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Parallel accessible domains and domains that are convex in some direction

Abstract: We show that univalent functions f of the unit disk $\mathbb{D}$ that have a range which is convex in some direction have an equivalent analytic representation of the form

$$\left|\arg e^{i\alpha}(1-xz)(1-yz)f'(z)\right| < \frac{\pi}{2}$$

for some $\alpha \in \mathbb{R}$, and $x, y \in \partial\mathbb{D}$. This question had been examined by Robertson ([11] – [12]), Hengartner and Schober ([2] – [3]) and others, and the above characterization had been mentioned by Pommerenke in [8] without proof, and finally established by Royster and Ziegler [13]. Rather than using geometric properties of level curves our approach uses an approximation of the domain by Schwarz-Christoffel mappings. Note that x and y may be equal; we give a geometric equivalent for that special case, i. e. for the condition

$$\left|\arg e^{i\alpha}(1-xz)^2 f'(z)\right| < \frac{\pi}{2} \quad (\alpha \in \mathbb{R},\ \ x \in \partial\mathbb{D}).$$

Finally we examine the geometric equivalents of the analytic conditions where in the above inequalities the right hand side is replaced by $\beta\frac{\pi}{2}$ for some $\beta \in [0,1]$.

To get our results we use a method that was developed in [7] which may be regarded as a general reference.

1. Introduction

We consider functions that are analytic in the unit disk $\mathbb{D} := \{z \in \mathbb{C} \mid |z| < 1\}$. A function is called *univalent* if it is one-to-one. The Riemann mapping theorem guarantees the existence of a univalent map $f : \mathbb{D} \to F$ for each simply connected plane domain $F \neq \mathbb{C}$. Moreover f with given $f(0)$ is uniquely determined except of the composition with rotations $z \mapsto e^{i\alpha}z$ of $\mathbb{D}$.

If we speak about convergence of a sequence (f_n) of analytic functions, we mean locally uniform convergence and write $f_n \to f$. The family A of *analytic functions of* $\mathbb{D}$ together with this topology is a Fréchet space, i. e. a locally convex complete metrizable linear space.

A sequence of univalent functions not converging locally uniformly to ∞ is normal, and there is a convergent subsequence. The limit function is univalent or constant. The geometric equivalent of convergence $f_n \to f$ for the images $f_n(\mathbb{D}) \to f(\mathbb{D})$ was characterized by Carathéodory and is called *Carathéodory kernel convergence.*

Let P denote the subset of A of functions p with positive real part that are normalized by $p(0) = 1$.

A function of the form

$$p(z) = \int_{\partial\mathbb{D}} \frac{1+xz}{1-xz}\, d\mu(x)\ ,$$

where μ denotes a Borel probability measure on $\partial\mathbb{D}$, clearly has positive real part, because the kernel functions have this property. The famous *Herglotz Representation Theorem* states that the converse is also true.

A compact family which is similar to P is the class $\widetilde{P}$ of functions p normalized by $p(0) = 1$ for which there is some $\alpha \in \mathbb{R}$ such that the real part of $e^{i\alpha}p$ is positive. The author deduced the following approximation result for functions $p \in \widetilde{P}$ from Herglotz's Theorem (see [7], Lemma 2.3).

Lemma 1 *The functions p_n $(n \in \mathbb{N})$ with a representation of the form*

$$p_n(z) = \prod_{k=1}^{n} \frac{1 - y_k z}{1 - x_k z}, \tag{1}$$

where

$$|x_k| = |y_k| = 1 \qquad (k = 1, \ldots, n), \tag{2}$$

and

$$\arg \overline{x_1} < \arg \overline{y_1} < \arg \overline{x_2} < \arg \overline{y_2} < \cdots < \arg \overline{x_n} < \arg \overline{y_n} < \arg \overline{x_1} + 2\pi \tag{3}$$

form a dense subset of $\widetilde{P}$.

2. Polygons and Schwarz-Christoffel mappings

Let $f \in A$ be continuous in $\overline{\mathbb{D}}$ and have a Riemann surface F as image domain whose boundary consists of a finite number of linear arcs, such that the boundary correspondence $\partial\mathbb{D} \to \partial F$ is one-to-one. Then F is called a *polygon*. Let F have n vertices of inner angles $\alpha_k\pi$ $(k = 1, \ldots, n)$. For a bounded vertex the relation

$$\alpha_k > 0 \tag{4}$$

holds. If a vertex lies at infinity we measure the angle on the Riemann sphere and have

$$\alpha_k \geq 0\,, \tag{5}$$

where $\alpha_k = 0$ is a zero angle which corresponds to two parallel rays of ∂F.

Let now x_k be the *prevertices*, i. e. the preimages under f of the vertices $f(x_k)$. Then the *Schwarz-Christoffel formula* is the representation

$$\frac{f''}{f'}(z) = -2\sum_{k=1}^{n} \frac{\mu_k}{z - x_k}\,, \tag{6}$$

where

$$2\mu_k\pi := \begin{cases} (1 - \alpha_k)\pi & \text{if } f(x_k) \text{ is bounded} \\ (1 + \alpha_k)\pi & \text{if } f(x_k) \text{ is unbounded} \end{cases} \tag{7}$$

denote the *outer angles*. The formula

$$\sum_{k=1}^{n} \mu_k = 1 \tag{8}$$

corresponds in the bounded (univalent) case both to the rule for the sum of angles in an n-gon and to the fact that the increment of the tangent direction is exactly 2π when surrounding the polygon on ∂F one time.

On the other hand, if f fulfills (6) and (8) with $x_k \in \partial\mathbb{D}$ for $k = 1, \ldots, n$, then the Riemann image surface $f(\mathbb{D})$ is a polygon.

A function $f \in A$ is called *convex* if it maps $\mathbb{D}$ univalently onto a convex domain. Therefore it is necessary and sufficient that

$$1 + z\frac{f''}{f'} \in P\,.$$

With regard to the Schwarz-Christoffel formula, a function is convex if and only if it can be approximated by convex polygons. Convex polygons have outer angles $\mu_k \geq 0$.

3. Parallel accessible domains and domains convex in some direction

Functions with domains that are convex in some direction have extensively studied in the literature, see e. g. [11] – [12], and [2] – [3]. It turned out that the analytic representation of the family of functions whose ranges are convex in some fixed direction, is rather difficult ([2] and [1]). On the other hand, if the direction is not fixed, the result is pretty much simpler [13], and is that which one would expect ([8], p. 297). We will give here another proof of the result of Royster and Ziegler [13] by a rather general approach that will enable us to solve some more problems of a similar kind.

A domain F is said to be *convex in the direction* ζ ($\zeta \in \partial\mathbb{D}$) if for all $z \in F$ and $w = z + u\zeta \in F$ ($u \in \mathbb{R}$) the segment

$$s(z, w) := \{tz + (1-t)w \mid t \in [0,1]\} \subset F\,.$$

A domain F is said to be *convex in some direction* if there is some $\zeta \in \partial\mathbb{D}$ such that F is convex in the direction ζ.

We will now give another geometric characterization for those domains that is more on the lines of [7]. We call a domain F *parallel accessible* if there is some direction $\zeta \in \partial\mathbb{D}$ such that the complement of F is the union

$$\mathbb{C} \setminus F = \bigcup_{t \in T} \gamma_t \cup \bigcup_{u \in U} \ell_u\,, \tag{9}$$

where γ_t are rays that are pairwise parallel having direction ζ or $-\zeta$, ℓ_u are lines of direction ζ, and T, U are appropriately chosen parameter sets. Here a ray γ_t has direction ζ if

$$\gamma_t = \{z_0 + v\zeta \mid v \geq 0\}\,. \tag{10}$$

F is called *strongly parallel accessible* if there is a representation of the form (9) such that all the rays γ_t are parallel in the strong sense (10), i. e. if they are not only parallel but have the same direction, too.

We will show that the notions of convexity in some direction and parallel accessibility agree. Therefore we utilize

Lemma 2 *Let F be convex in the direction ζ. Then the intersection of each line ℓ of direction ζ with F*

(a) *is either empty,*

(b) *is an open ray, or*

(c) *is an open segment.*

The simple proof of this lemma is left to the reader. It follows

Lemma 3 *A plane domain is convex in some direction if and only if it is parallel accessible (in the weak sense).*

Proof: "$\Rightarrow$" Let F be convex in direction ζ. We write $\mathbb{C} = \bigcup_{u \in U} \ell_u$ where ℓ_u are all the lines of direction ζ. So we get a representation

$$\mathbb{C} \setminus F = \bigcup_{u \in U} \left((\mathbb{C} \setminus F) \cap \ell_u \right) . \tag{11}$$

Now, if for some $u \in U$ we have $\ell_u \cap F = \emptyset$, then obviously $\ell_u \subset \mathbb{C} \setminus F$, and so ℓ_u is one of the representating lines of F corresponding to (9). In the other case, by Lemma 2, ℓ_u is divided into either two rays, one of them part of F, and the second of the complement $\mathbb{C} \setminus F$, and so of type γ_t in (9), or in three parts, one segment lying in F, and two of them of type γ_t in (9). Note that the last case produces rays that are not strongly parallel. So finally the whole complement of F (11) is written in the form (9).
"$\Leftarrow$" If (9) is given, then one easily sees that $F = \bigcup_{t \in T} s_t$ with (finite or infinite) open segments s_t of direction ζ. Therefore clearly F is convex in direction ζ. □

Similarly, one gets the following interior domain characterization in the strongly parallel case.

Lemma 4 *A plane domain F is strongly parallel accessible if and only if there is a direction $\zeta \in \partial\mathbb{D}$ such that for each $z \in F$ and $v > 0$ the point $z + v\zeta \in F$.*

Note that this notion for a fixed ζ was introduced in [2] where also an idea of an analytic equivalent was given.

We shall now give analytic equivalents of both the strong and the weak notions of parallel accessibility. First we consider the strongly parallel case.

Theorem 1 *A function $f \in A$ with $f'(0) = 1$ is univalent, and $f(\mathbb{D})$ is strongly parallel accessible, if and only if there is a representation of the form*

$$(1 - xz)^2 f' = p \tag{12}$$

for some $x \in \partial\mathbb{D}$ and some function $p \in \widetilde{P}$.

Proof: First we observe that both

(i) the functions with a representation of form (12), and

(ii) the domains F that are strongly parallel accessible,

are closed sets with respect to the corresponding topologies, i. e. with respect to locally uniform and Carathéodory kernel convergence, respectively, (see e. g. [7], proofs of Theorems 5.1 and 5.2).

Suppose now, $f(\mathbb{D})$ is strongly parallel accessible. Then by the geometric definition we have

$$f(\mathbb{D}) = \mathbb{C} \setminus \left(\bigcup_{t\in T} \gamma_t \cup \bigcup_{u\in U} \ell_u \right), \tag{13}$$

where γ_t are strongly parallel rays of direction ζ, say, ℓ_u are lines of direction ζ, and T, U are suitably chosen parameter sets that are separable (e. g. $\subset \mathbb{R}^3$). Choose a dense subset $\{t_n \in T | \, n \in \mathbb{N}\}$ of T and define f_n by

$$F_n = f_n(\mathbb{D}) := \mathbb{C} \setminus \bigcup_{k=1}^{n} \gamma_{t_k} \, . \tag{14}$$

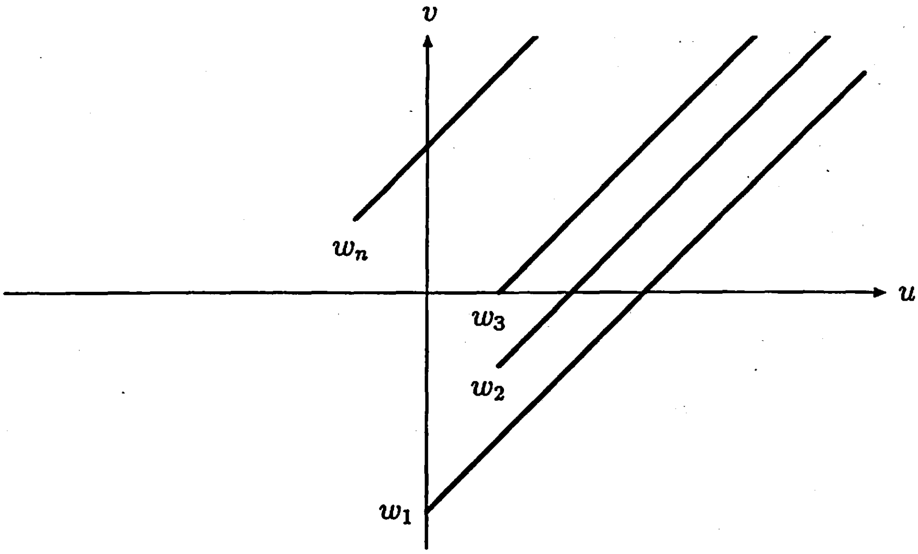

Figure 1: The complement of a finite number of strongly parallel rays

It may happen that it is necessary to include some of the lines ℓ_u into this procedure. In the case of a parallel strip, e. g., the complement may be considered to consist only of lines without rays to appear. In such a case ∂F contains straight line segments, and we can produce some rays of direction ζ out of that line segments.

In either case, we may declare F_n as the complement of n strongly parallel rays such that $F_n \to F$ with respect to Carathéodory kernel convergence, and so $f_n \to f$. Because of (i) it is so enough to show representation (12) for functions satisfying (14).

Observe that f_n given by (14) is a Schwarz-Christoffel mapping with n finite vertices at the points $w_k =: f_n(\overline{y_k})$, say. The interior angle at each of those hairpin vertices is 2π. The other n vertices alternate with w_k and lie at $\infty =: f_n(\overline{x_k})$, say. Note that so the numbers x_k, y_k $(k = 1, \ldots, n)$ fulfill conditions (2) and (3). As all rays are parallel in the strong sense, the interior angle at all of those vertices but one is zero whereas at one of them the angle is 2π, see Figure 1 (where the vertex at ∞ of angle 2π lies between w_n and w_1) .

So the Schwarz-Christoffel formula yields using (7)

$$\begin{aligned}\frac{f_n''}{f_n'}(z) &= -2\sum_{k=1}^{n}\frac{-1/2}{z-\overline{y_k}} - 2\sum_{k=1}^{n-1}\frac{1/2}{z-\overline{x_k}} - 2\frac{3/2}{z-\overline{x_n}} \\ &= \sum_{k=1}^{n}\left(\frac{1}{z-\overline{y_k}} - \frac{1}{z-\overline{x_k}}\right) - 2\frac{1}{z-\overline{x_n}}\,.\end{aligned}$$

The choice (1) gives a function $p_n \in \widetilde{P}$ by Lemma 1 as (2) and (3) are fulfilled, and so an integration gives (12) with $x = x_n$.

Now suppose, $f \in A$ has a representation of the form (12), so that

$$\frac{f''}{f'} = \frac{p'}{p} - 2\frac{1}{z-\overline{x}}\,. \tag{15}$$

Then by Kaplan's argument [5], f is close-to-convex, hence univalent. By (ii) it is enough to show the geometric property for a dense subfamily. Therefore we may replace p in (15) by p_n of Lemma 1, and get

$$\frac{f_n''}{f_n'} = \frac{p_n'}{p_n} - 2\frac{1}{z-\overline{x}} = -2\sum_{k=1}^{n}\frac{1/2}{z-\overline{x_k}} - 2\sum_{k=1}^{n}\frac{-1/2}{z-\overline{y_k}} - 2\frac{1}{z-\overline{x}}\,, \tag{16}$$

where the numbers x_k, y_k alternate with each other on $\partial\mathbb{D}$. Observe that f_n with a representation (16) is a Schwarz-Christoffel mapping again (as (8) is satisfied).

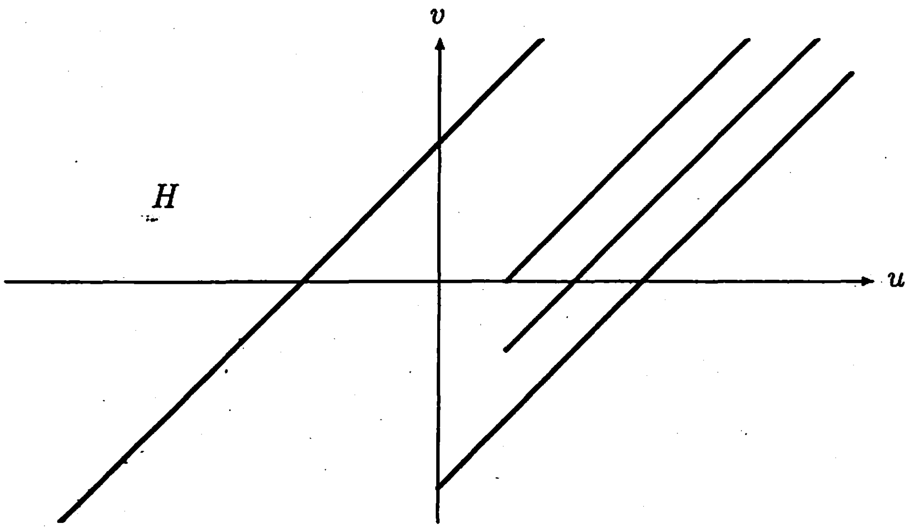

Figure 2: A polygonal domain with representation (16)

By construction we know a priori that f_n is close-to-convex, hence univalent, so that $F_n = f_n(\mathbb{D})$ represents a univalent polygonal domain. Without loss of generality we may assume that x is pairwise different from x_k and y_k $(k = 1, \ldots, n)$. Then one sees from

(16) that F_n has exactly n finite vertices of angle 2π, and alternately n vertices at ∞ of angle zero. Moreover there is one more vertex at ∞ of angle π. Figure 2 gives an example of the situation.

What we have to show is that this kind of polygon satisfies the geometric description considered. But this is trivially seen as by definition all angles at ∞ are zero, and so the corresponding rays are strictly parallel. The existence of one vertex at ∞ of angle π leads to a half-plane H (see Figure 2) that fully is part of the complement of F, and may be written as union of lines. □

The family of functions with a representation of form (12) had been introduced by Kaplan [4], and had lead him to the definition of close-to-convex functions [5]. Note that Theorem 1 shows in particular that this family is linearly invariant ([9] – [10]), a fact that also may be proved directly (see e. g. [6]). Note that this is not true, e. g., for the family of functions $f \in A$ for which $f' \in \widetilde{P}$, that is also of considerable interest, so that here a geometric description of the image domains $f(\mathbb{D})$ cannot be given.

Another consequence of Theorem 1 is that all functions with a representation (12) are unbounded. This follows easily from the geometric description given in Lemma 4, and has as a further consequence that none of the level domains ($r \in (0,1)$)

$$F_r := \{f(z) \mid |z| < r\}$$

of any function f with a representation (12) has the same geometrical property, i. e. is strongly parallel accessible. This results as clearly all level domains are bounded, and it should be compared with the observation of Hengartner and Schober [3] that the level domains of functions that are convex in some direction do not have to have the same property.

The family of functions convex in some direction is another linearly invariant family that we consider now.

Theorem 2 *A function $f \in A$ with $f'(0) = 1$ is univalent, and $f(\mathbb{D})$ is (weakly) parallel accessible, if and only if there is a representation of the form*

$$(1 - xz)(1 - yz)f' = p \tag{17}$$

for some $x, y \in \partial\mathbb{D}$ and some function $p \in \widetilde{P}$.

Proof: As in the proof of Theorem 1 we observe that both

(i) the functions with a representation of form (17), and

(ii) the domains F that are weakly parallel accessible,

are closed sets with respect to the corresponding topologies. Here we must realize in (i) that $x = y$ is allowed, and in (ii) that strongly parallel accessibility is allowed, too.

Suppose now, $f(\mathbb{D})$ is weakly parallel accessible. If in fact $f(\mathbb{D})$ is strongly parallel accessible, Theorem 1 applies, and we get a representation of form (17) with $x = y$.

So we may assume now that $f(\mathbb{D})$ is not strongly parallel accessible, but parallelity in both directions ζ and $-\zeta$ indeed occurs. Then by the geometric definition we have a representation (13) with weakly parallel rays γ_t of directions ζ and $-\zeta$, and lines ℓ_u of direction ζ. For a dense subset $\{t_n \in T|\, n \in \mathbb{N}\}$ of T define F_n and f_n by (14). As before, include if necessary some of the lines ℓ_u into this procedure, such that $F_n \to F$ with respect to Carathéodory kernel convergence, and so $f_n \to f$. Because of (i) it is then enough to show (17) for functions satisfying (14).

In the present case, f_n is a Schwarz-Christoffel mapping with n finite vertices at the points $w_k =: f_n(\overline{y_k})$, say. The interior angle at each of those hairpin vertices is 2π. The other n vertices alternate with w_k and lie at $\infty =: f_n(\overline{x_k})$, say. As all rays are parallel in the weak sense, the interior angle at all of those vertices but two is zero whereas at two of them the angle is π, see Figure 3.

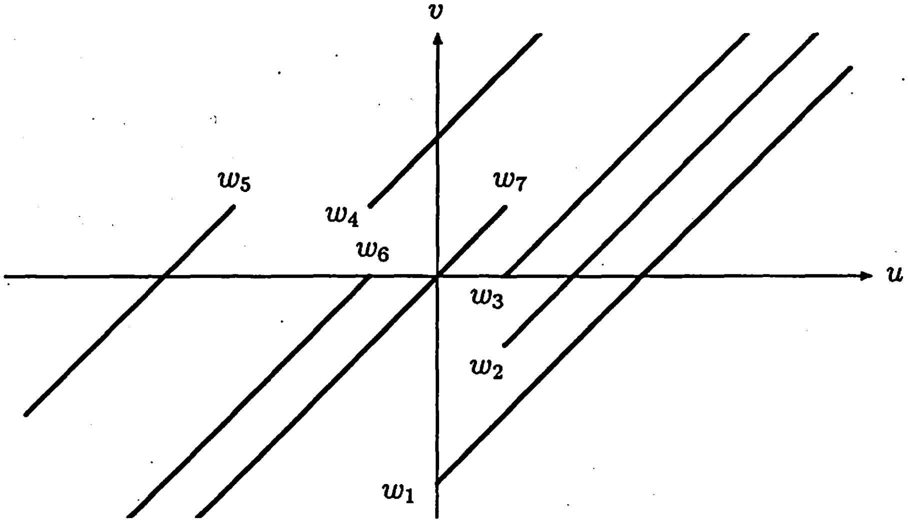

Figure 3: The complement of a finite number of weakly parallel rays

So the Schwarz-Christoffel formula yields using (7)

$$\begin{aligned} \frac{f_n''}{f_n'}(z) &= -2\sum_{k=1}^{n} \frac{-1/2}{z-\overline{y_k}} - 2 \sum_{\substack{k=1\\ k\neq j,m}}^{n} \frac{1/2}{z-\overline{x_k}} - 2\frac{1}{z-\overline{x_j}} - 2\frac{1}{z-\overline{x_m}} \\ &= \sum_{k=1}^{n} \left(\frac{1}{z-\overline{y_k}} - \frac{1}{z-\overline{x_k}}\right) - \left(\frac{1}{z-\overline{x_j}} + \frac{1}{z-\overline{x_m}}\right). \end{aligned}$$

The choice (1) gives a function $p_n \in \widetilde{P}$ by Lemma 1 as (2) and (3) are fulfilled, and so an integration gives (17) with $x = x_j$ and $y = x_m$.

Now suppose, $f \in A$ has a representation of the form (12), so that

$$\frac{f''}{f'} = \frac{p'}{p} - \frac{1}{z-\overline{x}} - \frac{1}{z-\overline{y}}. \tag{18}$$

Then f is close-to-convex, hence univalent, again. By (ii) it is enough to show the geometric property for a dense subfamily, so we replace p in (15) by p_n of Lemma 1, and get

$$\frac{f_n''}{f_n'} = \frac{p_n'}{p_n} - \frac{1}{z-\overline{x}} - \frac{1}{z-\overline{y}} = -2\sum_{k=1}^{n} \frac{1/2}{z-\overline{x_k}} - 2\sum_{k=1}^{n} \frac{-1/2}{z-\overline{y_k}} - 2\frac{1/2}{z-\overline{x}} - 2\frac{1/2}{z-\overline{y}}, \tag{19}$$

where the numbers x_k, y_k alternate with each other on $\partial\mathbb{D}$. Again f_n is a Schwarz-Christoffel mapping, and we know a priori that f_n is univalent, so that $F_n = f_n(\mathbb{D})$ represents a univalent polygonal domain. Without loss of generality we may assume that x and y are pairwise different from x_k and y_k ($k = 1, \ldots, n$), and different from each other (the case $x = y$ was considered in Theorem 1). Then one sees from (19) that F_n has exactly n finite vertices of angle 2π, and alternately n vertices at ∞ of angle zero. Moreover there are two more vertices at ∞ of zero angle.

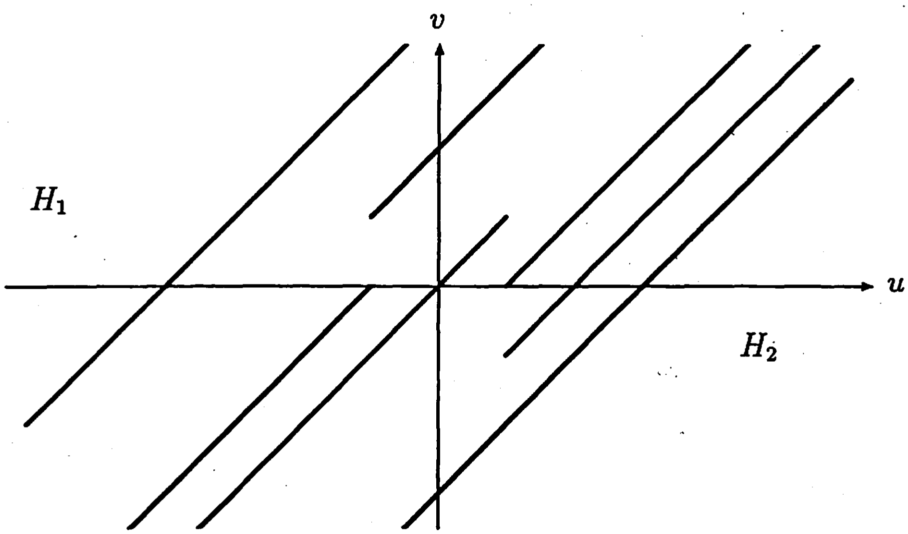

Figure 4: A polygonal domain with representation (19)

Figure 4 indicates how the situation looks. Again trivially the complement consists of weakly parallel rays, and the existence of two more vertices at ∞ of zero angle π leads to two half-planes $H_{1,2}$ that fully lie in the complement of F, and may be written as union of lines. □

We note that the content of Theorem 2 was stated by Pommerenke ([8], p. 297) without proof (having the results in [11] and compactness in mind). Royster and Ziegler [13] finally published a proof of the result.

4. Parallel accessible domains of order β

We call a domain F *(strongly) parallel accessible of order* β if it is (strongly) parallel accessible, and if for each ray γ_t of the corresponding representation of the complement (9) the sector S_t of angle $(1-\beta)\pi$ whose bisector is γ_t fully lies in the complement of F.

Similarly as in the case of close-to-convex functions of order β (see e. g. [7]) we get the following analytic characterizations.

Theorem 3 *Let $\beta \in (0,1)$. Then a function $f \in A$ with $f'(0) = 1$ is univalent, and $f(\mathbb{D})$ is*

(a) *strongly parallel accessible of order β,*

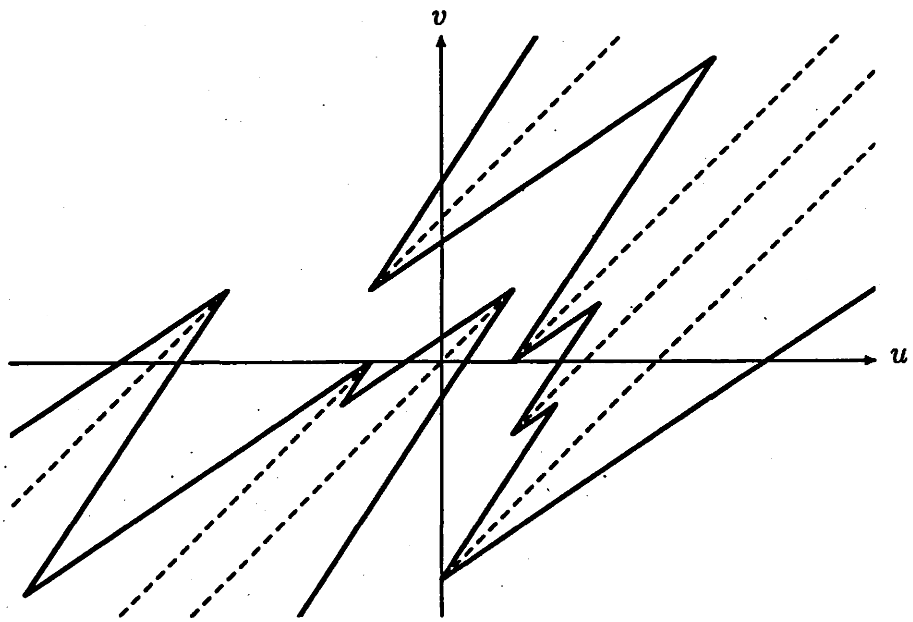

Figure 6: The complement of weakly parallel sectors

We give the calculations for the case (a). Here we observe that f_n given by (20) is a Schwarz-Christoffel mapping with n finite vertices of interior angle $(1+\beta)\pi$ at the points $w_k =: f_n(\overline{y_k})$, say. The other n vertices alternate with w_k and lie at $W_k =: f_n(\overline{x_k})$, say. Note that so the numbers x_k, y_k $(k = 1, \ldots, n)$ fulfill conditions (2) and (3). The interior angles at W_k $(k = 1, \ldots, n-1)$ all are $(1-\beta)\pi$, whereas one of the vertices, W_n, say, lies at ∞, and has an angle $(1+\beta)\pi$, see Figure 5.

So the Schwarz-Christoffel formula yields

$$\frac{f_n''}{f_n'}(z) = \beta \sum_{k=1}^{n} \left(\frac{1}{z - \overline{y_k}} - \frac{1}{z - \overline{x_k}} \right) - 2\frac{1}{z - \overline{x_n}} \,.$$

The choice (1) gives a function $p_n \in \widetilde{P}$ by Lemma 1 as (2) and (3) are fulfilled, and an integration gives (i) with $x = x_n$.

A similar calculation shows that (b) implies (ii).

Suppose now, one of the analytic conditions (i) or (ii) holds for f. Then we approximate f with the aid of Lemma 1 by functions f_n for which

$$\frac{f_n''}{f_n'} = \beta\frac{p_n'}{p_n} - \frac{2}{z - \overline{x}} = -2\sum_{k=1}^{n} \frac{\beta/2}{z - \overline{x_k}} - 2\sum_{k=1}^{n} \frac{-\beta/2}{z - \overline{y_k}} - 2\frac{1}{z - \overline{x}} \,, \tag{21}$$

or

$$\frac{f_n''}{f_n'} = \beta\frac{p_n'}{p_n} - \frac{1}{z-\overline{x}} - \frac{1}{z-\overline{y}} = -2\sum_{k=1}^{n} \frac{\beta/2}{z-\overline{x_k}} - 2\sum_{k=1}^{n} \frac{-\beta/2}{z-\overline{y_k}} - 2\frac{1/2}{z-\overline{x}} - 2\frac{1/2}{z-\overline{y}} \,, \tag{22}$$

respectively, with values x_k, y_k $(k = 1, \ldots, n)$ that alternate with each other. In the case (i) we have the geometric situation of Figure 7 for which condition (a) easily is deduced.

(b) *parallel accessible of order β,*

if and only if there is a representation of the form

(i) $(1-xz)^2 f' = p^\beta$ *for some* $x \in \partial\mathbb{D}$ *and some* $p \in \widetilde{P}$,

(ii) $(1-xz)(1-yz)f' = p^\beta$ *for some* $x, y \in \partial\mathbb{D}$ *and some* $p \in \widetilde{P}$,

respectively.

Proof: All families (a), (b), (i), and (ii) are closed. Suppose first, one of (a) or (b) holds. Then (13) follows with rays γ_t that are strongly or weakly parallel, and are such that for each γ_t the corresponding symmetric sector S_t of angle $(1-\beta)\pi$ lies in the complement of $f(\mathbb{D})$. In this situation we define

$$F_n = f_n(\mathbb{D}) := \mathbb{C} \setminus \bigcup_{k=1}^{n} S_{t_k} . \tag{20}$$

for some dense subset $\{t_n \in T \mid n \in \mathbb{N}\}$ of T. In both cases (a) and (b) it turns out that $f(\mathbb{D})$ can be approximated by appropriate polygons (20).

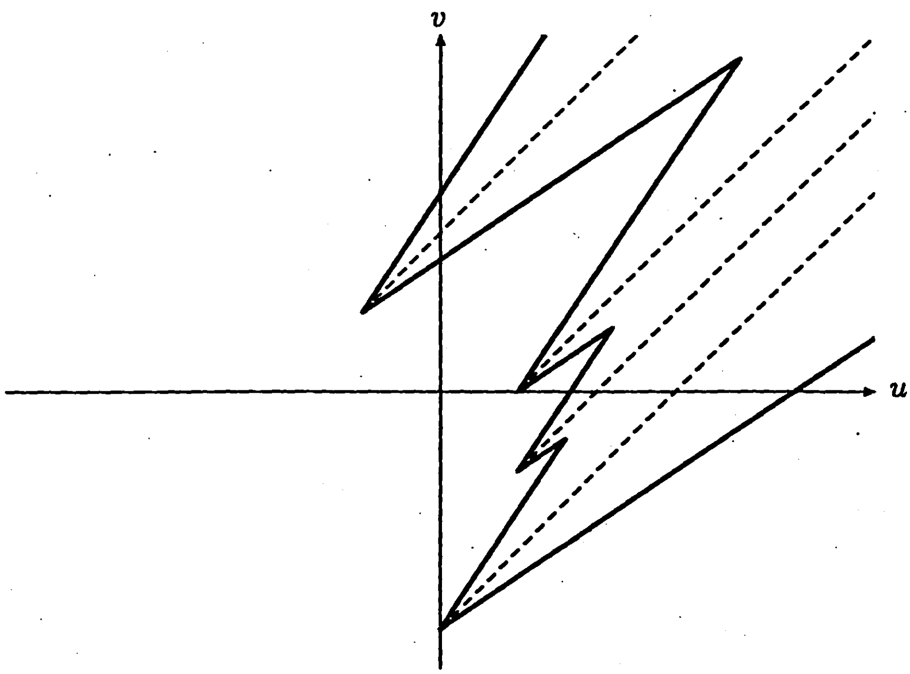

Figure 5: The complement of strongly parallel sectors

Figure 5 shows the situation (a) whereas Figure 6 shows situation (b) (where we assume (a) not to be satisfied). In both cases the analytic conditions (i) and (ii), respectively, may be verified as in Theorems 1 and 2 using Lemma 1 (see also [7], Theorem 5.1).

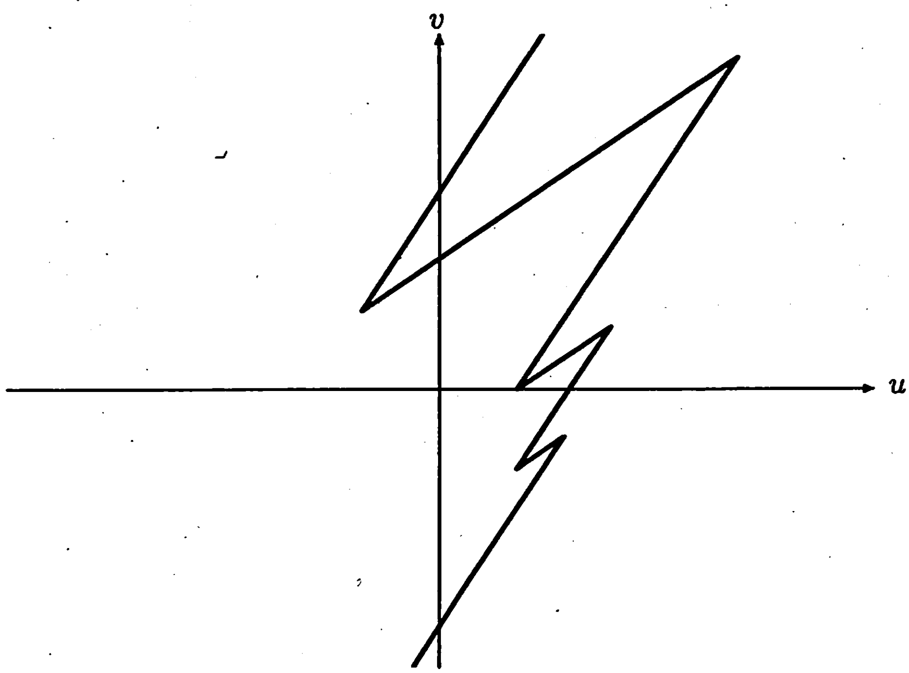

Figure 7: A polygonal domain with representation (21)

Therefore one must only observe that the construction given in [7], Theorem 5.2, produces strongly parallel rays.

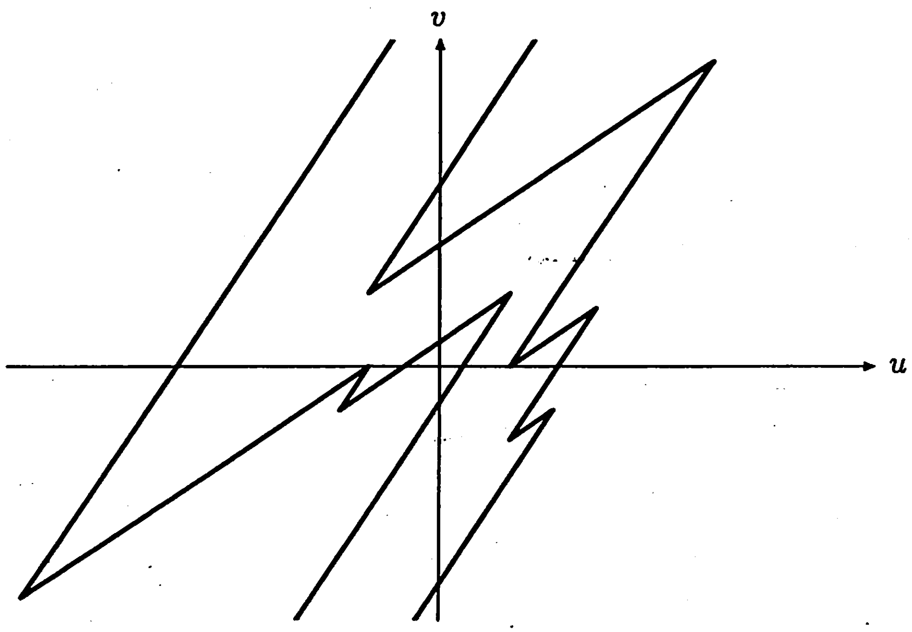

Figure 8: A polygonal domain with representation (22)

A similar examination shows that (b) follows from (ii), see Figure 8. □

We remark that for $\beta \to 0$ the families considered shrink to the parallel-strip domain mappings and the half-plane mappings, respectively. This may be seen either analytically or geometrically.

For $\beta = 0$ we observe again that no level domain has the same geometric property.

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