q^{-\binom{n}{2} - n(\alpha+1)} (q^{\alpha+1};q)_n, \tag{66}$$

for the continuous orthogonality.

(d) The q-Bessel polynomials

$$\mu_n = \frac{(-aq;q)_\infty}{(-aq;q)_n}. \tag{67}$$

(e) The Stieltjes-Wigert polynomials

$$\mu_n = -\ln q (q;q)_\infty q^{-\binom{n+1}{2}}. \tag{68}$$

Note that these moments appeared in [3, P. 91] and [10, P. 223].

Proof. Since the polynomials involved in this theorem are represented in the power basis, their canonical moments are easy to compute, by just taking the zeroth inversion coefficients in the inversion formulas, multiplied by μ_0 which comes from the orthogonality relations. $\square$

Theorem 42 (see [31])**.** *The Al Salam-Carlitz I and the Discrete q-Hermite I fulfil the following inversion formulas, respectively*

$$(x \ominus 1)_q^n = \sum_{m=0}^{n} a^{n-m} \begin{bmatrix} n \\ m \end{bmatrix}_q U_m^{(a)}(x;q), \tag{69}$$

$$(x \ominus 1)_q^n = \sum_{m=0}^{n} (-1)^{n-m} \begin{bmatrix} n \\ m \end{bmatrix}_q h_m(x;q).$$

Theorem 43. *The canonical moments of the Al Salam-Carlitz I and the Discrete q-Hermite I polynomials have the following representations, respectively*

$$\mu_n = (1-q)(q,a,a^{-1}q;q)_\infty \sum_{i=0}^{n} \begin{bmatrix} n \\ i \end{bmatrix}_q a^i, \tag{70}$$

$$\mu_n = (1-q)(q,-1,-q;q)_\infty \frac{1+(-1)^n}{2}(q;q^2)_{n/2}, \quad \textit{compare with [3, P. 91].} \tag{71}$$

Proof. Since the Discrete q-Hermite I are the Al Salam-Carlitz I polynomials for $a = -1$, it is sufficient to prove the result for the Al Salam-Carlitz I case.

The Al-Salam-Carlitz I polynomials $U_n^{(a)}(x;q)$ fulfil the following orthogonality relation [26, P. 534]

$$\int_a^1 (qx, a^{-1}qx;q)_\infty U_m^{(a)}(x;q) U_n^{(a)}(x;q) d_q x$$
$$= (-a)^n (1-q)(q;q)_n (q,a,a^{-1}q;q)_\infty q^{\binom{n}{2}} \delta_{mn}, \quad a < 0.$$

With $m = n = 0$, it follows that

$$\mu_0 = (1-q)(q,a,a^{-1}q;q)_\infty.$$

From the inversion formulas (69) for $\theta_n(x) = (x \ominus 1)_q^n$, we have the zeroth inversion coefficient

$$I_0(n) = a^n.$$

Hence, the Al Salam-Carlitz I generalized moments with respect to the q-power basis are

$$\mu_n((x \ominus 1)_q^k) = (1-q)(q,a,a^{-1}q;q)_\infty a^n,$$

takng into account Eq. (25). Finally, using the connection formula (11) and Corollary 29, (70) follows. □

Remark 44. *From the q-hypergeometric representation of the Al-Salam Carlitz I polynomials given in item (l) page 16 we realized using the relation $x^n(x^{-1};q)_n = (x \ominus 1)_q^n$, that the basis $(x \ominus 1)_q^n$ is the appropriate and natural basis to be used. This remark can be emphasized by the fact that the inversion formula given above is in term of this basis.*

4.5. Orthogonal polynomials on quadratic lattices

Note that by $W_m(x^2;a,b,c,d)$, $S_m(x^2;a,b,c)$, $p_m(x;a,b,c,d)$, $P_m^{(\lambda)}(x;\phi)$, we denote, respectively, the Wilson, the Continuous Dual Hahn, the Continuous Hahn, and the Meixner-Pollaczek polynomials. Their hypergeometric representation (see [26]) are:

$$\begin{aligned}
\frac{W_n(x^2;a,b,c,d)}{(a+b)_n(a+c)_n(a+d)_n} &= {}_4F_3\left(\begin{array}{c} -n, n+a+b+c+d-1, a+ix, a-ix \\ a+b, a+c, a+d \end{array}\middle| 1\right) \\
\frac{S_n(x^2;a,b,c)}{(a+b)_n(a+c)_n} &= {}_3F_2\left(\begin{array}{c} -n, a-ix, a+ix \\ a+b, a+c \end{array}\middle| 1\right) \\
p_n(x;a,b,c,d) &= i^n\frac{(a+c)_n(a+d)_n}{n!}{}_3F_2\left(\begin{array}{c} -n, n+a+b+c+d-1, a+ix \\ a+c, a+d \end{array}\middle| 1\right) \\
P_n^{(\lambda)}(x;\phi) &= \frac{(2\lambda)_n}{n!}e^{in\phi}{}_2F_1\left(\begin{array}{c} -n, \lambda+ix \\ 2\lambda \end{array}\middle| 1-e^{-2i\phi}\right),
\end{aligned}$$

where

$$\lambda(x) = x(x+\gamma+\delta+1).$$

These polynomials fulfil the following inversion formulas (for details see [31]).

Theorem 45 (see [31])**.** *The following inversion formulas are valid*

$$\begin{aligned}
\theta_n(x) &= \sum_{m=0}^{n}\binom{n}{m}\frac{(-1)^m(a+b+m)_{n-m}(a+c+m)_{n-m}(a+d+m)_{n-m}}{(a+b+c+d+m-1)_m(a+b+c+d+2m)_{n-m}}W_m(x^2;a,b,c,d), \quad (72)\\
\theta_n(x) &= \sum_{m=0}^{n}(-1)^m\binom{n}{m}(a+b+m)_{n-m}(a+c+m)_{n-m}S_m(x^2;a,b,c),
\end{aligned}$$

where

$$\theta_n(x) = (a-ix)_n(a+ix)_n,$$

$$\begin{aligned}
(a+ix)_n &= \sum_{m=0}^{n}\binom{n}{m}\frac{(-i)^m m!(a+c+m)_{n-m}(a+d+m)_{n-m}}{(a+b+c+d-1+m)_m(a+b+c+d+2m)_{n-m}}p_m(x;a,b,c,d), \quad (73)\\
(\lambda+ix)_n &= \sum_{m=0}^{n}\binom{n}{m}\frac{(-1)^m m!(2\lambda+m)_{n-m}}{(1-e^{-2i\phi})^n e^{im\phi}}P_m^{(\lambda)}(x;\phi).
\end{aligned}$$

Theorem 46. *The following representations for the canonical moments are valid for:*

(a) the Wilson polynomials

$$\mu_n = \mu_0\sum_{k=0}^{n}\sum_{l=0}^{k}\frac{(-k)_l}{k!l!}\frac{(a+b)_k(a+c)_k(a+d)_k}{(a+b+c+d)_k}\frac{(-2a-2k+2l)}{(-2a-2k+l)_{k+1}}(a+k-l)^{2n}, \quad (74)$$

with

$$\mu_0 = 2\pi\frac{\Gamma(a+b)\Gamma(a+c)\Gamma(b+c)\Gamma(b+d)\Gamma(c+d)}{\Gamma(a+b+c+d)};$$

(b) the Continuous Dual Hahn polynomials

$$\mu_n = \mu_0\sum_{k=0}^{n}\sum_{l=0}^{k}\frac{(-k)_l}{k!l!}\frac{(-2a-2k+2l)(a+c)_k(a+d)_k}{(-2a-2k+l)_{k+1}}(a+k-l)^{2n}, \quad (75)$$

with

$$\mu_0 = \Gamma(a+b)\Gamma(a+c)\Gamma(b+c);$$

(c) the Continuous Hahn polynomials

$$\mu_n = \mu_0\sum_{k=0}^{n}\sum_{l=0}^{k}\frac{(-1)^l}{k!}\binom{k}{l}\frac{(a+c)_k(a+d)_k}{(a+b+c+d)_k}((a+l)i)^n, \quad (76)$$

with

$$\mu_0 = \frac{\Gamma(a+c)\Gamma(a+d)\Gamma(b+c)\Gamma(b+d)}{\Gamma(a+b+c+d)};$$

(d) the Meixner-Pollaczek polynomials

$$\mu_n = \frac{2\pi\Gamma(2\lambda)}{(2\sin\phi)^{2\lambda}} \sum_{k=0}^{n}\sum_{l=0}^{k} \frac{(-1)^l}{k!}\binom{k}{l}\frac{(2\lambda)_k((a+l)i)^n}{(1-e^{-2i\phi})^k}. \tag{77}$$

Proof. The Wilson polynomials fulfil the orthogonality relation [26, P. 186]

$$\int_0^{\infty}\left|\frac{\Gamma(a+ix)\Gamma(b+ix)\Gamma(c+ix)\Gamma(d+ix)}{\Gamma(2ix)}\right|^2 W_m(x^2;a,b,c,d)W_n(x^2;a,b,c,d)dx$$
$$= \frac{2\pi\Gamma(n+a+b)\Gamma(n+a+c)\Gamma(n+b+c)\Gamma(n+b+d)\Gamma(n+c+d)n!}{\Gamma(2n+a+b+c+d)(n+a+b+c+d-1)_n^{-1}}\delta_{mn}.$$

With $m=n=0$, it follows that

$$\mu_0 = 2\pi\frac{\Gamma(a+b)\Gamma(a+c)\Gamma(b+c)\Gamma(b+d)\Gamma(c+d)}{\Gamma(a+b+c+d)}.$$

From the inversion formula (72), it follows that the zeroth inversion coefficient is

$$I_0(n) = \frac{(a+b)_n(a+c)_n(a+d)_n}{(a+b+c+d)_n}.$$

Application of Theorem 28 gives the following Wilson generalized moments

$$\mu_n(\theta_n(a,x)) = 2\pi\frac{\Gamma(a+b)\Gamma(a+c)\Gamma(b+c)\Gamma(b+d)\Gamma(c+d)}{\Gamma(a+b+c+d)}\frac{(a+b)_n(a+c)_n(a+d)_n}{(a+b+c+d)_n},$$

where $\theta_n(a,x)$ is defined as in Theorem 45. Combining these generalized moments with the connection formula (14), and using Corollary 29, we obtain (74). The other moments are obtained similarly. □

4.6. Orthogonal polynomials on q-quadratic lattices

These polynomials have the following q-hypergeometric representations (see [26]):

(a) The Askey-Wilson polynomials

$$\frac{a^n p_n(x;a,b,c,d|q)}{(ab,ac,ad;q)_n} = {}_4\phi_3\left(\begin{matrix} q^{-n}, abcdq^{n-1}, ae^{i\theta}, ae^{-i\theta} \\ ab, ac, ad\end{matrix}\middle| q;q\right), \quad x=\cos\theta.$$

(b) The Continuous Dual q-Hahn polynomials

$$\frac{a^n p_n(x;a,b,c|q)}{(ab,ac;q)_n} = {}_3\phi_2\left(\begin{matrix} q^{-n}, ae^{i\theta}, ae^{-i\theta} \\ ab, ac\end{matrix}\middle| q,q\right), \quad x=\cos\theta.$$

(c) The Continuous q-Hahn polynomials

$$\frac{(ae^{i\phi})^n p_n(x;a,b,c,d|q)}{(abe^{2i\theta}, ac, ad;q)_n} = {}_4\phi_3\left(\begin{matrix} q^{-n}, abcdq^{n-1}, ae^{i(\theta+2\phi)}, ae^{-i\theta} \\ abe^{2i\phi}, ac, ad\end{matrix}\middle| q,q\right), \quad x=\cos(\theta+\phi).$$

(d) The Al-Salam-Chihara polynomials

$$Q_n(x;a,b|q) = \frac{(ab;q)_n}{a^n}{}_3\phi_2\left(\begin{matrix} q^{-n}, ae^{i\theta}, ae^{-i\theta} \\ ab, 0\end{matrix}\middle| q,q\right), \quad x=\cos\theta.$$

(e) The q-Meixner-Pollaczek polynomials

$$P_n(x;a|q) = a^{-n}e^{-in\phi}\frac{(a^2;q)_n}{(q;q)_n}{}_3\phi_2\left(\begin{array}{c} q^{-n}, ae^{i(\theta+2\phi)}, ae^{-i\theta} \\ a^2, 0 \end{array}\middle| q,q\right), \quad x=\cos(\theta+\phi).$$

(f) The continuous q-Jacobi polynomials

$$P_n^{(\alpha,\beta)}(x|q) = \frac{(q^{\alpha+1};q)_n}{(q;q)_n}{}_4\phi_3\left(\begin{array}{c} q^{-n}, q^{n+\alpha+\beta+1}, q^{\frac{1}{2}\alpha+\frac{1}{4}}e^{i\theta}, q^{\frac{1}{2}\alpha+\frac{1}{4}}e^{-i\theta} \\ q^{\alpha+1}, -q^{\frac{1}{2}(\alpha+\beta+1)}, -q^{\frac{1}{2}(\alpha+\beta+2)} \end{array}\middle| q;q\right), \quad x=\cos\theta.$$

(g) The continuous big q-Hermite polynomials

$$H_n(x;a,|q) = a^{-n}{}_3\phi_2\left(\begin{array}{c} q^{-n}, ae^{i\theta}, ae^{-i\theta} \\ 0,0 \end{array}\middle| q,q\right), \quad x=\cos\theta.$$

(h) The continuous q-Laguerre polynomials

$$P_n^{(\alpha)}(x|q) = \frac{(q^{\alpha+1};q)_n}{(q;q)_n}{}_3\phi_2\left(\begin{array}{c} q^{-n}, q^{\frac{1}{2}\alpha+\frac{1}{4}}e^{i\theta}, q^{\frac{1}{2}\alpha+\frac{1}{4}}e^{-i\theta} \\ q^{\alpha+1}, 0 \end{array}\middle| q,q\right), \quad x=\cos\theta.$$

(i) The continuous q-Hermite polynomials

$$H_n(x|q) = e^{in\theta}{}_2\phi_0\left(\begin{array}{c} q^{-n}, 0 \\ - \end{array}\middle| q, q^n e^{-2i\theta}\right), \quad x=\cos\theta.$$

Note that these polynomials fulfill the following inversion formulas.

Theorem 47 (see [31])**.** *The Askey-Wilson, Continuous Dual q-Hahn, the Al-Salam-Chihara and the Continuous big q-Hermite polynomials fulfil the following inversion formulas, respectively*

$$\mathcal{B}_n(x) = \sum_{m=0}^{n}\begin{bmatrix} n \\ m \end{bmatrix}_q \frac{q^{\binom{m}{2}}(-a)^m(abq^m, acq^m, adq^m;q)_{n-m}}{(abcdq^{m-1};q)_m(abcdq^{2m};q)_{n-m}}p_m(x;a,b,c,d), (see[1,\ 5,\ 15,\ 31,\ 35]) \quad (78)$$

$$\mathcal{B}_n(x) = \sum_{m=0}^{n}(-a)^m\begin{bmatrix} n \\ m \end{bmatrix}_q q^{\binom{m}{2}}(abq^m, acq^m;q)_{n-m}p_m(x;a,b,c|q),$$

$$\mathcal{B}_n(x) = \sum_{m=0}^{n}(-a)^m\begin{bmatrix} n \\ m \end{bmatrix}_q q^{\binom{m}{2}}(abq^m;q)_{n-m}Q_m(x;a,b|q),$$

$$\mathcal{B}_n(x) = \sum_{m=0}^{n}(-a)^m\begin{bmatrix} n \\ m \end{bmatrix}_q q^{\binom{m}{2}}H_m(x;a|q),$$

where

$$\mathcal{B}_n(x) = (ae^{i\theta}, ae^{-i\theta};q)_n, \quad x=\cos\theta.$$

Remark 48. *It should be noted that from the inversion formula for the Askey-Wilson polynomials* (78)*, the remaining inversion coefficients for the polynomials of the Askey scheme may be obtained by appropriate limit transitions.*

Theorem 49. *The following representations are valid for the canonical moments of:*

(a) the Askey-Wilson polynomials

$$\mu_n = \frac{2\pi(abcd;q)_\infty}{(q;ab,ac,ad,bc,bd,cd;q)_\infty}\sum_{k=0}^{n}\sum_{j=0}^{k}\frac{(ab,ac,ad;q)_k}{(abcd;q)_k}\frac{q^k q^{-j^2}a^{-2j}(aq^j+a^{-1}q^{-j})^k}{(q,q^{1+2j}a^2;q)_{k-j}(q,q^{-1-2j}a^{-2};q)_j}. \tag{79}$$

(b) the Continuous Dual q-Hahn polynomials

$$\mu_n = \frac{2\pi}{(q,ab,ac,bc;q)_\infty}\sum_{k=0}^{n}\sum_{j=0}^{k}\frac{q^k q^{-j^2}a^{-2j}(aq^j+a^{-1}q^{-j})^k}{(q,q^{1+2j}a^2;q)_{k-j}(q,q^{-1-2j}a^{-2};q)_j}(ab,ac;q)_k. \tag{80}$$

(c) the Al-Salam-Chihara polynomials

$$\mu_n = \frac{2\pi}{(q,ab;q)_\infty}\sum_{k=0}^{n}\sum_{j=0}^{k}\frac{q^k q^{-j^2}a^{-2j}(aq^j+a^{-1}q^{-j})^k}{(q,q^{1+2j}a^2;q)_{k-j}(q,q^{-1-2j}a^{-2};q)_j}(ab;q)_k. \tag{81}$$

(d) the Continuous Big q-Hermite polynomials

$$\mu_n = \frac{2\pi}{(q;q)_\infty}\sum_{k=0}^{n}\sum_{j=0}^{k}\frac{q^k q^{-j^2}a^{-2j}(aq^j+a^{-1}q^{-j})^k}{(q,q^{1+2j}a^2;q)_{k-j}(q,q^{-1-2j}a^{-2};q)_j}. \tag{82}$$

Proof. From the Askey-Wilson orthogonality relation (see e.g [26]), with $m=n=0$, it follows that

$$\mu_0 = \frac{2\pi(abcd;q)_\infty}{(q;ab,ac,ad,bc,bd,cd;q)_\infty}.$$

From the inversion formula (78), we get the zeroth inversion coefficient

$$I_0(n) = \frac{(ab,ac,ad;q)_n}{(abcd;q)_n}.$$

Hence, using Eq. (25), the generalized Askey-Wilson moments with respect to $\mathcal{B}_n(x) = (ae^{i\theta}, ae^{-i\theta};q)_n$ have the representation

$$\mu_n(\mathcal{B}_n(x)) = \frac{2\pi(abcd;q)_\infty}{(q;ab,ac,ad,bc,bd,cd;q)_\infty}\frac{(ab,ac,ad;q)_n}{(abcd;q)_n}. \tag{83}$$

Finally, using (20) and Corollary 29, we obtain (79). The other canonical moments are computed using similar arguments as in the proof of Theorem 46. □

Note that formula (79) appeared in [12].

Remark 50. *Note also that by appropriate limit transitions, one may obtain the canonical moments of the remaining polynomials from the moments of the Askey-Wilson polynomials* (79)*. For example, the moments of the Continuous Dual q-Hahn polynomials given in* (80) *is obtaining by taking* $d=0$ *in the Askey-Wilson moments given by*(79)*.*

Theorem 51. *The Continuous q-Hahn and the q-Meixner-Pollaczek polynomials fulfil the following inversion formulas, respectively*

$$\mathcal{B}_n(x) = \sum_{m=0}^{n}\begin{bmatrix} n \\ m \end{bmatrix}_q \frac{(-ae^{i\phi})^m q^{\binom{m}{2}}(abq^m e^{2i\phi}, acq^m, adq^m;q)_{n-m}}{(abcdq^m;q)_m(abcdq^{2m};q)_{n-m}} P_m(x;a,b,c,d|q), \quad \textit{see}[31]$$

$$\mathcal{B}_n(x) = \sum_{m=0}^{n}(-ae^{i\phi})^m\begin{bmatrix} n \\ m \end{bmatrix}_q q^{\binom{m}{2}}(q;q)_m(a^2q^m;q)_{n-m}P_m(x;a|q), \quad \textit{see}[31]$$

where

$$\mathcal{B}_n(x) = (ae^{i(\theta+2\phi)}, ae^{-i\theta};q)_n, \quad x=\cos(\theta+\phi).$$

Theorem 52. *The Continuous q-Hahn and the q-Meixner canonical moments have the following representations, respectively.*

$$\mu_n = \mu_0 \sum_{k=0}^{n}\sum_{j=0}^{k} \frac{(abe^{2i\phi}, ac, ad; q)_k}{(abcd; q)_k} \frac{q^k q^{-j^2} a^{-2j} e^{2i\phi}(ae^{-i\phi}q^j + a^{-1}e^{i\phi}q^{-j})^k}{(q, q^{1+2j}a^2e^{-2i\phi}; q)_{k-j}(q, q^{-1-2j}a^{-2}e^{2i\phi}; q)_j}, \tag{84}$$

where

$$\mu_0 = \frac{4\pi(abcd; q)_\infty}{(q, abe^{2i\phi}, ac, ad, bc, bd, cde^{-2i\phi}; q)_\infty}$$

and

$$\mu_n = \frac{2\pi}{(a^2, q; q)_\infty} \sum_{k=0}^{n}\sum_{j=0}^{k} \frac{q^k q^{-j^2} a^{-2j} e^{2i\phi}(ae^{-i\phi}q^j + a^{-1}e^{i\phi}q^{-j})^k}{(q, q^{1+2j}a^2e^{-2i\phi}; q)_{k-j}(q, q^{-1-2j}a^{-2}e^{2i\phi}; q)_j}(a^2, q; q)_k. \tag{85}$$

Proof. The proof this theorem is similar to the one of Theorem 49. □

Theorem 53. *The Continuous q-Jacobi and the Continuous q-Laguerre polynomials fulfil the following inversion formulas, respectively, see [31].*

$$\mathcal{B}_n(x) = \sum_{m=0}^{n} \begin{bmatrix} n \\ m \end{bmatrix}_q \frac{(-1)^m (q;q)_m q^{\binom{m}{2}} (q^{\alpha+1+m}; q)_{n-m} (-q^{\frac{1}{2}(\alpha+\beta+1)}, -q^{\frac{1}{2}(\alpha+\beta+2)}; q)_n}{(q^{m+\alpha+\beta+1}; q)_m (q^{2m+\alpha+\beta+2}; q)_{n-m}} P_m^{(\alpha,\beta)}(x|q),$$

$$\mathcal{B}_n(x) = \sum_{m=0}^{n} (-1)^m \begin{bmatrix} n \\ m \end{bmatrix}_q q^{\binom{m}{2}} (q;q)_m (q^{\alpha+1+m}; q)_{n-m} P_m^{(\alpha)}(x|q),$$

where

$$\mathcal{B}_n(x) = (q^{\frac{1}{2}\alpha+\frac{1}{4}}e^{i\theta}, q^{\frac{1}{2}\alpha+\frac{1}{4}}e^{-i\theta}; q)_n, \quad x = \cos\theta.$$

Theorem 54. *The Continuous q-Jacobi and the Continuous q-Laguerre canonical moments have the following representations.*

$$\mu_n = \mu_0 \sum_{k=0}^{n}\sum_{j=0}^{k} \frac{q^{k-j^2-(\alpha+\frac{1}{2})j}(q^{j+\frac{\alpha}{2}+\frac{1}{4}} + q^{-j-\frac{\alpha}{2}-\frac{1}{4}})^k}{(q, q^{2j+\alpha+\frac{3}{2}}; q)_{k-j}(q, q^{-2j-\alpha-\frac{3}{2}}; q)_j} \frac{(q^{\alpha+1}, -q^{\frac{1}{2}(\alpha+\beta+1)}, -q^{\frac{1}{2}(\alpha+\beta+2)}; q)_k}{(q^{\alpha+\beta+2}; q)_k}, \tag{86}$$

where

$$\mu_0 = \frac{2\pi(q^{\frac{1}{2}(\alpha+\beta+2)}, q^{\frac{1}{2}(\alpha+\beta+3)}; q)_\infty}{(q, q^{\alpha+1}, q^{\beta+1}, -q^{\frac{1}{2}(\alpha+\beta+1)}, -q^{\frac{1}{2}(\alpha+\beta+2)}; q)_\infty}$$

and

$$\mu_n = \frac{2\pi}{(q, q^{\alpha+1}; q)_\infty} \sum_{k=0}^{n}\sum_{j=0}^{k} \frac{q^k q^{-j^2} q^{-(\alpha+\frac{1}{2})j}(q^{j+\frac{\alpha}{2}+\frac{1}{4}} + q^{-j-\frac{\alpha}{2}-\frac{1}{4}})^k (q^{\alpha+1}; q)_k}{(q, q^{2j+\alpha+\frac{3}{2}}; q)_{k-j}(q, q^{-2j-\alpha-\frac{3}{2}}; q)_j}. \tag{87}$$

Proof. The proof this theorem is similar to the one of Theorem 49. □

Theorem 55. *The Continuous q-Hermite polynomials have the following representation:*

$$\mu_{2n+1} = 0, \quad \mu_{2n} = \frac{\pi(-1)^n}{(q;q)_\infty} \sum_{k=0}^{n} (-1)^k \binom{2n}{k} (1+q^{n-k}) q^{\frac{(n-k)(n-k-1)}{2}}, \quad n = 0, 1, 2, \ldots \tag{88}$$

In order to prove this theorem, we need the following lemma.

Lemma 56 (See Lemma 13.1.4 in [18])**.** *The following relation is valid*

$$\int_0^\pi e^{2ij\theta}(e^{2i\theta}, e^{-2i\theta}; q)_\infty d\theta = \frac{\pi(-1)^j}{(q;q)_\infty}(1+q^j)q^{j(j-1)/2}. \tag{89}$$

Proof of the theorem. Note that $\mu_n = 0$ when n is odd. We start by writing

$$\cos^n \theta \quad = \quad \frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k}e^{i(n-2k)}.$$

Next, we use the the relation (89) to get:

$$\begin{aligned}
\mu_{2n} &= \int_0^{\pi}(\cos\theta)^{2n}(e^{2i\theta}, e^{-2i\theta}; q)_\infty d\theta \\
&= \frac{1}{2^{2n}}\sum_{k=0}^{2n}\binom{2n}{k}\int_0^{\pi} e^{2i(n-k)}(e^{2i\theta}, e^{-2i\theta}; q)_\infty d\theta \\
&= \frac{\pi(-1)^n}{(q;q)_\infty}\sum_{k=0}^{n}(-1)^k\binom{2n}{k}(1+q^{n-k})q^{\frac{(n-k)(n-k-1)}{2}}.
\end{aligned}$$

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