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Gröbner Bases and Triangles

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Abstract:

It is well-known that by polynomial elimination methods, in particular by the computation of Gröbner bases, proofs for geometric theorems can be automatically generated. On the other hand, it is much less known that Gröbner bases, in combination with rational factorization, can be even used to *find* new geometric theorems.

In this article such a method is described, and some new theorems on plane triangles are deduced.

1 Polynomial System Describing a Triangle

In this section, we introduce the elementary polynomial equations describing a plane triangle by giving the connection between its sidelengths a, b and c, the radii r and R of its incircle and circumcircle, respectively, and its area A and circumference $2s$.

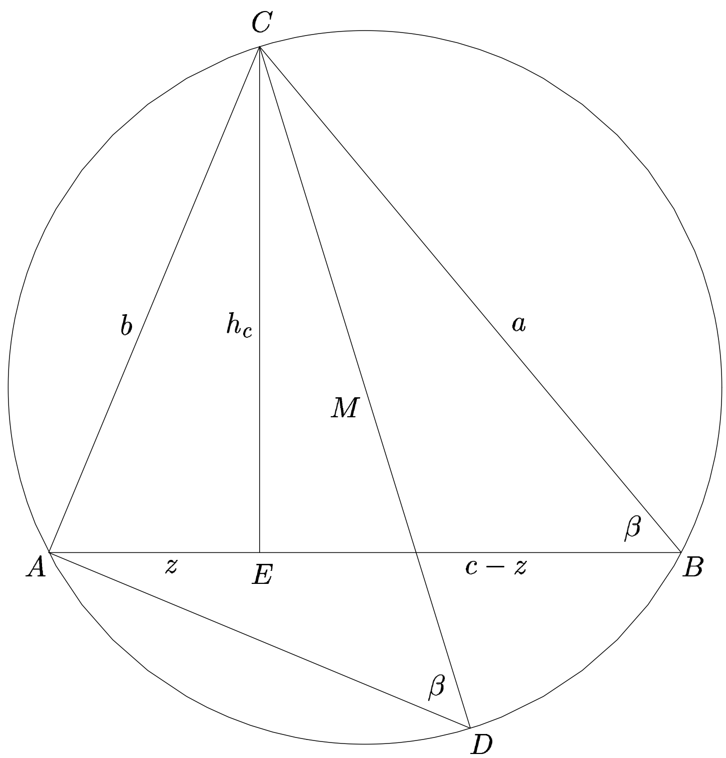

Figure 1: A general triangle ABC

Obviously we have

$$p_1 := 2\,s - a - b - c = 0 \tag{1}$$

for the circumference, and for the area one has the elementary result

$$p_2 := A - r\,s = 0\;. \tag{2}$$

To receive more equations, we use the auxiliary variables h_c and z, see Figure 1. In terms of these variables, the area satisfies the simple equation

$$p_3 := c\,h_c - 2\,A = 0\;. \tag{3}$$

By the Pythagorean theorem, we have moreover

$$p_4 := b^2 - h_c^2 - z^2 = 0 \tag{4}$$

and

$$p_5 := a^2 - h_c^2 - (c-z)^2 = 0\;. \tag{5}$$

We deduce a final equation completing our knowledge about the connections between our variables (see [4]–[5]). By Thales' theorem, the triangle CAD in Figure 1 is right-angled. Hence the triangles CAD and CEB are similar, and we get the sidelength ratio

$$\frac{a}{h_c} = \frac{2R}{b}\;.$$

Therefore the polynomial identity

$$p_6 := a\,b - 2\,R\,h_c = 0 \tag{6}$$

is valid.

We conclude that Equations (1)–(6) are six polynomial equations completely characterizing the triangle. Since h_c and z for us are only auxiliary variables, we could try to eliminate them. Such an elimination can be done by a Gröbner basis computation which we will be do in the next section.

Four independent equations will survive, and we can eliminate three more variables to deduce an identity between four remaining arbitrarily chosen variables, in particular we can write any of s, A, r and R in terms of a, b and c only.

2 Elimination by Gröbner Bases

Here, we need the concept of a *Gröbner basis*. If one applies *Gauß' algorithm* to a linear system, the variables are eliminated iteratively, resulting in an equivalent system which is simpler in the sense that it contains some equations that are free of some variables involved. Note that connected with an application of Gauß' algorithm is a certain order of the variables. *Buchberger's algorithm* is an elimination process, given a certain term order for the variables, with which a polynomial system (rather than a linear one) is transformed, resulting in an equivalent system (i.e., constituting the same ideal) for which the terms that are largest with respect to the term order, are eliminated as far as possible. Note that—in contrast to the linear case—the resulting equivalent system may contain more polynomials than the original one. Such a rewritten system is called a Gröbner basis of the ideal generated by the polynomial system given. For the purposes of this article we will only need the *lexicographical term order* which has the *elimination property*: the variables having highest order are eliminated as far as

possible. On the other hand this procedure obviously can increase the degrees of the remaining polynomials considerably. We will see this effect soon in our example case.
In computer algebra systems like Axiom, Macsyma, Maple, Mathematica, MuPAD or REDUCE the computation of Gröbner bases is accessible. We used REDUCE [6] for the computations of this paper since REDUCE has quite an efficient Gröbner basis implementation [10] (procedure `groebner`).
Note that a Gröbner basis $\mathcal{G}$ constitutes a *normal form* for an ideal $\mathcal{I}$ given by a set $\mathcal{P}$ of polynomials, and using $\mathcal{G}$ it can be easily checked whether or not a given polynomial p is a member of $\mathcal{I}$, i.e. whether the vanishing of the polynomials of $\mathcal{P}$ implies the vanishing of p (in REDUCE: procedure `preduce`).
Looking at our given set of polynomials, we can ask the question to eliminate the auxiliary variables h_c and z. We get

Theorem 1 The following is a system of equations equivalent to Equations (1)–(6) with the variables h_c and z eliminated, hence describing the triangle by presenting the connection between a, b, c, r, R, s and A:

$$\begin{aligned} p_1 &= 2\,s - a - b - c = 0\,, \\ p_2 &= A - r\,s = 0\,, \\ p_8 &:= 4\,A\,R - a\,b\,c = 0\,, \\ p_{11} &:= (b + c - a)\,(a + b - c)\,(a - b + c) - 8\,A\,r = 0\,. \end{aligned}$$

Proof: We compute a Gröbner basis of $\{p_1, p_2, p_3, p_4, p_5, p_6\}$ with respect to the lexicographical variable order $\{z, h_c, R, r, A, s, a, b, c\}$. This term order eliminates the variables h_c and z as far as possible (and writes everything using a, b and c as much as possible). The resulting Gröbner basis turns out to be the set of polynomials

$$\begin{aligned} \Big\{ & {h_c}^2 + z^2 - b^2, (a+b)\,(a-b)\,h_c - 2\,(c - 2\,z)\,A, (3\,c - 2\,z)\,b^2 - c^3 + (c + 2\,z)\,a^2 - 8\,A\,h_c, \\ & -(c - 2\,z)\,c - b^2 + a^2, 2\,R\,h_c - a\,b, \\ & 2\,((2\,b - c - a)\,a - (b + c)\,b + c^2 + 4\,h_c\,r)\,A + (a + b)\,(a - b)^2\,h_c, -2\,A + c\,h_c, \\ & -((2\,b + 2\,c - a)\,a - b^2 + 2\,b\,c - c^2 - 4\,r^2 - 16\,R\,r)\,(b + c)^2\,b^2\,c, 4\,A\,R - a\,b\,c, \\ & (a + b + c)\,(a + b - c)\,(a - b + c)\,(a - b - c)\,R + 4\,A\,a\,b\,c, \\ & -((b + c - a)\,(a + b - c)\,(a - b + c)\,R + 2\,(b + c)\,b\,c\,r - 4\,A\,b\,c)\,(b + c)\,b, \\ & -((b + c - a)\,(a + b - c)\,(a - b + c) - 8\,A\,r)\,(b + c)\,b, (a + b + c)\,r - 2\,A, \\ & (a + b + c)\,(a + b - c)\,(a - b + c)\,(a - b - c) + 16\,A^2, -c + 2\,s - b - a \Big\}\,. \end{aligned}$$

Note that we have factored the polynomials (rational factorization is algorithmically accessible in computer algebra systems) which gives much more insight and the factorizations can be used to simplify our output further. Whereas from the point of view of ideal theory the polynomials in the Gröbner basis cannot be lowered in degree, from the point of view of geometry they can. The factors $(b+c)^2\,b^2\,c$ of the eighth polynomial of the given Gröbner basis, for example, cannot be zero: neither can a sidelength of a triangle equal zero, nor can the sum of two of them. Hence such factors can be divided out in our geometric situation.
The first seven polynomials contain still the auxiliary variables h_c and z, hence they are not of interest to us. The rest of the polynomials, divided by factors that cannot take the value

zero in a triangle, is given by

$$\Big\{\, 2\,a\,b + 2\,a\,c + 2\,b\,c - a^2 - b^2 - c^2 - 4\,r^2 - 16\,R\,r = 0, \tag{7}$$

$$4\,A\,R - a\,b\,c = 0, \tag{8}$$

$$(\,a + b + c\,)\,(\,a + b - c\,)\,(\,a - b + c\,)\,(\,a - b - c\,)\,R + 4\,A\,a\,b\,c = 0, \tag{9}$$

$$(\,b + c - a\,)\,(\,a + b - c\,)\,(\,a - b + c\,)\,R + 2\,(\,b + c\,)\,b\,c\,r - 4\,A\,b\,c = 0, \tag{10}$$

$$(\,b + c - a\,)\,(\,a + b - c\,)\,(\,a - b + c\,) - 8\,A\,r = 0, \tag{11}$$

$$(\,a + b + c\,)\,r - 2\,A = 0, \tag{12}$$

$$(\,a + b + c\,)\,(\,a + b - c\,)\,(\,a - b + c\,)\,(\,a - b - c\,) + 16\,A^2 = 0, \tag{13}$$

$$2\,s - a - b - c = 0 \,\Big\}\ . \tag{14}$$

Obviously some of the equations are redundant. Eight equations remain, and only four of them are independent. This is typical for Gröbner basis computations.
It can be shown (by hand calculations, or by Gröbner bases, again, with a normalzation using `preduce`) that the equations (8), (11), (12), and (14) imply the other four.[1] Note that, using `preduce`, polynomial (7) has to be multiplied by $(b+c)\,b\,c$ to be simplified to zero. Hence this polynomial is not equal to zero from the point of view of ideal theory, but is is zero from the point of view of triangle geometry since $(b+c)\,b\,c$ cannot equal zero. □

Note that the highest degree of our input polynomials (1)–(6) is two, whereas after elimination of h and z the highest degree polynomial of the Gröbner basis has degree seven. After reducing factors referring to geometrically impossible situations, and after omitting redundant equations, degree three polynomials remain.
As another comment we mention that the Gröbner basis computation has automatically deduced *Heron's formula* for the area of a triangle

$$A^2 = s(s-a)(s-b)(s-c)\ ,$$

here given in the form (13)

$$(\,a + b + c\,)\,(\,a + b - c\,)\,(\,a - b + c\,)\,(\,a - b - c\,) + 16\,A^2 = 0\ .$$

This form of Heron's formula can also be regarded as a way to write the area in terms of the sidelengths a, b and c, only. Here the other variables r, R and s have been eliminated. This is so since we had chosen the term order such that A, s, a, b, c are eliminated as little as possible. Note that the last equation (14) which is one of our input equations, gives such a relation between the last four of our chosen variables s, a, b, c.
To eliminate another triple of variables, and to obtain another identity between an arbitrarily chosen quadruple of $\{a, b, c, r, R, s, A\}$, one chooses a different term order. The Gröbner basis calculations give these identities automatically. We are mainly interested to write any of s, A, r and R in terms of a, b, c. Here are the results.

Theorem 2 The following are the identities defining any of s, A, r and R in terms of a, b, c:

$$2\,s - a - b - c = 0 \quad \text{or} \quad s = \frac{a + b + c}{2}\ ,$$

[1] With a little more effort, it can be also shown that (8), (12), (13), and (14) imply the other four, but (13) has higher degree than (11).

$$(a+b+c)(a+b-c)(a-b+c)(a-b-c)+16A^2=0$$

or

$$A^2=\frac{(a+b+c)(b+c-a)(a+b-c)(a-b+c)}{16},$$

$$4(a+b+c)r^2-(b+c-a)(a+b-c)(a-b+c)=0$$

or

$$r^2=\frac{(b+c-a)(a+b-c)(a-b+c)}{4(a+b+c)},$$

and

$$(a+b+c)(b+c-a)(a+b-c)(a-b+c)R^2-a^2b^2c^2=0$$

or

$$R^2=\frac{a^2b^2c^2}{(a+b+c)(b+c-a)(a+b-c)(a-b+c)}. \qquad \square$$

In the next section we will consider how theorems about triangles can be proved using elimination by Gröbner bases computations in combination with rational factorization.

3 Algorithm to Prove Implications

It is well-known that by polynomial elimination methods, in particular by the computation of Gröbner bases, proofs for geometric theorems can be automatically generated. Several monographs are devoted to this technique. In [2] a sound theoretical introduction is given, and [1] is an immense collection of classical theorems proven by Wu's method [12]–[13].
Here, we would like to show how new theorems can be *deduced* if Gröbner bases are combined with rational factorization. As an example we start with a well-known theorem from triangle geometry, see [3].

Theorem 3 A triangle is right-angled if and only if the relation

$$p_{15}:=2R+r-s=0 \tag{15}$$

is valid.

By the Pythagorean theorem a triangle is right-angled if and only if

$$p_{16}:=(a^2-b^2-c^2)(b^2-c^2-a^2)(c^2-a^2-b^2)=0\,. \tag{16}$$

How can we prove a theorem like the given one by Gröbner bases techniques? Here is the general algorithm:

1. **(Prove (16) $\Rightarrow$ (15)):** Use a suitable term order to eliminate all but the variables R, r and s occurring in (15). Calculate a Gröbner basis $\mathcal{G}$ with respect to the term order chosen for the polynomial system $\{p_1, p_2, p_8, p_{11}, p_{16}\}$, and find a polynomial $p \in \mathcal{G}$ that does only depend on R, r and s. Factorize p, and show that the only geometrically possible factor is p_{15}.

2. **(Prove (15) $\Rightarrow$ (16)):** Use a suitable term order to eliminate all but the variables a, b and c occurring in (16). Calculate a Gröbner basis $\mathcal{G}$ with respect to the term order chosen for the polynomial system $\{p_1, p_2, p_8, p_{11}, p_{15}\}$, and find a polynomial $p \in \mathcal{G}$ that does only depend on a, b and c. Factorize p, and show that the only geometrically possible factor is p_{16}.

It is obvious how to generalize this method. If successful, the two steps prove the equivalence of the two geometric statements.
In the given case, we use first the variable ordering $\{A, a, b, c, s, R, r\}$, and compute the Gröbner basis $\mathcal{G}$ of $\{p_1, p_2, p_8, p_{11}, p_{16}\}$ with regard to the corresponding lexicographical term order. $\mathcal{G}$ contains the (factorized) polynomial

$$(2\,R + r + s)\,(2\,R + r - s)\,r^2\,s^2$$

in the variables R, r and s, only. Since neither of r, s or $2\,R + r + s$ can be zero (actually all our variables a, b, c, r, R, s, A are nonnegative), the only possibility is that $2\,R + r - s = 0$, hence (15) must be valid.
On the other hand, the Gröbner basis of $\{p_1, p_2, p_8, p_{11}, p_{15}\}$ with respect to the variable ordering $\{R, r, A, s, a, b, c\}$ contains the polynomial p_{16}. This finishes our proof of Theorem 3.
In the next section, we will show how the same method can be used to deduce new geometric theorems.

4 Deduction of New Theorems

Equation (15) gives a description of a right-angled triangle in terms of the variables r, R and s. Our next question will be, to find a similar description of an isosceles and of an equilateral triangle.
Let us first assume that our triangle is isosceles. This is obviously equivalent to the polynomial identity

$$p_{16} := (a - b)(a - c)(b - c) = 0\,.$$

What is the corresponding relation in terms of r, R and s? We compute the Gröbner basis $\mathcal{G}$ of $\{p_1, p_2, p_8, p_{11}, p_{16}\}$ with respect to the variable ordering $\{A, a, b, c, s, R, r\}$. $\mathcal{G}$ contains the polynomial

$$\left(64\,R^3\,r + 48\,R^2\,r^2 - 4\,R^2\,s^2 + 12\,R\,r^3 - 20\,R\,r\,s^2 + r^4 + 2\,r^2\,s^2 + s^4\right)r^2\,s^2\,.$$

Since neither s nor r are zero, we get

$$p_{15} := 64\,R^3\,r + 48\,R^2\,r^2 - 4\,R^2\,s^2 + 12\,R\,r^3 - 20\,R\,r\,s^2 + r^4 + 2\,r^2\,s^2 + s^4 = 0\,. \qquad (17)$$

The question remains open whether or not (17) implies that the triangle is isosceles. To prove this assertion, we compute the Gröbner basis $\mathcal{G}$ of $\{p_1, p_2, p_8, p_{11}, p_{15}\}$ with respect to the variable ordering $\{A, s, R, r, a, b, c\}$. $\mathcal{G}$ contains the polynomial

$$(a - b)^2\,(a - c)^2\,(b + c)^2\,(b - c)^2\,b^2\,c\,,$$

hence we are done.

Note that (17) is a much more difficult polynomial identity than (15): In terms of r, R and s the fact that the triangle is isosceles is not expressible by a polynomial of degree less than four. If we allow square roots, we get the identity

$$s^2 = 2\sqrt{R\,(R-2\,r)^3} + 2\,R^2 + 10\,R\,r - r^2\,. \tag{18}$$

As a byproduct, we have therefore proved that for an isosceles triangle the inequality

$$R - 2\,r \geq 0 \tag{19}$$

must be valid. Due to Euler one knows that this inequality holds in each triangle [7]. We will come back to (19) soon.
Hence we have deduced and proved

Theorem 4 A triangle is isosceles if and only if

$$64\,R^3\,r + 48\,R^2\,r^2 - 4\,R^2\,s^2 + 12\,R\,r^3 - 20\,R\,r\,s^2 + r^4 + 2\,r^2\,s^2 + s^4 = 0$$

or

$$s^2 = 2\sqrt{R\,(R-2\,r)^3} + 2\,R^2 + 10\,R\,r - r^2\,.$$ □

Note that the Taylor series expansion of s, given by (18), at $r = 0$ gives

$$s = 2\,R\ + r + \frac{r^2}{4R} + \frac{r^3}{8R^2} + O(r^4)\,.$$

In terms of Theorem 3 this can be interpreted as the fact that for small r an isosceles triangle is almost rectangular, and how far away from rectangularity it is.
Next, we assume that our triangle is equilateral. This is obviously equivalent to the polynomial identities

$$p_{16} := a - b = 0 \qquad \text{and} \qquad p_{17} := a - c = 0\,. \tag{20}$$

To obtain an equivalent statement in terms of r, R and s, we compute the Gröbner basis $\mathcal{G}$ of $\{p_1, p_2, p_8, p_{11}, p_{16}, p_{17}\}$ with respect to the variable ordering $\{a, b, c, A, s, R, r\}$. $\mathcal{G}$ contains the polynomial

$$(R - 2\,r)\,r\,s\,,$$

hence

$$p_{15} := R - 2\,r = 0\,. \tag{21}$$

We would like to show that (21) implies that the triangle is equilateral. It is kind of magic to assume this could work since our input is only one equation, and our output would be two, namely (20). From the point of view of ideal theory this should not work. Let's nevertheless try.
We compute the Gröbner basis $\mathcal{G}$ of $\{p_1, p_2, p_8, p_{11}, p_{15}\}$ with respect to the variable ordering $\{R, A, s, r, a, b, c\}$. $\mathcal{G}$ contains the polynomial

$$b\,(b+c)\left(a^3 + b^3 + c^3 - a^2\,b - a\,b^2 - a^2\,c - a\,c^2 - b^2\,c - b\,c^2 + 3ab\right),$$

hence we get the identity

$$a^3 + b^3 + c^3 - a^2\,b - a\,b^2 - a^2\,c - a\,c^2 - b^2\,c - b\,c^2 + 3abc = 0\,.$$

Is this equivalent to the fact that the triangle is equilateral? Yes! This, however, is not a result of Gröbner bases computations, and cannot be deduced by polynomial arithmetic alone, but depends heavily on the fact that triangles live in the subset $(a, b, c) \in (0, \infty)^3$.
The above computations (or a computation based on the formulas of Theorem 2, see [9]) show easily that

$$R - 2r \geq 0$$

if and only if

$$G(a, b, c) := a^3 + b^3 + c^3 - a^2\, b - a\, b^2 - a^2\, c - a\, c^2 - b^2\, c - b\, c^2 + 3abc = 4\, A\, (R - 2r) \geq 0\ .$$

We will now show that $G(a, b, c) \geq 0$ for all $a, b, c \geq 0$ with equality if and only if $a = b = c$, hence if the triangle is equilateral. In particular, this gives an elementary proof of Euler's inequality (19).
An easy calculation shows that the identity

$$G(x + w, y + w, z + w) = \frac{w}{2}\Big((x - y)^2 + (x - z)^2 + (y - z)^2\Big) + G(x, y, z) \tag{22}$$

is valid.
By symmetry considerations, it is enough to assume that $a \geq b \geq c \geq 0$. We substitute $z = 0, w = c$, and therefore $x = a - c \geq 0, y = b - c \geq 0$ in (22), and get the representation

$$\begin{aligned} G(a, b, c) &= \frac{c}{2}\Big((a - b)^2 + (a - c)^2 + (b - c)^2\Big) + G(a - c, b - c, 0) \\ &= \frac{c}{2}\Big((a - b)^2 + (a - c)^2 + (b - c)^2\Big) + (a - b)^2(a + b - 2c)\ . \end{aligned} \tag{23}$$

This representation shows that $G(a, b, c) \geq 0$ because all the summands have this property. Since $G(a, b, c)$ in (23) is the sum of four nonnegative terms, $G(a, b, c) = 0$ if and only if all of these are zero, implying $a = b = c$, because $c \neq 0$. Hence the triangle is equilateral.
Therefore we have deduced

Theorem 5 A triangle is equilateral if and only if

$$R = 2r\ .$$

Any other triangle satisfies the inequality

$$R > 2r\ .$$

□

Next, we are interested to find an equaivalent statement in terms of r, R and s, hopefully linear as (15), for the fact that a triangle has an angle of value α. Note that Theorem 3 gives the answer to this question for $\alpha = \pi/2$.
By the law of cosines, a triangle has an angle of value α if and only if

$$p_{16} := (a^2 - b^2 - c^2 + 2\, \cos(\alpha)\, b\, c)\, (b^2 - a^2 - c^2 + 2\, \cos(\alpha)\, a\, c)\, (c^2 - a^2 - b^2 + 2\, \cos(\alpha)\, a\, b) = 0\ . \tag{24}$$

We compute the Gröbner basis $\mathcal{G}$ of $\{p_1, p_2, p_8, p_{11}, p_{16}\}$ with respect to the variable ordering $\{A, a, b, c, s, R, r\}$. $\mathcal{G}$ contains the polynomial

$$s^2\, r^2\, \Big(4\, R^2\, \cos^3(\alpha) - 4\, R\, (R + r)\, \cos^2(\alpha) + (r^2 + s^2 - 4\, R^2)\, \cos(\alpha) + 4\, R^2 + 4\, R\, r + r^2 - s^2\Big),$$

and since neither s nor r are zero, we conclude that

$$p := 4\,R^2\,\cos^3(\alpha) - 4\,R\,(R+r)\,\cos^2(\alpha) + (r^2+s^2-4\,R^2)\,\cos(\alpha) + 4\,R^2 + 4\,R\,r + r^2 - s^2 = 0\,.$$

Unfortunately rational factorization is not successful to find linear factors as in the case of Theorem 3. This comes from the fact that trigonometric expressions appear. With rational trigonometric factorization [11], implemented in the REDUCE package `trigsimp` [8], one finds the factorization

$$p = 2\,\left(4\,\cos\frac{\alpha}{2}\,\sin^2\frac{\alpha}{2}\,R + \cos\frac{\alpha}{2}\,r + \sin\frac{\alpha}{2}\,s\right)\,\left(4\,\cos\frac{\alpha}{2}\,\sin^2\frac{\alpha}{2}\,R + \cos\frac{\alpha}{2}\,r - \sin\frac{\alpha}{2}\,s\right)\,,$$

however. By geometric considerations, $0 \le \alpha \le \pi/2$, hence all trigonometric expressions occurring are nonnegative. Therefore, the first factor cannot equal zero, and we get

$$p_{15} := 4\,\cos\frac{\alpha}{2}\,\sin^2\frac{\alpha}{2}\,R + \cos\frac{\alpha}{2}\,r - \sin\frac{\alpha}{2}\,s = 0\,. \tag{25}$$

Let's check whether (25) implies (24). To prove this assertion, we compute the Gröbner basis $\mathcal{G}$ of $\{p_1, p_2, p_8, p_{11}, p_{15}\}$ with respect to the variable ordering $\{A, s, R, r, a, b, c\}$. $\mathcal{G}$ contains the polynomial p_{16}, hence we are done.
We therefore have deduced the

Theorem 6 A triangle has an angle of value α if and only if the relation

$$r + 2\,(1-\cos\alpha)\,R = \tan\frac{\alpha}{2}\cdot s$$

is valid.

5 Some More Results

In this section we will present some more results on plane triangles. This time we are interested in the connection between the heights h_a, h_b and h_c and the sidelengths a, b, c. To the polynomials $p_1, \ldots, p_6$ of §1 we add the cyclic equations corresponding to (3)-(6) involving the heights h_a, h_b and their corresponding segments x of a and y of b. This results in 14 equations. Since we are not interested in the segments x, y and z, we eliminate them, and arrive at the following 11 equations.

Theorem 7 The following 11 equations describe the relations between the variables $a, b, c, h_a, h_b, h_c, s, A, r, R$ of a triangle

$$\big\{2\,R\,h_a - b\,c = 0, -2\,A\,h_a + 2\,A\,r + b\,h_a\,r + c\,h_a\,r = 0, -2\,A + a\,h_a = 0,$$
$$2\,R\,h_b - a\,c = 0, -2\,A + b\,h_b = 0, 2\,R\,h_c - a\,b = 0, -2\,A + c\,h_c = 0,$$
$$16\,R\,r + a^2 - 2\,a\,b - 2\,a\,c + b^2 - 2\,b\,c + c^2 + 4\,r^2 = 0, 4\,A\,R - a\,b\,c = 0,$$
$$-2\,A + a\,r + b\,r + c\,r = 0, -a - b - c + 2\,s = 0\,\big\}\,.$$

Note that, again, the equations have at most degree 3 since we have divided out factors that cannot equal zero. □

Now we can deduce equivalent statements as we did in the previous sections. Using Theorem 7, the following geometrical statements are automatically deduced.

Theorem 8 A triangle has a right angle at C if and only

$$ {h_a}^2{h_b}^2 - \left({h_a}^2 + {h_b}^2\right){h_c}^2 = 0 \, . $$

If a triangle has a right angle at C then the relations

$$ h_a = b, \qquad h_b = a, \qquad c\,h_c = a\,b $$

are valid. Similar statements hold if a triangle has a right angle at the other vertices. In particular, a triangle is right-angled if and only if

$$ \left({h_a}^2{h_b}^2 - {h_a}^2{h_c}^2 - {h_b}^2{h_c}^2\right)\left({h_b}^2{h_c}^2 - {h_b}^2{h_a}^2 - {h_c}^2{h_a}^2\right)\left({h_c}^2{h_a}^2 - {h_c}^2{h_b}^2 - {h_a}^2{h_b}^2\right) = 0 \quad \square $$

Note that some of these statements are rather trivial from the point of view of elementary geometry. A little less trivial is the following statement describing angles of value α, though.

Theorem 9 A triangle has an angle of value α at C if and only

$$ 2\,\cos(\alpha) h_a h_b {h_c}^2 + {h_a}^2{h_b}^2 - {h_a}^2{h_c}^2 - {h_b}^2{h_c}^2 = 0 \, . $$

If a triangle has an angle of value α at C then one has

$$ h_a = b\,\sin\alpha \qquad h_b = a\,\sin\alpha \qquad c\,h_c = a\,b\,\sin\alpha \, . $$

Similar statements hold if a triangle has an angle of value α at the other vertices. $\square$

One can describe the right-angle property also by

Theorem 10 A triangle is right-angled if and only if

$$ (a - h_b)\,(a - h_c)\,(b - h_a)\,(b - h_c)\,(c - h_a)\,(c - h_b) = 0 \, . $$

In particular: Either *one* of the identities

$$ h_b = a, \quad h_c = a, \quad h_a = b, \quad h_c = b, \quad h_a = c, \qquad \text{or} \qquad h_b = c $$

implies that the triangle is right-angled. $\square$

This theorem can be regarded as the limit case of the following result of elementary trigonometry which is also easily deduced by our technique.

Theorem 11 A triangle has an angle of value α if and only

$$ (a\sin\alpha - h_b)\,(a\sin\alpha - h_c)\,(b\sin\alpha - h_a)\,(b\sin\alpha - h_c)\,(c\sin\alpha - h_a)\,(c\sin\alpha - h_b) = 0 \, . $$

In particular: Either *one* of the identities

$$ h_b = a\sin\alpha, \quad h_c = a\sin\alpha, \quad h_a = b\sin\alpha, \quad h_c = b\sin\alpha, \quad h_a = c\sin\alpha, \quad \text{or} \quad h_b = c\sin\alpha $$

implies that the triangle has an angle of value α. $\square$

Whereas the above results can be also obtained by elementary geometric observations, results of the following type are not easliy deduced.

Theorem 12 In a plane triangle the following are equivalent:

(a) $h_c = \alpha\, c$,

(b) $a^4 + b^4 + c^4 - 2\,a^2b^2 - 2\,a^2c^2 - 2\,b^2c^2 + 4\,\alpha^2c^4 = 0$,

(c) ${h_a}^4{h_b}^4 + {h_a}^4{h_c}^4 + {h_b}^4{h_c}^4 - 2\,{h_a}^4{h_b}^2{h_c}^2 - 2\,{h_a}^2{h_b}^4{h_c}^2 - 2\,{h_a}^2{h_b}^2{h_c}^4 + 4\,\alpha^2{h_a}^4{h_b}^4 = 0$,

(d) $8R^2rs\alpha^3 + \left(8Rrs^2 - 16R^2r^2 - 8Rr^3 - r^4 - 2r^2s^2 - s^4\right)\alpha^2 + \left(4rs^3 - 16Rr^2s - 4r^3s\right)\alpha - 4r^2s^2 = 0.$

Analogous results are valid for the identities $h_b = \alpha\, b$ and $h_a = \alpha\, a$. $\square$

6 Conclusion

In this article we presented an algorithm based on the computation of Gröbner bases in combination with rational factorization that is capable to generate new geometric theorems. Several theorems had been deduced using this method. The use of an efficient implementation of Buchberger's algorithm is essential in this regard. We used REDUCE for this purpose.
We did not use coordinate geometry as usual, but instead we used equations between the side-lengths a, b and c of a triangle, the radii r and R of its incircle and circumcircle, respectively, and its area A and circumference $2s$. The method can be extended in an obvious way to deal with other systems of polynomial equations describing other situations as well.
We remark that some of our results had been published in [9] in a less automatic way, without the use of Gröbner bases.

Acknowledgments

I would like to thank Herbert Melenk and Karin Gatermann who taught me about Gröbner bases. The argument to deduce the elementary proof of Euler's inequality based on (22) is due to Dieter Schmersau.

Appendix

In this appendix we collect the REDUCE code with which we received the results of this article.

```
input_case nil;
load groebner;

% describing polynomials
p1:=2*s-a-b-c;
p2:=A-r*s;
p3:=c*h-2*A;
p4:=b^2-h^2-z^2;
p5:=a^2-h^2-(c-z)^2;
p6:=a*b-2*R*h;

% elimination of z and h, Theorem 1
base:=groebner({p1,p2,p3,p4,p5,p6},{z,h,R,r,A,s,a,b,c});
for i:=1:length(base) collect factorize(part(base,i));

% independent polynomials are p1, p2, p8 and p11
p8:=4*A*R-a*b*c;
p11:=(b+c-a)*(a+b-c)*(a-b+c)-8*A*r;

% reductions
torder ({R,r,s,A,a,b,c},lex);
base:=groebner({p1,p2,p8,p11});
p7:=2*a*b+2*a*c+2*b*c-a^2-b^2-c^2-4*r^2-16*r*R;
preduce(p7*(b+c)*b*c,base);
p9:=(a+b+c)*(a+b-c)*(a-b+c)*(a-b-c)*R+4*A*a*b*c;
preduce(p9,base);
```

```
p10:=(b+c-a)*(a+b-c)*(a-b+c)*R+2*(b+c)*b*c*r-4*A*b*c;
preduce(p10,base);
p13:=(a+b+c)*(a+b-c)*(a-b+c)*(a-b-c)+16*A^2;
preduce(p13,base);

% representation of r and R, Theorem 2
base:=groebner({p1,p2,p3,p4,p5,p6},{z,h,A,s,R,r,a,b,c});
factorize(first(reverse(base)));
on factor;
base:=groebner({p1,p2,p3,p4,p5,p6},{z,h,A,s,r,R,a,b,c});
factorize(first(reverse(base)));
off factor;

% Theorem 3
p16:=(a^2-b^2-c^2)*(a^2+b^2-c^2)*(a^2-b^2+c^2);
base:=groebner({p1,p2,p8,p11,p16},{A,a,b,c,s,R,r});
factorize(first(reverse(base)));
p15:=2*R+r-s;
base:=groebner({p1,p2,p8,p11,p15},{A,s,R,r,a,b,c});
factorize(first(reverse(base)));

% Theorem 4
p16:=(a-b)*(a-c)*(b-c);
base:=groebner({p1,p2,p8,p11,p16},{A,a,b,c,s,R,r});
factorize(first(reverse(base)));
p15:=r^4+12*r^3*R+2*r^2*s^2+48*r^2*R^2-20*r*s^2*R+64*r*R^3+s^4-4*s^2*R^2;
base:=groebner({p1,p2,p8,p11,p15},{A,s,R,r,a,b,c});
factorize(first(reverse(base)));

% Theorem 5
p16:=a-b;
p17:=a-c;
base:=groebner({p1,p2,p8,p11,p16,p17},{A,a,b,c,s,R,r});
factorize(first(reverse(base)));
p15:=2*r-R;
base:=groebner({p1,p2,p8,p11,p15},{A,s,R,r,a,b,c});
factorize(first(reverse(base)));

p16:=a^3+b^3+c^3-a^2*b-a^2*c-a*b^2+3*a*b*c-a*c^2-b^2*c-b*c^2;
p15:=R-2*r;
torder ({R,r,s,A,a,b,c},lex);
base:=groebner({p1,p2,p8,p11});
preduce(4*A*p15-p16,base);

% Theorem 6
p16:=(a^2-b^2-c^2+2*cos(alpha)*b*c)*(b^2-a^2-c^2+2*cos(alpha)*a*c)*
     (c^2-a^2-b^2+2*cos(alpha)*a*b);
base:=groebner({p1,p2,p8,p11,p16},{A,a,b,c,s,R,r});
term:=factorize(first(reverse(base)));
p:=part(term,5);
load(trigsimp);
trigfactorize(p,alpha/2);
p15:=4*cos(alpha/2)*sin(alpha/2)^2*R+cos(alpha/2)*r-sin(alpha/2)*s;
```

```
base:=groebner({p1,p2,p8,p11,p15},{s,r,R,A,a,b,c});
trigfactorize(first(reverse(base)),alpha/2);
trigsimp(ws,combine);

% describing polynomials
p1:=2*s-a-b-c;
p2:=A-r*s;

p3:=c*hc-2*A;
p4:=b^2-hc^2-z^2;
p5:=a^2-hc^2-(c-z)^2;
p6:=a*b-2*R*hc;

p7:=b*hb-2*A;
p8:=a^2-hb^2-y^2;
p9:=c^2-hb^2-(b-y)^2;
p10:=c*a-2*R*hb;

p11:=a*ha-2*A;
p12:=c^2-ha^2-x^2;
p13:=b^2-ha^2-(a-x)^2;
p14:=b*c-2*R*ha;

% eliminate x,y,z, Theorem 7
base:=groebner({p1,p2,p3,p4,p5,p6,p7,p8,p9,p10,p11,p12,p13,p14},
      {x,y,z,ha,hb,hc,R,r,A,s,a,b,c});

BASE:={
-4*A*a*b*hc-4*A*a*c*hb-4*A*b*c*ha+8*A*ha*hb*hc+a^3*hb*hc+b^3*ha*hc+c^3*ha*hb,
4*A*a*b-2*A*a*c-2*A*b*c+4*A*c^2-8*A*ha*hb-a^3*hb+a^2*c*hb+a*c^2*hb-
b^3*ha+b^2*c*ha+b*c^2*ha-c^3*ha-c^3*hb+4*c*ha*hb*r,
2*R*ha-b*c,
2*A*a^2-2*A*a*b-2*A*a*c-2*A*b^2+4*A*b*c-2*A*c^2+8*A*ha*r+b^3*ha-
b^2*c*ha-b*c^2*ha+c^3*ha,
-2*A*ha+2*A*r+b*ha*r+c*ha*r,
-2*A+a*ha,
2*R*hb-a*c,
-2*A*a^2-2*A*a*b+4*A*a*c+2*A*b^2-2*A*b*c-2*A*c^2+8*A*hb*r+a^3*hb-
a^2*c*hb-a*c^2*hb+c^3*hb,
-2*A+b*hb,
2*R*hc-a*b,
-2*A*a^2+4*A*a*b-2*A*a*c-2*A*b^2-2*A*b*c+2*A*c^2+8*A*hc*r+a^3*hc-
a^2*b*hc-a*b^2*hc+b^3*hc,
-2*A+c*hc,
16*R*r+a^2-2*a*b-2*a*c+b^2-2*b*c+c^2+4*r^2,
4*A*R-a*b*c,
4*A*a*b*c+R*a^4-2*R*a^2*b^2-2*R*a^2*c^2+R*b^4-2*R*b^2*c^2+R*c^4,
4*A*b*c+R*a^3-R*a^2*b-R*a^2*c-R*a*b^2+2*R*a*b*c-R*a*c^2+R*b^3-R*b^2*c-
R*b*c^2+R*c^3-2*b^2*c*r-2*b*c^2*r,
8*A*r+a^3-a^2*b-a^2*c-a*b^2+2*a*b*c-a*c^2+b^3-b^2*c-b*c^2+c^3,
-2*A+a*r+b*r+c*r,
16*A^2+a^4-2*a^2*b^2-2*a^2*c^2+b^4-2*b^2*c^2+c^4,
-a-b-c+2*s};
```

```
reducedBASE:={
2*R*ha-b*c,
-2*A*ha+2*A*r+b*ha*r+c*ha*r,
-2*A+a*ha,
2*R*hb-a*c,
-2*A+b*hb,
2*R*hc-a*b,
-2*A+c*hc,
16*R*r+a^2-2*a*b-2*a*c+b^2-2*b*c+c^2+4*r^2,
4*A*R-a*b*c,
-2*A+a*r+b*r+c*r,
-a-b-c+2*s};

torder ({ha,hb,hc,R,r,A,s,a,b,c},lex);
reducedbase:=groebner(reducedBASE);
preduce(part(BASE,1),reducedbase);
preduce(part(BASE,2),reducedbase);
preduce(part(BASE,3),reducedbase);
preduce(part(BASE,4),reducedbase);
preduce(part(BASE,5),reducedbase);
preduce(part(BASE,6),reducedbase);
preduce(part(BASE,7),reducedbase);
preduce(part(BASE,8),reducedbase);
preduce(part(BASE,9),reducedbase);
preduce(part(BASE,10),reducedbase);
preduce(part(BASE,11),reducedbase);
preduce(part(BASE,12),reducedbase);
preduce(part(BASE,12),reducedbase);
preduce(part(BASE,13),reducedbase);
preduce(part(BASE,14),reducedbase);
preduce(part(BASE,15),reducedbase);
preduce(part(BASE,16),reducedbase);
preduce(part(BASE,17),reducedbase);
preduce(part(BASE,18),reducedbase);
preduce(part(BASE,19),reducedbase);
preduce(part(BASE,20),reducedbase);

% Theorem 8
p16:=2*R+r-s;
base:=groebner(append(reducedBASE,{p16}),{A,r,s,R,a,b,c,ha,hb,hc});
term:=factorize(first(reverse(base)));
p16:=(ha^2*hb^2-ha^2*hc^2-hb^2*hc^2);
base:=groebner(append(reducedBASE,{p16}),{a,b,c,ha,hb,hc,A,r,s,R});
term:=factorize(first(reverse(base)));
p16:=(hb^2*hc^2-hb^2*ha^2-hc^2*ha^2);
base:=groebner(append(reducedBASE,{p16}),{a,b,c,ha,hb,hc,A,r,s,R});
term:=factorize(first(reverse(base)));
p16:=(hc^2*ha^2-hc^2*hb^2-ha^2*hb^2);
base:=groebner(append(reducedBASE,{p16}),{a,b,c,ha,hb,hc,A,r,s,R});
term:=factorize(first(reverse(base)));
p16:=(ha^2*hb^2-ha^2*hc^2-hb^2*hc^2);
base:=groebner(append(reducedBASE,{p16}),{b,c,ha,hc,A,R,r,s,a,hb});
term:=factorize(first(reverse(base)));
```

```
% Theorem 9
p16:=(c^2-a^2-b^2+2*cos(alpha)*a*b);
base:=groebner(append(reducedBASE,{p16}),{A,s,R,r,a,b,c,ha,hb,hc});
term:=factorize(first(reverse(base)));
base:=groebner(append(reducedBASE,{p16}),{b,c,ha,hc,A,R,r,s,a,hb});
term:=factorize(first(reverse(base)));
factorize(trigsimp(part(term,4)));
p16:=2*cos(alpha)*ha*hb*hc^2+ha^2*hb^2-ha^2*hc^2-hb^2*hc^2;
base:=groebner(append(reducedBASE,{p16}),{A,s,R,r,ha,hb,hc,a,b,c});
term:=factorize(first(reverse(base)));
base:=groebner(append(reducedBASE,{p16}),{b,c,ha,hc,A,R,r,s,a,hb});
term:=factorize(first(reverse(base)));

% Theorem 10
p16:=(a-hb)*(a-hc)*(b-ha)*(b-hc)*(c-ha)*(c-hb);
base:=groebner(append(reducedBASE,{p16}),{a,b,c,ha,hb,hc,A,R,r,s});
term:=factorize(first(reverse(base)));
p16:=a^2+b^2-c^2;
base:=groebner(append(reducedBASE,{p16}),{b,c,ha,hc,A,R,r,s,a,hb});
term:=factorize(first(reverse(base)));

% Theorem 11
p16:=sin(alpha)*a-hb;
base:=groebner(append(reducedBASE,{p16}),{ha,hb,hc,A,R,r,s,a,b,c});
term:=factorize(first(reverse(base)));
factorize(trigsimp(part(term,1),cos));
p16:=c^2-a^2-b^2+2*cos(alpha)*a*b;
base:=groebner(append(reducedBASE,{p16}),{b,c,ha,hc,A,R,r,s,a,hb});
term:=factorize(first(reverse(base)));
factorize(trigsimp(part(term,4)));

% Theorem 12
p16:=(alpha*c-hc);
base:=groebner(append(reducedBASE,{p16}),{ha,hb,hc,A,R,r,s,a,b,c});
term:=factorize(first(reverse(base)));
p16:=part(term,1);
base:=groebner(append(reducedBASE,{p16}),{a,b,ha,hb,A,R,r,s,c,hc});
term:=factorize(first(reverse(base)));

p16:=(alpha*c-hc);
base:=groebner(append(reducedBASE,{p16}),{a,b,c,A,R,r,s,ha,hb,hc});
term:=factorize(first(reverse(base)));
p16:=part(term,4);
base:=groebner(append(reducedBASE,{p16}),{a,b,ha,hb,A,R,r,s,c,hc});
term:=factorize(first(reverse(base)));

p16:=(alpha*c-hc);
base:=groebner(append(reducedBASE,{p16}),{a,b,c,ha,hb,hc,A,R,r,s});
term:=factorize(first(reverse(base)));
p16:=part(term,1);
base:=groebner(append(reducedBASE,{p16}),{a,b,ha,hb,A,R,r,s,c,hc});
term:=factorize(first(reverse(base)));
```

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