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Positivity and monotony properties of the de Branges functions

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Abstract

In his 1984 proof of the Bieberbach and Milin conjectures de Branges used a positivity result of special functions $\tau_k^n(t)$ which follows from an identity about Jacobi polynomial sums that was published by Askey and Gasper in 1976.

In 1991 Weinstein presented another proof of the Bieberbach and Milin conjectures, also using a special function system $\Lambda_k^n(t)$ which (by Todorov and Wilf) was realized to be directly connected with de Branges', $\dot\tau_k^n(t) = -k\Lambda_k^n(t)$, and the positivity results in both proofs $\dot\tau_k^n(t) \leqslant 0$ are essentially the same.

By the relation $\dot\tau_k^n(t) \leqslant 0$, the de Branges functions $\tau_k^n(t)$ are monotonic, and $\tau_k^n(t) \geqslant 0$ follows. In this article, we reconsider the de Branges and Weinstein functions, find more relations connecting them with each other, and make the above positivity and monotony result more precise, e.g., by showing $\tau_k^n(t) \geqslant (n-k+1)\mathrm{e}^{-kt}$.

Keywords: de Branges functions; Weinstein functions; Hypergeometric functions; Bieberbach conjecture

1. Introduction

Let S denote the family of analytic and univalent functions $f(z)=z+a_2z^2+\cdots$ of the unit disk $\mathbb{D}$. S is compact with respect to the topology of locally uniform convergence so that $k_n := \max_{f\in S} |a_n(f)|$ exists. In 1916 Bieberbach [5] proved that $k_2 = 2$, with equality if and only if f is a rotation of the *Koebe function*

$$K(z) := \frac{z}{(1-z)^2} = \frac{1}{4}\left(\left(\frac{1+z}{1-z}\right)^2 - 1\right) = \sum_{n=1}^{\infty} nz^n \tag{1}$$

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and in a footnote he mentioned “Vielleicht ist überhaupt $k_n = n$”. This statement is known as the *Bieberbach conjecture*.

In 1923 Löwner [15] proved the Bieberbach conjecture for $n = 3$. His method was to embed a univalent function $f(z)$ into a *Löwner chain*, i.e., a family $\{f(z,t) \mid t \geqslant 0\}$ of univalent functions of the form

$$f(z,t) = \mathrm{e}^t z + \sum_{n=2}^{\infty} a_n(t) z^n, \quad (z \in \mathbb{D}, t \geqslant 0, a_n(t) \in \mathbb{C} \ (n \geqslant 2))$$

which start with f

$$f(z,0) = f(z)$$

and for which the relation

$$\mathrm{Re}\ p(z,t) = \mathrm{Re}\left(\frac{\dot{f}(z,t)}{z f'(z,t)}\right) > 0 \quad (z \in \mathbb{D}) \tag{2}$$

is satisfied. Here $'$ and $\dot{}$ denote the partial derivatives with respect to z and t, respectively. Eq. (2) is referred to as the *Löwner differential equation*, and geometrically it states that the image domains of f_t expand as t increases.

The history of the Bieberbach conjecture showed that it was easier to obtain results about the *logarithmic coefficients* of a univalent function f, i.e., the coefficients d_n of the expansion

$$\varphi(z) = \ln \frac{f(z)}{z} =: \sum_{n=1}^{\infty} d_n z^n$$

rather than for the coefficients a_n of f itself. So Lebedev and Milin [14] in the mid-1960s developed methods to exponentiate such information. They proved that if for $f \in S$ the *Milin conjecture*

$$\sum_{k=1}^{n} (n+1-k)\left(k|d_k|^2 - \frac{4}{k}\right) \leqslant 0$$

on its logarithmic coefficients is satisfied for some $n \in \mathbb{N}$, then the Bieberbach conjecture for the index $n+1$ follows.

In 1984 de Branges [6] verified the Milin, and therefore the Bieberbach conjecture, and in 1991, Weinstein [19] gave a different proof. A reference other than [6] concerning de Branges’ proof is [7], and a German language summary of the history of the Bieberbach conjecture and its proofs was given in [9].

Both proofs use the positivity of special function systems, and independently Todorov [17] and Wilf [20] showed that (the t-derivatives of the) de Branges functions and Weinstein’s functions essentially are the same (see also [12]),

$$\dot{\tau}_k^n(t) = -k \Lambda_k^n(t),$$

$\tau_k^n(t)$ denoting the de Branges functions and $\Lambda_k^n(t)$ denoting the Weinstein functions, respectively.

Whereas de Branges applied an identity of Askey and Gasper [2] to his function system, Weinstein applied an addition theorem for Legendre polynomials to his function system to deduce the positivity result needed.

By the relation $\dot{\tau}_k^n(t) \leqslant 0$, the de Branges functions $\tau_k^n(t)$ are monotonic, and $\tau_k^n(t) \geqslant 0$ follows. In this article, we reconsider the de Branges and Weinstein functions, find more relations connecting them with each other, and make the above positivity and monotony result more precise, e.g., by showing $\tau_k^n(t) \geqslant (n-k+1)\mathrm{e}^{-kt}$.

2. The Löwner chain of the Koebe function

We consider the Löwner chain

$$w(z,t) := K^{-1}(\mathrm{e}^{-t}K(z)) \quad (z \in \mathbb{D},\ t \geqslant 0) \tag{3}$$

of bounded univalent functions in the unit disk $\mathbb{D}$ which is defined in terms of the Koebe function (1). Since K maps the unit disk onto the entire plane slit along the negative x-axis in the interval $(-\infty, -\frac{1}{4}]$, the image $w(\mathbb{D}, t)$ is the unit disk with a radial slit on the negative x-axis increasing with t.

The function $w(z,t)$ is implicitly given by the equation

$$K(w(z,t)) = \mathrm{e}^{-t}K(z),$$

and satisfies therefore [1]

$$K'(w) \cdot w' = \mathrm{e}^{-t}K'(z).$$

This gives

$$\frac{1+w}{(1-w)^3} \cdot w'(z,t) = \mathrm{e}^{-t}\,\frac{1+z}{(1-z)^3}$$

and therefore

$$\begin{aligned} w'(z,t) &= \frac{(1-w)^3}{1+w}\,\mathrm{e}^{-t}\,\frac{1+z}{(1-z)^3} \\ &= \frac{(1-w)^4}{1-w^2}\,\mathrm{e}^{-t}\,\frac{1-z^2}{(1-z)^4} \\ &= \frac{w^2}{1-w^2}\,\frac{(1-w)^4}{w^2}\,\mathrm{e}^{-t}\,\frac{1-z^2}{(1-z)^4} \\ &= \frac{w^2}{1-w^2}\,\frac{1}{K(w)^2}\,\mathrm{e}^{-t}\,\frac{1-z^2}{(1-z)^4} \\ &= \frac{w^2}{1-w^2}\,\mathrm{e}^{2t}\,\frac{1}{K(z)^2}\,\mathrm{e}^{-t}\,\frac{1-z^2}{(1-z)^4} \\ &= \frac{w^2}{1-w^2}\,\mathrm{e}^{t}\,\frac{1-z^2}{z^2}. \end{aligned} \tag{4}$$

[1] In the following deductions, for simplicity we omit the arguments.

A calculation shows moreover that $w(z,t)$ has the explicit representations [10]

$$w(z,t) = \frac{(1-z)\sqrt{1-2xz+z^2} - 1 + (1+x)z - z^2}{z(x-1)} = \frac{4e^{-t}z}{(1-z+\sqrt{1-2xz+z^2})^2}. \tag{5}$$

In this article we use interchangeably the variables t, x and y that are related by $y = e^{-t} = (1-x)/2$. The interval $t \in (0,\infty)$ corresponds to the intervals $y \in (0,1)$ and $x \in (-1,1)$, respectively.[2]

From the left-hand representation (5) of $w(z,t)$ we obtain the simple equation

$$\frac{1+w}{1-w} = \frac{\sqrt{1-2xz+z^2}}{1-z} \tag{6}$$

that we will need later.

The Löwner chain of the Koebe function $w(z,t)$ is a hypergeometric function and has hypergeometric Taylor coefficients

$$w(z,t) = -\frac{e^t}{2K(z)} \cdot {}_1F_0\left(\begin{matrix} -\frac{1}{2} \\ - \end{matrix}\,\middle|\, -4e^{-t}K(z)\right) = \sum_{n=0}^{\infty} n e^{-t} {}_2F_1\left(\begin{matrix} 1-n, 1+n \\ 3 \end{matrix}\,\middle|\, e^{-t}\right) z^n$$

(see e.g. [12] and for a computer generated proof [13]).

The function

$${}_pF_q\left(\begin{matrix} a_1,\ldots,a_p \\ b_1,\ldots,b_q \end{matrix}\,\middle|\, x\right) := \sum_{k=0}^{\infty} A_k x^k = \sum_{k=0}^{\infty} \frac{(a_1)_k \cdots (a_p)_k}{(b_1)_k \cdots (b_q)_k} \frac{x^k}{k!},$$

where $(a)_k = a(a+1)\cdots(a+k-1)$ denotes the *Pochhammer symbol*, is called the *generalized hypergeometric series*. Its coefficient term ratio

$$\frac{A_{k+1}x^{k+1}}{A_k x^k} = \frac{(k+a_1)\cdots(k+a_p)}{(k+b_1)\cdots(k+b_q)} \frac{x}{(k+1)}$$

is a general rational function, in factorized form. More informations about generalized hypergeometric functions can be found in [4] or [10].

3. The de Branges and Weinstein functions

de Branges [6] showed that the Milin conjecture is valid if for all $n \geqslant 1$ the *de Branges functions* $\tau_k^n : \mathbb{R}^+ \to \mathbb{R}$ $(k = 1,\ldots,n)$ defined by the system of differential equations

$$\tau_{k+1}^n(t) - \tau_k^n(t) = \frac{\dot{\tau}_k^n(t)}{k} + \frac{\dot{\tau}_{k+1}^n(t)}{k+1} \quad (k = 1,\ldots,n) \tag{7}$$

$$\tau_{n+1}^n \equiv 0 \tag{8}$$

with the initial values

$$\tau_k^n(0) = n+1-k \tag{9}$$

[2] Sometimes, we use sloppy notation when changing arguments according to these rules, e.g., using the notation $\tau_k^n(y)$ instead of the correct composition $\tau_k^n(-\ln y)$, considering τ_k^n as function of the variable y. This applies in particular to our graphs that we generally plot w.r.t. the y-variable in the interval $(0,1)$.

have the properties

$$\lim_{t\to\infty} \tau_k^n(t) = 0 \tag{10}$$

and

$$\dot{\tau}_k^n(t) \leqslant 0 \quad (t \in \mathbb{R}^+). \tag{11}$$

Relation (10) is easily checked using standard methods for ordinary differential equations, whereas (11) is a deep result.

de Branges gave the explicit representation

$$\tau_k^n(t) = \mathrm{e}^{-kt} \binom{n+k+1}{2k+1} {}_4F_3\left(\begin{matrix} k+1/2, n+k+2, k, k-n \\ k+1, 2k+1, k+3/2 \end{matrix} \middle| \mathrm{e}^{-t} \right) \tag{12}$$

[6,8,16], with which the proof of the de Branges theorem was completed as soon as de Branges realized that (11) was a theorem previously proved by Askey and Gasper [2].

On the other hand, Weinstein [19] used the Löwner chain (3), and showed the validity of Milin's conjecture if for all $n \geqslant 1$ the *Weinstein functions* $\Lambda_k^n : \mathbb{R}^+ \to \mathbb{R}$ $(k = 1, \ldots, n)$ defined by

$$\frac{\mathrm{e}^t w(z,t)^{k+1}}{1 - w^2(z,t)} =: \sum_{n=k}^{\infty} \Lambda_k^n(t) z^{n+1} = W_k(z,t) \tag{13}$$

satisfy the relations

$$\Lambda_k^n(t) \geqslant 0 \quad (t \in \mathbb{R}^+,\ k, n \in \mathbb{N}). \tag{14}$$

Weinstein did not identify the functions $\Lambda_k^n(t)$, but was able to prove (14) without an explicit representation.

Independently, both Todorov [17] and Wilf [20] proved—using the explicit representation (12) of the de Branges functions—that

$$\dot{\tau}_k^n(t) = -k \Lambda_k^n(t), \tag{15}$$

i.e., the (t-derivatives of the) de Branges functions and the Weinstein functions essentially are the same, and the main inequalities (11) and (14) are identical. In [10] another proof of (15) was given that does not use the explicit representation of the de Branges functions. Note further that in [13], we deduced result (14) using a version of the addition theorem for the Gegenbauer polynomials whose simple proof is contained in the same article.

In this article, we will use inequality (14) which is equivalent to the Askey–Gasper inequality stated in [2], as well as the inequality

$$\tau_k^n(t) \geqslant 0 \tag{16}$$

which easily follows from (14) by (15). Actually, we will refine statement (16).

Note that identity (15) yields the representation

$$\Lambda_k^n(z,t) = \mathrm{e}^{-kt} \binom{k+n+1}{1+2k} {}_3F_2\left(\begin{matrix} k+1/2, k-n, 2+k+n \\ 1+2k, k+3/2 \end{matrix} \middle| \mathrm{e}^{-t} \right) \tag{17}$$

which however can be also detected directly from the defining relations (13), see [11].

4. Inequalities for the de Branges functions

The following theorem states a recurrence relation between the de Branges functions $\tau_k^n(t)$ and the Weinstein functions $\Lambda_k^n(t)$.

Theorem 1. *We have* $\tau_k^k(t) = \Lambda_k^k(t) = \mathrm{e}^{-kt}$, *and for* $n \geqslant k+1$ *the relation*

$$\frac{1}{n+1}\,\tau_k^n(t) - \frac{1}{n}\,\tau_k^{n-1}(t) = \frac{k}{n(n+1)}\,\Lambda_k^n(t) + \frac{k}{n(n+1)}\,\Lambda_k^{n-1}(t)$$

is valid.

Proof. We consider the generating functions

$$B_k(z,t) := K(z)w(z,t)^k = \sum_{n=k}^{\infty} \tau_k^n(t) z^{n+1} \tag{18}$$

(see [12, Theorem 3]), and (13), i.e.

$$W_k(z,t) = \frac{\mathrm{e}^t w(z,t)^{k+1}}{1-w^2(z,t)} = \sum_{n=k}^{\infty} \Lambda_k^n(t) z^{n+1},$$

of $\tau_k^n(t)$ and $\Lambda_k^n(t)$, respectively. It is easy to check the differential equation

$$(1-z)\,zB_k'(z) - (1+z)\,B_k(z) = (1+z)\,kW_k(z) \tag{19}$$

relating $B_k(z)$ and $W_k(z)$.[3] Writing as Taylor expansions and equating the coefficients of z^n yields

$$n\tau_k^n(t) - (n+1)\tau_k^{n-1}(t) = k\Lambda_k^n(t) + k\Lambda_k^{n-1}(t).$$

This finishes the proof. □

The theorem has some immediate consequences.

Corollary 2. *For* $n \geqslant k$, $0 \leqslant t < \infty$ *we have* (Figs. 1 and 2):

(a) $\tau_k^n(t)$ *increases w.r.t.* n, i.e. $\tau_k^n(t) > \tau_k^{n-1}(t)$;
(b) *the Taylor expansion of the function* $\frac{w(z,t)^k}{1-z}$ *has nonnegative Taylor coefficients*;
(c) $\tau_k^n(t) \geqslant \tau_k^k(t) = \mathrm{e}^{-kt}$;
(d) $\frac{1}{n+1}\,\tau_k^n(t)$ *increases w.r.t.* n, i.e. $\frac{1}{n+1}\,\tau_k^n(t) > \frac{1}{n}\,\tau_k^{n-1}(t)$;
(e) $\tau_k^n(t) \geqslant \frac{n+1}{(k+1)}\,\mathrm{e}^{-kt}$.

[3] Using a computer algebra system, e.g., the difference of left- and right-hand side of (19) easily simplifies to 0 after replacing $B_k(z)$ by (18), $W_k(z)$ by (13), $w'(z) \to w^2/(1-w^2)(1-z^2)/yz^2$ by (4) and finally $K(z) \to z/(1-z)^2$.

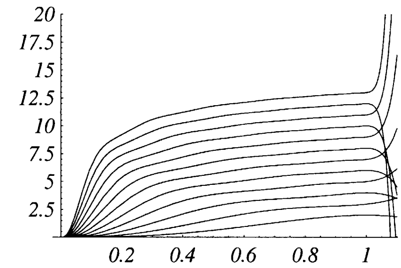

Fig. 1. This shows Corollary 2(a) in the form: $\tau_k^n(y)$ increases w.r.t. n, for $k=3$.

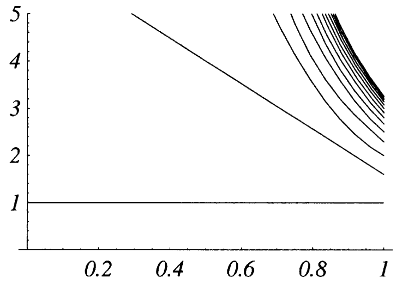

Fig. 2. This shows Corollary 2(e) for $k=3$ in the form $(k+1)/(n+1)y^{-k}\tau_k^n(y) \geqslant 1$.

Proof. The equation

$$\tau_k^n(t) - \tau_k^{n-1}(t) = \frac{1}{n}\,\tau_k^{n-1}(t) + \frac{k}{n}\,\Lambda_k^n(t) + \frac{k}{n}\,\Lambda_k^{n-1}(t)$$

restates Theorem 1, hence (a) follows by (14) and (16). Inequality (a) and

$$\frac{z}{1-z}\,w(z,t)^k = (1-z)B_k(z) = \sum_{n=k}^{\infty}(\tau_k^n(t) - \tau_k^{n-1}(t))z^{n+1}$$

implies (b). Induction applied to (a) yields (c). Finally, the inequality

$$\frac{1}{n+1}\,\tau_k^n(t) > \frac{1}{n}\,\tau_k^{n-1}(t)$$

is also an immediate consequence of Theorem 1, implying (d). From this, (e) follows by induction, again. □

Summing the identity of Theorem 1, we get moreover:

Corollary 3. *For* $n \geqslant k+1,\ 0 \leqslant t < \infty$ *we have* (Figs. 3 and 4):

(a) $\tau_k^n(t) = \frac{k}{n}\Lambda_k^n(t) + 2k(n+1)\sum_{j=k}^{n-1}\frac{1}{j(j+2)}\Lambda_k^j(t)$;
(b) $e^{nt}\tau_k^n(t)$ *increases w.r.t.* t, i.e. $\frac{d}{dt}(e^{nt}\tau_k^n(t)) \geqslant 0$;
(c) $\tau_k^n(t) \geqslant e^{-nt}(n-k+1)$.

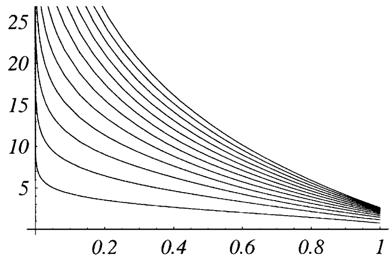

Fig. 3. This shows Corollary 3(b) in the form: $\ln(y^{-n}\tau_k^n(y))$ decreases w.r.t. y, for $k=3$.

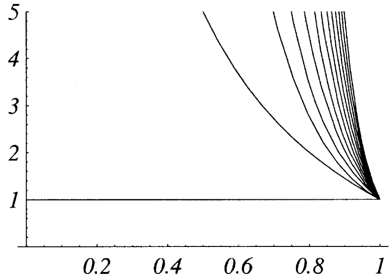

Fig. 4. This shows Corollary 3(c) for $k=3$ in the form $(y^{-n}/(n-k+1))\tau_k^n(y) \geqslant 1$.

Proof. (a) Follows by replacing n by j in Theorem 1 and summing for $j=k+1,\ldots,n$. (b) From this formula, it follows

$$n\tau_k^n(t) - k\Lambda_k^n(t) = 2kn(n+1)\sum_{j=k}^{n-1} \frac{1}{j(j+2)} \cdot \Lambda_k^j(t)$$

and by using (15), moreover

$$n\tau_k^n(t) + \dot{\tau}_k^n(t) = 2kn(n+1)\sum_{j=k}^{n-1} \frac{1}{j(j+2)}\, \Lambda_k^j(t).$$

Multiplying by e^{nt}, we get (b) by using the positivity statement (14).

(c) By (b), the function $e^{nt}\tau_k^n(t)$ is increasing w.r.t. t, hence by using the boundary value $\tau_k^n(0)$ we get (c). □

Next, we give a recurrence-differential equation which is valid for both $\tau_k^n(t)$ and $\Lambda_k^n(t)$ keeping k fixed.

Theorem 4. *The de Branges functions $\tau_k^n(t)$ and the Weinstein functions $\Lambda_k^n(t)$ both satisfy the following recurrence-differential equation*

$$(n+1)\tau_k^{n-1}(t) - n\tau_k^n(t) = \dot{\tau}_k^{n-1}(t) + \dot{\tau}_k^n(t), \quad (n \geqslant k+1),\ \tau_k^k(t) = e^{-kt}, \tag{20}$$

and

$$(n+1)\Lambda_k^{n-1}(t) - n\Lambda_k^n(t) = \dot{\Lambda}_k^{n-1}(t) + \dot{\Lambda}_k^n(t), \quad (n \geqslant k+1), \;\; \Lambda_k^k(t) = \mathrm{e}^{-kt}.$$

Proof. Using Theorem 1 and (15) gives the result for $\tau_k^n(t)$. Differentiating this result w.r.t. t yields the result for $\Lambda_k^n(t)$. □

Using de Branges' original differential equations, the following is a consequence of Theorem 4.

Corollary 5 (Recursive computation of the de Branges and Weinstein functions I).
(a) *For the de Branges functions the following recurrence relation is valid*

$$k(n+k)\tau_{k-1}^{n-1}(t) + (k-1)(n-k+1)\tau_k^{n-1}(t) = (k-1)(n+k)\tau_k^n(t) + k(n-k+1)\tau_{k-1}^n(t).$$

(b) *This yields the recursive scheme*

$$(k-1)\tau_k^n(t) = -\frac{n-k+1}{n+k}\, k\tau_{k-1}^n(t) + 2k(2k-1) \times \frac{(n-k+1)!}{(n+k)!} \sum_{j=k-1}^{n-1} \frac{(j+k-1)!}{(j-k+2)!}\,(j+1)\tau_{k-1}^j(t).$$

(c) *For the Weinstein functions the following recurrence relation is valid*

$$(n+k)\Lambda_{k-1}^{n-1}(t) + (n-k+1)\Lambda_k^{n-1}(t) = (n+k)\Lambda_k^n(t) + (n-k+1)\Lambda_{k-1}^n(t).$$

(d) *This yields the recursive scheme*

$$\Lambda_k^n(t) = -\frac{n-k+1}{n+k}\,\Lambda_{k-1}^n(t) + 2(2k-1)\,\frac{(n-k+1)!}{(n+k)!} \sum_{j=k-1}^{n-1} \frac{(j+k-1)!}{(j-k+2)!}\,(j+1)\Lambda_{k-1}^j(t).$$

Proof. Writing (20) for k and for $k-1$, and writing the de Branges system (7) for n and for $n-1$ gives four equations from which the three derivative terms can be eliminated by linear algebra. This yields (a). Differentiating and using (15) yields (c).

To get (d), one writes (c) for $n,\ n-1,\dots,k$ which yields a system of linear equations for the unknowns $\Lambda_k^j(t)$, $(j=k,\dots,n)$. Solving this linear system yields (d). The deduction of (b) follows in a similar manner. □

Note that by an application of Zeilberger's algorithm [10], one gets recurrence relations with a finite number of terms (independent of n) that also enable the recursive computation of $\tau_k^n(t)$ and $\Lambda_k^n(t)$ which is asymptotically more efficient. These relations, however, look much more difficult, their structure is much less symmetric and their coefficients contain $y=\mathrm{e}^{-t}$. These results are collected in

Theorem 6 (Recursive computation of the de Branges and Weinstein functions II).
(a) *For the de Branges functions the following recurrence relation w.r.t. n is valid*

$$0 = (2n-1)(n+k-2)(n-k-2)\tau_k^{n-4}(t)$$

$$+2(4yn^3-4n^3+15n^2-12yn^2+11yn-17n+5+k^2-3y)\tau_k^{n-3}(t)$$
$$-4(n-1)(4yn^2-3n^2-8yn+6n-k^2-2+3y)\tau_k^{n-2}(t)$$
$$+2(4yn^3-4n^3-12yn^2+9n^2-5n+11yn-3y-k^2+1)\tau_k^{n-1}(t)$$
$$+(-3+2n)(n-k)(n+k)\tau_k^n(t).$$

(b) *For the de Branges functions the following recurrence relation w.r.t. k is valid*:

$$0=y(k-1)(2k-3)(n+k-2)(n-k+4)\tau_{k-4}^n(t)-2(k-1)$$
$$\times(4k^3-4yk^3+29yk^2-28k^2-67yk+63k+6yn+3yn^2+51y-45)\tau_{k-3}^n(t)$$
$$+2(k-3)$$
$$\times(4yk^3-4k^3-19yk^2+20k^2+27yk-31k+6yn+3yn^2-9y+15)\tau_{k-1}^n(t)$$
$$+y(2k-5)(k-3)(n+k)(n-k+2)\tau_k^n(t)-2(2yk^2n^2-8ykn^2+3yn^2$$
$$+4yk^2n-16ykn+6yn+6yk^4-8k^4-48yk^3+64k^3+137yk^2-182k^2$$
$$-164yk+216k+66y-90)\tau_{k-2}^n(t).$$

(c) *For the Weinstein functions the following recurrence relation w.r.t. n is valid*:

$$0=-(n+k-1)(n-k-1)n\Lambda_k^{n-3}(t)$$
$$-(-3n^2+4yn^2-2yn+2n-k^2)(n-1)\Lambda_k^{n-2}(t)$$
$$+n(-3n^2+4n+4yn^2-6yn+2y-1-k^2)\Lambda_k^{n-1}(t)$$
$$+(n-k)(n+k)(n-1)\Lambda_k^n(t).$$

(d) *For the Weinstein functions the following recurrence relation w.r.t. k is valid*:

$$0=-y(k-1)(n+k-1)(n+3-k)\Lambda_{k-3}^n(t)$$
$$-(k-2)(3yk^2-4k^2-8yk+10k+yn^2+2yn+6y-6)\Lambda_{k-2}^n(t)$$
$$+(k-1)(3yk^2-4k^2-10yk+14k+yn^2+9y-12+2yn)\Lambda_{k-1}^n(t)$$
$$+y(k-2)(n+k)(n-k+2)\Lambda_k^n(t).$$

Proof. These computations were done with the Maple `sumtools` package[4] by the author which contains an implementation (`sumrecursion`) of Zeilberger's algorithm. Note that only equation (b) is hard and time consuming to obtain, the other three computations take only some seconds. □

[4] Similarly the package `hsum6.mpl` [10] can be used.

In the sequel we would like to strengthen the inequalities that were obtained in Corollaries 2 and 3. For this purpose, we use the identity

$$B_k(z,t) = \frac{1}{1-z}\sqrt{1-2xz+z^2}\,W_k(z,t) \tag{21}$$

between the generating functions of $\tau_k^n(t)$ and $\Lambda_k^n(t)$, which is easily verified by (6) using the defining equations (18) and (13).

We get

Theorem 7. *For $n \geqslant k$, $0 \leqslant t < \infty$ we have (Figs. 5 and 6)*

(a) $e^{kt}\tau_k^n(t)$ *increases w.r.t.* t, i.e. $\frac{d}{dt}(e^{kt}\tau_k^n(t)) \geqslant 0$;
(b) $(n-k+1)e^{-kt} \leqslant \tau_k^n(t) \leqslant \binom{n+k+1}{2k+1} e^{-kt}$.

Proof. Since

$$\sqrt{1-2xz+z^2} = \sum_{j=0}^{\infty} C_j^{-1/2}(x) z^j,$$

$C_j^{\alpha}(x)$ denoting the Gegenbauer polynomials (see e.g. [1]), we get

$$\frac{1}{1-z}\sqrt{1-2xz+z^2} = \sum_{n=0}^{\infty} s_n(x) z^n$$

with

$$s_n(x) = \sum_{j=0}^{n} C_j^{-1/2}(x).$$

As a lemma, we will prove that $s_n(x)$ is non-negative for $x \in [-1,1]$. Since for every non-negative integer $n \in \mathbb{N}$ we have $\sum_{j=0}^{n} P_j(x) \geqslant 0$ for the Legendre polynomials $P_n(x)$ (see e.g. [3, Theorem 2]) and since [18, (7.11)]

$$\frac{d}{dx} C_j^{-1/2}(x) = -C_{j-1}^{1/2}(x) = -P_{j-1}(x),$$

we get

$$s_n'(x) = -\sum_{j=0}^{n-1} P_j(x) \leqslant 0,$$

hence $s_n(x)$ is decreasing. The boundary values $s_n(-1) = 2$ and $s_n(1) = 0$ give $0 \leqslant s_n(x) \leqslant 2$ which proves our lemma (Figs. 7 and 8).

By equating coefficients in (21), we get

$$\tau_k^n(t) = \sum_{j=k}^{n-1} s_{n-j}(x) \cdot \Lambda_k^j(t) + \Lambda_k^n(t) \quad (n \geqslant k+1).$$

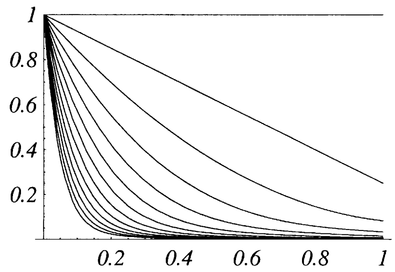

Fig. 5. This shows the first inequality of Theorem 7 for $k=3$ in the form: $y^{-k}/\binom{n+k+1}{2k+1}\tau_k^n(y) \leqslant 1$.

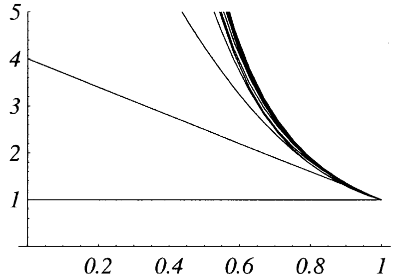

Fig. 6. This shows the second inequality of Theorem 7 for $k=3$ in the form: $y^{-k}/(n-k+1)\tau_k^n(y) \geqslant 1$.

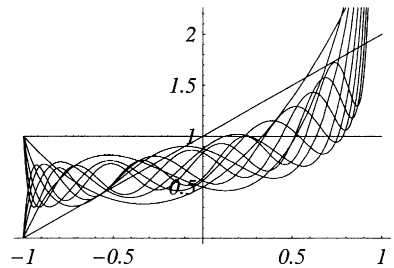

Fig. 7. This shows the non-negativity of $\sum_{j=0}^{n} P_j(x)$ for $n=0,\ldots,10$.

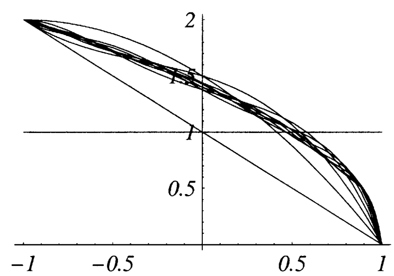

Fig. 8. This shows the non-negativity and monotony of $s_n(x)$ for $n=0,\ldots,10$.

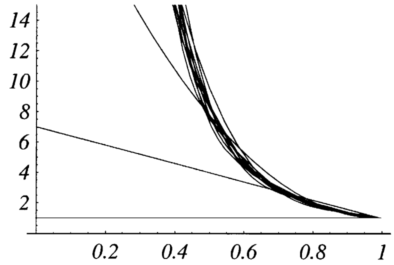

Fig. 9. This shows Theorem 8 for $k=3$ in the form: $y^{-k}(\tau_k^n(y)-\tau_k^{n-1}(y))\geqslant 1$ and decreasing.

This yields

$$k\tau_k^n(t)-k\Lambda_k^n(t)=k\sum_{j=k}^{n-1}s_{n-j}(x)\cdot\Lambda_k^j(t),$$

or by using (15) again,

$$k\tau_k^n(t)+\dot\tau_k^n(t)=k\sum_{j=k}^{n-1}s_{n-j}(x)\cdot\Lambda_k^j(t).$$

Multiplying by e^{kt} gives

$$\frac{d}{dt}(e^{kt}\tau_k^n(t))=ke^{kt}\sum_{j=k}^{n-1}s_{n-j}(x)\cdot\Lambda_k^j(t),$$

which by (14) and the lemma yields (a).

(b) The initial value $\tau_k^n(0)=n-k+1$ and (a) give the left inequality of (b). The right inequality is a consequence of the limit relation

$$\lim_{t\to\infty}e^{kt}\tau_k^n(t)=\binom{n+k+1}{2k+1},$$

which follows easily from the hypergeometric representation (12). □

Note that, since $n\geqslant k$, Theorem 7(b) is stronger than Corollary 3(d). The following theorem strengthens Theorem 7.

Theorem 8. *For $n\geqslant k$, $0\leqslant t<\infty$ we have (Fig. 9)*

(a) $e^{kt}(\tau_k^n(t)-\tau_k^{n-1}(t))$ *increases w.r.t. t*, i.e. $\frac{d}{dt}(e^{kt}(\tau_k^n(t)-\tau_k^{n-1}(t)))\geqslant 0$;
(b) $\tau_k^n(t)-\tau_k^{n-1}(t)\geqslant e^{-kt}$.

Proof. From the hypergeometric representation (12) of $\tau_k^n(t)$ one deduces by an elementary computation

$$\tau_k^n(t)-\tau_k^{n-1}(t)=e^{-kt}\binom{n+k}{n-k}\,{}_3F_2\left(\left.\begin{matrix}n+k+1,k,k-n\\ k+1,\ 2k+1\end{matrix}\right|e^{-t}\right),\tag{22}$$

an explicit hypergeometric representation for the difference $\tau_k^n(t) - \tau_k^{n-1}(t)$.[5] By differentiating the right hand hypergeometric function of (22) one gets

$$\frac{\mathrm{d}}{\mathrm{d}t}\,{}_3F_2\left(\left.\begin{matrix} n+k+1, k, k-n \\ k+1, 2k+1 \end{matrix}\right| \mathrm{e}^{-t}\right) = \mathrm{e}^{-t}\frac{(n+k+1)k(n-k)}{(k+1)(2k+1)}\,{}_3F_2\left(\left.\begin{matrix} n+k+2, k+1, k+1-n \\ k+2, 2k+2 \end{matrix}\right| \mathrm{e}^{-t}\right)$$

which is of the Askey–Gasper type [2]

$$\frac{(\alpha+2)_N}{N!}\cdot {}_3F_2\left(\left.\begin{matrix} -N,\ N+2+\alpha,\ (\alpha+1)/2 \\ \alpha+1,\ (\alpha+3)/2 \end{matrix}\right| x\right)$$

$$= \sum_{j=0}^{\lfloor (N-k)/2 \rfloor} \frac{\left(\frac{1}{2}\right)_j \left(\frac{\alpha}{2}+1\right)_{N-j} \left(\frac{(\alpha+3)}{2}\right)_{N-2j} (\alpha+1)_{N-2j}}{j!\left(\frac{(\alpha+3)}{2}\right)_{N-j}\left(\frac{(\alpha+1)}{2}\right)_{N-2j}(N-2j)!}$$

$$\times\, {}_3F_2\left(\left.\begin{matrix} 2j-N,\ N-2j+\alpha+1, (\alpha+1)/2 \\ \alpha+1, (\alpha+2)/2 \end{matrix}\right| x\right)$$

for $\alpha = 2k+1,\ N = n-k-1$, and therefore can be rewritten as non-negative linear combination. This yields (a). Relation (b) is an immediate consequence thereof. □

5. Non-negative hypergeometric functions

In this section we will combine the two main relations $\tau_k^n(k) \geqslant 0$ (16) and $\Lambda_k^n(t) \geqslant 0$ (14) to get new non-negativity results for certain hypergeometric functions. This will extract also some interesting informations about the de Branges and Weinstein functions. In particular, both sum and difference of the de Branges and Weinstein functions turn out to be non-negative, and are only shifts w.r.t. k of each other.

For this section we use the notation

$$S_k^n(t) := \mathrm{e}^{-kt}\binom{n+k+1}{2k+1}\,{}_4F_3\left(\left.\begin{matrix} k, k+1/2, k-n, n+k+2 \\ k+1, k+3/2, 2k \end{matrix}\right| \mathrm{e}^{-t}\right).$$

These functions turn out to be non-negative. We get the following results

Theorem 9. *For $n \geqslant k,\ 0 \leqslant t < \infty$ we have* (Figs. 10 and 11):

(a) $S_k^n(t) = \frac{1}{2}(\tau_k^n(t) + \Lambda_k^n(t))$ *and* $S_{k+1}^n(t) = \frac{1}{2}(\tau_k^n(t) - \Lambda_k^n(t))$;
(b) $\tau_k^n(t) = S_k^n(t) + S_{k+1}^n(t)$ *and* $\Lambda_k^n(t) = S_k^n(t) - S_{k+1}^n(t)$;
(c) $S_k^n(t) \geqslant 0$.

[5] Such computations can be done automatically, e.g., by the `sumtohyper` command of the Maple `sumtools` package by the first author.

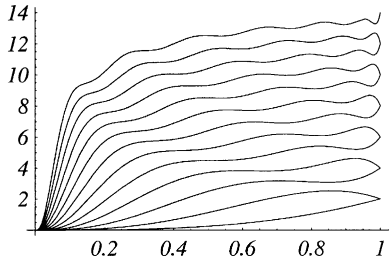

Fig. 10. This shows Theorem 9 for $k=3$ in the form: $2S_k^n(t)=\tau_k^n(y)+\Lambda_k^n(y)\geqslant 0$.

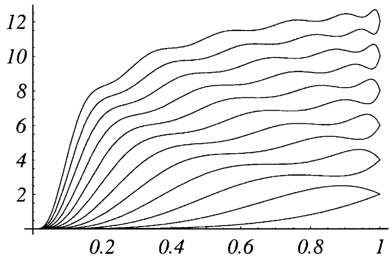

Fig. 11. This shows Theorem 9 for $k=3$ in the form: $\tau_k^n(y)-\Lambda_k^n(y)\geqslant 0$.

Proof. A simple computation[6] shows that

$$\tfrac{1}{2}(\tau_t^n(t)+\Lambda_k^n(t))=S_k^n(t). \tag{23}$$

Similarly the equation

$$\tfrac{1}{2}(\tau_k^n(t)-\Lambda_k^n(t))=S_{k+1}^n(t)$$

is deduced. This gives (a), whereas (b) follows by linear algebra from (a). Note that the identity $\tau_k^n(t)-\Lambda_k^n(t)=\tau_{k+1}^n(t)+\Lambda_{k+1}^n(t)$ follows also easily from the de Branges differential equations (7) and (15).

By (23) and the non-negativity of $\tau_k^n(t)$ and $\Lambda_k^n(t)$, (c) follows. □

Appendix: Maple code

The authors provide a Maple worksheet containing the computations of this article. This worksheet can be downloaded from the web site `http://www.mathematik.uni-kassel.de/~koepf/Publikationen`.

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[6] For example, using the `sumtohyper` command of the Maple `sumtools` package.

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