```
> restart;
> read "hsum17.mpl";
```

Package "Hypergeometric Summation", Maple V - Maple 17
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```
> Digits:=100:
```

The first procedure for the classical orthogonal polynomials of a continuous and discrete variable

```
> Mixedrec:=proc(F,k,Sn,shift0,alpha,shift1,beta,shift2)
  local n,S,bb,sigma,rat,p,q,r,upd,deg,f,i,j,jj,l,var,req,sol,num,
  den,J,a;
  if type(Sn,function) then S:=op(0,Sn); n:=op(1,Sn) else n:=Sn end
  if;
    a:=subs({n=n-shift0, alpha=alpha+shift1,beta=beta+shift2},F)-
  sigma[1]*F-sigma[2]*subs(n=n-1,F);
    rat:=ratio(a,k);
    if not type(rat,ratpoly(anything,k)) then
      error `Algorithm not applicable`
    end if;
    p:=1: q:=subs(k=k-1,numer(rat)): r:=subs(k=k-1,denom(rat)):
    upd:=update(p,q,r,k);
    p:=op(1,upd): q:=op(2,upd): r:=op(3,upd):
    deg:=degreebound(p,q,r,k);
    if deg>=0 then
      f:=add(bb[j]*k^j,j=0..deg);
      var:={seq(sigma[jj],jj=1..2),seq(bb[jj],jj=0..deg)};
      req:=collect(subs(k=k+1,q)*f-r*subs(k=k-1,f)-p,k);
      sol:={solve({coeffs(req,k)},var)};
      if not(sol={} or {seq(op(2,op(l,op(1,sol))),l=1..nops(op(1,
  sol)))}={0}) then
        req:=sigma[1]*S(n,alpha,beta)+sigma[2]*S(n-1,alpha,beta);
        req:=subs(op(1,sol),req);
        return subs(op(1,sol),S(n-shift0,alpha+shift1,beta+shift2))
  =map(factor,req);
      end if;
    end if;
  error cat(`Algorithm fails`);
  end:
```

the Laguerre polynomials

```
> Laguerre :=  (n, alpha, x) -> pochhammer(alpha+1, n)/n!*add
  (hyperterm([-n],[alpha+1],x,k), k = 0 .. n);
```

$$Laguerre := (n, \alpha, x) \rightarrow \frac{\mathrm{pochhammer}(\alpha+1, n)\ add\left(hyperterm\left([-n], [\alpha+1], x, k\right), k=0..n\right)}{n!} \quad \textbf{(1.1)}$$

```
> FLag:=pochhammer(alpha+1, n)/n!*(hyperterm([-n],[alpha+1],x,k))
  ;
```

(1.2)

$$FLag := \frac{\mathrm{pochhammer}(\alpha+1, n)\ \mathrm{pochhammer}(-n, k)\ x^k}{n!\ \mathrm{pochhammer}(\alpha+1, k)\ k!} \tag{1.2}$$

The weight function

```
> rhoLaguerre:=(alpha)->exp(-x)*x^alpha
```

$$rhoLaguerre := \alpha \rightarrow \mathrm{e}^{-x}\, x^{\alpha} \tag{1.3}$$

```
> cLag:=simplify(rhoLaguerre(alpha+s)/rhoLaguerre(alpha))
```

$$cLag := x^s \tag{1.4}$$

The mixed recurrence equation for m=3, k=6 (equation (6) of the manuscript)

```
> recLag:=Mixedrec(FLag,k,L(n),3,alpha,6,0,0):
> recLag1:=denom(rhs(recLag))*lhs(recLag)   =collect(numer(rhs
  (recLag)),[L(n, alpha, 0),L(n-1, alpha, 0),x],factor)
```

$$recLag1 := x^6 L(n-3, \alpha+6, 0) = \Big(-(n-2)(n-1)\,n\,x^3 - 3\,n\,(n-2)(n-1)(\alpha+3)\,x^2 + n\,(\alpha+4)(\alpha+3)(\alpha+2)(\alpha+4\,n-3)\,x - n\,(\alpha+5)(\alpha+4)(\alpha+3)(\alpha+2)(\alpha+1)\Big)\,L(n, \alpha, 0) + \Big((\alpha+3)(\alpha+n)\big(\alpha^2+3\,\alpha\,n+3\,n^2+3\,\alpha+3\,n+2\big)\,x^2 - 2\,(\alpha+4)(\alpha+3)(\alpha+2)(\alpha+1+2\,n)(\alpha+n)\,x + (\alpha+5)(\alpha+4)(\alpha+3)(\alpha+2)(\alpha+1)(\alpha+n)\Big)\,L(n-1, \alpha, 0) \tag{1.5}$$

The polynomial coefficient of $L(n-1, \alpha, 0)$

```
> GLag[3,6]:=op([2,2],recLag1)/L(n-1, alpha, 0)
```

$$GLag_{3,\,6} := (\alpha+3)(\alpha+n)\big(\alpha^2+3\,\alpha\,n+3\,n^2+3\,\alpha+3\,n+2\big)\,x^2 - 2\,(\alpha+4)(\alpha+3)(\alpha+2)(\alpha+1+2\,n)(\alpha+n)\,x + (\alpha+5)(\alpha+4)(\alpha+3)(\alpha+2)(\alpha+1)(\alpha+n) \tag{1.6}$$

The bound (7) of the manuscript which is accurate for x_{n,1}

```
> bound81:=(-coeff(GLag[3,6],x,1)/(2*coeff(GLag[3,6],x,2))-sqrt((
  (coeff(GLag[3,6],x,1)^2-4*coeff(GLag[3,6],x,0)*coeff(GLag[3,6],
  x,2))))/(2*coeff(GLag[3,6],x,2)))
```

$$bound81 := \frac{(\alpha+4)(\alpha+2)(\alpha+1+2\,n)}{\alpha^2+3\,\alpha\,n+3\,n^2+3\,\alpha+3\,n+2} - \big(\alpha^9 n + 3\,\alpha^8 n^2 + 3\,\alpha^7 n^3 + \alpha^6 n^4 - \alpha^9 + 17\,\alpha^8 n + 55\,\alpha^7 n^2 + 55\,\alpha^6 n^3 + 18\,\alpha^5 n^4 - 16\,\alpha^8 + 128\,\alpha^7 n + 444\,\alpha^5 n^3 + 142\,\alpha^4 n^4 - 106\,\alpha^7 + 554\,\alpha^6 n + 2050\,\alpha^5 n^2 + 2014\,\alpha^4 n^3 + 624\,\alpha^3 n^4 - 376\,\alpha^6 + 1457\,\alpha^5 n + 5627\,\alpha^4 n^2 + 5379\,\alpha^3 n^3 + 1585\,\alpha^2 n^4 - 769\,\alpha^5 + 2213\,\alpha^4 n + 8899\,\alpha^3 n^2 + 8083\,\alpha^2 n^3 + 2166\,\alpha\,n^4 - 904\,\alpha^4 + 1582\,\alpha^3 n + 7100\,\alpha^2 n^2 + 5838\,\alpha\,n^3 + 1224\,n^4 - 564\,\alpha^3 + 96\,\alpha^2 n + 1884\,\alpha\,n^2 + 1224\,n^3 - 144\,\alpha^2 - 288\,\alpha\,n - 144\,n^2\big)^{1/2} \Big/ \Big((\alpha+3)(\alpha+n)\big(\alpha^2+3\,\alpha\,n+3\,n^2+3\,\alpha+3\,n+2\big)\Big)$$

The mixed recurrence equation for m=4, k=0

```
> recLag2:=Mixedrec(FLag,k,L(n),4,alpha,0,0,0)
```

$$recLag2 := L(n-4, \alpha, 0) = \tag{1.8}$$

$$-\frac{n\left(\alpha^2+3\alpha n-2\alpha x+3n^2-4nx+x^2-6\alpha-12n+8x+11\right)L(n,\alpha,0)}{(\alpha-3+n)(\alpha-2+n)(\alpha-1+n)}$$
$$+\frac{1}{(\alpha-3+n)(\alpha-2+n)(\alpha-1+n)}\Big(\big(\alpha^3+4\alpha^2 n-3\alpha^2 x+6\alpha n^2-10\alpha n x$$
$$+3\alpha x^2+4n^3-10n^2x+6nx^2-x^3-6\alpha^2-18\alpha n+15\alpha x-18n^2+30nx$$
$$-9x^2+11\alpha+22n-18x-6\big)L(n-1,\alpha,0)\Big)$$

```
> recLag21:=denom(rhs(recLag2))*lhs(recLag2)   =collect(numer(rhs
  (recLag2)),[L(n, alpha, 0),L(n-1, alpha, 0),x],factor)
```

$$recLag21 := (\alpha-3+n)(\alpha-2+n)(\alpha-1+n)L(n-4,\alpha,0)=\Big(-nx^2+2n(\alpha+2n-4)x-n\left(\alpha^2+3\alpha n+3n^2-6\alpha-12n+11\right)\Big)L(n,\alpha,0)+\Big(-x^3+(3\alpha+6n-9)x^2+\left(-3\alpha^2-10\alpha n-10n^2+15\alpha+30n-18\right)x+(\alpha-3+2n)\left(\alpha^2+2\alpha n+2n^2-3\alpha-6n+2\right)\Big)L(n-1,\alpha,0) \quad \textbf{(1.9)}$$

The polynomial (9) of the manuscript whose largest zero is a lower bound for x_{n,n}

```
> EQ7:=op([2,2],recLag21)/L(n-1, alpha, 0)
```

$$EQ7 := -x^3+(3\alpha+6n-9)x^2+\left(-3\alpha^2-10\alpha n-10n^2+15\alpha+30n-18\right)x+(\alpha-3+2n)\left(\alpha^2+2\alpha n+2n^2-3\alpha-6n+2\right) \quad \textbf{(1.10)}$$

The mixed recurrence equation for m=4, k=8

```
> recLag3:=Mixedrec(FLag,k,L(n),4,alpha,8,0,0):
> recLag31:=denom(rhs(recLag3))*lhs(recLag3)   =collect(numer(rhs
  (recLag3)),[L(n, alpha, 0),L(n-1, alpha, 0),x],factor)
```

$$recLag31 := x^8 L(n-4,\alpha+8,0)=\Big((n-3)(n-2)(n-1)nx^4+4n(n-3)(n-2)(n-1)(\alpha+4)x^3-n(\alpha+5)(\alpha+4)(\alpha+3)\left(\alpha^2+5\alpha n+10n^2-2\alpha-20n+12\right)x^2+2n(\alpha+5)(\alpha+4)(\alpha+3)(\alpha+2)(\alpha+6)(\alpha+3n-2)x-n(\alpha+7)(\alpha+6)(\alpha+5)(\alpha+4)(\alpha+3)(\alpha+2)(\alpha+1)\Big)L(n,\alpha,0)+\Big(-(\alpha+4)(\alpha+1+2n)(\alpha+n)\left(\alpha^2+2\alpha n+2n^2+5\alpha+2n+6\right)x^3+(\alpha+5)(\alpha+4)(\alpha+3)(\alpha+n)\left(3\alpha^2+10\alpha n+10n^2+9\alpha+10n+6\right)x^2-3(\alpha+5)(\alpha+4)(\alpha+3)(\alpha+2)(\alpha+6)(\alpha+1+2n)(\alpha+n)x+(\alpha+7)(\alpha+6)(\alpha+5)(\alpha+4)(\alpha+3)(\alpha+2)(\alpha+1)(\alpha+n)\Big)L(n-1,\alpha,0) \quad \textbf{(1.11)}$$

The polynomial (8) whose smallest zero is an upper bound for x_{n,1}

```
> EQ48:=op([2,2],recLag31)/L(n-1, alpha, 0)
```

$$EQ48 := -(\alpha+4)(\alpha+1+2n)(\alpha+n)\left(\alpha^2+2\alpha n+2n^2+5\alpha+2n+6\right)x^3+(\alpha+5)(\alpha+4)(\alpha+3)(\alpha+n)\left(3\alpha^2+10\alpha n+10n^2+9\alpha+10n+6\right)x^2-3(\alpha+5)(\alpha+4)(\alpha+3)(\alpha+2)(\alpha+6)(\alpha+1+2n)(\alpha+n)x+(\alpha+7)(\alpha+6)(\alpha+5)(\alpha+4)(\alpha+3)(\alpha+2)(\alpha+1)(\alpha+n) \quad \textbf{(1.12)}$$

The mixed recurrence equation for m=7, k=0

```
> recLag4:=Mixedrec(FLag,k,L(n),7,alpha,0,0,0):
> recLag41:=denom(rhs(recLag4))*lhs(recLag4)   =collect(numer(rhs
```

```
(recLag4)),[L(n, alpha, 0),L(n-1, alpha, 0),x],factor)
```

$$recLag41 := (\alpha-6+n)(\alpha-5+n)(\alpha-4+n)(\alpha-3+n)(\alpha-2+n)(\alpha-1+n)\,L(n-7,\alpha,0) = \big(n x^5 - 5 n (\alpha+2 n-7) x^4 + 2 n (5 \alpha^2 + 18 \alpha n + 18 n^2 - 63 \alpha - 126 n + 208) x^3 - 2 n (\alpha+2 n-7)(5 \alpha^2 + 14 \alpha n + 14 n^2 - 49 \alpha - 98 n + 144) x^2 + n (5 \alpha^4 + 28 \alpha^3 n + 63 \alpha^2 n^2 + 70 \alpha n^3 + 35 n^4 - 98 \alpha^3 - 441 \alpha^2 n - 735 \alpha n^2 - 490 n^3 + 733 \alpha^2 + 2443 \alpha n + 2443 n^2 - 2548 \alpha - 5096 n + 3708) x - n (\alpha+2 n-7)(\alpha^4 + 4 \alpha^3 n + 7 \alpha^2 n^2 + 6 \alpha n^3 + 3 n^4 - 14 \alpha^3 - 49 \alpha^2 n - 63 \alpha n^2 - 42 n^3 + 77 \alpha^2 + 203 \alpha n + 203 n^2 - 196 \alpha - 392 n + 252)\big) L(n,\alpha,0) + \big(720 - 3528 n - 1764 \alpha + 1624 \alpha^2 + 4872 \alpha n + 4872 n^2 + 21 \alpha^4 n^2 + 35 \alpha^3 n^3 + 21 \alpha n^5 - 126 \alpha^4 n - 315 \alpha^3 n^2 - 420 \alpha^2 n^3 - 315 \alpha n^4 + 875 \alpha^3 n + 1750 \alpha^2 n^2 + 1750 \alpha n^3 - 2940 \alpha^2 n - 4410 \alpha n^2 + 35 \alpha^2 n^4 + x^6 + 175 \alpha^4 - 735 \alpha^3 + 7 n^6 - 126 n^5 + 875 n^4 - 2940 n^3 + (15 \alpha^4 + 90 \alpha^3 n + 216 \alpha^2 n^2 + 252 \alpha n^3 + 126 n^4 - 270 \alpha^3 - 1296 \alpha^2 n - 2268 \alpha n^2 - 1512 n^3 + 1785 \alpha^2 + 6246 \alpha n + 6246 n^2 - 5130 \alpha - 10260 n + 5400) x^2 - 2 (\alpha-6+2 n)(3 \alpha^4 + 14 \alpha^3 n + 28 \alpha^2 n^2 + 28 \alpha n^3 + 14 n^4 - 42 \alpha^3 - 168 \alpha^2 n - 252 \alpha n^2 - 168 n^3 + 213 \alpha^2 + 658 \alpha n + 658 n^2 - 462 \alpha - 924 n + 360) x + (15 \alpha^2 + 55 \alpha n + 55 n^2 - 165 \alpha - 330 n + 450) x^4 + (-6 \alpha - 12 n + 36) x^5 - 20 (\alpha-6+2 n)(\alpha^2 + 3 \alpha n + 3 n^2 - 9 \alpha - 18 n + 20) x^3 + \alpha^6 - 21 \alpha^5 + 7 \alpha^5 n\big) L(n-1,\alpha,0) \quad (1.13)$$

The polynomial whose largest zero is a lower bound for x_{n,n}

```
> EQ71:=op([2,2],recLag41)/L(n-1, alpha, 0)
```

$$EQ71 := 720 - 3528 n - 1764 \alpha + 1624 \alpha^2 + 4872 \alpha n + 4872 n^2 + 21 \alpha^4 n^2 + 35 \alpha^3 n^3 + 21 \alpha n^5 - 126 \alpha^4 n - 315 \alpha^3 n^2 - 420 \alpha^2 n^3 - 315 \alpha n^4 + 875 \alpha^3 n + 1750 \alpha^2 n^2 + 1750 \alpha n^3 - 2940 \alpha^2 n - 4410 \alpha n^2 + 35 \alpha^2 n^4 + x^6 + 175 \alpha^4 - 735 \alpha^3 + 7 n^6 - 126 n^5 + 875 n^4 - 2940 n^3 + (15 \alpha^4 + 90 \alpha^3 n + 216 \alpha^2 n^2 + 252 \alpha n^3 + 126 n^4 - 270 \alpha^3 - 1296 \alpha^2 n - 2268 \alpha n^2 - 1512 n^3 + 1785 \alpha^2 + 6246 \alpha n + 6246 n^2 - 5130 \alpha - 10260 n + 5400) x^2 - 2 (\alpha-6+2 n)(3 \alpha^4 + 14 \alpha^3 n + 28 \alpha^2 n^2 + 28 \alpha n^3 + 14 n^4 - 42 \alpha^3 - 168 \alpha^2 n - 252 \alpha n^2 - 168 n^3 + 213 \alpha^2 + 658 \alpha n + 658 n^2 - 462 \alpha - 924 n + 360) x + (15 \alpha^2 + 55 \alpha n + 55 n^2 - 165 \alpha - 330 n + 450) x^4 + (-6 \alpha - 12 n + 36) x^5 - 20 (\alpha-6+2 n)(\alpha^2 + 3 \alpha n + 3 n^2 - 9 \alpha - 18 n + 20) x^3 + \alpha^6 - 21 \alpha^5 + 7 \alpha^5 n \quad (1.14)$$

Comparison between the bounds and the extreme zeros

```
> xnlag0:= sort([solve(expand(Laguerre(10, -0.5, x)),x)]):
> extzerolag:=evalf[15]([min(xnlag0),max(xnlag0)])
```

(1.15)

$$extzerolag := [0.0601920631495879, 29.0249503402362] \quad (1.15)$$

```
> lag48:=unapply(EQ48,[n,alpha,x]):
> evalf[15](min([solve(lag48(10,-0.5,x),x)]))
```

$$0.0601920633241655 \quad (1.16)$$

```
> lag36:=unapply(bound81,[n,alpha]):
> evalf[15](lag36(10,-0.5))
```

$$0.0601928943871 40 \quad (1.17)$$

```
> boundGupta:=(n,alpha)->(alpha+1)*(alpha+2)*(alpha+4)*(2*n+
  alpha+1)/((alpha+1)^2*(alpha+2)+(5*alpha+11)*n*(n+alpha+1) )
```

$$boundGupta := (n, \alpha) \rightarrow \frac{(\alpha+1)\,(\alpha+2)\,(\alpha+4)\,(\alpha+1+2\,n)}{(\alpha+1)^2\,(\alpha+2)+(5\,\alpha+11)\,n\,(n+\alpha+1)} \quad (1.18)$$

```
> evalf[15](boundGupta(10, -0.5))
```

$$0.0602687946241075 \quad (1.19)$$

```
> lag40:=unapply(EQ7,[n,alpha,x]):
> evalf[15](max([solve(lag40(10,-0.5,x),x)]))
```

$$28.4690066850468 \quad (1.20)$$

```
> lag70:=unapply(EQ71,[n,alpha,x]):
> evalf[15](max([solve(lag70(10,-0.5,x),x)]))
```

$$29.0247866350679 \quad (1.21)$$

the Jacobi polynomials

```
> Jacobi := (n, alpha, beta, x) -> pochhammer(alpha+1, n)/n!*add
  (hyperterm([-n,n+alpha+beta+1],[alpha+1],(1-x)/2,k), k = 0 ..
  n);
```

$$Jacobi := (n, \alpha, \beta, x) \rightarrow \frac{1}{n!}\left(\mathrm{pochhammer}(\alpha+1, n)\ add\left(hyperterm\left([-n, n+\alpha+\beta+1], [\alpha+1], \frac{1}{2}-\frac{1}{2}\,x, k\right), k=0..n\right)\right) \quad (2.1)$$

```
> FJac:=pochhammer(alpha+1, n)/n!*(hyperterm([-n,n+alpha+beta+1],
  [alpha+1],(1-x)/2,k))
```

The weight function

```
> rhoJacobi:=(alpha,beta)->(1-x)^alpha*(1+x)^beta
```

$$rhoJacobi := (\alpha, \beta) \rightarrow (1-x)^{\alpha}\,(1+x)^{\beta} \quad (2.2)$$

```
> cJac:=simplify(rhoJacobi(alpha+s,beta+t)/rhoJacobi(alpha,beta))
```

$$cJac := (1-x)^{s}\,(1+x)^{t} \quad (2.3)$$

The mixed recurrence equation involving $L(n-3, \alpha, \beta+6)$ giving an upper bound of x_(n, 1)

```
> recJac1:=Mixedrec(FJac,k,L(n),3,alpha,0,beta,6):
> recJac11:=denom(rhs(recJac1))*lhs(recJac1)   =collect(numer(rhs
  (recJac1)),[L(n, alpha, beta),L(n-1, alpha, beta),x],factor)
```

$$recJac11 := (1+x)^6\,(\alpha-2+n)\,(\alpha-1+n)\,(2\,n+\alpha+\beta)\,(n+3+\alpha+\beta)\,(n+\alpha+\beta+2)\,(n+\alpha+\beta+1)\,L(n-3, \alpha, \beta+6) = \big(8\,n\,(n-1)\,(n-2)\,(\alpha-2+n)\,(\alpha-1+n)\,(2\,n+\alpha+\beta)\,x^3+24\,n\,(n-1)\,(n-2)\,(\alpha-2+n)\,(\alpha-1 \quad (2.4)$$

$$+n)\ (\alpha+3\beta+2n+6)\ x^2+8n\left(3\alpha^3n^2+15\alpha^2\beta n^2+12\alpha^2n^3-4\alpha\beta^4\right.$$
$$-16\alpha\beta^3n+30\alpha\beta n^3+15\alpha n^4-4\beta^5-8\beta^4n-16\beta^3n^2+15\beta n^4+6n^5-9\alpha^3n$$
$$-45\alpha^2\beta n-9\alpha^2n^2-24\alpha\beta^3-144\alpha\beta^2n-135\alpha\beta n^2-48\beta^4-48\beta^3n$$
$$-144\beta^2n^2-90\beta n^3+6\alpha^3+30\alpha^2\beta-57\alpha^2n+4\alpha\beta^2-221\alpha\beta n-207\alpha n^2$$
$$-236\beta^3+8\beta^2n-221\beta n^2-138n^3+54\alpha^2+126\alpha\beta+12\alpha n-624\beta^2+252\beta n$$
$$\left.+12n^2+84\alpha-852\beta+168n-432\right)x+8n\left(\alpha^3n^2+7\alpha^2\beta n^2+4\alpha^2n^3-4\alpha\beta^4\right.$$
$$-16\alpha\beta^3n+14\alpha\beta n^3+5\alpha n^4+4\beta^5-8\beta^4n-16\beta^3n^2+7\beta n^4+2n^5-3\alpha^3n$$
$$-21\alpha^2\beta n+3\alpha^2n^2-24\alpha\beta^3-144\alpha\beta^2n-63\alpha\beta n^2+12\alpha n^3+72\beta^4-48\beta^3n$$
$$-144\beta^2n^2-42\beta n^3+6n^4+2\alpha^3+14\alpha^2\beta-37\alpha^2n+4\alpha\beta^2-325\alpha\beta n$$
$$-123\alpha n^2+444\beta^3+8\beta^2n-325\beta n^2-82n^3+30\alpha^2+174\alpha\beta-174\alpha n$$
$$\left.\left.+1176\beta^2+348\beta n-174n^2+184\alpha+1308\beta+368n+456\right)\right)L(n,\alpha,\beta)$$
$$+\left(16\ (\beta+3)\ (n+\beta)\ \left(\alpha^2\beta^2+3\alpha^2\beta n+3\alpha^2n^2+2\alpha\beta^3+7\alpha\beta^2n+9\alpha\beta n^2\right.\right.$$
$$+6\alpha n^3+\beta^4+4\beta^3n+7\beta^2n^2+6\beta n^3+3n^4+3\alpha^2\beta+3\alpha^2n+11\alpha\beta^2+24\alpha\beta n$$
$$+9\alpha n^2+8\beta^3+22\beta^2n+24\beta n^2+6n^3+2\alpha^2+19\alpha\beta+23\alpha n+23\beta^2+38\beta n$$
$$\left.+23n^2+10\alpha+28\beta+20n+12\right)x^2+32\ (\beta+3)\ (n+\beta)\ \left(\alpha^2\beta^2+3\alpha^2\beta n\right.$$
$$+3\alpha^2n^2+3\alpha\beta^2n+9\alpha\beta n^2+6\alpha n^3-\beta^4+3\beta^2n^2+6\beta n^3+3n^4+3\alpha^2\beta$$
$$+3\alpha^2n-3\alpha\beta^2+9\alpha n^2-12\beta^3-6\beta^2n+6n^3+2\alpha^2-9\alpha\beta-9\alpha n-47\beta^2$$
$$\left.-18\beta n-9n^2-6\alpha-72\beta-12n-36\right)x+16\ (\beta+3)\ (n+\beta)\ \left(\alpha^2\beta^2+3\alpha^2\beta n\right.$$
$$+3\alpha^2n^2-2\alpha\beta^3-\alpha\beta^2n+9\alpha\beta n^2+6\alpha n^3+\beta^4-4\beta^3n-\beta^2n^2+6\beta n^3+3n^4$$
$$+3\alpha^2\beta+3\alpha^2n-17\alpha\beta^2-24\alpha\beta n+9\alpha n^2+16\beta^3-34\beta^2n-24\beta n^2+6n^3$$
$$\left.\left.+2\alpha^2-37\alpha\beta-41\alpha n+79\beta^2-74\beta n-41n^2-22\alpha+140\beta-44n+76\right)\right)L(n$$
$$-1,\alpha,\beta)$$

```
> GJac[3,0,6]:=simplify(op([2,2],recJac11)/L(n-1, alpha, beta)/(
  (16*(beta+3))*(beta+n))):
> bound1Jac:=(-coeff(GJac[3,0,6],x,1)-sqrt(((coeff(GJac[3,0,6],x,
  1)^2-4*coeff(GJac[3,0,6],x,0)*coeff(GJac[3,0,6],x,2)))))/(2*
  coeff(GJac[3,0,6],x,2))
```

$$bound1Jac := \frac{1}{2}\left(-6n^4-(12\beta+12\alpha+12)\ n^3-\left(6\alpha^2+18\alpha\beta+6\beta^2+18\alpha\right.\right. \quad \textbf{(2.5)}$$
$$\left.-18\right)n^2-6\left((\alpha-2)\ \beta+\alpha-4\right)\ (\alpha+\beta+1)\ n+2\ (\beta+2)\ (\beta+1)\ (\beta+\alpha$$
$$+3)\ (\beta+6-\alpha)$$
$$-4\left(\alpha^2\beta^5n+\alpha^2\beta^4n^2+\alpha\beta^6n+3\alpha\beta^5n^2+2\alpha\beta^4n^3+\beta^6n^2+2\beta^5n^3+\beta^4n^4\right.$$
$$-\alpha^2\beta^5+13\alpha^2\beta^4n+12\alpha^2\beta^3n^2-\alpha\beta^6+12\alpha\beta^5n+39\alpha\beta^4n^2+24\alpha\beta^3n^3$$

$$- 2 \beta^6 n + 12 \beta^5 n^2 + 26 \beta^4 n^3 + 12 \beta^3 n^4 - 10 \alpha^2 \beta^4 + 73 \alpha^2 \beta^3 n + 61 \alpha^2 \beta^2 n^2 - 11 \alpha \beta^5 + 66 \alpha \beta^4 n + 219 \alpha \beta^3 n^2 + 122 \alpha \beta^2 n^3 + 2 \beta^6 - 22 \beta^5 n + 66 \beta^4 n^2 + 146 \beta^3 n^3 + 61 \beta^2 n^4 - 37 \alpha^2 \beta^3 + 211 \alpha^2 \beta^2 n + 150 \alpha^2 \beta n^2 - 47 \alpha \beta^4 + 210 \alpha \beta^3 n + 633 \alpha \beta^2 n^2 + 300 \alpha \beta n^3 + 26 \beta^5 - 94 \beta^4 n + 210 \beta^3 n^2 + 422 \beta^2 n^3 + 150 \beta n^4 - 64 \alpha^2 \beta^2 + 286 \alpha^2 \beta n + 136 \alpha^2 n^2 - 101 \alpha \beta^3 + 369 \alpha \beta^2 n + 858 \alpha \beta n^2 + 272 \alpha n^3 + 134 \beta^4 - 202 \beta^3 n + 369 \beta^2 n^2 + 572 \beta n^3 + 136 n^4 - 52 \alpha^2 \beta + 136 \alpha^2 n - 116 \alpha \beta^2 + 318 \alpha \beta n + 408 \alpha n^2 + 350 \beta^3 - 232 \beta^2 n + 318 \beta n^2 + 272 n^3 - 16 \alpha^2 - 68 \alpha \beta + 104 \alpha n + 488 \beta^2 - 136 \beta n + 104 n^2 - 16 \alpha + 344 \beta - 32 n + 96 \Big)^{1/2} \Big) \Big/ \Big(3 n^4 + (6 \beta + 6 \alpha + 6) n^3 + \left(7 \beta^2 + (9 \alpha + 24) \beta + 3 \alpha^2 + 9 \alpha + 23\right) n^2 + 4 (\alpha + \beta + 1) \left(\beta^2 + \left(\frac{3}{4} \alpha + \frac{9}{2}\right) \beta + \frac{3}{4} \alpha + 5\right) n + (\beta + 2) (\beta + 1) (\beta + \alpha + 3) (\beta + \alpha + 2) \Big)$$

The mixed recurrence equation involving $L(n-3, \alpha+6, \beta)$ giving a lower bound of x_(n, n)

```
> recJac2:=Mixedrec(FJac,k,L(n),3,alpha,6,beta,0):
> recJac21:=denom(rhs(recJac2))*lhs(recJac2)   =collect(numer(rhs
  (recJac2)),[L(n, alpha, beta),L(n-1, alpha, beta),x],factor)
```

$$recJac21 := (-1 + x)^6 (\beta + n - 1) (\beta + n - 2) (n + 3 + \alpha + \beta) (n + \alpha + \beta + 2) (n + \alpha + \beta + 1) (2 n + \alpha + \beta) L(n - 3, \alpha + 6, \beta) = \Big(8 n (n-1) (n-2) (\beta + n - 1) (\beta + n - 2) (2 n + \alpha + \beta) x^3 - 24 n (n-1) (n-2) (\beta + n - 1) (\beta + n - 2) (3 \alpha + \beta + 2 n + 6) x^2 - 8 n \big(4 \alpha^5 + 4 \alpha^4 \beta + 8 \alpha^4 n + 16 \alpha^3 \beta n + 16 \alpha^3 n^2 - 15 \alpha \beta^2 n^2 - 30 \alpha \beta n^3 - 15 \alpha n^4 - 3 \beta^3 n^2 - 12 \beta^2 n^3 - 15 \beta n^4 - 6 n^5 + 48 \alpha^4 + 24 \alpha^3 \beta + 48 \alpha^3 n + 144 \alpha^2 \beta n + 144 \alpha^2 n^2 + 45 \alpha \beta^2 n + 135 \alpha \beta n^2 + 90 \alpha n^3 + 9 \beta^3 n + 9 \beta^2 n^2 + 236 \alpha^3 - 4 \alpha^2 \beta - 8 \alpha^2 n - 30 \alpha \beta^2 + 221 \alpha \beta n + 221 \alpha n^2 - 6 \beta^3 + 57 \beta^2 n + 207 \beta n^2 + 138 n^3 + 624 \alpha^2 - 126 \alpha \beta - 252 \alpha n - 54 \beta^2 - 12 \beta n - 12 n^2 + 852 \alpha - 84 \beta - 168 n + 432\big) x - 8 n \big(4 \alpha^5 - 4 \alpha^4 \beta - 8 \alpha^4 n - 16 \alpha^3 \beta n - 16 \alpha^3 n^2 + 7 \alpha \beta^2 n^2 + 14 \alpha \beta n^3 + 7 \alpha n^4 + \beta^3 n^2 + 4 \beta^2 n^3 + 5 \beta n^4 + 2 n^5 + 72 \alpha^4 - 24 \alpha^3 \beta - 48 \alpha^3 n - 144 \alpha^2 \beta n - 144 \alpha^2 n^2 - 21 \alpha \beta^2 n - 63 \alpha \beta n^2 - 42 \alpha n^3 - 3 \beta^3 n + 3 \beta^2 n^2 + 12 \beta n^3 + 6 n^4 + 444 \alpha^3 + 4 \alpha^2 \beta + 8 \alpha^2 n + 14 \alpha \beta^2 - 325 \alpha \beta n - 325 \alpha n^2 + 2 \beta^3 - 37 \beta^2 n - 123 \beta n^2 - 82 n^3 + 1176 \alpha^2 + 174 \alpha \beta + 348 \alpha n \quad \textbf{(2.6)}$$

$+ 30\,\beta^2 - 174\,\beta\,n - 174\,n^2 + 1308\,\alpha + 184\,\beta + 368\,n + 456)\big)\,L(n, \alpha, \beta) + \big(16\,(\alpha$
$+ 3)\,(\alpha + n)\,\big(\alpha^4 + 2\,\alpha^3\,\beta + 4\,\alpha^3\,n + \alpha^2\,\beta^2 + 7\,\alpha^2\,\beta\,n + 7\,\alpha^2\,n^2 + 3\,\alpha\,\beta^2\,n + 9\,\alpha\,\beta\,n^2$
$+ 6\,\alpha\,n^3 + 3\,\beta^2\,n^2 + 6\,\beta\,n^3 + 3\,n^4 + 8\,\alpha^3 + 11\,\alpha^2\,\beta + 22\,\alpha^2\,n + 3\,\alpha\,\beta^2 + 24\,\alpha\,\beta\,n$
$+ 24\,\alpha\,n^2 + 3\,\beta^2\,n + 9\,\beta\,n^2 + 6\,n^3 + 23\,\alpha^2 + 19\,\alpha\,\beta + 38\,\alpha\,n + 2\,\beta^2 + 23\,\beta\,n + 23\,n^2$
$+ 28\,\alpha + 10\,\beta + 20\,n + 12\big)\,x^2 + 32\,(\alpha + 3)\,(\alpha + n)\,\big(\alpha^4 - \alpha^2\,\beta^2 - 3\,\alpha^2\,\beta\,n$
$- 3\,\alpha^2\,n^2 - 3\,\alpha\,\beta^2\,n - 9\,\alpha\,\beta\,n^2 - 6\,\alpha\,n^3 - 3\,\beta^2\,n^2 - 6\,\beta\,n^3 - 3\,n^4 + 12\,\alpha^3 + 3\,\alpha^2\,\beta$
$+ 6\,\alpha^2\,n - 3\,\alpha\,\beta^2 - 3\,\beta^2\,n - 9\,\beta\,n^2 - 6\,n^3 + 47\,\alpha^2 + 9\,\alpha\,\beta + 18\,\alpha\,n - 2\,\beta^2 + 9\,\beta\,n$
$+ 9\,n^2 + 72\,\alpha + 6\,\beta + 12\,n + 36\big)\,x + 16\,(\alpha + 3)\,(\alpha + n)\,\big(\alpha^4 - 2\,\alpha^3\,\beta - 4\,\alpha^3\,n$
$+ \alpha^2\,\beta^2 - \alpha^2\,\beta\,n - \alpha^2\,n^2 + 3\,\alpha\,\beta^2\,n + 9\,\alpha\,\beta\,n^2 + 6\,\alpha\,n^3 + 3\,\beta^2\,n^2 + 6\,\beta\,n^3 + 3\,n^4$
$+ 16\,\alpha^3 - 17\,\alpha^2\,\beta - 34\,\alpha^2\,n + 3\,\alpha\,\beta^2 - 24\,\alpha\,\beta\,n - 24\,\alpha\,n^2 + 3\,\beta^2\,n + 9\,\beta\,n^2 + 6\,n^3$
$+ 79\,\alpha^2 - 37\,\alpha\,\beta - 74\,\alpha\,n + 2\,\beta^2 - 41\,\beta\,n - 41\,n^2 + 140\,\alpha - 22\,\beta - 44\,n + 76\big)\big)\,L(n$
$- 1, \alpha, \beta)$

```
> Gjac[3,6,0]:=simplify(op([2,2],recJac21)/L(n-1, alpha, beta)/(
  (16*(alpha+3))*(alpha+n))):
> bound2Jac:=(-coeff(Gjac[3,6,0],x,1)+sqrt(((coeff(Gjac[3,6,0],x,
  1)^2-4*coeff(Gjac[3,6,0],x,0)*coeff(Gjac[3,6,0],x,2)))))/(2*
  coeff(Gjac[3,6,0],x,2))
```

$bound2Jac := \frac{1}{2}\,\big(6\,n^4 - (-12\,\beta - 12\,\alpha - 12)\,n^3 - \big(-6\,\alpha^2 - 18\,\alpha\,\beta - 6\,\beta^2 - 18\,\beta$ **(2.7)**
$+ 18\big)\,n^2 + 6\,(\alpha + \beta + 1)\,((\beta - 2)\,\alpha + \beta - 4)\,n + 2\,(\alpha + 2)\,(\alpha + 1)\,(\beta + \alpha$
$+ 3)\,(\beta - 6 - \alpha)$
$+ 4\,\big(\alpha^6\,\beta\,n + \alpha^6\,n^2 + \alpha^5\,\beta^2\,n + 3\,\alpha^5\,\beta\,n^2 + 2\,\alpha^5\,n^3 + \alpha^4\,\beta^2\,n^2 + 2\,\alpha^4\,\beta\,n^3$
$+ \alpha^4\,n^4 - \alpha^6\,\beta - 2\,\alpha^6\,n - \alpha^5\,\beta^2 + 12\,\alpha^5\,\beta\,n + 12\,\alpha^5\,n^2 + 13\,\alpha^4\,\beta^2\,n + 39\,\alpha^4\,\beta\,n^2$
$+ 26\,\alpha^4\,n^3 + 12\,\alpha^3\,\beta^2\,n^2 + 24\,\alpha^3\,\beta\,n^3 + 12\,\alpha^3\,n^4 + 2\,\alpha^6 - 11\,\alpha^5\,\beta - 22\,\alpha^5\,n - 10\,\alpha^4\,\beta^2$
$+ 66\,\alpha^4\,\beta\,n + 66\,\alpha^4\,n^2 + 73\,\alpha^3\,\beta^2\,n + 219\,\alpha^3\,\beta\,n^2 + 146\,\alpha^3\,n^3 + 61\,\alpha^2\,\beta^2\,n^2$
$+ 122\,\alpha^2\,\beta\,n^3 + 61\,\alpha^2\,n^4 + 26\,\alpha^5 - 47\,\alpha^4\,\beta - 94\,\alpha^4\,n - 37\,\alpha^3\,\beta^2 + 210\,\alpha^3\,\beta\,n$
$+ 210\,\alpha^3\,n^2 + 211\,\alpha^2\,\beta^2\,n + 633\,\alpha^2\,\beta\,n^2 + 422\,\alpha^2\,n^3 + 150\,\alpha\,\beta^2\,n^2 + 300\,\alpha\,\beta\,n^3$
$+ 150\,\alpha\,n^4 + 134\,\alpha^4 - 101\,\alpha^3\,\beta - 202\,\alpha^3\,n - 64\,\alpha^2\,\beta^2 + 369\,\alpha^2\,\beta\,n + 369\,\alpha^2\,n^2$
$+ 286\,\alpha\,\beta^2\,n + 858\,\alpha\,\beta\,n^2 + 572\,\alpha\,n^3 + 136\,\beta^2\,n^2 + 272\,\beta\,n^3 + 136\,n^4 + 350\,\alpha^3$
$- 116\,\alpha^2\,\beta - 232\,\alpha^2\,n - 52\,\alpha\,\beta^2 + 318\,\alpha\,\beta\,n + 318\,\alpha\,n^2 + 136\,\beta^2\,n + 408\,\beta\,n^2$

$+ 272 n^3 + 488 \alpha^2 - 68 \alpha \beta - 136 \alpha n - 16 \beta^2 + 104 \beta n + 104 n^2 + 344 \alpha - 16 \beta - 32 n + 96)^{1/2}\Big) \Big/ \Big(3 n^4 + (6 \beta + 6 \alpha + 6) n^3 + \big(7 \alpha^2 + (9 \beta + 24) \alpha + 3 \beta^2 + 9 \beta + 23\big) n^2 + 3 \left(\frac{4}{3} \alpha^2 + (\beta + 6) \alpha + \beta + \frac{20}{3}\right) (\alpha + \beta + 1) n + (\alpha + 2) (\alpha + 1) (\beta + \alpha + 3) (\beta + \alpha + 2)\Big)$

Comparison between the bounds and the extreme zeros

```
> xnJac0:= sort([solve(expand(Jacobi(12, 30.9,-0.8, x)),x)]):
> extzerolag:=evalf[15]([min(xnJac0),max(xnJac0)])
```

$$extzerolag := [-0.999156791323282, 0.141417391206758] \quad \textbf{(2.8)}$$

```
> Jac306:=unapply(bound1Jac,[n,alpha,beta]):
> evalf[15](Jac306(12,30.9,-0.8))
```

$$-0.999156790939300 \quad \textbf{(2.9)}$$

```
> Jac360:=unapply(bound2Jac,[n,alpha,beta]):
> evalf[15](Jac360(12,30.9,-0.8))
```

$$0.108318796166866 \quad \textbf{(2.10)}$$

the Gegenbauer polynomials

```
> Gegenbauer :=(n, alpha, x) ->  pochhammer(alpha, n)*2^n*x^n/n!*
  add(hyperterm([-n/2,-n/2+1/2],[-n-alpha+1],1/x^2,k), k = 0 ..
  n);
```

$$Gegenbauer := (n, \alpha, x) \rightarrow \frac{1}{n!} \left(\mathrm{pochhammer}(\alpha, n)\, 2^n x^n\, add\left(hyperterm\left(\left[-\frac{1}{2} n, -\frac{1}{2} n + \frac{1}{2}\right], [-n - \alpha + 1], \frac{1}{x^2}, k\right), k = 0..n\right)\right) \quad \textbf{(3.1)}$$

```
> FGe:=pochhammer(alpha, n)*2^n*x^n/n!*(hyperterm([-n/2,-
  n/2+1/2],[-n-alpha+1],1/x^2,k));
```

The weight function

```
> rhoGegenbauer:=(lambda)->(1-x^2)^(lambda-1/2)
```

$$rhoGegenbauer := \lambda \rightarrow (-x^2 + 1)^{\lambda - \frac{1}{2}} \quad \textbf{(3.2)}$$

```
> cGegen:=simplify(rhoGegenbauer(lambda+s)/rhoGegenbauer(lambda))
```

$$cGegen := (-x^2 + 1)^s \quad \textbf{(3.3)}$$

Mixed recurrence equation involving $L(n - 6, \alpha + 6, 0)$ giving a lower bound of x_(n,n): The zeros are symetric about the origin when n is odd

```
> recGeg1:=Mixedrec(FGe,k,L(n),6,alpha,6,0,0):
> recGeg11:=denom(rhs(recGeg1))*lhs(recGeg1)   =collect(numer(rhs
  (recGeg1)),[L(n, alpha, 0),L(n-1, alpha, 0),x],factor)
```

$$recGeg11 := 64 (-1 + x)^6 (1 + x)^6 (\alpha + 5) (\alpha + 4) (\alpha + 3) (\alpha + 2) (\alpha + 1) \alpha L(n - 6, \alpha + 6, 0) = \big((n - 5) (n - 4) (n - 3) (n - 2) (n - 1) n x^6 - n \big(32 \alpha^5 + 48 \alpha^4 n \quad \textbf{(3.4)}$$

$$+40\alpha^3 n^2+12\alpha^2 n^3+6\alpha n^4+3n^5+256\alpha^4+240\alpha^3 n+192\alpha^2 n^2-12\alpha n^3$$
$$-30n^4+816\alpha^3+264\alpha^2 n+488\alpha n^2+150n^3+1376\alpha^2-552\alpha n-30n^2$$
$$+1474\alpha-333n+600\big)x^4+3n\big(16\alpha^4 n+16\alpha^3 n^2+8\alpha^2 n^3+4\alpha n^4+n^5-48\alpha^4$$
$$+64\alpha^3 n+48\alpha^2 n^2-8\alpha n^3-5n^4-320\alpha^3+72\alpha^2 n+136\alpha n^2+15n^3-744\alpha^2$$
$$-152\alpha n+65n^2-592\alpha-181n-75\big)x^2-n(n-2)(n-4)(n+4+2\alpha)(n$$
$$+2+2\alpha)(n+2\alpha)\big)L(n,\alpha,0)+\big((2\alpha+5)(n+2\alpha-1)\big(16\alpha^4+32\alpha^3 n$$
$$+28\alpha^2 n^2+12\alpha n^3+3n^4+80\alpha^3+100\alpha^2 n+50\alpha n^2+140\alpha^2+90\alpha n+45n^2$$
$$+100\alpha+24\big)x^5-2(2\alpha+5)(n+2\alpha-1)\big(16\alpha^3 n+20\alpha^2 n^2+12\alpha n^3+3n^4$$
$$-40\alpha^3+20\alpha^2 n+10\alpha n^2-160\alpha^2+6\alpha n+3n^2-190\alpha-60\big)x^3+3(2\alpha$$
$$+5)(n+2\alpha-1)\big(4\alpha^2 n^2+4\alpha n^3+n^4-20\alpha^2 n-10\alpha n^2+20\alpha^2-26\alpha n$$
$$-13n^2+40\alpha+15\big)x\big)L(n-1,\alpha,0)$$

```
> GGegen[5]:=collect(op([2,2],recGeg11)/L(n-1, alpha, 0)/((2*
  alpha+5)*(n+2*alpha-1)*x),[x,alpha],factor):
> boundGegen:=sqrt((-coeff(GGegen[5],x,2)+sqrt(((coeff(GGegen[5],
  x,2)^2-4*coeff(GGegen[5],x,0)*coeff(GGegen[5],x,4)))))/(2*coeff
  (GGegen[5],x,4)))
```

$$boundGegen := \frac{1}{2}\sqrt{2}\Big(\Big(-(-32n+80)\alpha^3-(-40n^2-40n+320)\alpha^2-(-24n^3-20n^2-12n$$
$$+380)\alpha+6(n-2)(n+2)(n^2+5)$$
$$+2\big(64\alpha^6 n^2+64\alpha^5 n^3+16\alpha^4 n^4-320\alpha^6 n+480\alpha^5 n^2+640\alpha^4 n^3$$
$$+160\alpha^3 n^4+640\alpha^6-2592\alpha^5 n+1552\alpha^4 n^2+2848\alpha^3 n^3+712\alpha^2 n^4+6080\alpha^5$$
$$-8160\alpha^4 n+2160\alpha^3 n^2+6240\alpha^2 n^3+1560\alpha n^4+22080\alpha^4-13360\alpha^3 n$$
$$-1892\alpha^2 n^2+4788\alpha n^3+1197n^4+39200\alpha^3-10740\alpha^2 n-5370\alpha n^2+35560\alpha^2$$
$$-2898\alpha n-1449n^2+15420\alpha+2520\big)^{1/2}\Big)\Big/\big(16\alpha^4+(32n+80)\alpha^3+(28n^2$$
$$+100n+140)\alpha^2+2(2n+5)(3n^2+5n+10)\alpha+3n^4+45n^2+24\big)\Big)^{1/2} \quad \textbf{(3.5)}$$

Comparison between the bounds and the extreme zeros

```
> xngeg0:= sort([solve(expand(Gegenbauer(51, 30.9, x)),x)]):
> xngeg1:=evalf[15](xngeg0):
> extzerogeg:=[min(op(xngeg1)),max(op(xngeg1))]
```

$$extzerogeg := [-0.897416627596833, 0.897416627596833] \quad \textbf{(3.6)}$$

```
> gegen66:=unapply(boundGegen,[n,alpha]):
> evalf[15](gegen66(51,30.9))
```

$$0.889358028044289 \quad \textbf{(3.7)}$$

the Hermite polynomials

```
> Hermitep := (n, x) -> 2^n*x^n*add(hyperterm([-n/2,-n/2+1/2],[],
  -1/x^2,k), k = 0 .. n) ;
```

$$Hermitep := (n, x) \rightarrow 2^n x^n\, add\left(hyperterm\left(\left[-\frac{1}{2}\, n, -\frac{1}{2}\, n+\frac{1}{2}\right], [\,], -\frac{1}{x^2}, k\right), k=0..n\right) \quad \textbf{(4.1)}$$

```
> FHerm:=2^n*x^n*(hyperterm([-n/2,-n/2+1/2],[],-1/x^2,k)) ;
```

Mixed recurrence equation for m=6

```
> recHerm1:=Mixedrec(FHerm,k,L(n),6,0,0,0,0):
> recHerm11:=denom(rhs(recHerm1))*lhs(recHerm1)  =collect(numer
  (rhs(recHerm1)),[L(n, 0, 0),L(n-1, 0, 0),x],factor)
```

$$recHerm11 := 8\,(n-5)\,(n-4)\,(n-3)\,(n-2)\,(n-1)\,L(n-6,0,0) = \left(-4\,x^4+(6\,n-18)\,x^2-(n-2)\,(n-4)\right) L(n,0,0) + \left(8\,x^5+(-16\,n+40)\,x^3+\left(6\,n^2-30\,n+30\right) x\right) L(n-1,0,0) \quad \textbf{(4.2)}$$

```
> GHerm[6]:=op([2,2],recHerm11)/L(n-1, 0, 0)/x
```

$$GHerm_6 := \frac{8\,x^5+(-16\,n+40)\,x^3+\left(6\,n^2-30\,n+30\right) x}{x} \quad \textbf{(4.3)}$$

```
> boundHerm:=sqrt((-coeff(GHerm[6],x,2)+sqrt(((coeff(GHerm[6],x,
  2)^2-4*coeff(GHerm[6],x,0)*coeff(GHerm[6],x,4)))))/(2*coeff
  (GHerm[6],x,4)))
```

$$boundHerm := \frac{1}{2}\sqrt{4\,n-10+2\sqrt{n^2-5\,n+10}} \quad \textbf{(4.4)}$$

Comparison between the bounds and the extreme zeros

```
> xnHer0:= sort([solve(expand(Hermitep(50,x)),x)]):
> extzeroHerm:=evalf[15]([min(xnHer0),max(xnHer0)])
```

$$extzeroHerm := [-9.18240695812930, 9.18240695812930] \quad \textbf{(4.5)}$$

```
> Herm:=unapply(boundHerm,n):
> evalf[15](Herm(50))
```

$$8.44214005143300 \quad \textbf{(4.6)}$$

the Bessel polynomials

```
> Bessel :=  (n, alpha, x) -> add(hyperterm([-n,n+alpha+1],[],-
  x/2,k), k = 0 .. n) ;
```

$$Bessel := (n, \alpha, x) \rightarrow add\left(hyperterm\left([-n, n+\alpha+1], [\,], -\frac{1}{2}\, x, k\right), k=0..n\right) \quad \textbf{(5.1)}$$

```
> FBess:=hyperterm([-n,n+alpha+1],[],-x/2,k)
```

The weight function

```
> rhoBessel:=alpha->x^alpha*exp(-2/x)
```

$$rhoBessel := \alpha \rightarrow x^{\alpha}\, e^{-\frac{2}{x}} \quad \textbf{(5.2)}$$

```
> cBess:=simplify(rhoBessel(alpha+s)/rhoBessel(alpha))
```

$$cBess := x^s \quad (5.3)$$

Mixed recurrence equation for m=3, k=0 giving a lower bound of x_(n,n)

```
> recBess1:=Mixedrec(FBess,k,L(n),3,alpha,0,0,0):
> recBess11:=denom(rhs(recBess1))*lhs(recBess1)  =collect(numer
  (rhs(recBess1)),[L(n, alpha, 0),L(n-1, alpha, 0),x],factor)
```

$$recBess11 := 4\,(n-1)\,(n-2)\,(\alpha+2\,n)\,L(n-3,\alpha,0) = \big(-2\,(\alpha+n)\,(2\,n-2+\alpha)\,(\alpha-3+2\,n)\,(\alpha+2\,n-4)\,x-4\,(\alpha-3+2\,n)\,(\alpha+n)\,\alpha\big)\,L(n,\alpha,0) + \Big(\big((\alpha+2\,n)\,(2\,n+\alpha-1)\,(2\,n-2+\alpha)\,(\alpha-3+2\,n)\,(\alpha+2\,n-4)\,x^2 + 4\,\alpha\,(2\,n+\alpha-1)\,(2\,n-2+\alpha)\,(\alpha-3+2\,n)\,x+4\,(2\,n-2+\alpha)\,(\alpha^2+\alpha\,n+n^2-\alpha-2\,n)\Big)\,L(n-1,\alpha,0) \quad (5.4)$$

```
> GBess[3,0]:=simplify(op([2,2],recBess11)/L(n-1, alpha, 0)/(2*
  n-2+alpha)):
> boundBess1:=-coeff(GBess[3,0],x,1)/(2*coeff(GBess[3,0],x,2))+
  sqrt(((coeff(GBess[3,0],x,1)^2-4*coeff(GBess[3,0],x,0)*coeff
  (GBess[3,0],x,2))))/(2*coeff(GBess[3,0],x,2))
```

$$boundBess1 := -\frac{1}{2}\,\frac{\alpha}{\left(n-2+\frac{1}{2}\,\alpha\right)\left(n+\frac{1}{2}\,\alpha\right)} + \frac{1}{8}\,\big(-\alpha^5\,n-9\,\alpha^4\,n^2-32\,\alpha^3\,n^3-56\,\alpha^2\,n^4-48\,\alpha\,n^5-16\,n^6+\alpha^5+18\,\alpha^4\,n+96\,\alpha^3\,n^2+224\,\alpha^2\,n^3+240\,\alpha\,n^4+96\,n^5-5\,\alpha^4-71\,\alpha^3\,n-275\,\alpha^2\,n^2-408\,\alpha\,n^3-204\,n^4+7\,\alpha^3+102\,\alpha^2\,n+264\,\alpha\,n^2+176\,n^3-3\,\alpha^2-48\,\alpha\,n-48\,n^2\big)^{1/2}\Big/\left(\left(n-2+\frac{1}{2}\,\alpha\right)\left(n-\frac{1}{2}+\frac{1}{2}\,\alpha\right)\left(n-\frac{3}{2}+\frac{1}{2}\,\alpha\right)\left(n+\frac{1}{2}\,\alpha\right)\right) \quad (5.5)$$

Mixed recurrence equation for m=3, k=6 giving an upper bound of x_(n,1)

```
> recBess2:=Mixedrec(FBess,k,L(n),3,alpha,6,0,0):
> recBess21:=denom(rhs(recBess2))*lhs(recBess2)  =collect(numer
  (rhs(recBess2)),[L(n, alpha, 0),L(n-1, alpha, 0),x],factor)
```

$$recBess21 := x^6\,(n-1)\,(n-2)\,(\alpha+3+n)\,(\alpha+n+2)\,(n+\alpha+1)\,(\alpha+2\,n)\,L(n-3,\alpha+6,0) = \big(-64+8\,(n-1)\,(n-2)\,(\alpha+2\,n)\,x^3-48\,(n-1)\,(n-2)\,x^2+(-32\,\alpha+64\,n-192)\,x\big)\,L(n,\alpha,0) + \big(64+(16\,\alpha^2+16\,\alpha\,n+16\,n^2+80\,\alpha+16\,n+96)\,x^2+(64\,\alpha+192)\,x\big)\,L(n-1,\alpha,0) \quad (5.6)$$

```
> GBess[3,6]:=(op([2,2],recBess21)/L(n-1, alpha, 0))
```

$$GBess_{3,6} := 64+(16\,\alpha^2+16\,\alpha\,n+16\,n^2+80\,\alpha+16\,n+96)\,x^2+(64\,\alpha+192)\,x \quad (5.7)$$

```
> boundBess2:=(-coeff(GBess[3,6],x,1)-sqrt(((coeff(GBess[3,6],x,
  1)^2-4*coeff(GBess[3,6],x,0)*coeff(GBess[3,6],x,2)))))/(2*coeff
  (GBess[3,6],x,2))
```

$$boundBess2 := \frac{1}{2}\,\frac{-64\,\alpha-192-64\,\sqrt{-\alpha\,n-n^2+\alpha-n+3}}{16\,\alpha^2+16\,\alpha\,n+16\,n^2+80\,\alpha+16\,n+96} \quad (5.8)$$

Comparison between the bounds and the extreme zeros

```
> xnBes0:= sort([solve(simpcomb(Bessel(10,-305,x)),x)]):
> extzeroHerm:=evalf[15]([min(xnBes0),max(xnBes0)])
```

$$extzeroHerm := [0.00523390794247464, 0.00927526873340181] \quad (5.9)$$

```
> Bess1:=unapply(boundBess1,[n,alpha]):
> evalf[15](Bess1(10,-305))
```

$$0.00866036554136563 \quad (5.10)$$

```
> Bess2:=unapply(boundBess2,[n,alpha]):
> evalf[15](Bess2(10,-305))
```

$$0.00565992674042895 \quad (5.11)$$

the Hahn polynomials

```
> Hahn:=(n,x,alpha,beta,N)->add(hyperterm([-n,-x,n+1+alpha+beta],
  [alpha+1,-N],1,m),m=0..n);
```

$$Hahn := (n, x, \alpha, \beta, N) \rightarrow add(hyperterm([-n, -x, n+1+\alpha+\beta], [\alpha+1, -N], 1, m), m = 0..n) \quad (6.1)$$

```
> FHahn:=(hyperterm([-n,-x,n+1+alpha+beta],[alpha+1,-N],1,k));
```

The weight function

```
> rhoHahn:=(alpha,beta)->binomial(alpha+x,x)*binomial(beta+N-x,N-
  x)
```

$$rhoHahn := (\alpha, \beta) \rightarrow \mathrm{binomial}(\alpha+x, x)\ \mathrm{binomial}(\beta+N-x, N-x) \quad (6.2)$$

```
> cHahn:=pochhammer(alpha+x+1,s)/pochhammer(alpha+1,s)*pochhammer
  (beta+N-x+1,t)/pochhammer(beta+1,t);
```

$$cHahn := \frac{\mathrm{pochhammer}(\alpha+x+1, s)\ \mathrm{pochhammer}(\beta+N-x+1, t)}{\mathrm{pochhammer}(\alpha+1, s)\ \mathrm{pochhammer}(\beta+1, t)} \quad (6.3)$$

```
> simplify(rhoHahn(alpha+s,beta+t)/rhoHahn(alpha,beta)-cHahn)
```

$$0 \quad (6.4)$$

Mixed recurrence equation involving $L(n-3, \alpha, \beta+6)$ giving a lower bound of x_(n,n)

```
> recHahn1:=Mixedrec(FHahn,k,L(n),3,alpha,0,beta,6):
> recHahn11:=denom(rhs(recHahn1))*lhs(recHahn1)   =collect(numer
  (rhs(recHahn1)),[L(n, alpha, beta),L(n-1, alpha, beta),x],
  factor):
> GHahn[3,0,6]:=collect(op([2,2],recHahn11)/L(n-1, alpha, beta)/(
  (beta+n)*(n+1+N+alpha+beta)),[x,n],factor):
> boundHahn1:=(-coeff(GHahn[3,0,6],x,1)+sqrt(((coeff(GHahn[3,0,
  6],x,1)^2-4*coeff(GHahn[3,0,6],x,0)*coeff(GHahn[3,0,6],x,2)))))
  /(2*coeff(GHahn[3,0,6],x,2)):
```

Mixed recurrence equation for $(L(n-3, \alpha+6, \beta))$ giving an upper bound of x_(n,1)

```
> recHahn2:=Mixedrec(FHahn,k,L(n),3,alpha,6,beta,0):
Warning,  computation interrupted
> recHahn21:=denom(rhs(recHahn2))*lhs(recHahn2)   =collect(numer
  (rhs(recHahn2)),[L(n, alpha, beta),L(n-1, alpha, beta),x],
```

```
factor)
```

Comparison between the bounds and the extreme zeros

```
> xnHahn0:= sort([solve(simpcomb(Hahn(5,x,200,2,30)),x)]):
> xnHahn1:=evalf[15](xnHahn0):
> extzeroHahn:=[min(op(xnHahn1)),max(op(xnHahn1))]
```

$$extzeroHahn := [23.2193756526935, 29.9984563719298] \quad (6.5)$$

```
> Hahn306:=unapply(boundHahn1,[n,alpha,beta,N]):
> evalf[15](Hahn306(5,200,2,30) )
```

$$28.4168327719773 \quad (6.6)$$

```
> Hahn360:=unapply(recHahn21,[n,alpha,beta,N]):
> evalf[15](Hahn306(5,200,2,30) )
```

the Meixner polynomials

```
> Meixner:=(n,x,gamma,mu)->add(hyperterm([-n,-x],[gamma],1-1/mu,
  m),m=0..n );
```

$$Meixner := (n, x, \gamma, \mu) \to add\left(hyperterm\left([-n, -x], [\gamma], 1 - \frac{1}{\mu}, m\right), m = 0..n\right) \quad (7.1)$$

```
> FMeix:=(hyperterm([-n,-x],[gamma],1-1/mu,k) )
```

The weight function

```
> rhoMeix:=(beta,c)->pochhammer(beta,x)*c^x/x!
```

$$rhoMeix := (\beta, c) \to \frac{\mathrm{pochhammer}(\beta, x)\, c^x}{x!} \quad (7.2)$$

```
> cMeix:=pochhammer(beta+x,s)/pochhammer(beta,s)
```

$$cMeix := \frac{\mathrm{pochhammer}(\beta + x, s)}{\mathrm{pochhammer}(\beta, s)} \quad (7.3)$$

```
> simpcomb(rhoMeix(beta+s,c)/rhoMeix(beta,c)- cMeix)
```

$$0 \quad (7.4)$$

Mixed recurrence equation involving $M(n-3, \gamma, \mu)$ giving a lower bound of x_(n,n)

```
> recMeix1:=Mixedrec(FMeix,k,M(n),3,gamma,0,mu,0):
> recMeix11:=denom(rhs(recMeix1))*lhs(recMeix1)   =collect(numer
  (rhs(recMeix1)),[M(n, gamma, mu),M(n-1, gamma, mu),x],factor)
```

$$recMeix11 := (n-1)(n-2)\, M(n-3, \gamma, \mu) = \big(-\mu(\mu-1)(n-1+\gamma)\, x - \mu(n-1+\gamma)(\gamma\mu + \mu n - 2\mu + n - 2)\big) M(n, \gamma, \mu) + \big((\mu-1)^2 x^2 + (\mu-1)(2\gamma\mu + 2\mu n - 3\mu + 2n - 3)\, x + \gamma^2\mu^2 + 2\gamma\mu^2 n + \mu^2 n^2 - 3\gamma\mu^2 + \gamma\mu n - 3\mu^2 n + \mu n^2 - 2\gamma\mu + 2\mu^2 - 3n\mu + n^2 + 2\mu - 3n + 2\big) M(n-1, \gamma, \mu) \quad (7.5)$$

```
> GMeix[3,0,0]:=op([2,2],recMeix11)/M(n-1, gamma, mu)
```

$$GMeix_{3,0,0} := (\mu-1)^2 x^2 + (\mu-1)(2\gamma\mu + 2\mu n - 3\mu + 2n - 3)\, x + \gamma^2\mu^2 + 2\gamma\mu^2 n + \mu^2 n^2 - 3\gamma\mu^2 + \gamma\mu n - 3\mu^2 n + \mu n^2 - 2\gamma\mu + 2\mu^2 - 3n\mu + n^2 + 2\mu - 3n + 2 \quad (7.6)$$

```
> boundMeix1:=-coeff(GMeix[3,0,0],x,1)/(2*coeff(GMeix[3,0,0],x,2)
)+sqrt(((coeff(GMeix[3,0,0],x,1)^2-4*coeff(GMeix[3,0,0],x,0)*
coeff(GMeix[3,0,0],x,2))))/(2*coeff(GMeix[3,0,0],x,2))
```

$$boundMeix1 := -\frac{1}{2}\,\frac{2\,\gamma\mu+2\,\mu n-3\,\mu+2\,n-3}{\mu-1} + \frac{1}{2}\,\frac{1}{(\mu-1)^2}\Big((\mu-1)^2\,(2\,\gamma\mu+2\,\mu n-3\,\mu+2\,n-3)^2-4\,\big(\gamma^2\mu^2 + 2\,\gamma\mu^2 n+\mu^2 n^2-3\,\gamma\mu^2+\gamma\mu n-3\,\mu^2 n+\mu n^2-2\,\gamma\mu+2\,\mu^2-3\,\mu n+n^2+2\,\mu - 3\,n+2\big)\,(\mu-1)^2\Big)^{1/2} \tag{7.7}$$

Mixed recurrence equation involving $M(n-3,\gamma+6,\mu)$ giving an upper bound of x_(n,1)

```
> recMeix2:=Mixedrec(FMeix,k,M(n),3,gamma,6,mu,0):
> recMeix21:=denom(rhs(recMeix2))*lhs(recMeix2)  =collect(numer
(rhs(recMeix2)),[M(n, gamma, mu),M(n-1, gamma, mu),x],factor):
> GMeix[3,6]:= collect(op([2,2],recMeix21)/M(n-1, gamma, mu)/
(gamma*(gamma+5)*(gamma+4)*(gamma+3)*(gamma+2)*(gamma+1)),x,
factor)
```

$$GMeix_{3,6} := 4\,\gamma+4\,n+78\,\gamma\mu^2 n-34\,\gamma\mu n+n^5-5\,n^3-\mu^5 n^5+5\,\mu^4 n^5-10\,\mu^3 n^5 + 10\,\mu^2 n^5-5\,\mu n^5+11\,\gamma\mu^4 n^4+3\,\gamma\mu^5 n^3+6\,\gamma^2\mu^4 n^3+3\,\gamma^2\mu^5 n^2-2\,\gamma\mu^5 n^4-\gamma^2\mu^5 n^3 + 6\,\gamma^2-34\,\gamma^2\mu-16\,\gamma^3\mu+3\,\gamma n^4-2\,\gamma^4\mu+49\,\gamma^3\mu^2+12\,\gamma^4\mu^2+\gamma^5\mu^2+46\,\gamma\mu n^2 + 5\,\gamma^2\mu n+16\,\gamma\mu n^3-78\,\gamma\mu^3 n-81\,\gamma\mu^2 n^2+18\,\gamma^2\mu^2 n+30\,\gamma^2\mu n^2+12\,\gamma^3\mu n - 14\,\gamma\mu n^4+69\,\gamma\mu^3 n^2-33\,\gamma\mu^2 n^3+34\,\gamma\mu^4 n-54\,\gamma^2\mu^2 n^2-12\,\gamma^2\mu n^3-30\,\gamma^2\mu^3 n - 9\,\gamma^3\mu^2 n-2\,\gamma^3\mu n^2+\gamma^4\mu n-24\,\gamma\mu^3 n^4-16\,\gamma\mu^4 n^3+5\,\gamma\mu^5 n^2-18\,\gamma^2\mu^4 n^2 - 15\,\gamma^2\mu^3 n^3+(\mu-1)^2\,\big(-\mu^3 n^3+3\,\mu^3 n^2-2\,\mu^3 n+\gamma^3+3\,\gamma^2 n+3\,\gamma n^2+n^3+3\,\gamma^2 + 6\,\gamma n+3\,n^2+2\,\gamma+2\,n\big)\,x^2+(\mu-1)\,\big(-2\,\gamma\mu^4 n^3-2\,\mu^4 n^4+6\,\gamma\mu^4 n^2+6\,\gamma\mu^3 n^3 + 3\,\mu^4 n^3+4\,\mu^3 n^4-4\,\gamma\mu^4 n-18\,\gamma\mu^3 n^2+5\,\mu^4 n^2-3\,\mu^3 n^3+2\,\gamma^4\mu+2\,\gamma^3\mu n - 6\,\gamma^2\mu n^2+12\,\gamma\mu^3 n-10\,\gamma\mu n^3-6\,\mu^4 n-19\,\mu^3 n^2-4\,\mu n^4+15\,\gamma^3\mu+2\,\gamma^3 n + 27\,\gamma^2\mu n+6\,\gamma^2 n^2+9\,\gamma\mu n^2+6\,\gamma n^3+18\,\mu^3 n-3\,\mu n^3+2\,n^4-3\,\gamma^3+31\,\gamma^2\mu - 3\,\gamma^2 n+50\,\gamma\mu n+3\,\gamma n^2+19\,\mu n^2+3\,n^3-9\,\gamma^2+18\,\gamma\mu-14\,\gamma n+18\,\mu n-5\,n^2 - 6\,\gamma-6\,n\big)\,x-2\,\gamma^2\mu^5 n+5\,\mu^5 n^3-25\,\mu^4 n^3-4\,\mu^5 n+50\,\mu^3 n^3+20\,\mu^4 n-50\,\mu^2 n^3 - 40\,\mu^3 n+25\,\mu n^3+2\,\gamma^3+26\,\gamma\mu^2 n^4+33\,\gamma\mu^3 n^3-6\,\gamma\mu^5 n-29\,\gamma\mu^4 n^2+19\,\gamma^2\mu^2 n^3 + 45\,\gamma^2\mu^3 n^2+12\,\gamma^2\mu^4 n+\gamma^3\mu^2 n^2-\gamma^4\mu^2 n+\gamma^3 n^2+3\,\gamma^2 n^3-3\,\gamma^3 n-6\,\gamma^2 n^2-3\,\gamma n^3 - 3\,\gamma^2 n-10\,\gamma n^2+6\,\gamma n+78\,\gamma^2\mu^2+40\,\gamma\mu^2+40\,\mu^2 n-20\,\gamma\mu-20\,n\mu \tag{7.8}$$

```
> boundMeix2:=(-coeff(GMeix[3,6],x,1)-sqrt(((coeff(GMeix[3,6],x,
1)^2-4*coeff(GMeix[3,6],x,0)*coeff(GMeix[3,6],x,2)))))/(2*coeff
(GMeix[3,6],x,2))
```

(7.9)

$$boundMeix2 := \frac{1}{2} \Big(-(\mu-1) \left(-2\gamma\mu^4 n^3 - 2\mu^4 n^4 + 6\gamma\mu^4 n^2 + 6\gamma\mu^3 n^3 + 3\mu^4 n^3 \right. \tag{7.9}$$

$$+ 4\mu^3 n^4 - 4\gamma\mu^4 n - 18\gamma\mu^3 n^2 + 5\mu^4 n^2 - 3\mu^3 n^3 + 2\gamma^4 \mu + 2\gamma^3 \mu n - 6\gamma^2 \mu n^2$$

$$+ 12\gamma\mu^3 n - 10\gamma\mu n^3 - 6\mu^4 n - 19\mu^3 n^2 - 4\mu n^4 + 15\gamma^3 \mu + 2\gamma^3 n + 27\gamma^2 \mu n$$

$$+ 6\gamma^2 n^2 + 9\gamma\mu n^2 + 6\gamma n^3 + 18\mu^3 n - 3\mu n^3 + 2n^4 - 3\gamma^3 + 31\gamma^2 \mu - 3\gamma^2 n$$

$$+ 50\gamma\mu n + 3\gamma n^2 + 19\mu n^2 + 3n^3 - 9\gamma^2 + 18\gamma\mu - 14\gamma n + 18\mu n - 5n^2 - 6\gamma$$

$$\left. - 6n \right)$$

$$- \Big((\mu-1)^2 \left(-2\gamma\mu^4 n^3 - 2\mu^4 n^4 + 6\gamma\mu^4 n^2 + 6\gamma\mu^3 n^3 + 3\mu^4 n^3 \right.$$

$$+ 4\mu^3 n^4 - 4\gamma\mu^4 n - 18\gamma\mu^3 n^2 + 5\mu^4 n^2 - 3\mu^3 n^3 + 2\gamma^4 \mu + 2\gamma^3 \mu n - 6\gamma^2 \mu n^2$$

$$+ 12\gamma\mu^3 n - 10\gamma\mu n^3 - 6\mu^4 n - 19\mu^3 n^2 - 4\mu n^4 + 15\gamma^3 \mu + 2\gamma^3 n + 27\gamma^2 \mu n$$

$$+ 6\gamma^2 n^2 + 9\gamma\mu n^2 + 6\gamma n^3 + 18\mu^3 n - 3\mu n^3 + 2n^4 - 3\gamma^3 + 31\gamma^2 \mu - 3\gamma^2 n$$

$$+ 50\gamma\mu n + 3\gamma n^2 + 19\mu n^2 + 3n^3 - 9\gamma^2 + 18\gamma\mu - 14\gamma n + 18\mu n - 5n^2 - 6\gamma$$

$$\left. - 6n \right)^2 - 4 \left(-\gamma^2 \mu^5 n^3 - 2\gamma\mu^5 n^4 - \mu^5 n^5 + 3\gamma^2 \mu^5 n^2 + 6\gamma^2 \mu^4 n^3 + 3\gamma\mu^5 n^3 \right.$$

$$+ 11\gamma\mu^4 n^4 + 5\mu^4 n^5 - 2\gamma^2 \mu^5 n - 18\gamma^2 \mu^4 n^2 - 15\gamma^2 \mu^3 n^3 + 5\gamma\mu^5 n^2 - 16\gamma\mu^4 n^3$$

$$- 24\gamma\mu^3 n^4 + 5\mu^5 n^3 - 10\mu^3 n^5 + \gamma^5 \mu^2 - \gamma^4 \mu^2 n + \gamma^3 \mu^2 n^2 + 12\gamma^2 \mu^4 n + 45\gamma^2 \mu^3 n^2$$

$$+ 19\gamma^2 \mu^2 n^3 - 6\gamma\mu^5 n - 29\gamma\mu^4 n^2 + 33\gamma\mu^3 n^3 + 26\gamma\mu^2 n^4 - 25\mu^4 n^3 + 10\mu^2 n^5$$

$$+ 12\gamma^4 \mu^2 + \gamma^4 \mu n - 9\gamma^3 \mu^2 n - 2\gamma^3 \mu n^2 - 30\gamma^2 \mu^3 n - 54\gamma^2 \mu^2 n^2 - 12\gamma^2 \mu n^3$$

$$+ 34\gamma\mu^4 n + 69\gamma\mu^3 n^2 - 33\gamma\mu^2 n^3 - 14\gamma\mu n^4 - 4\mu^5 n + 50\mu^3 n^3 - 5\mu n^5 - 2\gamma^4 \mu$$

$$+ 49\gamma^3 \mu^2 + 12\gamma^3 \mu n + \gamma^3 n^2 + 18\gamma^2 \mu^2 n + 30\gamma^2 \mu n^2 + 3\gamma^2 n^3 - 78\gamma\mu^3 n - 81\gamma\mu^2 n^2$$

$$+ 16\gamma\mu n^3 + 3\gamma n^4 + 20\mu^4 n - 50\mu^2 n^3 + n^5 - 16\gamma^3 \mu - 3\gamma^3 n + 78\gamma^2 \mu^2 + 5\gamma^2 \mu n$$

$$- 6\gamma^2 n^2 + 78\gamma\mu^2 n + 46\gamma\mu n^2 - 3\gamma n^3 - 40\mu^3 n + 25\mu n^3 + 2\gamma^3 - 34\gamma^2 \mu - 3\gamma^2 n$$

$$+ 40\gamma\mu^2 - 34\gamma\mu n - 10\gamma n^2 + 40\mu^2 n - 5n^3 + 6\gamma^2 - 20\gamma\mu + 6\gamma n - 20\mu n + 4\gamma$$

$$\left. + 4n \right) (\mu-1)^2 \left(-\mu^3 n^3 + 3\mu^3 n^2 - 2\mu^3 n + \gamma^3 + 3\gamma^2 n + 3\gamma n^2 + n^3 + 3\gamma^2 + 6\gamma n \right.$$

$$+3n^2+2\gamma+2n)\Big)^{1/2}\Big) \Big/ \Big((\mu-1)^2\big(-\mu^3 n^3+3\mu^3 n^2-2\mu^3 n+\gamma^3+3\gamma^2 n+3\gamma n^2 + n^3+3\gamma^2+6\gamma n+3n^2+2\gamma+2n\big)\Big)$$

Comparison between the bounds and the extreme zeros

```
> xnMeix0:= sort([solve(simpcomb(Meixner(10,x,0.09,0.99)),x)]):
> extzeroMeix:=evalf[15]([min(xnMeix0),max(xnMeix0)])
```

$$extzeroMeix := [0.886751385278167, 2814.03598753338] \quad \textbf{(7.10)}$$

```
> Meix300:=unapply(boundMeix1 ,[n,gamma,mu]):
> evalf[15](Meix300(10,0.09,0.99) )
```

$$2555.23118013068 \quad \textbf{(7.11)}$$

```
> Meix360:=unapply(boundMeix2 ,[n,gamma,mu]):
> evalf[15](Meix360(10,0.09,0.99) )
```

$$0.886768650875350 \quad \textbf{(7.12)}$$

the Charlier polynomials

```
> Charlier:=(n,x,alpha)->add(hyperterm([-n,-x],[],-1/alpha,k) ,k=
  0..n );
```

$$Charlier := (n, x, \alpha) \rightarrow add\left(hyperterm\left([-n, -x], [\,], -\frac{1}{\alpha}, k\right), k=0..n\right) \quad \textbf{(8.1)}$$

```
> FChar:=(hyperterm([-n,-x],[],-1/alpha,k));
```

$$FChar := \frac{\mathrm{pochhammer}(-n, k)\ \mathrm{pochhammer}(-x, k)\left(-\frac{1}{\alpha}\right)^k}{k!} \quad \textbf{(8.2)}$$

The weight function

```
> rhoCharlier:=alpha->alpha^x/x!
```

$$rhoCharlier := \alpha \rightarrow \frac{\alpha^x}{x!} \quad \textbf{(8.3)}$$

```
> cCharlier:=simplify(rhoCharlier(alpha+s)/rhoCharlier(alpha))
```

$$cCharlier := (\alpha+s)^x\,\alpha^{-x} \quad \textbf{(8.4)}$$

cCharlier is not a polynomial of the variable x. Therefore Theorem 1 can not be applied. We can only use Equation (2)

Mixed recurrence equation for m=3, k=0

```
> recChar:=Mixedrec(FChar,k,S(n),3,alpha,0,0,0):
> recChar1:=denom(rhs(recChar))*lhs(recChar)   =collect(numer(rhs
  (recChar)),[S(n, alpha, 0),S(n-1, alpha, 0),x],factor)
```

$$recChar1 := (n-1)\,(n-2)\,S(n-3, \alpha, 0) = \big(x\,\alpha-\alpha\,(\alpha+n-2)\big)\,S(n, \alpha, 0) + \big(x^2 + (-2\,\alpha-2\,n+3)\,x+\alpha^2+\alpha\,n+n^2-2\,\alpha-3\,n+2\big)\,S(n-1, \alpha, 0) \quad \textbf{(8.5)}$$

```
> GChar[3,0,0]:=op([2,2],recChar1)/S(n-1, alpha, 0)
```

$$GChar_{3,\,0,\,0} := x^2+(-2\,\alpha-2\,n+3)\,x+\alpha^2+\alpha\,n+n^2-2\,\alpha-3\,n+2 \quad \textbf{(8.6)}$$

```
> boundChar:=[solve(GChar[3,0,0],x)]
```

$$boundChar := \left[\alpha + n - \frac{3}{2} + \frac{1}{2}\sqrt{4\,\alpha\,n - 4\,\alpha + 1}, \alpha + n - \frac{3}{2} - \frac{1}{2}\sqrt{4\,\alpha\,n - 4\,\alpha + 1}\right] \quad (8.7)$$

Comparison of the bounds and the extreme zeros

```
> xnChar:= sort([solve(simpcomb(Charlier(20,x,0.5)),x)]):
> extzeroChar:=evalf[15]([min(xnChar),max(xnChar)])
```

$$extzeroChar := [4.62976142954428\ 10^{-24}, 23.2132663026693] \quad (8.8)$$

```
> Char300:=unapply(boundChar ,[n,alpha]):
> sort(evalf[15](Char300(20,0.5) ))
```

$$[15.8775010008008, 22.1224989991992] \quad (8.9)$$

the Krawtchouk polynomials

```
> Krawtchouk:=(n,x,p,N)->add(hyperterm([-n,-x],[-N],1/p,k)   ,k=0.
  .n );
```

$$Krawtchouk := (n, x, p, N) \rightarrow add\left(hyperterm\left([-n, -x], [-N], \frac{1}{p}, k\right), k = 0..n\right) \quad (9.1)$$

```
> FKrawt:=(hyperterm([-n,-x],[-N],1/p,k));
```

$$FKrawt := \frac{\text{pochhammer}(-n, k)\ \text{pochhammer}(-x, k)\left(\frac{1}{p}\right)^k}{\text{pochhammer}(-N, k)\ k!} \quad (9.2)$$

For this polynomial family, 0<p<1 and 0<= x<=N, so we cannot do a shift on p or N. Theorem 1 is therefore not applicable and we use only Equation (2).

```
> recKraw:=Mixedrec(FKrawt,k,M(n),3,p,0,N,0):
> recKraw1:=denom(rhs(recKraw))*lhs(recKraw)   =collect(numer(rhs
  (recKraw)),[M(n, p, N),M(n-1, p, N),x],factor)
```

$$recKraw1 := (-1+p)^2\,(n-1)\,(n-2)\,M(n-3, p, N) = ((-n+1+N)\,p\,x - p\,(-n+1+N)\,(N\,p - 2\,n\,p + n + 4\,p - 2))\,M(n, p, N) + \big(x^2 + (-2\,N\,p + 4\,n\,p - 2\,n - 6\,p + 3)\,x + N^2\,p^2 - 3\,N\,n\,p^2 + 3\,n^2\,p^2 + N\,n\,p + 5\,N\,p^2 - 3\,n^2\,p - 9\,n\,p^2 - 2\,N\,p + n^2 + 9\,n\,p + 6\,p^2 - 3\,n - 6\,p + 2\big)\,M(n-1, p, N) \quad (9.3)$$

```
> GKraw[3,0,0]:=op([2,2],recKraw1)/M(n-1, p, N)
```

$$GKraw_{3,0,0} := x^2 + (-2\,N\,p + 4\,n\,p - 2\,n - 6\,p + 3)\,x + N^2\,p^2 - 3\,N\,n\,p^2 + 3\,n^2\,p^2 + N\,n\,p + 5\,N\,p^2 - 3\,n^2\,p - 9\,n\,p^2 - 2\,N\,p + n^2 + 9\,n\,p + 6\,p^2 - 3\,n - 6\,p + 2 \quad (9.4)$$

```
> boundKraw:=[solve(GKraw[3,0,0],x)]
```

$$boundKraw := \Big[N\,p - 2\,n\,p + n + 3\,p - \frac{3}{2} + \frac{1}{2}\,(-4\,N\,n\,p^2 + 4\,n^2\,p^2 + 4\,N\,n\,p + 4\,N\,p^2 - 4\,n^2\,p - 12\,n\,p^2 - 4\,N\,p + 12\,n\,p + 12\,p^2 - 12\,p + 1)^{1/2}, N\,p - 2\,n\,p + n + 3\,p - \frac{3}{2} \quad (9.5)$$

$$-\frac{1}{2}\left(-4 N n p^2+4 n^2 p^2+4 N n p+4 N p^2-4 n^2 p-12 n p^2-4 N p+12 n p+12 p^2-12 p+1\right)^{1/2}\Big]$$

Comparison of the bounds and the extreme zeros

```
> xnKraw:= sort([solve(simpcomb(Krawtchouk(25,x,0.5,40)),x)]):
> extzeroKraw:=evalf[15]([min(xnKraw),max(xnKraw)])
```

$$extzeroKraw := [0.188252439775629, 39.8117475602244] \quad (9.6)$$

```
> Kraw300:=unapply(boundKraw ,[n,p,N]):
> sort(evalf[15](Kraw300(25,0.5,40) ))
```

$$[9.9004950616379, 30.0995049383621] \quad (9.7)$$

```
> read `qsum17.mpl`:
```

Package "q-Hypergeometric Summation", Maple V-17

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The second procedure for the classical q-orthogonal polynomials

```
> _qsum_local_specialsolution:= false:
> qMixRec:=proc(F,q,k,Sn,shift0,alpha,shift1,beta,shift2)
          local zeit,pp,qq,rr,Rat,evalS,z,lo,hi,sigma,sigmasol,
  Poly,K,j,J,\
                    PQR,f,rec,S,n;
    zeit:= time();
      lo:=1; hi:=5;
      S:=op(0,Sn); n:=op(1,Sn);
          sigmasol:= NULL;
          for J from 1 to hi while (sigmasol = NULL) do
        Poly:=qsimpcomb(subs({n=n-shift0, alpha=alpha*q^shift1,
  beta=beta*q^shift2},F)-add(sigma[j]*subs(n=n-j,F),j=0..J));
                    Rat:= `power/subs`({q^k=K,q^n=N,q^(-n)=1/N},
  qratio(Poly,k));
                    if has(Rat,{k,qpochhammer}) then
                            ERROR(`Algorithm not applicable.`);
                    fi;
                    if (J < lo) then next; fi;
        pp:=1: qq:=numer(Rat): rr:=denom(Rat):
                    PQR:= `qgosper/update`(pp,qq,rr,q,K);
                    f:= `qgosper/findf`(op(PQR),q,K,[seq(sigma
  [j],j=0..J)],'sigmasol');
          od;
          if (sigmasol = NULL) then
               ERROR(cat(`Found no q-derivative rule of order
  smaller than `,J,`.`));
          fi;
          rec:= subs(sigmasol, add(sigma[j]*S(n-j,alpha,beta), j=
```

```
0..J-1));
    rec:=combine(map(factor, subs({N=q^n,K=q^k},rec)),power);
        if (_qsum_profile) then
            printf(`CPU-time: %.1f seconds`, time()-zeit);
        fi;
        RETURN(S(n-shift0,alpha*q^shift1,beta*q^shift2)=rec);
end:
```

the big q-Jacobi polynomials

```
> bqj:=(n,alpha,beta,gamma,x,q)->add(qphihyperterm([q^(-n),alpha*beta*q^(n+1),x],[alpha*q,gamma*q],q,q,k),k=0..n);
```

$$bqj := (n, \alpha, \beta, \gamma, x, q) \rightarrow add\left(qphihyperterm\left(\left[q^{-n}, \alpha\,\beta\,q^{n+1}, x\right], \left[\alpha\,q, \gamma\,q\right], q, q, k\right), k=0..n\right) \quad \textbf{(10.1)}$$

```
> Fbqj:=(qphihyperterm([q^(-n),alpha*beta*q^(n+1),x],[alpha*q,gamma*q],q,q,k));
```

$$Fbqj := qphihyperterm\left(\left[q^{-n}, \alpha\,\beta\,q^{n+1}, x\right], \left[\alpha\,q, \gamma\,q\right], q, q, k\right) \quad \textbf{(10.2)}$$

The weight function

```
> rhoBQJ:=(alpha,beta,gamma)->qpochhammer(x/alpha,q,infinity)*qpochhammer(x/gamma,q,infinity)/qpochhammer(x,q,infinity)/qpochhammer(beta*x/gamma,q,infinity)
```

$$rhoBQJ := (\alpha, \beta, \gamma) \rightarrow \frac{qpochhammer\left(\frac{x}{\alpha}, q, \infty\right) qpochhammer\left(\frac{x}{\gamma}, q, \infty\right)}{qpochhammer(x, q, \infty)\, qpochhammer\left(\frac{\beta\,x}{\gamma}, q, \infty\right)} \quad \textbf{(10.3)}$$

```
> cBQJ:=(s,t)->rhoBQJ(alpha*q^s,beta*q^t,gamma)/rhoBQJ(alpha,beta,gamma)
```

$$cBQJ := (s, t) \rightarrow \frac{rhoBQJ\left(\alpha\,q^{s}, \beta\,q^{t}, \gamma\right)}{rhoBQJ(\alpha, \beta, \gamma)} \quad \textbf{(10.4)}$$

```
> cBQJ1:=(s,t)->qpochhammer(x/(alpha*q^s), q, s)*qpochhammer(beta*x/gamma, q, t)
```

$$cBQJ1 := (s, t) \rightarrow qpochhammer\left(\frac{x}{\alpha\,q^{s}}, q, s\right) qpochhammer\left(\frac{\beta\,x}{\gamma}, q, t\right) \quad \textbf{(10.5)}$$

```
> qsimpcomb([seq(seq(cBQJ1(s,t)-qsimpcomb(cBQJ(s,t)),s=0..5),t=0..5)])
```

[0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0] **(10.6)**

Mixed recurrence equation involving $S\left(n-2, \alpha, \beta\,q^{4}\right)$ giving an upper bound of x_(n,1)

```
> recBQJ1:=qMixRec(Fbqj,q,k,S(n),2,alpha,0,beta,4):
> recBQJ11:=denom(rhs(recBQJ1))*lhs(recBQJ1)   =collect(numer(rhs(recBQJ1)),[S(n, alpha, beta),S(n-1, alpha, beta),x],qsimpcomb)
:
> boundBQJ1:=combine((solve(op([2,2],recBQJ11)/S(n-1, alpha, beta),x)),power)
```

$$boundBQJ1 := \left(q^{n-1}\left(-\gamma^{3}\alpha^{3}\beta^{2}q^{3n+3} - \gamma^{2}\alpha^{3}\beta^{2}q^{3n+3} - \gamma\alpha^{3}\beta^{2}q^{3n+3} + \gamma^{3}\alpha^{2}\beta\,q^{2n+1} + \gamma^{2}\alpha^{2}\beta\,q^{2n+1} + \gamma\alpha^{3}\beta^{4}q^{2n+9} + \alpha^{3}\beta^{4}q^{2n+9} + \gamma\alpha^{3}\beta^{4}q^{2n+8} + \alpha^{3}\beta^{4}q^{2n+8}\right.\right. \quad \textbf{(10.7)}$$

$$-\gamma^2\alpha^3\beta^3 q^{2n+7}+\gamma\alpha^3\beta^4 q^{2n+7}+\alpha^3\beta^4 q^{2n+7}+\gamma^2\alpha^3\beta^2 q^{2n+4}+\gamma^2\alpha^2\beta^2 q^{n+6}$$
$$+\gamma\alpha^3\beta^2 q^{2n+4}+\alpha^3\beta^2 q^{2n+4}+2\gamma^2\alpha^2\beta^2 q^{n+5}+2\gamma^2\alpha^2\beta^2 q^{n+4}-\gamma^3\alpha^2\beta q^{3+n}$$
$$+\gamma^2\alpha^2\beta^2 q^{3+n}-\gamma^3\alpha^2\beta q^{n+2}+2\gamma^2\alpha^2\beta^3 q^{3n+6}-\gamma^2\alpha^2\beta^3 q^{2n+7}+\gamma^2\alpha^3\beta^3 q^{3n+4}$$
$$-2\gamma^2\alpha^2\beta^3 q^{2n+6}-\gamma\alpha^2\beta^3 q^{2n+7}-\alpha^3\beta^5 q^{2n+10}-\gamma^2\alpha^2\beta q^{3+n}-\gamma^2\alpha^2\beta q^{n+2}$$
$$-\gamma\alpha^2\beta^3 q^{n+8}-\gamma\alpha^2\beta^3 q^{n+7}-\gamma^2\alpha^3\beta^3 q^{2n+6}-\gamma\alpha^3\beta^3 q^{2n+7}-\alpha^3\beta^3 q^{2n+7}$$
$$-\gamma\alpha^2\beta^3 q^{n+6}-\gamma\alpha^2\beta^3 q^{n+5}+\gamma\alpha^2\beta^2 q^{n+6}+2\gamma\alpha^2\beta^2 q^{n+5}+2\gamma\alpha^2\beta^2 q^{n+4}$$
$$+\gamma\alpha^2\beta^2 q^{3+n}-\gamma\alpha^2\beta q^{3+n}-\gamma\alpha^2\beta q^{n+2}-\gamma^2\alpha^3\beta^3 q^{2n+5}-2\gamma\alpha^3\beta^3 q^{2n+6}$$
$$-\alpha^3\beta^3 q^{2n+6}+\gamma^3\alpha^3\beta^2 q^{2n+4}-\gamma\alpha^3\beta^3 q^{2n+5}-\alpha^3\beta^3 q^{2n+5}+\gamma^3 q$$
$$-\gamma^2\alpha^2\beta^3 q^{2n+5}-\gamma^3\alpha^2\beta^2 q^{3n+3}+\gamma^3\alpha^2\beta^2 q^{2n+4}-\gamma^2\alpha^2\beta^2 q^{3n+3}-\gamma^3\alpha\beta^2 q^{3n+3}$$
$$-\gamma^2\alpha\beta^3 q^{2n+7}-\gamma^2\alpha^2\beta^2 q^{2n+2}+\gamma\alpha^3\beta^3 q^{3n+4}-\gamma^3\alpha q^{n+1}+\alpha\gamma^2 q-q^n\alpha\gamma^3$$
$$-\gamma\alpha^3\beta^4 q^{3n+6}+\gamma\alpha^2\beta^4 q^{2n+8}+2\gamma^2\alpha\beta^2 q^{n+4}+2\gamma^3\alpha\beta q^{2n+2}+\gamma^2\alpha\beta^2 q^{3+n}$$
$$-\gamma^3\alpha\beta q^{3+n}+\gamma^3\alpha\beta q^{2n+1}-\gamma^3\alpha\beta q^{n+2}-\gamma^2\alpha\beta q^{3+n}-\gamma^2\alpha\beta q^{n+2}+q\alpha\gamma^3$$
$$-2\gamma^2\alpha^2\beta^2 q^{2n+4}+\gamma^3\alpha\beta^2 q^{2n+4}+\gamma^3\beta^2 q^{2n+4}+\gamma^2\alpha\beta^2 q^{n+6}+\gamma\alpha^2\beta^2 q^{2n+4}$$
$$+\gamma^2\alpha\beta^2 q^{2n+4}+2\gamma^2\alpha\beta^2 q^{n+5}-\gamma^3\beta q^{3+n}-\gamma^3\beta q^{n+2}+\gamma\alpha^2\beta^4 q^{2n+7}$$
$$+2\gamma^2\alpha^2\beta^3 q^{3n+5}+\gamma^2\alpha^2\beta^3 q^{3n+4}-2\gamma\alpha^2\beta^3 q^{2n+6}-\gamma^2\alpha\beta^3 q^{2n+6}$$
$$-\gamma\alpha^2\beta^3 q^{2n+5}-\gamma^2\alpha\beta^3 q^{2n+5}-\alpha\beta\gamma^2 q^4-\alpha\beta\gamma^2 q^3-\alpha\beta\gamma^2 q^2+2\gamma^3\alpha^2\beta q^{2n+2}$$
$$+\gamma^2\alpha^2\beta q^{2n+3}+\gamma^3\alpha\beta q^{2n+3}+2\gamma^2\alpha^2\beta q^{2n+2}-\gamma\alpha^3\beta^4 q^{3n+8}-\gamma^2\alpha^3\beta^3 q^{4n+6}$$
$$+\gamma^2\alpha^3\beta^3 q^{3n+7}-\gamma^3\alpha^2\beta^2 q^{3n+4}-\gamma^2\alpha^3\beta^2 q^{3n+4}-\gamma^2\alpha^2\beta^2 q^{2n+6}-\gamma\alpha^3\beta^2 q^{3n+4}$$
$$-\gamma^3\alpha\beta^2 q^{3n+4}-2\gamma^2\alpha^2\beta^2 q^{2n+5}-\gamma^3\alpha^2\beta q^{3n+2}+\gamma^3\alpha^2\beta q^{2n+3}-\gamma^3\alpha^2\beta q^{3n+1}$$
$$+\gamma^2\alpha^3\beta^2 q^{4n+3}+\gamma^3\alpha^2\beta^2 q^{4n+3}+\gamma^3\alpha^3\beta^2 q^{4n+3}-\gamma^3\alpha^3\beta^2 q^{3n+4}-\gamma^2\alpha^2\beta^2 q^{3n+4}$$
$$+2\gamma\alpha^3\beta^3 q^{3n+6}+2\gamma\alpha^3\beta^3 q^{3n+5}-\gamma\alpha^3\beta^4 q^{3n+9}-\gamma^2\alpha^3\beta^3 q^{4n+5}$$
$$+2\gamma^2\alpha^3\beta^3 q^{3n+6}+\gamma\alpha^3\beta^3 q^{3n+7}+\gamma^2\alpha^2\beta^3 q^{3n+7}-\gamma^2\alpha^3\beta^3 q^{4n+4}$$
$$+2\gamma^2\alpha^3\beta^3 q^{3n+5}+\gamma\alpha^2\beta^4 q^{2n+9}-\gamma\alpha^3\beta^4 q^{3n+7}-2\gamma^2\alpha^2\beta^2 q^{2n+3}))\Big/\big(\gamma^2$$
$$-\gamma^2\alpha^3\beta^3 q^{5n+3}+\gamma^2\alpha^2\beta^2 q^{4n+2}-\gamma\alpha\beta^3 q^{3n+4}-\gamma\alpha^2\beta^2 q^{3n+2}-\gamma^2\alpha\beta^2 q^{3n+2}$$
$$+\gamma\alpha\beta^2 q^{2n+2}+\gamma^2\alpha\beta q^{2n+1}-\gamma\alpha\beta q^{n+1}+\gamma^2\alpha^3\beta^3 q^{6n+3}-\gamma\alpha^3\beta^3 q^{5n+3}$$
$$-\gamma^2\alpha^2\beta^3 q^{5n+3}+\gamma^2\alpha\beta^3 q^{4n+4}+\gamma\alpha^2\beta^3 q^{4n+3}-\gamma^2\alpha^2\beta^2 q^{3n+3}+\gamma\alpha^3\beta^3 q^{4n+3}$$
$$-\gamma\alpha^3\beta^3 q^{3n+4}-\alpha^3\beta^3 q^{3n+4}+\alpha^3\beta^4 q^{3n+6}+\gamma\alpha^2\beta^4 q^{3n+7}+\alpha^2\beta^4 q^{3n+7}$$
$$+2\gamma^2\alpha\beta^2 q^{2n+3}+\gamma^2\beta^2 q^{2n+3}+\gamma\alpha^2\beta^2 q^{2n+2}+\gamma^2\alpha\beta^2 q^{2n+2}-\gamma^2\alpha\beta q^{n+2}$$
$$-\gamma^2\alpha\beta q^{n+1}-\gamma^2\beta q^{n+1}-\gamma^2\beta q^{n+2}-\gamma^2\alpha^2\beta^2 q^{3n+2}+\gamma\alpha^2\beta^2 q^{2n+4}$$
$$+\gamma^2\alpha\beta^2 q^{2n+4}+2\gamma\alpha^2\beta^2 q^{2n+3}+\alpha^2\beta^2 q^{2n+3}+\alpha^3\beta^3 q^{4n+4}-\alpha^3\beta^3 q^{3n+5}$$
$$+\alpha^3\beta^4 q^{3n+7}-\alpha^2\beta^3 q^{3n+4}-\gamma\alpha^2\beta^3 q^{3n+3}-\alpha^3\beta^5 q^{4n+8}+\gamma\alpha^2\beta^4 q^{3n+6}$$

$$+2\gamma^2\alpha^2\beta^3 q^{4n+4}-\gamma^2\alpha^2\beta^3 q^{3n+5}+\gamma^2\alpha^2\beta^3 q^{4n+3}-\gamma^2\alpha^2\beta^3 q^{3n+4}$$
$$-3\gamma\alpha^2\beta^3 q^{3n+5}-\gamma^2\alpha\beta^3 q^{3n+5}-3\gamma\alpha^2\beta^3 q^{3n+4}-\gamma^2\alpha\beta^3 q^{3n+4}-\alpha^2\beta^3 q^{3n+5}$$
$$+\alpha^2\beta^4 q^{3n+6}-\alpha^2\beta^4 q^{2n+7}+\gamma\alpha^3\beta^3 q^{4n+5}-\gamma\alpha^3\beta^3 q^{5n+4}-\gamma^2\alpha^2\beta^3 q^{5n+4}$$
$$+\gamma\alpha^2\beta^3 q^{4n+5}+2\gamma\alpha^2\beta^3 q^{4n+4}-\gamma\alpha\beta^3 q^{3n+5}-\gamma\alpha^2\beta^2 q^{3n+3}-\gamma^2\alpha\beta^2 q^{3n+3}$$
$$+\gamma\alpha\beta^2 q^{2n+4}+2\gamma\alpha\beta^2 q^{2n+3}-\gamma\alpha\beta q^{n+2}-\gamma^2\alpha^3\beta^3 q^{5n+4}+\gamma^2\alpha^2\beta^3 q^{4n+5}$$
$$+2\gamma\alpha^3\beta^3 q^{4n+4}-\gamma\alpha^3\beta^3 q^{3n+5}-\gamma\alpha^2\beta^3 q^{3n+6}+\gamma^2\alpha^3\beta^3 q^{4n+4}+\gamma^2\alpha^2\beta^2 q^{2n+3}\Big)$$

Mixed recurrence equation involving $S(n-2,\alpha q^4,\beta)$ giving a lower bound of x_(n,n)

```
> recBQJ2:=qMixRec(Fbqj,q,k,S(n),2,alpha,4,beta,0):
> recBQJ21:=denom(rhs(recBQJ2))*lhs(recBQJ2)   =collect(numer(rhs
  (recBQJ2)),[S(n, alpha, beta),S(n-1, alpha, beta),x],qsimpcomb)
  :
> boundBQJ2:=combine(solve(op([2,2],recBQJ21),x),power)
```

$$boundBQJ2 := \Big(q^n\big(q^7\alpha^3-\gamma\alpha^3 q^{n+6}-\alpha^3 q^{n+6}+\gamma^2\alpha^2 q^{3+n}-\gamma^2\alpha q^{n+4}-\gamma^2\alpha q^{3+n} \qquad (10.8)$$
$$-\gamma^2\alpha q^{n+2}-\alpha^3\beta^2 q^{n+7}-\alpha^3\beta^2 q^{n+6}-\alpha^3\beta q^{n+6}+\gamma\alpha^2 q^{3+n}-\alpha^4\beta q^{n+9}$$
$$-\alpha^4\beta q^{n+8}+\alpha^4\beta q^{2n+7}+\gamma^3\alpha^3 q^{2n+6}-\gamma^2\alpha^3 q^{n+7}-\gamma^3\alpha^2 q^{2n+5}-\gamma^2\alpha^3 q^{n+6}$$
$$-\gamma^3\alpha^2 q^{2n+4}+\gamma^2\alpha^2 q^{n+6}-\gamma^3\alpha^2 q^{2n+3}+2\gamma^2\alpha^2 q^{n+5}+\gamma^3\alpha q^{2n+3}+2\gamma^2\alpha^2 q^{n+4}$$
$$+\gamma^3\alpha q^{2n+2}+\alpha^4\beta^2 q^{2n+7}+2\alpha^4\beta^2 q^{2n+8}-\alpha^4\beta^3 q^{3n+8}-\alpha^4\beta^3 q^{3n+7}$$
$$+\alpha^4\beta^2 q^{2n+9}-\alpha^5\beta^2 q^{3n+9}-\alpha^5\beta^2 q^{3n+10}+\alpha^5\beta^3 q^{4n+9}-q^{2n}\gamma^3$$
$$+2\gamma^2\alpha^3\beta q^{3n+6}+\gamma\alpha^3\beta q^{3n+7}-\gamma\alpha^4\beta q^{3n+7}-\gamma^2\alpha^2 q^{2n+5}+\gamma\alpha^2\beta q^{3+n}$$
$$+\alpha^3\beta^3 q^{2n+6}+\gamma^2\alpha\beta q^{2n+1}-\gamma^2\alpha^2\beta q^{3n+5}+\alpha^3 q^{2n+6}-\gamma^2\alpha^2\beta q^{3n+4}$$
$$+2\gamma\alpha^3\beta q^{3n+6}-2\gamma\alpha^3\beta q^{2n+7}+2\gamma\alpha^3\beta q^{3n+5}+2\gamma\alpha^3\beta^2 q^{3n+6}+\gamma\alpha^3\beta q^{3n+4}$$
$$-2\gamma\alpha^2\beta q^{2n+4}+\gamma^2\alpha^3 q^{2n+6}+\gamma\alpha^3 q^{2n+6}-\gamma\alpha^3 q^{n+7}+\alpha^4\beta q^{2n+9}$$
$$+2\alpha^4\beta q^{2n+8}-\gamma\alpha^2 q^{2n+3}+2\gamma\alpha^2 q^{n+4}+\gamma^2\alpha q^{2n+1}+\gamma^3\alpha q^{2n+1}-\gamma^2\alpha q^{n+1}$$
$$-\gamma\alpha^2\beta q^{2n+5}-2\gamma\alpha^3\beta q^{2n+6}-\gamma\alpha^2\beta^2 q^{2n+4}-\gamma\alpha^2\beta^2 q^{2n+3}-\gamma^2\alpha^2 q^{2n+3}$$
$$+\gamma\alpha^3 q^7-2\gamma^2\alpha^2 q^{2n+4}-\gamma\alpha^2 q^{2n+5}+\gamma\alpha^2 q^{n+6}+\alpha^3\beta q^{2n+6}+\alpha^3\beta^2 q^{2n+6}$$
$$-\alpha^3\beta q^{n+7}-\alpha^3 q^{n+7}-\gamma\alpha^4\beta^2 q^{4n+6}-\gamma\alpha^4\beta^2 q^{4n+7}-\gamma\alpha^4\beta^2 q^{4n+8}$$
$$-\gamma\alpha^4\beta q^{3n+8}-\gamma\alpha^2 q^6+\alpha^5\beta^2 q^{4n+9}-\gamma\alpha^2 q^5-\gamma\alpha^2\beta q^{2n+3}+\gamma^2\alpha\beta q^{2n+3}$$
$$-\gamma^2\alpha^2\beta q^{3n+3}-\gamma^2\alpha^2\beta q^{3n+2}-\gamma^2\alpha^2\beta q^{2n+3}+\gamma\alpha^3\beta^2 q^{3n+4}-\gamma\alpha^2\beta^2 q^{2n+5}$$
$$+\gamma^2\alpha\beta q^{2n+2}+\gamma\beta\alpha^2 q^{n+6}+2\gamma\beta\alpha^2 q^{n+5}+2\gamma\beta\alpha^2 q^{n+4}-\gamma\alpha^3\beta q^{n+6}$$
$$-\gamma\alpha^3\beta q^{2n+4}-2\gamma^2\alpha^2\beta q^{2n+4}-\gamma\alpha^3\beta q^{n+7}-\gamma^2\alpha^2\beta q^{2n+5}+\gamma\alpha^3\beta^2 q^{2n+6}$$
$$+\gamma^2\alpha^3\beta q^{2n+6}-2\gamma\alpha^3\beta q^{2n+5}+2\gamma\alpha^3\beta^2 q^{3n+5}+2\gamma^2\alpha^3\beta q^{3n+5}+\gamma^2\alpha^3\beta q^{3n+4}$$
$$+\gamma\alpha^4\beta q^{2n+7}-\gamma\alpha^3\beta q^{2n+8}+2\gamma\alpha^4\beta q^{2n+8}+\gamma\alpha^3\beta^2 q^{3n+7}+\gamma^2\alpha^3\beta q^{3n+7}$$

$$-\gamma\alpha^4\beta^2 q^{3n+7}-\gamma^2\alpha^4\beta q^{3n+7}+\gamma\alpha^4\beta q^{2n+9}-\gamma\alpha^4\beta^2 q^{3n+8}-\gamma^2\alpha^4\beta q^{3n+8}$$
$$+\gamma\alpha^5\beta^2 q^{4n+9}+\alpha^3\beta q^7-\alpha^2\gamma q^4-\alpha^4\beta^2 q^{3n+8}-\alpha^4\beta q^{3n+8}-\alpha^4\beta^2 q^{3n+7}$$
$$-\alpha^4\beta q^{3n+7}-\gamma\alpha^2 q^{2n+4}+2\gamma\alpha^2 q^{n+5}+\gamma^2\alpha q^{2n+3}+\gamma^2\alpha q^{2n+2}\big)\big)\Big/\big(q^5\alpha^2$$
$$-\alpha^2 q^{n+5}-\alpha^2 q^{n+4}+2\alpha^2\beta q^{2n+4}+2\alpha^3\beta^2 q^{4n+5}+\alpha^3\beta^2 q^{4n+6}+\alpha^3\beta^2 q^{4n+4}$$
$$-\alpha^2\beta^2 q^{3n+4}-\alpha^2\beta q^{3n+4}+\alpha^2\beta q^{2n+5}-\alpha^2\beta q^{n+4}-\alpha^3\beta q^{3n+5}-\alpha^2\beta q^{n+5}$$
$$-\alpha^4\beta^3 q^{5n+6}-\alpha^4\beta^2 q^{5n+6}-\alpha^4\beta^3 q^{5n+7}-\alpha^4\beta^2 q^{5n+7}+\alpha^3\beta^3 q^{4n+5}$$
$$+\alpha^3\beta q^{4n+5}+\alpha^2\beta q^{2n+3}-\alpha^2\beta q^{3n+3}+2\gamma\alpha^3\beta q^{4n+5}+2\gamma\alpha^3\beta^2 q^{4n+5}$$
$$+\gamma^2\alpha^3\beta q^{4n+5}+\gamma\alpha^3\beta q^{4n+6}+\gamma\alpha^3\beta^2 q^{4n+6}+\alpha^5\beta^3 q^{6n+8}-\gamma^2\alpha^2\beta q^{3n+4}$$
$$-\gamma^2\alpha^2 q^{3n+4}-\gamma\alpha^2\beta q^{3n+5}-\gamma\alpha^3\beta q^{3n+6}+\gamma\alpha^3\beta q^{4n+4}-\gamma\alpha^3\beta q^{3n+5}$$
$$-\alpha^3\beta^2 q^{3n+6}-\gamma^2\alpha^2 q^{3n+3}-\gamma\alpha^4\beta^2 q^{5n+7}+2\gamma\alpha^2\beta q^{2n+4}+\gamma\alpha^2 q^{2n+3}$$
$$-\gamma\alpha^2 q^{n+4}-3\gamma\alpha^2\beta q^{3n+4}+\gamma\alpha^2\beta q^{2n+5}-3\gamma\alpha^2\beta q^{3n+3}+\gamma^2\alpha\beta q^{3n+2}$$
$$+\gamma^2\alpha\beta q^{3n+1}-\gamma\alpha^2 q^{3n+4}+\gamma^2\alpha^2 q^{2n+4}+\gamma\alpha^2 q^{2n+5}+\alpha^3\beta q^{2n+6}-\alpha^3\beta q^{3n+6}$$
$$-\gamma\alpha^4\beta^2 q^{5n+6}+\alpha^4\beta^2 q^{4n+7}+\gamma\alpha^2\beta q^{2n+3}-\gamma^2\alpha^2\beta q^{3n+3}+\gamma\alpha^3\beta^2 q^{4n+4}$$
$$-\gamma\alpha^2\beta^2 q^{3n+4}-\gamma\alpha^2\beta^2 q^{3n+3}-q^{2n}\gamma^2+\alpha^2\beta^2 q^{2n+4}+\alpha^2 q^{2n+4}-\alpha^3\beta^2 q^{3n+5}$$
$$+\gamma\alpha\beta q^{3n+1}+\gamma\alpha\beta q^{3n+2}-\alpha^2\beta^2 q^{3n+3}-\gamma\beta\alpha^2 q^{3n+2}-\gamma^2\alpha\beta q^{4n+1}$$
$$-\gamma\alpha^2 q^{3n+3}+2\gamma\alpha^2 q^{2n+4}-\gamma\alpha^2 q^{n+5}+\gamma^2\alpha q^{3n+2}+\gamma^2\alpha q^{3n+1}\big)$$

Comparison between the bounds and the extreme zeros

```
> xnbqj:= sort([solve(expand(bqj(25,0.1,0.2,-0.05,x,0.99)),x)]):
> xnbqj1:=evalf[15](xnbqj):
> extzerobqj:=[min(op(xnbqj1)),max(op(xnbqj1))]
```

extzerobqj := [−0.0220679578747795, 0.0882383597118014]   **(10.9)**

```
> Bqj204:=unapply(boundBQJ1,[n,alpha,beta,gamma,q]):
> evalf[15](Bqj204(25,0.1,0.2,-0.05,0.99))
```

0.0197659017069492   **(10.10)**

```
> Bqj240:=unapply(boundBQJ2,[n,alpha,beta,gamma,q]):
> evalf[15](Bqj240(25,0.1,0.2,-0.05,0.99))
```

0.0855694053208267   **(11.11)**

the q-Hahn polynomials

```
> QH:=(n,alpha,beta,N,x,q)->add(qphihyperterm([q^(-n),alpha*beta*
  q^(n+1),x],[alpha*q,q^(-N)],q,q,k),k=0..n);
```

$QH := (n,\alpha,\beta,N,x,q) \rightarrow add\left(qphihyperterm\left([q^{-n},\alpha\beta q^{n+1},x],[\alpha q,q^{-N}],q,q,k\right),k=0..n\right)$ **(11.1)**

```
> Fqh:=(qphihyperterm([q^(-n),alpha*beta*q^(n+1),x],[alpha*q,q^(-
  NN)],q,q,k));
```

The weight function

x in QH is q^{-x} in the weight function

```
> rhoQH:=(alpha,beta,N)->qpochhammer(alpha*q,q,x)*qpochhammer(q^(-N),q,x)/qpochhammer(q,q,x)/qpochhammer(1/beta/q^N,q,x)*(alpha*beta*q)^(-x)
```

$$rhoQH := (\alpha, \beta, N) \rightarrow \frac{qpochhammer(\alpha\, q, q, x)\; qpochhammer(q^{-N}, q, x)\; (\alpha\, \beta\, q)^{-x}}{qpochhammer(q, q, x)\; qpochhammer\left(\frac{1}{\beta\, q^{N}}, q, x\right)} \tag{11.2}$$

```
> cQH:=(s,t)->(qpochhammer(alpha*q*q^x,q,s)/qpochhammer(alpha*q,q,s))*qpochhammer(1/(beta*q^N)*q^(x-t),q,t)/qpochhammer(1/(beta*q^N)/q^t,q,t)*1/(q^(s*x)*q^(t*x))
```

$$cQH := (s, t) \rightarrow \frac{qpochhammer(\alpha\, q\, q^{x}, q, s)\; qpochhammer\left(\frac{q^{x-t}}{\beta\, q^{N}}, q, t\right)}{qpochhammer(\alpha\, q, q, s)\; qpochhammer\left(\frac{1}{\beta\, q^{N}\, q^{t}}, q, t\right) q^{s\,x}\, q^{t\,x}} \tag{11.3}$$

```
> qsimplify(rhoQH(alpha*q^s,beta*q^t,N)/rhoQH(alpha,beta,N)-cQH(s,t))
```

$$0 \tag{11.4}$$

cQH is a polynomial of degree s+t of the variable q^(-x). For example

```
> qsimpcomb(cQH(2,3))
```

$$\frac{(\alpha\, q\, q^{x} - 1)\, (\alpha\, q^{2}\, q^{x} - 1)\, (\beta\, q^{N}\, q^{3} - q^{x})\, (\beta\, q^{N}\, q^{2} - q^{x})\, (\beta\, q^{N}\, q - q^{x})}{(\alpha\, q - 1)\, (\alpha\, q^{2} - 1)\, (\beta\, q^{N}\, q^{3} - 1)\, (\beta\, q^{N}\, q^{2} - 1)\, (\beta\, q^{N}\, q - 1)\, (q^{x})^{5}} \tag{11.5}$$

Mixed recurrence equation involving $S(n-2, \alpha, \beta\, q^{4})$ giving a lower bound of x_(n,n)

```
> recQH1:=subs(NN=N,qMixRec(Fqh,q,k,S(n),2,alpha,0,beta,4)):
> recQH11:=denom(rhs(recQH1))*lhs(recQH1)   =collect(numer(rhs(recQH1)),[S(n, alpha, beta),S(n-1, alpha, beta),x],qsimpcomb):
> boundQH1:=combine((solve(op([2,2],recQH11)/S(n-1, alpha, beta),x)),power)
```

$$\begin{aligned} boundQH1 := -\Big(q^{n-N-2}\big(&-q + \alpha^{2}\beta^{2} q^{3n+4} - \alpha^{2}\beta\, q^{2n+3} - \alpha^{2}\beta^{3} q^{3n+5+N} \\ &+ \alpha^{2}\beta^{3} q^{2n+2N+9} - \alpha^{2}\beta^{2} q^{2n+2N+6} - 2\,\alpha^{2}\beta^{3} q^{3n+N+7} - 2\,\alpha^{2}\beta^{3} q^{3n+N+6} \\ &+ 2\,\alpha^{2}\beta^{2} q^{2n+N+5} + 2\,\alpha^{2}\beta^{2} q^{2n+N+4} - \beta^{2}\alpha\, q^{2n+N+5} - \alpha^{2}\beta^{4} q^{2n+2N+11} \\ &- \alpha^{2}\beta^{4} q^{2n+2N+10} - \alpha^{2}\beta^{4} q^{2n+2N+9} + \alpha^{2}\beta^{3} q^{2n+N+8} + 2\,\alpha^{2}\beta^{3} q^{2n+N+7} \\ &- \alpha^{3}\beta^{2} q^{2n+4} + \alpha\beta\, q^{3+n} - \alpha\beta\, q^{2n+3} + \alpha^{2}\beta\, q^{3n+2} + \alpha^{2}\beta\, q^{3n+1} + \beta^{2}\alpha\, q^{3n+4} \\ &- 2\,\alpha^{2}\beta\, q^{2n+2} + \alpha^{2}\beta\, q^{n+2N+4} - \alpha^{2}\beta^{2} q^{n+N+4} + \alpha^{2}\beta\, q^{n+N+4} + \alpha^{2}\beta^{3} q^{n+2N+8} \\ &- \alpha^{3}\beta^{2} q^{2n+2N+6} + \alpha^{3}\beta^{3} q^{2n+N+6} - \beta^{2} q^{2n+4} + \beta\, q^{3+n} + \beta\, q^{n+2} \\ &- 2\,\alpha^{2}\beta^{2} q^{n+2N+6} - \alpha^{2}\beta^{2} q^{n+N+7} - \alpha^{2}\beta^{2} q^{n+2N+5} - 2\,\alpha^{2}\beta^{2} q^{n+N+6} \\ &+ \alpha^{2}\beta\, q^{n+2N+5} - 2\,\alpha^{2}\beta^{2} q^{n+N+5} + \alpha^{2}\beta\, q^{3+n} + \alpha\beta\, q^{N+4} - \alpha^{2}\beta\, q^{2n+1} \\ &+ \alpha\beta\, q^{N+3} + \alpha^{2}\beta\, q^{n+N+3} - \alpha^{3}\beta^{2} q^{2n+N+5} + \alpha^{2}\beta\, q^{n+2} + \alpha\, q^{n} + \alpha\, q^{n+1} - \alpha\, q \\ &- \alpha^{2}\beta\, q^{2n+N+4} - 2\,\alpha^{2}\beta\, q^{2n+3+N} + \alpha^{2}\beta^{2} q^{3n+5+N} - 2\,\alpha^{3}\beta^{3} q^{3n+N+7} \\ &- \alpha^{3}\beta^{3} q^{3n+2N+9} - 2\,\alpha^{3}\beta^{3} q^{3n+2N+8} + \alpha^{3}\beta^{3} q^{4n+N+6} - \alpha^{2}\beta^{3} q^{3n+N+8} \end{aligned} \tag{11.6}$$

$$
\begin{aligned}
&-\alpha^3\beta^2 q^{4n+4+N}+\alpha^3\beta^2 q^{3n+5+N}+\alpha^3\beta^3 q^{4n+N+7}+\alpha^3\beta^2 q^{3n+2N+6}\\
&-2\alpha^3\beta^3 q^{3n+2N+7}+\alpha^3\beta^4 q^{3n+2N+11}-\alpha^3\beta^3 q^{3n+N+8}+\alpha^2\beta^2 q^{2n+N+7}\\
&+\alpha^3\beta^4 q^{3n+2N+10}+\alpha^3\beta^3 q^{4n+N+5}-2\alpha^3\beta^3 q^{3n+N+6}+2\alpha^2\beta^2 q^{2n+N+6}\\
&+\alpha^3\beta^3 q^{2n+2N+7}+\alpha^3\beta^3 q^{2n+N+8}+\alpha^2\beta^3 q^{n+2N+9}+\alpha^3\beta^3 q^{2n+N+7}\\
&+\alpha^3\beta^3 q^{2n+3N+9}-\alpha^3\beta^4 q^{2n+2N+9}+\alpha^3\beta^2 q^{3n+3}+\alpha\beta q^{N+5}+\beta^2\alpha q^{3n+3}\\
&-\beta^2\alpha q^{2n+4}-2\alpha\beta q^{2n+2}-\alpha\beta q^{2n+1}+\alpha\beta q^{n+2}-\alpha q^{N+2}+\alpha^2\beta^3 q^{n+2N+7}\\
&-\alpha^2\beta^2 q^{n+2N+8}-2\alpha^2\beta^2 q^{n+2N+7}+\alpha^2\beta^3 q^{n+2N+10}-\alpha^3\beta^2 q^{2n+3N+7}\\
&+\alpha\beta^3 q^{2n+N+8}+\alpha^2\beta^3 q^{2n+N+6}+\alpha\beta^3 q^{2n+N+7}+\alpha\beta^3 q^{2n+N+6}\\
&-\beta^2\alpha q^{n+N+7}-2\beta^2\alpha q^{n+N+6}-2\beta^2\alpha q^{n+N+5}-\beta^2\alpha q^{n+N+4}\\
&+\alpha^2\beta^3 q^{2n+2N+7}+\alpha^3\beta^4 q^{3n+2N+8}+2\alpha^2\beta^3 q^{2n+2N+8}+\alpha^2\beta^2 q^{3n+N+4}\\
&-\alpha^3\beta^3 q^{3n+2N+6}+\alpha\beta q^{n+N+4}+\alpha^3\beta^2 q^{3n+4}-\alpha^2\beta^2 q^{2n+4}+\alpha^2\beta^2 q^{3n+3}\\
&-\alpha^3\beta^2 q^{4n+3}-\alpha^2\beta^2 q^{4n+3}+\alpha^3\beta^3 q^{2n+3N+8}+\alpha^3\beta^3 q^{2n+2N+9}\\
&+2\alpha^3\beta^3 q^{2n+2N+8}-\alpha^3\beta^4 q^{2n+3N+11}-\alpha^3\beta^4 q^{2n+3N+10}-\alpha^3\beta^4 q^{2n+2N+11}\\
&+\alpha^3\beta^3 q^{2n+3N+10}-\alpha^3\beta^4 q^{2n+2N+10}+\alpha^3\beta^5 q^{2n+3N+13}-\alpha^3\beta^4 q^{2n+3N+12}\\
&+\alpha^3\beta^2 q^{3n+N+4}-\alpha^2\beta q^{2n+N+2}+\alpha^3\beta^2 q^{3n+2N+5}+\alpha^3\beta^4 q^{3n+2N+9}\\
&-\alpha^3\beta^3 q^{3n+5+N}+\alpha^2\beta^2 q^{2n+3+N}+\alpha\beta q^{n+N+3})\big)\Big/\big(1+\alpha^2\beta^4 q^{3n+2N+9}\\
&+2\beta^2\alpha q^{2n+N+4}+\alpha^2\beta^2 q^{2n+2N+5}+\alpha^3\beta^3 q^{4n+4+N}-3\alpha^2\beta^3 q^{3n+5+N}\\
&-\alpha\beta q^{n+N+2}-\alpha^2\beta^3 q^{3n+N+7}-\alpha^3\beta^5 q^{4n+2N+10}-3\alpha^2\beta^3 q^{3n+N+6}\\
&+\alpha^2\beta^2 q^{2n+N+5}+2\alpha^2\beta^2 q^{2n+N+4}+\beta^2\alpha q^{2n+N+5}-\alpha^2\beta^4 q^{2n+2N+9}\\
&+\alpha^2\beta^4 q^{3n+N+8}+\alpha^2\beta^4 q^{3n+N+7}-\alpha\beta q^{n+1}+\beta^2 q^{2n+3}-\beta q^{n+2}-\beta q^{n+1}\\
&-\alpha^3\beta^3 q^{5n+N+4}-\alpha^2\beta^3 q^{3n+2N+7}+\alpha^3\beta^3 q^{4n+N+6}+\alpha^2\beta^3 q^{4n+N+6}\\
&-\alpha^3\beta^3 q^{5n+N+5}-\alpha\beta^3 q^{3n+N+6}-\alpha^3\beta^3 q^{3n+2N+7}+\alpha^3\beta^3 q^{4n+2N+6}\\
&+2\alpha^2\beta^3 q^{4n+N+5}+2\alpha^3\beta^3 q^{4n+N+5}-\alpha^3\beta^3 q^{3n+N+6}+\alpha^2\beta^2 q^{2n+3}\\
&-\beta^2\alpha q^{3n+3}-\beta^2\alpha q^{3n+2}+\beta^2\alpha q^{2n+4}+2\beta^2\alpha q^{2n+3}+\alpha^3\beta^3 q^{4n+4}\\
&-\alpha^2\beta^2 q^{3n+2}+\alpha\beta q^{2n+1}-\alpha\beta q^{n+2}-\alpha^2\beta^3 q^{3n+4}+\beta^2\alpha q^{2n+2}+\alpha^2\beta^3 q^{4n+3}\\
&-\alpha\beta^3 q^{3n+4}-\alpha^2\beta^3 q^{3n+N+4}+\alpha^3\beta^4 q^{3n+2N+8}+\alpha^2\beta^4 q^{3n+2N+8}\\
&-\alpha^2\beta^2 q^{3n+N+4}-\alpha^2\beta^2 q^{3n+N+3}-\alpha^3\beta^3 q^{3n+2N+6}-\alpha^2\beta^3 q^{3n+2N+6}\\
&+\alpha^2\beta^3 q^{4n+4+N}-\alpha\beta^3 q^{3n+5+N}-\alpha^2\beta^3 q^{3n+5}-\alpha^2\beta^2 q^{3n+3}+\alpha^2\beta^2 q^{4n+2}\\
&-\alpha^3\beta^3 q^{5n+3}+2\alpha^2\beta^3 q^{4n+4}-\alpha^3\beta^3 q^{5n+4}+\alpha^3\beta^3 q^{6n+3}-\alpha^2\beta^3 q^{5n+4}\\
&-\alpha^2\beta^3 q^{5n+3}+\alpha\beta^3 q^{4n+4}+\alpha^2\beta^3 q^{4n+5}-\alpha\beta^3 q^{3n+5}+\alpha^3\beta^4 q^{3n+2N+9}\\
&-\alpha^3\beta^3 q^{3n+5+N}+\alpha^2\beta^2 q^{2n+3+N}-\alpha\beta q^{n+N+3}+\beta^2\alpha q^{2n+3+N}\big)
\end{aligned}
$$

Mixed recurrence equation involving $S(n-2,\alpha q^4,\beta)$ giving an upper bound of x_(n,1)

```
> recQH2:=subs(NN=N,qMixRec(Fqh,q,k,S(n),2,alpha,4,beta,0)):
> recQH21:=denom(rhs(recQH2))*lhs(recQH2)  =collect(numer(rhs
  (recQH2)),[S(n, alpha, beta),S(n-1, alpha, beta),x],qsimpcomb):
> boundQH2:=combine((solve(op([2,2],recQH21)/S(n-1, alpha, beta),
  x)),power)
```

$$boundQH2 := ((-\alpha^2\beta^2 q^{2n+2N+5} - \alpha^2\beta^2 q^{2n+2N+6} + \alpha q^{2n+1} - q^{2n} + \alpha^3\beta q^{3n+N+8} - \alpha^3\beta q^{n+2N+9} - 2\alpha^3\beta q^{2n+2N+7} + 2\alpha^3\beta q^{3n+N+6} - \alpha^3\beta q^{n+3N+9} + \alpha^2\beta q^{n+2N+5} - \alpha q^{n+N+5} + \alpha^3 q^{2n+6} - \alpha^3 q^{n+3N+9} - \alpha^3 q^{n+N+7} + \alpha^2 q^{n+2N+5} + \alpha^2 q^{n+N+4} - \alpha^4\beta q^{n+3N+11} - \alpha^3\beta^2 q^{n+3N+9} - \alpha^4\beta q^{3n+N+9} + \alpha^4\beta q^{2n+2N+9} - \alpha^3\beta q^{2n+2N+10} + 2\alpha^3\beta^2 q^{3n+2N+7} - \alpha^4\beta q^{3n+N+8} + \alpha^3\beta^2 q^{2n+2N+8} - \alpha^2\beta q^{3n+5+N} - \alpha^4\beta q^{3n+2N+9} - \alpha^4\beta^2 q^{3n+3N+10} - \alpha^4\beta q^{3n+3N+10} - \alpha^4\beta^2 q^{4n+2N+8} - 2\alpha^3\beta q^{2n+2N+9} - \alpha^4\beta q^{3n+2N+10} + \alpha^5\beta^2 q^{4n+3N+12} - \alpha^4\beta^2 q^{4n+2N+10} + 2\alpha^3\beta q^{3n+2N+7} + \alpha^3\beta q^{3n+2N+6} + \alpha^3\beta q^{3n+2N+9} + \alpha\beta q^{2n+N+2} - \alpha^2 q^{2n+2N+6} + \alpha^2 q^{n+2N+8} - 2\alpha^2 q^{2n+N+5} + \alpha q^{2n+N+4} + \alpha q^{2n+3+N} - \alpha^2 q^{2n+N+6} + 2\alpha^2 q^{n+2N+7} + \alpha^3 q^{2n+2N+8} + 2\alpha^2 q^{n+2N+6} - \alpha^2 q^{2n+N+4} - \alpha^3 q^{n+2N+9} + \alpha^3 q^{2n+N+7} + \alpha q^{2n+N+2} - \alpha^3 q^{n+3N+10} + \alpha^3 q^{2n+3N+9} - \alpha^2 q^{2n+2N+5} - \alpha^2 q^{2n+2N+7} - \alpha^2\beta q^{2n+N+4} - \alpha^2\beta^2 q^{2n+2N+7} + \alpha\beta q^{2n+N+4} + \alpha\beta q^{2n+3+N} - \alpha^2\beta q^{3n+N+4} - \alpha^2\beta q^{2n+2N+5} + \alpha^3\beta^2 q^{3n+2N+6} - \alpha^2\beta q^{3n+N+3} + \alpha^3\beta^3 q^{2n+3N+9} - \alpha^2 q^{2n+5} - \alpha^2 q^{2n+3} + \alpha q^{2n+3} + \alpha q^{2n+2} - \alpha^2 q^{2N+7} - \alpha^4\beta q^{3n+3N+11} - \alpha^4\beta^2 q^{3n+3N+11} + 2\alpha^3\beta q^{3n+N+7} + 2\alpha^3\beta^2 q^{3n+2N+8} + \alpha^4\beta q^{2n+3N+12} + 2\alpha^4\beta q^{2n+3N+11} - 2\alpha^3\beta q^{2n+2N+8} - \alpha^3\beta q^{n+3N+10} + \alpha^5\beta^3 q^{4n+3N+12} - \alpha^5\beta^2 q^{3n+3N+13} - \alpha^5\beta^2 q^{3n+3N+12} - \alpha q^{n+N+2} - \alpha^2 q^{2n+4} - \alpha^2 q^{2N+8} + \alpha^5\beta^2 q^{4n+2N+11} - \alpha^4\beta^3 q^{3n+3N+11} - \alpha^4\beta^3 q^{3n+3N+10} + \alpha^4\beta^2 q^{2n+3N+12} + 2\alpha^4\beta^2 q^{2n+3N+11} + \alpha^3\beta q^{3N+10} + \alpha^3 q^{3N+10} - 2\alpha^2\beta q^{2n+2N+6} - \alpha^4\beta^2 q^{4n+2N+9} - \alpha^2\beta q^{2n+2N+7} + 2\alpha^3\beta q^{3n+2N+8} - \alpha^2\beta q^{3n+N+6} + \alpha^4\beta q^{2n+3N+10} - \alpha^4\beta^2 q^{3n+2N+10} + \alpha^2\beta q^{n+2N+8} + \alpha^3\beta q^{3n+5+N} + 2\alpha^2\beta q^{n+2N+7} - \alpha^2\beta q^{2n+N+6} + 2\alpha^2\beta q^{n+2N+6} - 2\alpha^2\beta q^{2n+N+5} - \alpha^3 q^{n+N+8} + \alpha^2 q^{n+N+7} + 2\alpha^2 q^{n+N+6} + 2\alpha^2 q^{n+N+5} - \alpha q^{n+N+4} - \alpha q^{n+N+3} + \alpha^4\beta^2 q^{2n+3N+10} - \alpha^4\beta^2 q^{3n+2N+9} + \alpha^4\beta q^{2n+2N+11} + \alpha^3\beta^2 q^{3n+2N+9} + 2\alpha^4\beta q^{2n+2N+10} - \alpha^3\beta^2 q^{n+3N+10} - \alpha^3\beta q^{2n+2N+6} + \alpha^3\beta q^{2n+N+7} + \alpha^3\beta q^{2n+3N+9} + \alpha^3\beta^2 q^{2n+3N+9} \quad (11.7)$$

$$
-\alpha^4\beta q^{n+3N+12}+\alpha^3 q^{2N+9}-\alpha^2 q^{2N+6}-\alpha^3\beta q^{n+2N+8}-\alpha^3 q^{n+2N+8}\Big)
q^{n-N-1}\Big)\Big/\Big(-\alpha^2\beta q^{3n+4}+\alpha^3\beta q^{4n+5}-\alpha^2\beta q^{3n+3}+\alpha^2\beta^2 q^{2n+2N+6}
+\alpha\beta q^{3n+2}+\alpha\beta q^{3n+1}-q^{2n}-\alpha^3\beta q^{3n+N+6}+2\alpha^3\beta^2 q^{4n+N+6}
-\alpha^4\beta^3 q^{5n+2N+9}-\alpha^3\beta^2 q^{3n+2N+7}-3\alpha^2\beta q^{3n+5+N}+\alpha^3\beta q^{4n+N+5}
+\alpha^3\beta q^{4n+N+7}-\alpha^2\beta^2 q^{3n+2N+5}-\alpha^3\beta q^{3n+2N+7}+2\alpha^3\beta^2 q^{4n+2N+7}
-\alpha\beta q^{4n+1}+\alpha q^{3n+1}-\alpha^2 q^{3n+3}+\alpha q^{3n+2}-\alpha^2 q^{3n+4}+\alpha^2 q^{2n+2N+6}
+2\alpha^2 q^{2n+N+5}-\alpha^2 q^{3n+N+4}+\alpha^2 q^{2n+N+6}-\alpha^2 q^{3n+5+N}-\alpha^2 q^{n+2N+7}
-\alpha^2 q^{n+2N+6}+\alpha^2 q^{2n+N+4}+\alpha^2\beta q^{2n+N+4}-\alpha^2\beta^2 q^{3n+5+N}
-3\alpha^2\beta q^{3n+N+4}+\alpha^3\beta^2 q^{4n+N+5}+\alpha^2\beta q^{2n+2N+5}+\alpha^3\beta^2 q^{4n+2N+6}
-\alpha^2\beta^2 q^{3n+2N+6}+\alpha^3\beta^3 q^{4n+2N+7}-\alpha^2\beta q^{3n+N+3}+\alpha^2 q^{2N+7}
-\alpha^2\beta q^{3n+2N+6}-\alpha^3\beta q^{3n+N+7}-\alpha^4\beta^2 q^{5n+2N+9}-\alpha^4\beta^2 q^{5n+2N+8}
-\alpha^3\beta^2 q^{3n+2N+8}+2\alpha^3\beta q^{4n+N+6}+\alpha^3\beta q^{2n+2N+8}+q^{6n+2N+10}\alpha^5\beta^3
-\alpha^2\beta^2 q^{3n+N+4}+\alpha^2 q^{2n+4}+\alpha\beta q^{3n+2+N}+\alpha\beta q^{3n+N+3}+2\alpha^2\beta q^{2n+2N+6}
+\alpha^3\beta q^{4n+2N+7}+\alpha^4\beta^2 q^{4n+2N+9}+\alpha^2\beta q^{2n+2N+7}+\alpha^3\beta^2 q^{4n+2N+8}
-\alpha^2\beta q^{3n+2N+5}-\alpha^3\beta q^{3n+2N+8}-\alpha^2\beta q^{3n+N+6}-\alpha^4\beta^3 q^{5n+2N+8}
-\alpha^4\beta^2 q^{5n+N+8}-\alpha^4\beta^2 q^{5n+N+7}+\alpha^3\beta^2 q^{4n+N+7}-\alpha^2\beta q^{n+2N+7}
+\alpha^2\beta q^{2n+N+6}-\alpha^2\beta q^{n+2N+6}+2\alpha^2\beta q^{2n+N+5}-\alpha^2 q^{n+N+6}-\alpha^2 q^{n+N+5}\Big)
$$

Comparison between the bounds and the extreme zeros

```
> xnqh:= sort([solve(expand(QH(7,0.5,0.9,10,x,0.9)),x)]):
> evalf[15]([min(xnqh),max(xnqh)])
```

$$[1.16164046616789, 2.86796717091290] \tag{11.8}$$

```
> QH204:=unapply(boundQH1,[n,alpha,beta,N,q]):
> evalf[15](QH204(7,0.5,0.9,10,0.9))
```

$$1.83394847461674 \tag{11.9}$$

```
> QH240:=unapply(boundQH2,[n,alpha,beta,N,q]):
> evalf[15](QH240(7,0.5,0.9,10,0.9))
```

$$1.30810416837332 \tag{11.10}$$

the little q-Jacobi polynomials

```
> LQJ:=(n,alpha,beta,x,q)->add(qphihyperterm([q^(-n),alpha*beta*
  q^(n+1)],[alpha*q],q,q*x,k),k=0..n);
```

$$LQJ := (n,\alpha,\beta,x,q)\rightarrow add\big(qphihyperterm\big([q^{-n},\alpha\beta q^{n+1}],[\alpha q],q,q\,x,k\big),k=0..n\big) \tag{12.1}$$

```
> Flqj:=(qphihyperterm([q^(-n),alpha*beta*q^(n+1)],[alpha*q],q,q*
  x,k));
```

$$Flqj := \frac{qpochhammer\left(q^{-n}, q, k\right)\, qpochhammer\left(\alpha\,\beta\,q^{n+1}, q, k\right)\,(q\,x)^k}{qpochhammer(\alpha\,q, q, k)\; qpochhammer(q, q, k)} \quad \textbf{(12.2)}$$

The weight function

```
> rhoLQJ:=(alpha,beta)->qpochhammer(beta*q,q,k)*(alpha*q)
  ^k/qpochhammer(q,q,k)
```

$$rhoLQJ := (\alpha, \beta) \rightarrow \frac{qpochhammer(\beta\,q, q, k)\,(\alpha\,q)^k}{qpochhammer(q, q, k)} \quad \textbf{(12.3)}$$

```
> cLQJ1:=qsimplify(rhoLQJ(alpha*q^s,beta*q^t)/rhoLQJ(alpha,beta))
```

$$cLQJ1 := \frac{qpochhammer\left(\beta\,q^t\,q, q, k\right)\, q^{s\,k}}{qpochhammer(\beta\,q, q, k)} \quad \textbf{(12.4)}$$

```
> cLQJ:=qpochhammer(beta*q*q^k, q, t)*q^(s*k)/qpochhammer(beta*q,
  q,t)
```

$$cLQJ := \frac{qpochhammer\left(\beta\,q\,q^k, q, t\right)\, q^{s\,k}}{qpochhammer(\beta\,q, q, t)} \quad \textbf{(12.5)}$$

cLQJ is a polynomial of degree s+t of the variable q^k.

Mixed recurrence equation (11) of the manuscript involving $S\left(n-2, \alpha\,q^4, \beta\right)$ giving an upper bound of x_(n,1)

```
> recLQJ1:=qMixRec(Flqj,q,k,S(n),2,alpha,4,beta,0):
> recLQJ11:=combine(denom(rhs(recLQJ1))*lhs(recLQJ1)   =collect
  (numer(rhs(recLQJ1)),[S(n, alpha, beta),S(n-1, alpha, beta),x],
  qsimpcomb),power)
```

$$recLQJ11 := \left(q^n\beta-q\right)\alpha\left(\alpha\,q^{n+1}-1\right)\left(q^{2n}\alpha\,\beta-1\right)\left(\alpha\,\beta\,q^{2+n}-1\right)\left(q^n-q\right)\left(\alpha\,q^{2+n}-1\right)\left(\alpha\,\beta\,q^{n+1}-1\right)x^4\,S\left(n-2, \alpha\,q^4, \beta\right) = \Big(\left(q^n\beta-q\right)\alpha\left(\alpha\,q^3-1\right)\left(\alpha\,q^4-1\right)\left(q^{2n}\alpha\,\beta-1\right)\left(q^n-q\right)\left(\alpha\,q^2-1\right)q^{2n-3}x^2\left(\alpha\,q-1\right)+\left(q^n\beta-q\right)(q+1)\,\alpha\left(\alpha\,q^3-1\right)\left(\alpha\,q^4-1\right)\left(q^n-q\right)\left(\alpha\,q^2-1\right)^2q^{3n-5}x\left(\alpha\,q-1\right)-\left(\alpha\,q^3-1\right)^2\left(\alpha\,q^4-1\right)\left(\alpha\,q^2-1\right)^2q^{4n-6}\left(\alpha\,q-1\right)^2\Big)S(n, \alpha, \beta)+\Big(-\left(\alpha\,q^3-1\right)\left(\beta\,\alpha^2\,q^{2n+3}+\alpha\,\beta\,q^{2n+1}-\alpha\,\beta\,q^{2+n}-\alpha\,\beta\,q^{n+1}-\alpha\,q^{2+n}-\alpha\,q^{n+1}+\alpha\,q^2+1\right)\left(\alpha\,q^4-1\right)\left(\alpha\,q^2-1\right)^2q^{3n-5}x\left(\alpha\,q-1\right)+\left(\alpha\,q^3-1\right)^2\left(\alpha\,q^4-1\right)\left(\alpha\,q^2-1\right)^2q^{4n-6}\left(\alpha\,q-1\right)^2\Big)S(n-1, \alpha, \beta) \quad \textbf{(12.6)}$$

```
> GLQJ[2,4]:=op([2,2],recLQJ11)/S(n-1, alpha, beta)
```

$$GLQJ_{2,4} := -\left(\alpha\,q^3-1\right)\left(\beta\,\alpha^2\,q^{2n+3}+\alpha\,\beta\,q^{2n+1}-\alpha\,\beta\,q^{2+n}-\alpha\,\beta\,q^{n+1}-\alpha\,q^{2+n}-\alpha\,q^{n+1}+\alpha\,q^2+1\right)\left(\alpha\,q^4-1\right)\left(\alpha\,q^2-1\right)^2q^{3n-5}x\left(\alpha\,q-1\right)+\left(\alpha\,q^3-1\right)^2\left(\alpha\,q^4-1\right)\left(\alpha\,q^2-1\right)^2q^{4n-6}\left(\alpha\,q-1\right)^2 \quad \textbf{(12.7)}$$

Bound (12) of the manuscript

```
> Eq9:=combine(factor(solve(op([2,2],recLQJ11),x)),power)
```

$$Eq9 := \frac{\left(\alpha\,q^3-1\right)q^{n-1}\left(\alpha\,q-1\right)}{\beta\,\alpha^2\,q^{2n+3}+\alpha\,\beta\,q^{2n+1}-\alpha\,\beta\,q^{2+n}-\alpha\,\beta\,q^{n+1}-\alpha\,q^{2+n}-\alpha\,q^{n+1}+\alpha\,q^2+1} \quad \textbf{(12.8)}$$

Mixed recurrence equation involving $S(n-3, \alpha q^6, \beta)$ giving an upper bound of x_(n,1)

```
> recLQJ2:=qMixRec(Flqj,q,k,S(n),3,alpha,6,beta,0):
> recLQJ21:=combine(denom(rhs(recLQJ2))*lhs(recLQJ2)  =collect
  (numer(rhs(recLQJ2)),[S(n, alpha, beta),S(n-1, alpha, beta),x],
  qsimpcomb),power)
```

$$recLQJ21 := (-q^2+q^n)(\alpha q^{n+2}-1)(\alpha q^{3+n}-1)(q^n-q)(\alpha\beta q^{n+1}-1)\alpha^2(\alpha q^{n+1}-1)(q^n\beta-q)(\alpha\beta q^{n+2}-1)(\alpha\beta q^{3+n}-1)x^6(q^{2n}\alpha\beta-1)(q^n\beta-q^2)S(n-3, q^6\alpha, \beta) = \Big(-(-q^2+q^n)(q^n-q)\alpha^2(\alpha q-1)(\alpha q^6-1)(q^n\beta-q)(\alpha q^3-1)(\alpha q^4-1)(\alpha q^5-1)(\alpha q^2-1)q^{3n-6}x^3(q^{2n}\alpha\beta-1)(q^n\beta-q^2) - (-q^2+q^n)(q^n-q)\alpha^2(\alpha q-1)(\alpha q^6-1)(q^n\beta-q)(\alpha q^3-1)^2(\alpha q^4-1)(\alpha q^5-1)(\alpha q^2-1)(q^2+q+1)q^{4n-9}x^2(q^n\beta-q^2) + (\alpha q-1)(\alpha q^6-1)(\alpha q^3-1)^2(\alpha q^4-1)^2(\alpha q^5-1)(\alpha q^2-1)^2 q^{5n-11}x(\alpha^2\beta q^{2n+4}-\alpha\beta q^{n+4}+\alpha\beta q^{2n+2}-\alpha\beta q^{3+n}-\alpha q^{n+4}+q^5\alpha+\alpha\beta q^{2n+1}-\alpha\beta q^{n+2}-\alpha q^{3+n}+q^4\alpha+q^{2n}\alpha\beta-\alpha\beta q^{n+1}-\alpha q^{n+2}+q^3\alpha-\alpha q^{n+1}+q) - (\alpha q-1)^2(\alpha q^6-1)(\alpha q^3-1)^2(\alpha q^4-1)^2(\alpha q^5-1)^2(\alpha q^2-1)^2 q^{6n-12}\Big)S(n,\alpha,\beta) + \Big((\alpha q-1)(\alpha q^6-1)(\alpha q^3-1)^2(\alpha q^4-1)(\alpha q^5-1)(1+q^3\alpha-\alpha^2 q^{n+5}-\alpha^2 q^{n+4}+4\alpha^2\beta q^{2n+4}+\alpha^3\beta^2 q^{4n+5}-\alpha^2\beta^2 q^{3n+4}-\alpha^2\beta q^{3n+4}+2\alpha^2\beta q^{2n+5}-\alpha^2\beta q^{n+4}-\alpha^3\beta q^{3n+5}-\alpha^2\beta q^{n+5}+2\alpha^2\beta q^{2n+3}-\alpha^2\beta q^{3n+3}-\alpha\beta q^{3+n}-\alpha\beta q^{n+1}-\alpha^2\beta q^{3n+2}+\alpha^2\beta q^{2n+2}-\alpha^3\beta^2 q^{3n+7}-\alpha^3\beta^2 q^{3n+6}+\alpha^2\beta^2 q^{2n+5}-\beta\alpha^2 q^{n+6}-\alpha q^{n+1}-\alpha q^{n+2}+\alpha^2\beta^2 q^{2n+3}+\alpha^2 q^{2n+5}+\alpha^2 q^{2n+3}-\alpha^2\beta^2 q^{3n+2}+\alpha\beta q^{2n+1}-\alpha\beta q^{n+2}-\alpha^3\beta q^{3n+6}+\alpha^3\beta q^{2n+7}+\alpha^4\beta^2 q^{4n+8}-\alpha^2 q^{n+6}+\alpha^2\beta^2 q^{2n+4}+\alpha^2 q^{2n+4}-\alpha^3\beta^2 q^{3n+5}-\alpha^2\beta^2 q^{3n+3}+q^6\alpha^2+\alpha^2\beta^2 q^{4n+2}-\beta\alpha^3 q^{3n+7}-\alpha q^{3+n}+\beta\alpha^2 q^{2n+6})(\alpha q^2-1)q^{4n-9}x^2 - (\alpha q-1)(\alpha q^6-1)(\alpha q^3-1)^2(\alpha q^4-1)^2(\alpha q^5-1)(q+1)(\alpha q^2-1)^2 q^{5n-11}x(\alpha^2\beta q^{2n+4}-\alpha\beta q^{3+n}+\alpha\beta q^{2n+1}-\alpha q^{3+n}-\alpha\beta q^{n+1}+q^3\alpha-\alpha q^{n+1}+1) + (\alpha q-1)^2(\alpha q^6-1)(\alpha q^3-1)^2(\alpha q^4-1)^2(\alpha q^5-1)^2(\alpha q^2-1)^2 q^{6n-12}\Big)S(n-1,\alpha,\beta) \quad \mathbf{(12.9)}$$

```
> GLQJ[3,6]:=collect(op([2,2],recLQJ21)/S(n-1, alpha, beta)/(
  (alpha*q-1)*(alpha*q^2-1)*(alpha*q^4-1)*(alpha*q^6-1)*(alpha*
  q^5-1)*(alpha*q^3-1)^2),x,factor):
```

We get here the upper bound for x_{n,1} given by

```
> boundLQJC2:=(-coeff(GLQJ[3,6],x,1)-sqrt(((coeff(GLQJ[3,6],x,1)
  ^2-4*coeff(GLQJ[3,6],x,0)*coeff(GLQJ[3,6],x,2)))))/(2*coeff
  (GLQJ[3,6],x,2))
```

(12.10)

$$
\begin{aligned}
&boundLQJC2 := -\frac{1}{2}\Big(-(\alpha q^4-1)(q+1)(\alpha q^2-1)\,q^{5n-11}\big(-\alpha^2\beta q^{2n+4}-q^3\alpha \\
&+\alpha\beta q^{n+1}-\alpha\beta q^{2n+1}+\alpha\beta q^{3+n}+\alpha q^{n+1}+\alpha q^{3+n}-1\big) \\
&-\Big((\alpha q^4-1)^2(q+1)^2(\alpha q^2-1)^2(q^{5n-11})^2\big(-\alpha^2\beta q^{2n+4}-q^3\alpha \\
&+\alpha\beta q^{n+1}-\alpha\beta q^{2n+1}+\alpha\beta q^{3+n}+\alpha q^{n+1}+\alpha q^{3+n}-1\big)^2+4(\alpha q-1)(\alpha q^4 \\
&-1)(\alpha q^5-1)(\alpha q^2-1)\,q^{6n-12}\big(-1-q^3\alpha+\alpha^2 q^{n+5}+\alpha^2 q^{n+4} \\
&-4\alpha^2\beta q^{2n+4}-\alpha^3\beta^2 q^{4n+5}+\alpha^2\beta^2 q^{3n+4}+\alpha^2\beta q^{3n+4}-2\alpha^2\beta q^{2n+5} \\
&+\alpha^2\beta q^{n+4}+\alpha^3\beta q^{3n+5}+\alpha^2\beta q^{n+5}-2\alpha^2\beta q^{2n+3}+\alpha^2\beta q^{3n+3}+\alpha\beta q^{3+n} \\
&+\alpha\beta q^{n+1}+\alpha^2\beta q^{3n+2}-\alpha^2\beta q^{2n+2}+\alpha^3\beta^2 q^{3n+7}+\alpha^3\beta^2 q^{3n+6}-\alpha^2\beta^2 q^{2n+5} \\
&+\beta\alpha^2 q^{n+6}+\alpha q^{n+1}+\alpha q^{n+2}-\alpha^2\beta^2 q^{2n+3}-\alpha^2 q^{2n+5}-\alpha^2 q^{2n+3} \\
&+\alpha^2\beta^2 q^{3n+2}-\alpha\beta q^{2n+1}+\alpha\beta q^{n+2}+\alpha^3\beta q^{3n+6}-\alpha^3\beta q^{2n+7}-\alpha^4\beta^2 q^{4n+8} \\
&+\alpha^2 q^{n+6}-\alpha^2\beta^2 q^{2n+4}-\alpha^2 q^{2n+4}+\alpha^3\beta^2 q^{3n+5}+\alpha^2\beta^2 q^{3n+3}-q^6\alpha^2 \\
&-\alpha^2\beta^2 q^{4n+2}+\beta\alpha^3 q^{3n+7}+\alpha q^{3+n}-\beta\alpha^2 q^{2n+6}\big)\,q^{4n-9}\Big)^{1/2}\Big)\Big/\Big(\big(-1-q^3\alpha \\
&+\alpha^2 q^{n+5}+\alpha^2 q^{n+4}-4\alpha^2\beta q^{2n+4}-\alpha^3\beta^2 q^{4n+5}+\alpha^2\beta^2 q^{3n+4}+\alpha^2\beta q^{3n+4} \\
&-2\alpha^2\beta q^{2n+5}+\alpha^2\beta q^{n+4}+\alpha^3\beta q^{3n+5}+\alpha^2\beta q^{n+5}-2\alpha^2\beta q^{2n+3}+\alpha^2\beta q^{3n+3} \\
&+\alpha\beta q^{3+n}+\alpha\beta q^{n+1}+\alpha^2\beta q^{3n+2}-\alpha^2\beta q^{2n+2}+\alpha^3\beta^2 q^{3n+7}+\alpha^3\beta^2 q^{3n+6} \\
&-\alpha^2\beta^2 q^{2n+5}+\beta\alpha^2 q^{n+6}+\alpha q^{n+1}+\alpha q^{n+2}-\alpha^2\beta^2 q^{2n+3}-\alpha^2 q^{2n+5} \\
&-\alpha^2 q^{2n+3}+\alpha^2\beta^2 q^{3n+2}-\alpha\beta q^{2n+1}+\alpha\beta q^{n+2}+\alpha^3\beta q^{3n+6}-\alpha^3\beta q^{2n+7} \\
&-\alpha^4\beta^2 q^{4n+8}+\alpha^2 q^{n+6}-\alpha^2\beta^2 q^{2n+4}-\alpha^2 q^{2n+4}+\alpha^3\beta^2 q^{3n+5}+\alpha^2\beta^2 q^{3n+3} \\
&-q^6\alpha^2-\alpha^2\beta^2 q^{4n+2}+\beta\alpha^3 q^{3n+7}+\alpha q^{3+n}-\beta\alpha^2 q^{2n+6}\big)\,q^{4n-9}\Big)
\end{aligned}
\qquad \textbf{(12.10)}
$$

Comparison between the bounds and the extreme zeros

```
> xnLQJ:= sort([solve(expand(subs({alpha=0.5,beta=-10,q=0.6},LQJ
  (30,alpha,beta,x,q))),x)]):
> extzeroxnLQJ:=evalf[15]([min(xnLQJ) ,max(xnLQJ)]);
```

$$extzeroxnLQJ := [1.86417900023822\ 10^{-7}, 1.00000000000000] \qquad \textbf{(12.11)}$$

```
> LQj240:=unapply(Eq9 ,[n,alpha,beta,q]):
> evalf[15](LQj240(30,0.5,-10,0.6))
```

$$1.94969553321315\ 10^{-7} \qquad \textbf{(12.12)}$$

```
> LQj360:=unapply(boundLQJC2,[n,alpha,beta,q]):
> evalf[15](LQj360(30,0.5,-10,0.6))
```

$$1.86456827116844\ 10^{-7} \quad \textbf{(12.13)}$$

the q-Meixner polynomials

```
> QM:=(n,beta,gamma,x,q)->add(qphihyperterm([q^(-n),x],[beta*q],
  q,-q^(n+1)/gamma,k),k=0..n);
```

$$QM := (n, \beta, \gamma, x, q) \rightarrow add\left(qphihyperterm\left(\left[q^{-n}, x\right], [\beta q], q, -\frac{q^{n+1}}{\gamma}, k\right), k = 0..n\right) \quad \textbf{(13.1)}$$

```
> Fqm:=(qphihyperterm([q^(-n),x],[beta*q],q,-q^(n+1)/gamma,k));
```

The weight function

```
> rhoQM:=(beta,gamma)->qpochhammer(beta*q,q,x)*gamma^x*q^binomial
  (x,2)/qpochhammer(q,q,x)/qpochhammer(-beta*gamma*q,q,x)
```

$$rhoQM := (\beta, \gamma) \rightarrow \frac{qpochhammer(\beta q, q, x)\, \gamma^x\, q^{\mathrm{binomial}(x,\,2)}}{qpochhammer(q, q, x)\ qpochhammer(-\beta\gamma q, q, x)} \quad \textbf{(13.2)}$$

```
> cQM:=qsimplify(rhoQM(beta,gamma*q^(-s))/rhoQM(beta,gamma))
```

$$cQM := \frac{qpochhammer\left(-\frac{1}{\gamma\beta q^x}, q, s\right)}{qpochhammer\left(-\frac{1}{\beta\gamma}, q, s\right)} \quad \textbf{(13.3)}$$

cQM is a polynomial of degree s of the variable q^{-x}

```
> subs(q^x=1/x,cQM)
```

$$\frac{qpochhammer\left(-\frac{x}{\gamma\beta}, q, s\right)}{qpochhammer\left(-\frac{1}{\beta\gamma}, q, s\right)} \quad \textbf{(13.4)}$$

Mixed recurrence equation involving $S(n-3, \beta, \gamma)$ giving a lower bound of x_(n,n)

```
> recQM1:=qMixRec(Fqm,q,k,S(n),3,beta,0,gamma,0):
> recQM11:=combine(denom(rhs(recQM1))*lhs(recQM1)   =collect(numer
  (rhs(recQM1)),[S(n, beta, gamma),S(n-1, beta, gamma),x],
  qsimpcomb),power)
```

$$recQM11 := (q^n - q)\,(-q^2 + q^n)\,(\gamma q + q^n)\,(\gamma q^2 + q^n)\, S(n-3, \beta, \gamma) = \Big(-\gamma\,(q^n \beta - 1)\, q^{2n+1} x - \gamma q^3\,(q^n\beta - 1)\,(q^n\beta\gamma - \gamma q^2 + q^n\gamma - \gamma q - q^n)\Big)\, S(n, \beta, \gamma) + \Big(q^{4n} x^2 + (q+1)\, q^{2n+1} x\,(q^n\beta\gamma - \gamma q^2 + q^n\gamma - q^n - \gamma) - q^2\,\big(-\gamma^2\beta^2 q^{2n+1} + \gamma^2\beta q^{3+n} - \gamma^2\beta q^{2n+1} + \gamma^2\beta q^{n+2} + \gamma^2 q^{3+n} - \gamma^2 q^4 + q^{3n}\beta\gamma + \gamma\beta q^{2n+1} - \gamma^2 q^{2n+1} + \gamma^2\beta q^{n+1} + \gamma^2 q^{n+2} - \gamma q^{3+n} - \gamma^2 q^3 + \gamma q^{2n+1} + \gamma^2 q^{n+1} - \gamma q^{n+2} - \gamma^2 q^2 - q^{2n+1} - \gamma q^{n+1}\big)\Big)\, S(n-1, \beta, \gamma) \quad \textbf{(13.5)}$$

```
> GQM[3,0]:=op([2,2],recQM11)/S(n-1, beta, gamma):
```

```
> boundQM1:=combine((-coeff(GQM[3,0],x,1)+sqrt(((coeff(GQM[3,0],
  x,1)^2-4*coeff(GQM[3,0],x,0)*coeff(GQM[3,0],x,2)))))/(2*coeff
  (GQM[3,0],x,2)),power)
```

$$boundQM1 := \frac{1}{2} \Big(-(q+1)\, q^{2n+1} \left(q^n \beta\gamma - \gamma q^2 + q^n \gamma - q^n - \gamma\right) + \Big((q+1)^2 q^{4n+2} \left(q^n \beta\gamma - \gamma q^2 + q^n\gamma - q^n - \gamma\right)^2 + 4 \big(-\gamma^2 \beta^2 q^{2n+1} + \gamma^2 \beta q^{3+n} - \gamma^2 \beta q^{2n+1} + \gamma^2 \beta q^{n+2} + \gamma^2 q^{3+n} - \gamma^2 q^4 + q^{3n} \beta\gamma + \gamma\beta q^{2n+1} - \gamma^2 q^{2n+1} + \gamma^2 \beta q^{n+1} + \gamma^2 q^{n+2} - \gamma q^{3+n} - \gamma^2 q^3 + \gamma q^{2n+1} + \gamma^2 q^{n+1} - \gamma q^{n+2} - \gamma^2 q^2 - q^{2n+1} - \gamma q^{n+1} \big) q^{4n+2} \Big)^{1/2} \Big) q^{-4n} \qquad \textbf{(13.6)}$$

Mixed recurrence equation involving $S\left(n-3, \beta, \gamma q^{-6}\right)$ giving an upper bound of x_(n,1)

```
> recQM2:=qMixRec(Fqm,q,k,S(n),3,beta,0,gamma,-6):
> recQM21:=combine(denom(rhs(recQM2))*lhs(recQM2)   =collect(numer
  (rhs(recQM2)),[S(n, beta, gamma),S(n-1, beta, gamma),x],
  qsimpcomb),power):
> GQM[3,6]:=collect(op([2,2],recQM21)/S(n-1, beta, gamma)/(q^n+
  gamma),x,factor):
> boundQM2:=combine((-coeff(GQM[3,6],x,1)-sqrt(((coeff(GQM[3,6],
  x,1)^2-4*coeff(GQM[3,6],x,0)*coeff(GQM[3,6],x,2)))))/(2*coeff
  (GQM[3,6],x,2)),power)
```

$$boundQM2 := -\frac{1}{2} \Big(\Big(-\big(-\gamma^2 q^8 - \gamma^3 q^5 - \beta\gamma^3 q^5 - \beta\gamma^2 q^8 - \beta^3\gamma^4 q^3 - \beta^2\gamma^3 q^5 - \beta^2\gamma^4 q^3 \qquad \textbf{(13.7)}$$

$$- \beta\gamma^4 q^3 + 2\beta\gamma^3 q^{n+4} + \gamma^4 \beta q^{n+2} - \gamma q^{12} - \gamma^4 q^3 + \gamma^2 \beta q^{n+12} - \gamma q^{n+13} - \gamma q^{n+11}$$

$$+ \gamma\beta q^{2n+11} + \gamma^2 \beta q^{n+6} + \gamma^4 \beta^3 q^{n+1} + \gamma^4 \beta^2 q^{n+1} + \gamma^4 \beta q^{n+1} + \gamma\beta q^{2n+9}$$

$$+ \gamma\beta q^{2n+13} - q^{2n} \beta^3 \gamma^4 - q^{2n} \beta^2 \gamma^4 - \beta^3 \gamma^3 q^{4n+4} - q^{4n} \gamma^4 \beta^4 - q^{4n} \gamma^4 \beta^3 + \beta^2 \gamma^3 q^{3+n}$$

$$- 2\beta^3\gamma^3 q^{2n+5} - 2\beta^2\gamma^3 q^{2n+7} - 4\beta^2\gamma^3 q^{2n+6} + \gamma^3 q^{n+5} + \gamma^4 q^{3+n} + \beta^2\gamma^3 q^{3n+6}$$

$$+ 2q^{3n}\gamma^4\beta^3 + q^{3n}\gamma^4\beta^4 - \gamma^4 \beta q^{2n+2} - \gamma^4 \beta q^{2n+1} + \gamma^4 \beta^2 q^{3n+1} + \beta^2\gamma^3 q^{3n+2}$$

$$- \beta\gamma^3 q^{2n+4} + q^{3n}\gamma^4\beta^2 + \gamma^3 q^{n+7} + 3\gamma^4\beta^3 q^{3+n} + 2\gamma^4\beta^2 q^{n+4} + 4\beta^2\gamma^3 q^{n+5}$$

$$+ \gamma^4 \beta q^{n+5} + 4\beta\gamma^3 q^{n+6} + \gamma^2 \beta q^{n+7} + \beta^2\gamma^3 q^{3n+3} - 2\beta\gamma^3 q^{2n+5} - \beta^3\gamma^3 q^{4n+2}$$

$$+ \beta^3\gamma^3 q^{3n+2} + 2\gamma^3\beta^3 q^{3n+4} - 2\gamma^3\beta^2 q^{2n+3} - \gamma^3\beta^2 q^{2n+2} - \beta^2\gamma^3 q^{2n+8}$$

$$- \gamma^4\beta^4 q^{2n+1} + \gamma^4\beta^3 q^{n+5} - \gamma^3 \beta q^{2n+3} - 4\gamma^3\beta^2 q^{2n+4} - \beta^3\gamma^3 q^{2n+6} - \beta\gamma^3 q^{2n+7}$$

$$+ \beta^2\gamma^3 q^{n+9} - 2\gamma^4\beta^2 q^{2n+1} + 2\gamma^4\beta^2 q^{n+2} + 3\gamma^4 \beta q^{3+n} + \gamma^4\beta^4 q^{3+n} + \gamma^4\beta^3 q^{n+4}$$

$$+\beta^{3}\gamma^{3}q^{n+5}+\gamma^{4}\beta^{2}q^{n+5}+4\beta^{2}\gamma^{3}q^{n+6}-\gamma q^{14}-\gamma^{2}q^{12}-\gamma^{2}q^{11}-\gamma^{3}q^{9}-2\gamma^{2}q^{10}$$

$$-\gamma^{3}q^{8}-\gamma^{2}q^{9}-2\gamma^{3}q^{7}-\gamma^{4}q^{5}-\gamma^{3}q^{6}+\gamma^{3}\beta q^{3+n}+2\gamma^{2}\beta q^{n+10}+\gamma^{2}\beta q^{n+9}$$

$$+\gamma^{2}\beta q^{n+11}+\beta^{3}\gamma^{3}q^{3n+6}+\beta^{3}\gamma^{3}q^{3n+5}-\beta\gamma^{3}q^{2n+6}-2\gamma^{4}\beta^{3}q^{2n+1}$$

$$-3\gamma^{4}\beta^{2}q^{2n+2}+\gamma^{4}\beta^{3}q^{n+2}+3\gamma^{4}\beta^{2}q^{3+n}+2\beta^{2}\gamma^{3}q^{n+4}+\gamma^{4}\beta q^{n+4}+4\beta\gamma^{3}q^{n+5}$$

$$+2\gamma^{3}\beta^{2}q^{3n+4}+4\beta\gamma^{3}q^{n+7}+2\gamma^{2}\beta q^{n+8}-\gamma^{4}\beta q^{2n+3}+\gamma^{4}\beta^{3}q^{3n+1}-\gamma^{4}\beta^{4}q^{2n+3}$$

$$-\gamma^{4}\beta^{3}q^{2n+4}-q^{n+15}-\beta\gamma^{2}q^{12}-\beta^{2}\gamma^{3}q^{9}-\beta\gamma^{2}q^{11}-\beta^{2}\gamma^{3}q^{8}-\beta\gamma^{3}q^{9}-2\beta\gamma^{2}q^{10}$$

$$-\beta^{3}\gamma^{4}q^{5}-2\beta^{2}\gamma^{3}q^{7}-2\beta\gamma^{3}q^{8}-\beta\gamma^{2}q^{9}-\beta^{2}\gamma^{4}q^{5}-\beta^{2}\gamma^{3}q^{6}-3\beta\gamma^{3}q^{7}-\beta^{2}\gamma^{4}q^{4}$$

$$-\beta\gamma^{4}q^{5}-2\beta\gamma^{3}q^{6}-\beta\gamma^{4}q^{4}+\gamma^{4}\beta^{4}q^{3n+2}+\beta^{3}\gamma^{3}q^{3n+3}+\beta^{2}\gamma^{3}q^{3n+5}+\gamma^{4}\beta^{2}q^{3n+2}$$

$$+\gamma^{4}\beta^{4}q^{3n+1}+2\gamma^{4}\beta^{3}q^{3n+2}-\gamma^{4}\beta^{4}q^{2n+2}-3\gamma^{4}\beta^{3}q^{2n+2}-\beta^{3}\gamma^{3}q^{2n+3}$$

$$-2\gamma^{4}\beta^{2}q^{2n+3}-\beta^{3}\gamma^{3}q^{2n+7}+\beta^{3}\gamma^{3}q^{n+7}+2\beta^{2}\gamma^{3}q^{n+8}+4\beta^{2}\gamma^{3}q^{n+7}+\beta\gamma^{3}q^{n+9}$$

$$+2\beta\gamma^{3}q^{n+8}-\gamma\beta q^{n+11}-\gamma\beta q^{n+13}-2\gamma^{4}\beta^{3}q^{2n+3}-\beta^{3}\gamma^{3}q^{2n+4}-\gamma^{4}\beta^{2}q^{2n+4}$$

$$-4\beta^{2}\gamma^{3}q^{2n+5}\Big)\,(q+1)\,q^{-n+1}$$

$$-\Big(\Big(-\gamma^{2}q^{8}-\gamma^{3}q^{5}-\beta\gamma^{3}q^{5}-\beta\gamma^{2}q^{8}-\beta^{3}\gamma^{4}q^{3}-\beta^{2}\gamma^{3}q^{5}-\beta^{2}\gamma^{4}q^{3}$$

$$-\beta\gamma^{4}q^{3}+2\beta\gamma^{3}q^{n+4}+\gamma^{4}\beta q^{n+2}-\gamma q^{12}-\gamma^{4}q^{3}+\gamma^{2}\beta q^{n+12}-\gamma q^{n+13}-\gamma q^{n+11}$$

$$+\gamma\beta q^{2n+11}+\gamma^{2}\beta q^{n+6}+\gamma^{4}\beta^{3}q^{n+1}+\gamma^{4}\beta^{2}q^{n+1}+\gamma^{4}\beta q^{n+1}+\gamma\beta q^{2n+9}$$

$$+\gamma\beta q^{2n+13}-q^{2n}\beta^{3}\gamma^{4}-q^{2n}\beta^{2}\gamma^{4}-\beta^{3}\gamma^{3}q^{4n+4}-q^{4n}\gamma^{4}\beta^{4}-q^{4n}\gamma^{4}\beta^{3}+\beta^{2}\gamma^{3}q^{3+n}$$

$$-2\beta^{3}\gamma^{3}q^{2n+5}-2\beta^{2}\gamma^{3}q^{2n+7}-4\beta^{2}\gamma^{3}q^{2n+6}+\gamma^{3}q^{n+5}+\gamma^{4}q^{3+n}+\beta^{2}\gamma^{3}q^{3n+6}$$

$$+2q^{3n}\gamma^{4}\beta^{3}+q^{3n}\gamma^{4}\beta^{4}-\gamma^{4}\beta q^{2n+2}-\gamma^{4}\beta q^{2n+1}+\gamma^{4}\beta^{2}q^{3n+1}+\beta^{2}\gamma^{3}q^{3n+2}$$

$$-\beta\gamma^{3}q^{2n+4}+q^{3n}\gamma^{4}\beta^{2}+\gamma^{3}q^{n+7}+3\gamma^{4}\beta^{3}q^{3+n}+2\gamma^{4}\beta^{2}q^{n+4}+4\beta^{2}\gamma^{3}q^{n+5}$$

$$+\gamma^{4}\beta q^{n+5}+4\beta\gamma^{3}q^{n+6}+\gamma^{2}\beta q^{n+7}+\beta^{2}\gamma^{3}q^{3n+3}-2\beta\gamma^{3}q^{2n+5}-\beta^{3}\gamma^{3}q^{4n+2}$$

$$+\beta^{3}\gamma^{3}q^{3n+2}+2\gamma^{3}\beta^{3}q^{3n+4}-2\gamma^{3}\beta^{2}q^{2n+3}-\gamma^{3}\beta^{2}q^{2n+2}-\beta^{2}\gamma^{3}q^{2n+8}$$

$$-\gamma^{4}\beta^{4}q^{2n+1}+\gamma^{4}\beta^{3}q^{n+5}-\gamma^{3}\beta q^{2n+3}-4\gamma^{3}\beta^{2}q^{2n+4}-\beta^{3}\gamma^{3}q^{2n+6}-\beta\gamma^{3}q^{2n+7}$$

$$+\beta^2\gamma^3 q^{n+9}-2\gamma^4\beta^2 q^{2n+1}+2\gamma^4\beta^2 q^{n+2}+3\gamma^4\beta q^{3+n}+\gamma^4\beta^4 q^{3+n}+\gamma^4\beta^3 q^{n+4}$$

$$+\beta^3\gamma^3 q^{n+5}+\gamma^4\beta^2 q^{n+5}+4\beta^2\gamma^3 q^{n+6}-\gamma q^{14}-\gamma^2 q^{12}-\gamma^2 q^{11}-\gamma^3 q^9-2\gamma^2 q^{10}$$

$$-\gamma^3 q^8-\gamma^2 q^9-2\gamma^3 q^7-\gamma^4 q^5-\gamma^3 q^6+\gamma^3\beta q^{3+n}+2\gamma^2\beta q^{n+10}+\gamma^2\beta q^{n+9}$$

$$+\gamma^2\beta q^{n+11}+\beta^3\gamma^3 q^{3n+6}+\beta^3\gamma^3 q^{3n+5}-\beta\gamma^3 q^{2n+6}-2\gamma^4\beta^3 q^{2n+1}$$

$$-3\gamma^4\beta^2 q^{2n+2}+\gamma^4\beta^3 q^{n+2}+3\gamma^4\beta^2 q^{3+n}+2\beta^2\gamma^3 q^{n+4}+\gamma^4\beta q^{n+4}+4\beta\gamma^3 q^{n+5}$$

$$+2\gamma^3\beta^2 q^{3n+4}+4\beta\gamma^3 q^{n+7}+2\gamma^2\beta q^{n+8}-\gamma^4\beta q^{2n+3}+\gamma^4\beta^3 q^{3n+1}-\gamma^4\beta^4 q^{2n+3}$$

$$-\gamma^4\beta^3 q^{2n+4}-q^{n+15}-\beta\gamma^2 q^{12}-\beta^2\gamma^3 q^9-\beta\gamma^2 q^{11}-\beta^2\gamma^3 q^8-\beta\gamma^3 q^9-2\beta\gamma^2 q^{10}$$

$$-\beta^3\gamma^4 q^5-2\beta^2\gamma^3 q^7-2\beta\gamma^3 q^8-\beta\gamma^2 q^9-\beta^2\gamma^4 q^5-\beta^2\gamma^3 q^6-3\beta\gamma^3 q^7-\beta^2\gamma^4 q^4$$

$$-\beta\gamma^4 q^5-2\beta\gamma^3 q^6-\beta\gamma^4 q^4+\gamma^4\beta^4 q^{3n+2}+\beta^3\gamma^3 q^{3n+3}+\beta^2\gamma^3 q^{3n+5}+\gamma^4\beta^2 q^{3n+2}$$

$$+\gamma^4\beta^4 q^{3n+1}+2\gamma^4\beta^3 q^{3n+2}-\gamma^4\beta^4 q^{2n+2}-3\gamma^4\beta^3 q^{2n+2}-\beta^3\gamma^3 q^{2n+3}$$

$$-2\gamma^4\beta^2 q^{2n+3}-\beta^3\gamma^3 q^{2n+7}+\beta^3\gamma^3 q^{n+7}+2\beta^2\gamma^3 q^{n+8}+4\beta^2\gamma^3 q^{n+7}+\beta\gamma^3 q^{n+9}$$

$$+2\beta\gamma^3 q^{n+8}-\gamma\beta q^{n+11}-\gamma\beta q^{n+13}-2\gamma^4\beta^3 q^{2n+3}-\beta^3\gamma^3 q^{2n+4}-\gamma^4\beta^2 q^{2n+4}$$

$$-4\beta^2\gamma^3 q^{2n+5}\Big)^2 (q+1)^2 q^{-2n+2}-4\,\big(-q^{18}+\beta\gamma^3 q^{n+4}+\gamma^5\beta q^{n+1}+\gamma^4\beta q^{n+2}$$

$$-4\beta^2\gamma^3 q^{2n+9}+2\gamma^4\beta^3 q^{3n+3}+4\gamma^2\beta q^{n+12}+2\gamma^4\beta^4 q^{3n+4}-4\beta^3\gamma^3 q^{2n+5}$$

$$-6\beta^2\gamma^3 q^{2n+7}-5\beta^2\gamma^3 q^{2n+6}-2\gamma^2\beta^2 q^{2n+8}-\gamma^2\beta^2 q^{2n+7}+\gamma^5\beta^5 q^{3+n}$$

$$-\gamma^5\beta^2 q^{2n+1}+3\gamma^4\beta^3 q^{3+n}+\beta^3\gamma^3 q^{n+4}+5\gamma^4\beta^2 q^{n+4}+3\beta^2\gamma^3 q^{n+5}+4\gamma^4\beta q^{n+5}$$

$$+4\beta\gamma^3 q^{n+6}+\gamma^2\beta q^{n+7}-q^{2n}\beta^4\gamma^5-q^{2n}\beta^3\gamma^5-q^{2n}\beta^2\gamma^5+5\beta^3\gamma^3 q^{n+9}$$

$$+6\beta^2\gamma^3 q^{n+10}+4\gamma^2\beta^2 q^{n+11}-q^{2n}\beta^5\gamma^5+\gamma^3\beta^3 q^{3n+4}-\gamma^3\beta^2 q^{2n+3}$$

$$-5\beta^2\gamma^3 q^{2n+8}-\gamma^4\beta^4 q^{2n+1}+\gamma^4\beta^3 q^{3n+5}-\beta\gamma^4 q^8+\gamma^4\beta^2 q^{n+8}+\gamma^5\beta^5 q^{n+2}$$

$$+3\gamma^4\beta^4 q^{n+4}+6\gamma^4\beta^3 q^{n+5}+4\beta^3\gamma^3 q^{n+6}+5\gamma^4\beta^2 q^{n+6}+2\gamma^2\beta^2 q^{n+8}$$

$$+2\gamma^2\beta^2 q^{n+14}+\gamma\beta q^{n+16}-2\gamma^2\beta^2 q^{2n+12}+2\beta^3\gamma^3 q^{n+11}+3\gamma^2\beta^2 q^{n+13}$$

$$-2\gamma^3\beta^2 q^{2n+4}+\gamma\beta q^{n+12}+3\gamma^2\beta q^{n+13}+\gamma\beta q^{n+14}-5\beta^3\gamma^3 q^{2n+6}-\gamma^4\beta^2 q^{2n+7}$$

$$-\gamma^5\beta^4 q^{2n+2}+3\gamma^4\beta^2 q^{n+7}+4\gamma^4\beta^4 q^{n+5}+5\gamma^4\beta^3 q^{n+6}+8\beta^2\gamma^3 q^{n+9}$$
$$-\gamma^4\beta^2 q^{2n+1}+\gamma^5\beta^2 q^{n+1}+\gamma^4\beta^2 q^{n+2}+\gamma^5\beta q^{n+2}+2\gamma^4\beta q^{3+n}+\gamma^5\beta^4 q^{n+2}$$
$$+2\gamma^4\beta^4 q^{3+n}+5\gamma^4\beta^3 q^{n+4}+2\beta^3\gamma^3 q^{n+5}+6\gamma^4\beta^2 q^{n+5}+6\beta^2\gamma^3 q^{n+6}-\gamma^2 q^{15}$$
$$-\gamma q^{16}-\gamma^2 q^{14}-\gamma q^{15}-\gamma^3 q^{12}-2\gamma^2 q^{13}-\gamma q^{14}-\gamma^3 q^{11}-2\gamma^2 q^{12}-\gamma q^{13}-2\gamma^3 q^{10}$$
$$-2\gamma^2 q^{11}-2\gamma^3 q^9-\gamma^2 q^{10}-\gamma^4 q^7-2\gamma^3 q^8-\gamma^2 q^9-\gamma^4 q^6-\gamma^3 q^7-\gamma^4 q^5-\gamma^3 q^6$$
$$-\gamma^5 q^3-\gamma^4 q^4+q^{3n}\beta^5\gamma^5+q^{3n}\beta^4\gamma^5+q^{3n}\beta^3\gamma^5+4\gamma^2\beta q^{n+10}+3\gamma^2\beta q^{n+9}$$
$$+2\gamma^4\beta q^{n+7}+4\gamma^2\beta q^{n+11}+2\beta^3\gamma^3 q^{3n+6}+2\beta^3\gamma^3 q^{3n+5}+3\gamma^4\beta^4 q^{n+6}$$
$$+3\gamma^4\beta^3 q^{n+7}+6\beta^3\gamma^3 q^{n+8}-4\gamma^4\beta^4 q^{2n+4}-\gamma^4\beta^3 q^{2n+1}-2\gamma^4\beta^2 q^{2n+2}$$
$$+\gamma^5\beta^3 q^{n+1}+\gamma^4\beta^3 q^{n+2}+\gamma^5\beta^2 q^{n+2}+3\gamma^4\beta^2 q^{3+n}+\beta^2\gamma^3 q^{n+4}+3\gamma^4\beta q^{n+4}$$
$$+2\beta\gamma^3 q^{n+5}+\gamma^5\beta^4 q^{3+n}+\gamma^2\beta^2 q^{n+7}+3\gamma^4\beta q^{n+6}+5\beta\gamma^3 q^{n+7}+2\gamma^2\beta q^{n+8}$$
$$-3\gamma^4\beta^3 q^{2n+6}+\gamma^4\beta^3 q^{3n+1}-\gamma^4\beta^4 q^{4n+2}+2\gamma^4\beta^4 q^{3n+3}+2\gamma^4\beta^3 q^{3n+4}$$
$$-\beta^2\gamma^3 q^{2n+11}+\beta^2\gamma^3 q^{n+12}-4\gamma^4\beta^4 q^{2n+3}-6\gamma^4\beta^3 q^{2n+4}-5\gamma^4\beta^3 q^{2n+5}$$
$$-\gamma^5\beta^4 q^{2n+1}-\gamma^5\beta^3 q^{2n+2}+\beta^3\gamma^3 q^{3n+8}-\gamma^4\beta^4 q^{2n+7}-2\beta^2\gamma^3 q^{2n+10}$$
$$+2\gamma^4\beta^4 q^{n+7}+2\beta\gamma^3 q^{n+11}+2\gamma^2\beta q^{n+14}+\gamma\beta q^{n+15}-4\beta^3\gamma^3 q^{2n+9}$$
$$-2\gamma^2\beta^2 q^{2n+11}+4\beta^3\gamma^3 q^{n+10}+3\beta^2\gamma^3 q^{n+11}+4\gamma^2\beta^2 q^{n+12}-\gamma^5\beta^2 q^{2n+2}$$
$$-\beta^2\gamma^2 q^{15}-\beta\gamma q^{17}-\beta^3\gamma^3 q^{12}-\beta^2\gamma^2 q^{14}-\beta\gamma^2 q^{15}-\beta\gamma q^{16}-\beta^3\gamma^3 q^{11}-\beta^2\gamma^3 q^{12}$$
$$-2\beta^2\gamma^2 q^{13}-2\beta\gamma^2 q^{14}-\beta\gamma q^{15}-\beta^4\gamma^4 q^8-2\beta^3\gamma^3 q^{10}-2\beta^2\gamma^3 q^{11}-2\beta^2\gamma^2 q^{12}$$
$$-\beta\gamma^3 q^{12}-3\beta\gamma^2 q^{13}-\beta\gamma q^{14}-\beta^4\gamma^4 q^7-\beta^3\gamma^4 q^8-2\beta^3\gamma^3 q^9-4\beta^2\gamma^3 q^{10}$$
$$-2\beta^2\gamma^2 q^{11}-2\beta\gamma^3 q^{11}-4\beta\gamma^2 q^{12}-\beta\gamma q^{13}-\beta^4\gamma^4 q^6-2\beta^3\gamma^4 q^7-2\beta^3\gamma^3 q^8$$
$$-\beta^2\gamma^4 q^8-4\beta^2\gamma^3 q^9-\beta^2\gamma^2 q^{10}-4\beta\gamma^3 q^{10}-3\beta\gamma^2 q^{11}-\beta^5\gamma^5 q^3-\beta^4\gamma^4 q^5$$
$$-2\beta^3\gamma^4 q^6-\beta^3\gamma^3 q^7-2\beta^2\gamma^4 q^7-4\beta^2\gamma^3 q^8-\beta^2\gamma^2 q^9-4\beta\gamma^3 q^9-2\beta\gamma^2 q^{10}$$
$$-\beta^4\gamma^5 q^3-\beta^4\gamma^4 q^4-2\beta^3\gamma^4 q^5-\beta^3\gamma^3 q^6-3\beta^2\gamma^4 q^6-2\beta^2\gamma^3 q^7-2\beta\gamma^4 q^7-4\beta\gamma^3 q^8$$

$$- \beta \gamma^2 q^9 - \beta^3 \gamma^5 q^3 - \beta^3 \gamma^4 q^4 - 2 \beta^2 \gamma^4 q^5 - \beta^2 \gamma^3 q^6 - 2 \beta \gamma^4 q^6 - 2 \beta \gamma^3 q^7 - \beta^2 \gamma^5 q^3$$
$$- \beta^2 \gamma^4 q^4 - 2 \beta \gamma^4 q^5 - \beta \gamma^3 q^6 - \beta \gamma^5 q^3 - \beta \gamma^4 q^4 + 2 \gamma^4 \beta^4 q^{3n+2} + \beta^3 \gamma^3 q^{3n+3}$$
$$- \gamma^2 \beta^2 q^{2n+14} + \gamma^2 \beta^2 q^{n+15} + \gamma \beta q^{n+17} + 4 \gamma^2 \beta^2 q^{n+10} + 4 \beta \gamma^3 q^{n+10} - \gamma q^{17}$$
$$+ \gamma^4 \beta^4 q^{3n+1} + 2 \gamma^4 \beta^3 q^{3n+2} - 2 \gamma^4 \beta^4 q^{2n+2} + \gamma^4 \beta^4 q^{n+8} + \beta \gamma^3 q^{n+12}$$
$$- \gamma^5 \beta^5 q^{2n+2} - 5 \beta^3 \gamma^3 q^{2n+8} - 3 \gamma^2 \beta^2 q^{2n+10} - 2 \gamma^2 \beta^2 q^{2n+9} - 3 \gamma^4 \beta^3 q^{2n+2}$$
$$- \beta^3 \gamma^3 q^{2n+3} - 4 \gamma^4 \beta^2 q^{2n+3} + \gamma^5 \beta^4 q^{n+1} + \gamma^4 \beta^4 q^{n+2} + \gamma^5 \beta^3 q^{n+2} - \gamma^5 \beta^5 q^{2n+1}$$
$$- \gamma^2 \beta^2 q^{2n+13} + \beta^3 \gamma^3 q^{n+12} + \gamma^2 \beta q^{n+15} + \beta^3 \gamma^3 q^{3n+9} - 2 \beta^3 \gamma^3 q^{2n+10}$$
$$+ \gamma^4 \beta^4 q^{3n+5} - 2 \gamma^4 \beta^4 q^{2n+6} - \gamma^4 \beta^3 q^{2n+7} + \gamma^4 \beta^3 q^{n+8} - 2 \gamma^4 \beta^2 q^{2n+6} - \gamma^4 q^8$$
$$- 6 \beta^3 \gamma^3 q^{2n+7} - \beta^3 \gamma^3 q^{2n+11} + \gamma^5 \beta^3 q^{3+n} + \gamma^5 \beta q^{3+n} + 5 \beta^3 \gamma^3 q^{n+7} + 9 \beta^2 \gamma^3 q^{n+8}$$
$$+ 8 \beta^2 \gamma^3 q^{n+7} + 3 \gamma^2 \beta^2 q^{n+9} + 5 \beta \gamma^3 q^{n+9} + 6 \beta \gamma^3 q^{n+8} - 4 \gamma^4 \beta^2 q^{2n+5}$$
$$+ 2 \beta^3 \gamma^3 q^{3n+7} + \gamma \beta q^{n+11} + \gamma \beta q^{n+13} - 4 \gamma^4 \beta^4 q^{2n+5} - \gamma^5 \beta^3 q^{2n+1}$$
$$- 5 \gamma^4 \beta^3 q^{2n+3} - 2 \beta^3 \gamma^3 q^{2n+4} - 4 \gamma^4 \beta^2 q^{2n+4} - 4 \beta^2 \gamma^3 q^{2n+5} - \gamma^2 \beta^2 q^{2n+6}$$
$$+ \gamma^5 \beta^5 q^{n+1} + \gamma^4 \beta q^{n+8} + \gamma^5 \beta^2 q^{3+n} \big) q^{-2n+3} \big(-\gamma^2 q^8 - \gamma^2 q^7 - \gamma^3 q^5 - \gamma^3 q^4$$
$$- \beta \gamma^3 q^5 - \beta \gamma^3 q^4 + 3 \beta \gamma^3 q^{n+4} + \gamma^3 q^{3+n} - \gamma q^{n+11} - \gamma q^{n+10} - \gamma q^{n+9} - \gamma^3 q^{2n+3}$$
$$- \beta \gamma^3 q^{2n+1} + \beta^3 \gamma^3 q^{3n+1} + \beta^2 \gamma^3 q^{3+n} - \beta^2 \gamma^3 q^{2n+1} + \beta \gamma^3 q^{3n+2} + \beta \gamma^3 q^{3n+1}$$
$$- \beta^3 \gamma^3 q^{4n+1} - \beta^2 \gamma^3 q^{4n+2} - \beta^2 \gamma^3 q^{4n+1} + \beta \gamma^3 q^{3n+3} + 2 \beta^2 \gamma^3 q^{3n+1} - q^{4n} \beta^3 \gamma^3$$
$$+ q^{3n} \beta^2 \gamma^3 + q^{5n} \gamma^3 \beta^3 - q^{4n} \gamma^3 \beta^2 + \gamma^3 q^{n+5} + \gamma^3 q^{n+4} + 3 \beta^2 \gamma^3 q^{3n+2} - 2 \beta \gamma^3 q^{2n+4}$$
$$+ \beta^2 \gamma^3 q^{n+5} + \beta \gamma^3 q^{n+6} + 2 \beta^2 \gamma^3 q^{3n+3} - \beta \gamma^3 q^{2n+5} - \beta^3 \gamma^3 q^{4n+2} + \beta^3 \gamma^3 q^{3n+2}$$
$$- 3 \gamma^3 \beta^2 q^{2n+3} - 2 \gamma^3 \beta^2 q^{2n+2} - 3 \gamma^3 \beta q^{2n+3} - 2 \gamma^3 \beta q^{2n+2} - 2 \gamma^3 \beta^2 q^{2n+4} - \gamma^2 q^9$$
$$- \gamma^3 q^6 + 2 \gamma^3 \beta q^{3+n} + \gamma^3 \beta q^{n+2} + \beta^2 \gamma^3 q^{n+4} + 2 \beta \gamma^3 q^{n+5} + \gamma^3 \beta^2 q^{3n+4} - q^{2n+12}$$
$$- \beta \gamma^3 q^6 + \beta^3 \gamma^3 q^{3n+3} - \beta^3 \gamma^3 q^{2n+3} - \beta^2 \gamma^3 q^{2n+5} \big) \big)^{1/2} \Big) q^{n-3} \Big) \Big/ \big(-\gamma^2 q^8 - \gamma^2 q^7$$
$$- \gamma^3 q^5 - \gamma^3 q^4 - \beta \gamma^3 q^5 - \beta \gamma^3 q^4 + 3 \beta \gamma^3 q^{n+4} + \gamma^3 q^{3+n} - \gamma q^{n+11} - \gamma q^{n+10}$$

$-\gamma q^{n+9}-\gamma^3 q^{2n+3}-\beta\gamma^3 q^{2n+1}+\beta^3\gamma^3 q^{3n+1}+\beta^2\gamma^3 q^{3+n}-\beta^2\gamma^3 q^{2n+1}$
$+\beta\gamma^3 q^{3n+2}+\beta\gamma^3 q^{3n+1}-\beta^3\gamma^3 q^{4n+1}-\beta^2\gamma^3 q^{4n+2}-\beta^2\gamma^3 q^{4n+1}+\beta\gamma^3 q^{3n+3}$
$+2\beta^2\gamma^3 q^{3n+1}-q^{4n}\beta^3\gamma^3+q^{3n}\beta^2\gamma^3+q^{5n}\gamma^3\beta^3-q^{4n}\gamma^3\beta^2+\gamma^3 q^{n+5}+\gamma^3 q^{n+4}$
$+3\beta^2\gamma^3 q^{3n+2}-2\beta\gamma^3 q^{2n+4}+\beta^2\gamma^3 q^{n+5}+\beta\gamma^3 q^{n+6}+2\beta^2\gamma^3 q^{3n+3}-\beta\gamma^3 q^{2n+5}$
$-\beta^3\gamma^3 q^{4n+2}+\beta^3\gamma^3 q^{3n+2}-3\gamma^3\beta^2 q^{2n+3}-2\gamma^3\beta^2 q^{2n+2}-3\gamma^3\beta q^{2n+3}$
$-2\gamma^3\beta q^{2n+2}-2\gamma^3\beta^2 q^{2n+4}-\gamma^2 q^9-\gamma^3 q^6+2\gamma^3\beta q^{3+n}+\gamma^3\beta q^{n+2}+\beta^2\gamma^3 q^{n+4}$
$+2\beta\gamma^3 q^{n+5}+\gamma^3\beta^2 q^{3n+4}-q^{2n+12}-\beta\gamma^3 q^6+\beta^3\gamma^3 q^{3n+3}-\beta^3\gamma^3 q^{2n+3}$
$-\beta^2\gamma^3 q^{2n+5})$

Comparison between the bounds and the extreme zeros

```
> xnQM:= sort([solve(expand(subs({beta=0.1,gamma=0.5,q=0.9},QM
  (10,beta,gamma,x,q))),x)]):
> extzeroxnQM:=evalf[15]([min(xnQM)  ,max(xnQM)]);
```

$$extzeroxnQM := [1.04786345156205, 11.3462643915000] \quad (13.8)$$

```
> QM300:=unapply(boundQM1,[n,beta,gamma,q]):
> evalf[15](QM300(10,0.1,0.5,0.9) )
```

$$10.7793371973647 \quad (13.9)$$

```
> QM30m6:=unapply(boundQM2,[n,beta,gamma,q]):
> evalf[15](QM30m6(10,0.1,0.5,0.9) )
```

$$1.26513833534888 \quad (13.10)$$

the quantum q-Krawtchouk polynomials

```
> QQK:=(n,p,N,x,q)->add(qphihyperterm([q^(-n),x],[q^(-N)],q,p*q^
  (n+1),k),k=0..n);
```

$$QQK := (n,p,N,x,q) \rightarrow add\left(qphihyperterm\left([q^{-n},x],[q^{-N}],q,p\,q^{n+1},k\right),k=0..n\right) \quad (14.1)$$

```
> Fqqk:=(qphihyperterm([q^(-n),x],[q^(-NN)],q,p*q^(n+1),k));
```

$$Fqqk := \frac{qpochhammer(q^{-n},q,k)\ qpochhammer(x,q,k)\ (p\,q^{n+1})^k}{qpochhammer(q^{-NN},q,k)\ qpochhammer(q,q,k)} \quad (14.2)$$

For this polynomial family, $p > q^{-N}$ and $0 \le x \le N$, so we cannot do a shift on p or N. Theorem 1 is therefore not applicable and we use only Equation (2).

```
> recQQK:=subs(NN=N,qMixRec(Fqqk,q,k,S(n),3,p,0,NN,0)):
> recQQK1:=combine(denom(rhs(recQQK))*lhs(recQQK)   =collect(numer
  (rhs(recQQK)),[S(n, p, N),S(n-1, p, N),x],qsimpcomb),power)
```

$recQQK1 := (p\,q^n-q)\,(q^n-q)\,(-q^2+q^n)\,(p\,q^n-q^2)\,S(n-3,p,N)=((-q^{N+1}$ **(14.3)**
$+q^n)\,q^{2n-N}x\,p-(-q^{N+1}+q^n)\,(q^{n+N+1}p-q^{N+3}+q^{n+N+1}-q^{N+2}$
$+q^n)\,q^{1-2N})\,S(n,p,N)+(q^{4n}p^2x^2-(q^{n+N+1}p-q^{N+3}+q^{n+N+1}-q^{N+1}$
$+q^n)\,(q+1)\,q^{2n-N}x\,p+(q^{2n+2N+2}p^2-q^{n+2N+4}p+q^{2n+2N+2}p$

$$-q^{n+2N+3}p-q^{n+2N+4}+q^{2N+5}+q^{2n+2N+2}-q^{n+2N+2}p-q^{n+2N+3}+q^{2N+4}+q^{3n+N}p+q^{2n+N+1}p-q^{n+2N+2}-q^{n+3+N}+q^{2N+3}+q^{2n+N+1}-q^{n+N+2}-q^{n+N+1}+q^{2n}\big)\,q^{1-2N}\big)\,S(n-1,p,N)$$

```
> Gqqk[3,0]:=op([2,2],recQQK1)/S(n-1, p, N):
> boundqqk1:=combine((-coeff(Gqqk[3,0],x,1)-sqrt(((coeff(Gqqk[3,0],x,1)^2-4*coeff(Gqqk[3,0],x,0)*coeff(Gqqk[3,0],x,2)))))/(2*coeff(Gqqk[3,0],x,2)),power)
```

$$boundqqk1 := \frac{1}{2}\,\frac{1}{p^2}\Big(\Big(\big(q^{n+N+1}p-q^{N+3}+q^{n+N+1}-q^{N+1}+q^{n}\big)\,(q+1)\,q^{2n-N}p-\Big(\big(q^{n+N+1}p-q^{N+3}+q^{n+N+1}-q^{N+1}+q^{n}\big)^2\,(q+1)^2\,q^{4n-2N}p^2-4\,\big(q^{2n+2N+2}p^2-q^{n+2N+4}p+q^{2n+2N+2}p-q^{n+2N+3}p-q^{n+2N+4}+q^{2N+5}+q^{2n+2N+2}-q^{n+2N+2}p-q^{n+2N+3}+q^{2N+4}+q^{3n+N}p+q^{2n+N+1}p-q^{n+2N+2}-q^{n+3+N}+q^{2N+3}+q^{2n+N+1}-q^{n+N+2}-q^{n+N+1}+q^{2n}\big)\,q^{1-2N+4n}p^2\Big)^{1/2}\Big)\,q^{-4n}\Big) \quad \mathbf{(14.4)}$$

```
> boundqqk2:=combine((-coeff(Gqqk[3,0],x,1)+sqrt(((coeff(Gqqk[3,0],x,1)^2-4*coeff(Gqqk[3,0],x,0)*coeff(Gqqk[3,0],x,2)))))/(2*coeff(Gqqk[3,0],x,2)),power)
```

$$boundqqk2 := \frac{1}{2}\,\frac{1}{p^2}\Big(\Big(\big(q^{n+N+1}p-q^{N+3}+q^{n+N+1}-q^{N+1}+q^{n}\big)\,(q+1)\,q^{2n-N}p+\Big(\big(q^{n+N+1}p-q^{N+3}+q^{n+N+1}-q^{N+1}+q^{n}\big)^2\,(q+1)^2\,q^{4n-2N}p^2-4\,\big(q^{2n+2N+2}p^2-q^{n+2N+4}p+q^{2n+2N+2}p-q^{n+2N+3}p-q^{n+2N+4}+q^{2N+5}+q^{2n+2N+2}-q^{n+2N+2}p-q^{n+2N+3}+q^{2N+4}+q^{3n+N}p+q^{2n+N+1}p-q^{n+2N+2}-q^{n+3+N}+q^{2N+3}+q^{2n+N+1}-q^{n+N+2}-q^{n+N+1}+q^{2n}\big)\,q^{1-2N+4n}p^2\Big)^{1/2}\Big)\,q^{-4n}\Big) \quad \mathbf{(14.5)}$$

Comparison between the bounds and the extreme zeros

```
> hypqqk:=(n,p,N,q)->p > q^(-N) and n<=N
```

$$hypqqk := (n,p,N,q) \rightarrow q^{-N} < p \ \textbf{and}\ n \le N \quad \mathbf{(14.6)}$$

```
> hypqqk(20,50,30,0.9)
```

$$true \quad \mathbf{(14.7)}$$

```
> xnQQK:= sort([solve(expand(subs({p=50,N=30,q=0.9},qsimpcomb(QQK(20,p,N,x,q)))),x)]):
> extzeroxnQQK:=evalf[15]([min(xnQQK) ,max(xnQQK)]);
```

$$extzeroxnQQK := [1.00024027565834,\ 13.8854605013747] \quad \mathbf{(14.8)}$$

```
> QQK300:=unapply([boundqqk1,boundqqk2 ],[n,p,N,q]):
> evalf[15](QQK300(20,50,30,0.9) )
```

$$[5.05180302627350,\ 12.4799599687882] \quad \mathbf{(14.9)}$$

the q-Krawtchouk polynomials

```
> QK:=(n,p,N,x,q)->add(qphihyperterm([q^(-n),x,-p*q^n],[q^(-N),
  0],q,q,k),k=0..n);
```

$$QK := (n, p, N, x, q) \rightarrow add\left(qphihyperterm\left(\left[q^{-n}, x, -p\, q^{n}\right], \left[q^{-N}, 0\right], q, q, k\right), k = 0..n\right) \quad \textbf{(15.1)}$$

```
> Fqk:=(qphihyperterm([q^(-n),x,-p*q^n],[q^(-N),0],q,q,k));
```

The weight function

```
> rhoQK:=(p,N)->qpochhammer(q^(-N),q,x)*(-p)^(-x)/qpochhammer(q,
  q,x)
```

$$rhoQK := (p, N) \rightarrow \frac{qpochhammer\left(q^{-N}, q, x\right)\, (-p)^{-x}}{qpochhammer(q, q, x)} \quad \textbf{(15.2)}$$

```
> cQK:=qsimplify(rhoQK(p*q^s,N)/rhoQK(p,N));
```

$$cQK := \frac{1}{q^{s\,x}} \quad \textbf{(15.3)}$$

cQK is a polynomial of degree s in the variable q^{-x}

Mixed recurrence equation involving $(S(n-3, p, N)$ giving a lower bound of x_(n,n)

```
> recQK1:=subs(NN=N,qMixRec(subs(N=NN,Fqk),q,k,S(n),3,p,0,NN,0)):
> recQK11:=combine(denom(rhs(recQK1))*lhs(recQK1)  =collect(numer
  (rhs(recQK1)),[S(n, p, N),S(n-1, p, N),x],qsimpcomb),power)
```

$$recQK11 := \left(-q^{2} + q^{n}\right)\left(q^{n} - q\right)\left(q^{n+N} p + q^{2}\right)\left(q^{n+N} p + q\right)\left(q^{2n} p + q\right) p^{2}\, S(n-3, p, N) = \Big(\left(\left(q^{5} + q^{2n} p\right)\left(-q^{N+1} + q^{n}\right)\left(q^{4} + q^{2n} p\right)\left(p\, q^{n} + q\right) q^{N+1-4n} x \left(q^{2n} p + q^{3}\right) - \left(q^{n+N+3} p - q^{N+4} p + q^{2n+N} p^{2} + q^{n+N+2} p + q^{n+2} p + q^{4} - q^{2n} p + p\, q^{n+1}\right)\left(-q^{N+1} + q^{n}\right)\left(q^{4} + q^{2n} p\right)\left(p\, q^{n} + q\right) q^{3-3n}\right) S(n, p, N) + \Big(\left(q^{2n} p + q^{2}\right)\left(q^{5} + q^{2n} p\right)\left(q^{4} + q^{2n} p\right)\left(q^{2n} p + q\right) q^{2N-4n} x^{2}\left(q^{2n} p + q^{3}\right) - \left(q^{2n} p + q^{2}\right)\left(q^{n+N+3} p + q^{2n+N} p^{2} - q^{N+3} p + q^{n+N+1} p + q^{n+2} p - q^{2n} p + q^{3} + p\, q^{n}\right)\left(q^{4} + q^{2n} p\right)(q+1)\, q^{N+1-3n} x \left(q^{2n} p + q^{3}\right) + \big(q^{6} + 2\, q^{2n+N+2} p^{2} - q^{n+N+3} p^{2} + q^{n+N+4} p + q^{2n+N+1} p^{2} + q^{3n+2+N} p^{3} + q^{2N+6} p^{2} + q^{4n} p^{2} - q^{3n} p^{2} - q^{N+6} p + q^{n+5} p - q^{3n+1} p^{2} + q^{n+4} p + q^{2n+1} p^{2} + q^{3+n} p - q^{4n+N} p^{3} + q^{2n+2} p^{2} + q^{2n+3} p^{2} + q^{3n+N} p^{3} - q^{n+N+5} p^{2} + q^{2n+N+5} p^{2} + 2\, q^{2n+N+4} p^{2} - q^{3n+N+3} p^{2} - q^{2N+3+2n} p^{3} + q^{4n+2N} p^{4} - q^{3n+2} p^{2} + q^{3n+2N+3} p^{3} + q^{2n+2N+5} p^{2} - q^{n+2N+6} p^{2} + q^{3n+2N+2} p^{3} + q^{2n+2N+4} p^{2} - q^{n+2N+5} p^{2} + q^{3n+2N+1} p^{3} + q^{2N+3+2n} p^{2} - q^{n+2N+4} p^{2} + q^{3n+N+1} p^{3} + 4\, q^{2n+3+N} p^{2} - q^{n+N+4} p^{2} + q^{n+N+5} p - q^{3n+N+1} p^{2} + q^{n+N+6} p - q^{3n+2+N} p^{2} - q^{2n+3} p\big)\, q^{-2n+3}\left(q^{2n} p + q^{3}\right)\Big) S(n-1, p, N) \quad \textbf{(15.4)}$$

```
> GQK[3,0]:=collect(op([2,2],recQK11)/S(n-1, p, N)/(q^(2*n)*p+
  q^3),x,simplify):
> boundQK1:=(-coeff(GQK[3,0],x,1)+sqrt(((coeff(GQK[3,0],x,1)^2-4*
  coeff(GQK[3,0],x,0)*coeff(GQK[3,0],x,2)))))/(2*coeff(GQK[3,0],
  x,2))
```

$$boundQK1 := \frac{1}{2}\Big((q+1)\left(q^{N+7} p^{2} + q^{2N+8} p^{2} + q^{2N+4} p^{2} + q^{N+3} p^{2} + q^{N+10-3n}\right. \quad \textbf{(15.5)}$$

$$- q^{n+N+3} p^2 + 2 q^{2N+6} p^2 - q^{n+N+5} p^2 - q^{3n+N+1} p^3 + q^{n+N+4} p^2 + 2 q^{N+5} p^2$$
$$+ q^{3n+2N+1} p^4 - q^{n+2N+4} p^3 + q^{2n+N+1} p^3 + q^{n+2N+3} p^3 - q^{-n+2N+6} p^2$$
$$+ q^{-n+N+6} p + q^{n+2N+5} p^3 + q^{2n+3+N} p^3 - q^{2N+8-n} p^2 + q^{N+8-n} p$$
$$+ q^{2N+7-n} p^2 - q^{2N+10-3n} p + q^{N+9-2n} p - q^{N+7-n} p + q^{N+7-2n} p$$
$$+ q^{2N+10-2n} p + q^{2n+2N+4} p^3 + q^{2N-2n+8} p + q^{2n+2N+2} p^3)$$
$$+ \Big((q+1)^2 \big(q^{N+7} p^2 + q^{2N+8} p^2 + q^{2N+4} p^2 + q^{N+3} p^2 + q^{N+10-3n}$$
$$- q^{n+N+3} p^2 + 2 q^{2N+6} p^2 - q^{n+N+5} p^2 - q^{3n+N+1} p^3 + q^{n+N+4} p^2 + 2 q^{N+5} p^2$$
$$+ q^{3n+2N+1} p^4 - q^{n+2N+4} p^3 + q^{2n+N+1} p^3 + q^{n+2N+3} p^3 - q^{-n+2N+6} p^2$$
$$+ q^{-n+N+6} p + q^{n+2N+5} p^3 + q^{2n+3+N} p^3 - q^{2N+8-n} p^2 + q^{N+8-n} p$$
$$+ q^{2N+7-n} p^2 - q^{2N+10-3n} p + q^{N+9-2n} p - q^{N+7-n} p + q^{N+7-2n} p$$
$$+ q^{2N+10-2n} p + q^{2n+2N+4} p^3 + q^{2N-2n+8} p + q^{2n+2N+2} p^3\big)^2 - 4 \big(p^2 q^6$$
$$+ 2 q^{N+7} p^2 + q^{2N+8} p^2 + q^{N+8} p^2 + q^{N+4} p^2 - q^{2N+6} p^3 - q^{n+5} p^2 + p^2 q^4 - p q^6$$
$$+ q^{2N+6} p^2 + q^{2n+3} p^2 - q^{n+N+5} p^2 - q^{n+N+4} p^2 + 4 q^{N+6} p^2 + 2 q^{N+5} p^2$$
$$+ q^{-2n+9} + q^{n+2N+4} p^3 + q^{n+2N+5} p^3 - q^{2n+3+N} p^3 - q^{2N+8-n} p^2$$
$$+ q^{N+8-n} p - q^{2N+7-n} p^2 - q^{N+9-2n} p + q^{N+7-n} p + q^{-n+9+N} p$$
$$- q^{-n+N+6} p^2 + q^{-2n+9+2N} p^2 + q^{n+N+3} p^3 - q^{N+8-n} p^2 + q^{2N+3+2n} p^4$$
$$+ q^{n+N+5} p^3 + q^{n+N+4} p^3 - q^{N+7-n} p^2 + q^{n+2N+6} p^3 - q^{-n+2N+9} p^2$$
$$- q^{n+N+6} p^2 - q^{3+n} p^2 + q^{-n+8} p - q^{n+4} p^2 + q^{-n+7} p + q^{-n+6} p + q^5 p^2$$
$$+ q^{2N+7} p^2\big) \big(q^{2N+9} p^2 + q^{2n+2N+5} p^3 + q^{2N+10-2n} p + 2 q^{2N+6} p^2$$
$$+ q^{2n+2N+4} p^3 + q^{4n+2N} p^4 + q^{2N+5} p^2 + q^{2N+1+2n} p^3 + q^{2N+11-2n} p$$
$$+ q^{2N+7} p^2 + q^{2N-4n+12} + q^{2N-2n+8} p + q^{2n+2N+2} p^3 + q^{2N-2n+7} p$$
$$+ q^{2N+3} p^2\big)\big)^{1/2}\Big) \Big/ \big(q^{2N+9} p^2 + q^{2n+2N+5} p^3 + q^{2N+10-2n} p + 2 q^{2N+6} p^2$$
$$+ q^{2n+2N+4} p^3 + q^{4n+2N} p^4 + q^{2N+5} p^2 + q^{2N+1+2n} p^3 + q^{2N+11-2n} p$$
$$+ q^{2N+7} p^2 + q^{2N-4n+12} + q^{2N-2n+8} p + q^{2n+2N+2} p^3 + q^{2N-2n+7} p$$
$$+ q^{2N+3} p^2\big)$$

Mixed recurrence equation for $(S(n-3, p\, q^6, N))$ giving an upper bound of x_(n,1)

```
> recQK2:=subs(NN=N,qMixRec(subs(N=NN,Fqk),q,k,S(n),3,p,6,NN,0)):
> recQK21:=combine(denom(rhs(recQK2))*lhs(recQK2)   =collect(numer
  (rhs(recQK2)),[S(n, p, N),S(n-1, p, N),x],qsimpcomb),power)
```

$$recQK21 := (-q^2+q^n)(q^n-q)(1+q^{n+2}p)(1+p\,q^n)(q^{2n}p+q)(1+p\,q^{n+1})x^6p^2S(n-3,p\,q^6,N) = \Big(-(-q^2+q^n)(q^n-q)(-q^{N+1}+q^n)(-q^{N+2}+q^n)(q^{2n}p+q)(-q^{N+3}+q^n)q^{-6-3N}x^3p^2 - (-q^2+q^n)(q^n-q)(-q^{N+1}+q^n)(-q^{N+2}+q^n)(q^{N+3}p+1)(-q^{N+3}+q^n)(q^2+q+1)q^{n-8-4N}x^2p^2 + (-q^{N+1}+q^n)(q^{N+3}p+1)(q^{N+2}p+1)(q^{N+4}p+1)q^{2n-9-5N}x\,(q^{2n+N+4}p^2+q^{n+N+5}p-q^{N+6}p+q^{n+N+4}p-q^{N+5}p+q^{n+N+3}p+q^{n+4}p-q^{N+4}p-q^{2n+2}p+q^{n+N+2}p+q^{3+n}p-q^{2n+1}p+q^{n+2}p-q^{2n}p+p\,q^{n+1}+q^2) - (q^{N+5}p+1)(q^{N+1}p+1)(-q^{N+1}+q^n)(q^{N+3}p+1)(q^{N+2}p+1)(q^{N+4}p+1)q^{3n-9-6N}\Big)S(n,p,N) + \Big(\big((q^{N+3}p+1)q^{2n-6-4N}x^2(1+q^{n+N+3}p+q^{n+N+2}p+q^{n+2}p+p\,q^n+p\,q^{n+1}-q^{N+3}p+q^{n+N+1}p+2q^{2n+N+2}p^2-q^{n+N+3}p^2+q^{2n+N+1}p^2+q^{2N+6}p^2+q^{4n}p^2-q^{3n}p^2-q^{3n+1}p^2+q^{2n+1}p^2+q^{2n+2}p^2+q^{2n+3}p^2-q^{n+N+5}p^2+q^{2n+N+5}p^2+2q^{2n+N+4}p^2-q^{3n+N+3}p^2-q^{3n+2}p^2+q^{2n+2N+5}p^2-q^{n+2N+6}p^2+q^{2n+2N+4}p^2-q^{n+2N+5}p^2+q^{2N+3+2n}p^2-q^{n+2N+4}p^2+4q^{2n+3+N}p^2-q^{n+N+4}p^2-q^{3n+N+1}p^2-q^{3n+2+N}p^2-q^{2n}p+q^{3n+N+4}p^3+q^{3n+N+3}p^3+q^{3n+5+N}p^3+q^{3n+2N+4}p^3+q^{3n+2N+6}p^3+q^{3n+2N+5}p^3-q^{2n+2N+6}p^3+q^{4n+2N+6}p^4-q^{4n+N+3}p^3)(q^{n+N}p+1) - (q^{2n+3+N}p^2+q^{n+N+3}p-q^{N+3}p+q^{n+N+1}p+q^{n+2}p-q^{2n}p+p\,q^n+1)(q^{N+3}p+1)(q^{N+2}p+1)(q+1)(q^{N+4}p+1)q^{3n-8-5N}x\,(q^{n+N}p+1) + (q^{N+5}p+1)(q^{N+1}p+1)(q^{N+3}p+1)(q^{N+2}p+1)(q^{N+4}p+1)q^{4n-9-6N}(q^{n+N}p+1)\Big)S(n-1,p,N) \quad \textbf{(15.6)}$$

```
> GQK[3,6]:=collect(op([2,2],recQK21)/S(n-1, p, N)/((q^(N+3)*p+1)
  *(q^(n+N)*p+1)),x,qsimpcomb):
> boundQK2:=combine((-coeff(GQK[3,6],x,1)-sqrt(((coeff(GQK[3,6],
  x,1)^2-4*coeff(GQK[3,6],x,0)*coeff(GQK[3,6],x,2)))))/(2*coeff
  (GQK[3,6],x,2)),power)
```

$$boundQK2 := \frac{1}{2}\Big(\Big((q^{2n+3+N}p^2+q^{n+N+3}p-q^{N+3}p+q^{n+N+1}p+q^{n+2}p-q^{2n}p+p\,q^n+1)(q^{N+2}p+1)(q+1)(q^{N+4}p+1)q^{3n-8-5N} - \Big((q^{2n+3+N}p^2+q^{n+N+3}p-q^{N+3}p+q^{n+N+1}p+q^{n+2}p-q^{2n}p+p\,q^n+1)^2(q^{N+2}p+1)^2(q+1)^2(q^{N+4}p+1)^2q^{-16-10N+6n} - 4(q^{N+5}p+1)(q^{N+1}p+1)(q^{N+2}p+1)(q^{N+4}p+1)q^{-15-10N+6n}(1+q^{n+N+3}p+q^{n+N+2}p+q^{n+2}p+p\,q^n+p\,q^{n+1}-q^{N+3}p+q^{n+N+1}p+2q^{2n+N+2}p^2 \quad \textbf{(15.7)}$$

$$- q^{n+N+3} p^2 + q^{2n+N+1} p^2 + q^{2N+6} p^2 + q^{4n} p^2 - q^{3n} p^2 - q^{3n+1} p^2 + q^{2n+1} p^2$$
$$+ q^{2n+2} p^2 + q^{2n+3} p^2 - q^{n+N+5} p^2 + q^{2n+N+5} p^2 + 2 q^{2n+N+4} p^2$$
$$- q^{3n+N+3} p^2 - q^{3n+2} p^2 + q^{2n+2N+5} p^2 - q^{n+2N+6} p^2 + q^{2n+2N+4} p^2$$
$$- q^{n+2N+5} p^2 + q^{2N+3+2n} p^2 - q^{n+2N+4} p^2 + 4 q^{2n+3+N} p^2 - q^{n+N+4} p^2$$
$$- q^{3n+N+1} p^2 - q^{3n+2+N} p^2 - q^{2n} p + q^{3n+N+4} p^3 + q^{3n+N+3} p^3 + q^{3n+5+N} p^3$$
$$+ q^{3n+2N+4} p^3 + q^{3n+2N+6} p^3 + q^{3n+2N+5} p^3 - q^{2n+2N+6} p^3 + q^{4n+2N+6} p^4$$
$$- q^{4n+N+3} p^3))^{1/2}) q^{6+4N-2n}) \Big/ (1 + q^{n+N+3} p + q^{n+N+2} p + q^{n+2} p + p q^n$$
$$+ p q^{n+1} - q^{N+3} p + q^{n+N+1} p + 2 q^{2n+N+2} p^2 - q^{n+N+3} p^2 + q^{2n+N+1} p^2$$
$$+ q^{2N+6} p^2 + q^{4n} p^2 - q^{3n} p^2 - q^{3n+1} p^2 + q^{2n+1} p^2 + q^{2n+2} p^2 + q^{2n+3} p^2$$
$$- q^{n+N+5} p^2 + q^{2n+N+5} p^2 + 2 q^{2n+N+4} p^2 - q^{3n+N+3} p^2 - q^{3n+2} p^2$$
$$+ q^{2n+2N+5} p^2 - q^{n+2N+6} p^2 + q^{2n+2N+4} p^2 - q^{n+2N+5} p^2 + q^{2N+3+2n} p^2$$
$$- q^{n+2N+4} p^2 + 4 q^{2n+3+N} p^2 - q^{n+N+4} p^2 - q^{3n+N+1} p^2 - q^{3n+2+N} p^2$$
$$- q^{2n} p + q^{3n+N+4} p^3 + q^{3n+N+3} p^3 + q^{3n+5+N} p^3 + q^{3n+2N+4} p^3$$
$$+ q^{3n+2N+6} p^3 + q^{3n+2N+5} p^3 - q^{2n+2N+6} p^3 + q^{4n+2N+6} p^4 - q^{4n+N+3} p^3)$$

Comparison between the bounds and the extreme zeros

```
> xnQK:= sort([solve(expand(subs({p=1,N=20,q=0.85},QK(20,p,N,x,q)
  )),x)]):
> extzeroxnQK:=evalf[15]([min(xnQK) ,max(xnQK)]);
```

$$extzeroxnQK := [1.14087829494922, 25.8001057293371] \quad (15.8)$$

```
> QK300:=unapply(boundQK1,[n,p,N,q]):
> evalf[15](QK300(20,1,20,0.85))
```

$$1.56109321132226 \quad (15.9)$$

```
> QK360:=unapply(boundQK2,[n,p,N,q]):
> evalf[15](QK360(20,1,20,0.85))
```

$$1.14539498081829 \quad (15.10)$$

the affine q-Krawtchouk polynomials

```
> AQK:=(n,p,N,x,q)->add(qphihyperterm([q^(-n),x,0],[p*q,q^(-N)],
  q,q,k),k=0..n);
```

$$AQK := (n, p, N, x, q) \rightarrow add(qphihyperterm([q^{-n}, x, 0], [p\, q, q^{-N}], q, q, k), k = 0..n) \quad (16.1)$$

```
> Faqk:=(qphihyperterm([q^(-n),x,0],[p*q,q^(-NN)],q,q,k));
```

$$Faqk := \frac{qpochhammer(q^{-n}, q, k)\; qpochhammer(x, q, k)\; q^k}{qpochhammer(p\, q, q, k)\; qpochhammer(q^{-NN}, q, k)\; qpochhammer(q, q, k)} \quad (16.2)$$

The weight function

```
> rhoAQK:=(p,N)->qpochhammer(p*q,q,x)*qpochhammer(q,q,N)*(p*q)^(-x)/qpochhammer(q,q,x)/qpochhammer(q,q,N-x)
```

$$rhoAQK := (p, N) \rightarrow \frac{qpochhammer(p\,q, q, x)\ qpochhammer(q, q, N)\ (p\,q)^{-x}}{qpochhammer(q, q, x)\ qpochhammer(q, q, N-x)} \quad (16.3)$$

```
> cAQK:=qsimplify(rhoAQK(p*q^s,N)/rhoAQK(p,N))
```

$$cAQK := \frac{qpochhammer(q\,q^{x} p, q, s)}{qpochhammer(p\,q, q, s)\ q^{s\,x}} \quad (16.4)$$

cAQK is a polynomial of degree s of the variable q^{-x}

Mixed recurrence equation involving $S(n-3, p, N)$ giving a lower bound of x_(n,n)

```
> recAQK1:=subs(NN=N,qMixRec(Faqk,q,k,S(n),3,p,0,NN,0)):
> recAQK11:=combine(denom(rhs(recAQK1))*lhs(recAQK1)   =collect(numer(rhs(recAQK1)),[S(n, p, N),S(n-1, p, N),x],qsimpcomb),power)
```

$$recAQK11 := (q^{n}-q)\,(-q^{2}+q^{n})\,p^{2}\,S(n-3, p, N) = \big(-(p\,q^{n}-1)\,(-q^{N+1} + q^{n})\,q^{5+N-2n}\,x - (p\,q^{n}-1)\,(-q^{N+1}+q^{n})\,(-q^{N+3}p + p\,q^{n+1} - q^{2}p + p\,q^{n} - q^{2})\,q^{-n+1}\big)\,S(n, p, N) + \big(q^{6-2n+2N}x^{2} + (-q^{N+3}p + q^{n+2}p - q^{2}p + p\,q^{n} - q^{2})\,(q+1)\,q^{N+2-n}x + (q^{n+2N+5}p^{2} - q^{2n+N+4}p^{2} - q^{2n+N+3}p^{2} + q^{n+N+4}p^{2} + q^{3n+2}p^{2} - q^{2n+N+2}p^{2} - q^{2n+3}p^{2} + q^{n+N+4}p + q^{5+N}p + q^{3n+1}p^{2} - q^{2n+2}p^{2} - q^{2n+3}p + q^{n+3}p^{2} + q^{3n}p^{2} - q^{2n+1}p^{2} - q^{2n+2}p + q^{n+3}p - q^{2n+1}p + q^{n+3})\,q^{-n}\big)\,S(n-1, p, N) \quad (16.5)$$

```
> GAQK[3,0]:=collect(op([2,2],recAQK11)/S(n-1, p, N),x,simplify):
> boundAQK1:=(-coeff(GAQK[3,0],x,1)+sqrt(((coeff(GAQK[3,0],x,1)^2-4*coeff(GAQK[3,0],x,0)*coeff(GAQK[3,0],x,2)))))/(2*coeff(GAQK[3,0],x,2))
```

$$boundAQK1 := \frac{1}{2}\,\frac{1}{q^{6-2n+2N}}\Big(-(q+1)\,(-q^{5-n+2N}p + q^{N+4}p - q^{4-n+N}p + q^{N+2}p - q^{4-n+N}) + \big((q+1)^{2}\,(-q^{5-n+2N}p + q^{N+4}p - q^{4-n+N}p + q^{N+2}p - q^{4-n+N})^{2} - 4\,(q^{2n}p^{2} + q^{N+4}p^{2} + q^{2N+5}p^{2} - q^{n+3+N}p^{2} - q^{n+N+2}p^{2} - q^{n+N+4}p^{2} + q^{3}p^{2} - q^{n+1}p^{2} + q^{2n+1}p^{2} - q^{n+2}p^{2} + q^{2n+2}p^{2} - q^{n+3}p^{2} + q^{-n+N+5}p + q^{N+4}p + q^{3}p - p\,q^{n+1} - q^{n+2}p - q^{n+3}p + q^{3})\,q^{6-2n+2N}\big)^{1/2}\Big) \quad (16.6)$$

Mixed recurrence equation involving $(S(n-3, pq^{6}, N)$ giving an upper bound of x_(n,1)

```
> recAQK2:=subs(NN=N,qMixRec(Faqk,q,k,S(n),3,p,6,NN,0)):
> recAQK21:=combine(denom(rhs(recAQK2))*lhs(recAQK2)   =collect(numer(rhs(recAQK2)),[S(n, p, N),S(n-1, p, N),x],qsimpcomb),power):
> GAQK[3,6]:=collect(op([2,2],recAQK21)/S(n-1, p, N)/((p*q^6-1)*(p*q^5-1)*(p*q^2-1)*(p*q^3-1)*(p*q-1)*(p*q^4-1)),x,simplify):
> boundAQK2:=(-coeff(GAQK[3,6],x,1)+sqrt(((coeff(GAQK[3,6],x,1)^2-4*coeff(GAQK[3,6],x,0)*coeff(GAQK[3,6],x,2)))))/(2*coeff(GAQK
```

```
[3,6],x,2)):
```

Comparison between the bounds and the extreme zeros

```
> hypAQK:=(n,p,N,q)-> n<=N and 0<p*q and p*q<1
```

$$hypAQK := (n, p, N, q) \rightarrow n \leq N \text{ and } 0 < q\,p \text{ and } q\,p < 1 \quad (16.7)$$

```
> hypAQK(25,1.15,30,0.85)
```

$$true \quad (16.8)$$

```
> xnAQK:= sort([solve(expand(subs({p=1.15,N=30,q=0.85},AQK(20,p,
  N,x,q))),x)]):
> extzeroxnAQK:=evalf[15]([min(xnAQK) ,max(xnAQK)]);
```

$$extzeroxnAQK := [1.01196473131558, 131.048552582213] \quad (16.9)$$

```
> AQK300:=unapply(boundAQK1,[n,p,N,q]):
> evalf[15](AQK300(20,1.15,30,0.85))
```

$$19.5461986932290 \quad (16.10)$$

```
> AQK360:=unapply( boundAQK2 ,[n,p,N,q]):
> evalf[15](AQK360(20,1.15,30,0.85))
```

$$1.01371523462766 \quad (16.11)$$

the little q-Laguerre/Wall polynomials

```
> LQLW:=(n,alpha,x,q)->add(qphihyperterm([q^(-n),0],[alpha*q],q,
  q*x,k),k=0..n);
```

$$LQLW := (n, \alpha, x, q) \rightarrow add\left(qphihyperterm\left([q^{-n}, 0], [q\,\alpha], q, q\,x, k\right), k = 0..n\right) \quad (17.1)$$

```
> Flqlw:=(qphihyperterm([q^(-n),0],[alpha*q],q,q*x,k));
```

The weight function

```
> rhoLQLW:=alpha->(alpha*q)^k/qpochhammer(q,q,k)
```

$$rhoLQLW := \alpha \rightarrow \frac{(\alpha\,q)^k}{qpochhammer(q, q, k)} \quad (17.2)$$

```
> cLQLW:=qsimplify(rhoLQLW(alpha*q^s)/rhoLQLW(alpha))
```

$$cLQLW := q^{s\,k} \quad (17.3)$$

cLQLW is a polynomial of degree s of the variable q^k

Mixed recurrence equation involving $S(n-3, \alpha\,q^6, 0))$ giving an upper bound of x_(n,1)

```
> recLQLW1:=qMixRec(Flqlw,q,k,S(n),3,alpha,6,0,0):
> recLQLW11:=combine(denom(rhs(recLQLW1))*lhs(recLQLW1)  =collect
  (numer(rhs(recLQLW1)),[S(n, alpha, 0),S(n-1, alpha, 0),x],
  qsimpcomb),power)
```

$$recLQLW11 := (-q^2 + q^n)\,(\alpha\,q^{n+2} - 1)\,(\alpha\,q^{3+n} - 1)\,(q^n - q)\,\alpha^2\,(\alpha\,q^{n+1} - 1)\,x^6\,S(n-3, q^6\,\alpha, 0) = \Big((-q^2 + q^n)\,(q^n - q)\,\alpha^2\,(\alpha\,q - 1)\,(\alpha\,q^6 - 1)\,(\alpha\,q^3 - 1)\,(\alpha\,q^4 - 1)\,(\alpha\,q^5 - 1)\,(\alpha\,q^2 - 1)\,q^{3n-6}\,x^3 - (-q^2 + q^n)\,(q^n - q)\,\alpha^2\,(\alpha\,q - 1)\,(\alpha\,q^6 - 1)\,(\alpha\,q^3 - 1)^2\,(\alpha\,q^4 - 1)\,(\alpha\,q^5 - 1)\,(\alpha\,q^2 - 1)\,(q^2 + q + 1)\,q^{4n-9}\,x^2 - (\alpha\,q - 1)\,(\alpha\,q^6 - 1)\,(\alpha\,q^3 - 1)^2\,(\alpha\,q^4 - 1)^2\,(\alpha\,q^5 - 1)\,(\alpha\,q^2 - 1)^2\,q^{5n-13}\,x\,(\alpha\,q^{3+n} - q^4\,\alpha + \alpha\,q^{n+2} - q^3\,\alpha + \alpha\,q^{n+1} - q^2\,\alpha + \alpha\,q^n \quad (17.4)$$

$-1)-(\alpha q-1)^2(\alpha q^6-1)(\alpha q^3-1)^2(\alpha q^4-1)^2(\alpha q^5-1)^2(\alpha q^2-1)^2 q^{6n-15}\Big)S(n,\alpha,0)+\Big((\alpha q-1)(\alpha q^6-1)(\alpha q^3-1)^2(\alpha q^4-1)(\alpha^2 q^{2n+5}-\alpha^2 q^{n+6}+\alpha^2 q^{2n+4}-\alpha^2 q^{n+5}+q^6\alpha^2+\alpha^2 q^{2n+3}-\alpha^2 q^{n+4}-\alpha q^{3+n}-\alpha q^{n+2}+q^3\alpha-\alpha q^{n+1}+1)(\alpha q^5-1)(\alpha q^2-1)q^{4n-12}x^2+(\alpha q-1)(\alpha q^6-1)(\alpha q^3-1)^2(\alpha q^4-1)^2(\alpha q^5-1)(q+1)(\alpha q^{3+n}-q^3\alpha+\alpha q^{n+1}-1)(\alpha q^2-1)^2 q^{5n-14}x+(\alpha q-1)^2(\alpha q^6-1)(\alpha q^3-1)^2(\alpha q^4-1)^2(\alpha q^5-1)^2(\alpha q^2-1)^2 q^{6n-15}\Big)S(n-1,\alpha,0)$

```
> GLQLW[3,6]:=collect(op([2,2],recLQLW11)/S(n-1, alpha, 0)/(
  (alpha*q^6-1)*(alpha*q^5-1)),x,simplify):
> boundLQLW1:=(-coeff(GLQLW[3,6],x,1)+sqrt(((coeff(GLQLW[3,6],x,
  1)^2-4*coeff(GLQLW[3,6],x,0)*coeff(GLQLW[3,6],x,2)))))/(2*coeff
  (GLQLW[3,6],x,2))
```

$$boundLQLW1 := \frac{1}{2}\Big(-(\alpha q-1)(\alpha q^3-1)^2(\alpha q^4-1)^2(q+1)(\alpha q^2-1)^2(q^{6n-11}\alpha-q^{5n-11}\alpha+q^{6n-13}\alpha-q^{5n-14})+\big((\alpha q-1)^2(\alpha q^3-1)^4(\alpha q^4-1)^4(q+1)^2(\alpha q^2-1)^4(q^{6n-11}\alpha-q^{5n-11}\alpha+q^{6n-13}\alpha-q^{5n-14})^2-4(\alpha q-1)^3(\alpha q^3-1)^4(\alpha q^4-1)^3(\alpha q^5-1)(\alpha q^2-1)^3 q^{6n-15}(q^{6n-7}\alpha^2-q^{5n-6}\alpha^2+q^{6n-8}\alpha^2-q^{5n-7}\alpha^2+q^{4n-6}\alpha^2+q^{6n-9}\alpha^2-q^{5n-8}\alpha^2-q^{5n-9}\alpha-q^{5n-10}\alpha+q^{4n-9}\alpha-q^{5n-11}\alpha+q^{4n-12})\big)^{1/2}\Big)\Big/\Big((\alpha q-1)(\alpha q^3-1)^2(\alpha q^4-1)(\alpha q^2-1)(q^{6n-7}\alpha^2-q^{5n-6}\alpha^2+q^{6n-8}\alpha^2-q^{5n-7}\alpha^2+q^{4n-6}\alpha^2+q^{6n-9}\alpha^2-q^{5n-8}\alpha^2-q^{5n-9}\alpha-q^{5n-10}\alpha+q^{4n-9}\alpha-q^{5n-11}\alpha+q^{4n-12})\Big) \quad (17.5)$$

Comparison between the bounds and the extreme zeros

```
> xnLQLW:= sort([solve(expand(subs({alpha=0.1,q=0.9},LQLW(20,
  alpha,x,q))),x)]):
> extzeroxnLQLW:=evalf[15]([min(xnLQLW) ,max(xnLQLW)]);
```

$extzeroxnLQLW := [0.0904898209404331, 1.00000000000000]$ (17.6)

```
> LQLW360:=unapply(boundLQLW1,[n,alpha,q]):
> evalf[15](LQLW360(20,0.1,0.9))
```

0.0926461669560155 (17.7)

the q-Laguerre polynomials

```
> QL:=(n,alpha,x,q)->qpochhammer(q^(alpha+1),q,n)/qpochhammer(q,
  q,n)*add(qphihyperterm([q^(-n)],[q^(alpha+1)],q,-q^(n+alpha+1)*
```

```
x,k) ,k=0..n);
```

$$QL := (n, \alpha, x, q) \to \frac{1}{qpochhammer(q, q, n)} \left(qpochhammer(q^{\alpha+1}, q, n)\, add\left(qphihyperterm\left([q^{-n}], [q^{\alpha+1}], q, -q^{n+\alpha+1} x, k\right), k=0..n\right)\right) \quad \textbf{(18.1)}$$

```
> Fql:=qpochhammer(q^(alpha+1),q,n)/qpochhammer(q,q,n)*
  (qphihyperterm([q^(-n)],[q^(alpha+1)],q,-q^(n+alpha+1)*x,k));
```

The weight function

```
> rhoQL:=alpha->x^alpha/qpochhammer(-x,q,infinity)
```

$$rhoQL := \alpha \to \frac{x^{\alpha}}{qpochhammer(-x, q, \infty)} \quad \textbf{(18.2)}$$

```
> cQL:=qsimplify(rhoQL(alpha+s)/rhoQL(alpha))
```

$$cQL := x^{s} \quad \textbf{(18.3)}$$

Mixed recurrence equation involving $S(n-3, q^{\alpha}, 0)$ giving a lower bound of x_(n,n)

```
> recQL1:=subs(A=q^alpha,qMixRec(subs({q^alpha=A,q^(alpha*k)=A^k}
  ,qsimpcomb(Fql)),q,k,S(n),3,A,0,0,0)):
> recQL11:=combine(denom(rhs(recQL1))*lhs(recQL1)  =collect(numer
  (rhs(recQL1)),[S(n, q^alpha, 0),S(n-1, q^alpha, 0),x],
  qsimpcomb),power)
```

$$recQL11 := q^{3} \left(q^{\alpha+n} - q\right) \left(q^{\alpha+n} - q^{2}\right) S\left(n-3, q^{\alpha}, 0\right) = \Big(-\left(q^{n}-1\right) q^{\alpha+2n+1} x - q^{3} \left(q^{n}-1\right) \left(q^{\alpha+n} - q^{2} + q^{n} - q\right) \Big) S\left(n, q^{\alpha}, 0\right) + \Big(q^{4n+2\alpha} x^{2} + \left(q^{\alpha+n} - q^{2} + q^{n} - 1\right) (q+1)\, q^{\alpha+2n+1} x + q^{3} \left(q^{2n+2\alpha} - q^{\alpha+n+2} + q^{\alpha+2n} - q^{n+\alpha+1} - q^{n+2} + q^{3} + q^{2n} - q^{\alpha+n} - q^{n+1} + q^{2} - q^{n} + q\right) \Big) S\left(n-1, q^{\alpha}, 0\right) \quad \textbf{(18.4)}$$

```
> GQL[3,0]:=op([2,2],recQL11)/S(n-1, q^alpha, 0):
> boundQL1:=(-coeff(GQL[3,0],x,1)+sqrt(((coeff(GQL[3,0],x,1)^2-4*
  coeff(GQL[3,0],x,0)*coeff(GQL[3,0],x,2)))))/(2*coeff(GQL[3,0],
  x,2))
```

$$boundQL1 := \frac{1}{2} \frac{1}{q^{4n+2\alpha}} \Big(-\left(q^{\alpha+n} - q^{2} + q^{n} - 1\right) (q+1)\, q^{\alpha+2n+1} + \Big(\left(q^{\alpha+n} - q^{2} + q^{n} - 1\right)^{2} (q+1)^{2} \left(q^{\alpha+2n+1}\right)^{2} - 4 q^{3} \left(q^{2n+2\alpha} - q^{\alpha+n+2} + q^{\alpha+2n} - q^{n+\alpha+1} - q^{n+2} + q^{3} + q^{2n} - q^{\alpha+n} - q^{n+1} + q^{2} - q^{n} + q\right) q^{4n+2\alpha} \Big)^{1/2} \Big) \quad \textbf{(18.5)}$$

Mixed recurrence equation involving $S(n-3, q^{\alpha+6}, 0)$ giving an upper bound of x_(n,1)

```
> recQL2:=subs(A=q^alpha,qMixRec(subs({q^alpha=A,q^(alpha*k)=A^k}
  ,qsimpcomb(Fql)),q,k,S(n),3,A,6,0,0)):
> recQL21:=combine(denom(rhs(recQL2))*lhs(recQL2)  =collect(numer
  (rhs(recQL2)),[S(n, q^alpha, 0),S(n-1, q^alpha, 0),x],
  qsimpcomb),power)
```

$$recQL21 := x^{6} q^{6\alpha} S\left(n-3, q^{\alpha+6}, 0\right) = \Big(\left(-q^{2} + q^{n}\right) \left(q^{n} - q\right) \left(q^{n} - 1\right) q^{3\alpha-12} x^{3} - \Big(-q^{2} + q^{n}\right) \left(q^{n} - q\right) \left(q^{n} - 1\right) \left(q^{\alpha+3} - 1\right) \left(q^{2} + q + 1\right) q^{2\alpha-14-n} x^{2} - \left(q^{\alpha+2} - 1\right) \left(q^{n} - 1\right) \left(q^{\alpha+3} - 1\right) \left(q^{\alpha+4} - 1\right) q^{\alpha-15-n} x \left(q^{\alpha+4+n} - q^{4} + q^{n+2} - q^{3}\right.$$

$$+q^{n+1}-q^2+q^n-q\big)-\left(q^{\alpha+2}-1\right)\left(q^{\alpha+5}-1\right)\left(q^n-1\right)\left(q^{\alpha+3}-1\right)\left(q^{\alpha+1}-1\right)\left(q^{\alpha+4}-1\right)q^{-15-n}\Big)S(n,q^{\alpha},0)+\Big(\left(q^{\alpha+n}-1\right)\left(q^{\alpha+3}-1\right)\big(q^{2n+6+2\alpha}-q^{\alpha+5+n}+q^{2n+3+\alpha}-q^{\alpha+4+n}-q^{\alpha+3+n}-q^{n+2}+q^3+q^{2n}-q^{n+1}+q^2-q^n+q\big)q^{2\alpha-12-n}x^2+\left(q^{\alpha+3+n}-q^2+q^n-1\right)\left(q^{\alpha+2}-1\right)\left(q^{\alpha+n}-1\right)\left(q^{\alpha+3}-1\right)\left(q^{\alpha+4}-1\right)(q+1)q^{\alpha-14-n}x+\left(q^{\alpha+2}-1\right)\left(q^{\alpha+5}-1\right)\left(q^{\alpha+n}-1\right)\left(q^{\alpha+3}-1\right)\left(q^{\alpha+1}-1\right)\left(q^{\alpha+4}-1\right)q^{-15-n}\Big)S(n-1,q^{\alpha},0)$$

```
> GQL[3,6]:=collect(op([2,2],recQL21)/S(n-1, q^alpha, 0)/((q^(alpha+n)-1)*(q^(alpha+3)-1)),x,qsimpcomb):
> boundQL2:=combine((-coeff(GQL[3,6],x,1)-sqrt(((coeff(GQL[3,6],x,1)^2-4*coeff(GQL[3,6],x,0)*coeff(GQL[3,6],x,2)))))/(2*coeff(GQL[3,6],x,2)),power)
```

$$boundQL2 := \frac{1}{2}\Bigg(\Bigg(-\left(q^{\alpha+3+n}-q^2+q^n-1\right)\left(q^{\alpha+2}-1\right)\left(q^{\alpha+4}-1\right)(q+1)q^{\alpha-14-n}-\Big(\left(q^{\alpha+3+n}-q^2+q^n-1\right)^2\left(q^{\alpha+2}-1\right)^2\left(q^{\alpha+4}-1\right)^2(q+1)^2q^{2\alpha-28-2n}-4\left(q^{\alpha+2}-1\right)\left(q^{\alpha+5}-1\right)\left(q^{\alpha+1}-1\right)\left(q^{\alpha+4}-1\right)\left(q^{2n+6+2\alpha}-q^{\alpha+5+n}+q^{2n+3+\alpha}-q^{\alpha+4+n}-q^{\alpha+3+n}-q^{n+2}+q^3+q^{2n}-q^{n+1}+q^2-q^n+q\right)q^{-27-2n+2\alpha}\Big)^{1/2}\Bigg)q^{-2\alpha+12+n}\Bigg)\Big/\left(q^{2n+6+2\alpha}-q^{\alpha+5+n}+q^{2n+3+\alpha}-q^{\alpha+4+n}-q^{\alpha+3+n}-q^{n+2}+q^3+q^{2n}-q^{n+1}+q^2-q^n+q\right) \quad \textbf{(18.7)}$$

Comparison between the bounds and the extreme zeros

```
> xnQL:= sort([solve(expand(subs({alpha=-0.95,q=0.9},QL(40,alpha,x,q))),x)]):
> extzeroQL:=evalf[15]([min(xnQL) ,max(xnQL)]);
```

$$extzeroQL := [0.000547997704214058, 10086.1904727894] \quad \textbf{(18.8)}$$

```
> QL300   :=unapply(boundQL1  ,[n,alpha,q]):
> evalf[15](  QL300(40,-0.95,0.9))
```

$$9496.15922563230 \quad \textbf{(18.9)}$$

```
> QL360   :=unapply(boundQL2  ,[n,alpha,q]):
> evalf[15](  QL360(40,-0.95,0.9))
```

$$0.000547997705314085 \quad \textbf{(18.10)}$$

the alternative q-Charlier polynomials

```
> AQC:=(n,alpha,x,q)->add(qphihyperterm([q^(-n),-alpha*q^(n)],[0],q,q*x,k),k=0..n);
```

$$AQC := (n,\alpha,x,q)\rightarrow add\left(qphihyperterm\left(\left[q^{-n},-\alpha q^n\right],[0],q,q\,x,k\right),k=0..n\right) \quad \textbf{(19.1)}$$

```
> Faqc:=(qphihyperterm([q^(-n),-alpha*q^(n)],[0],q,q*x,k));
```

$$Faqc := \frac{qpochhammer(q^{-n}, q, k)\ qpochhammer(-\alpha q^{n}, q, k)\ (q x)^{k}}{qpochhammer(q, q, k)} \quad \textbf{(19.2)}$$

The weight function

```
> rhoAQC:=alpha->alpha^k*q^binomial(k+1,2)/qpochhammer(q,q,k)
```

$$rhoAQC := \alpha \to \frac{\alpha^{k}\ q^{\mathrm{binomial}(k+1,\ 2)}}{qpochhammer(q, q, k)} \quad \textbf{(19.3)}$$

```
> cAQC:=qsimplify(rhoAQC(alpha*q^s)/rhoAQC(alpha))
```

$$cAQC := q^{s k} \quad \textbf{(19.4)}$$

cAQC is a polynomial of degree s of the variable q^k

Mixed recurrence equation involving $S(n-2, \alpha q^{4}, 0)$ giving an upper bound of x_(n,1)

```
> recAQC1:=qMixRec(Faqc,q,k,S(n),2,alpha,4,0,0):
> recAQC11:=combine(denom(rhs(recAQC1))*lhs(recAQC1)  =collect
  (numer(rhs(recAQC1)),[S(n, alpha, 0),S(n-1, alpha, 0),x],
  qsimpcomb),power)
```

$$recAQC11 := \alpha\,(1+\alpha q^{n})\,(q^{n}-q)\,(1+\alpha q^{n+1})\,(\alpha q^{2n}+q)\,x^{4}\,S(n-2, \alpha q^{4}, 0) = \left(\alpha\,(q^{n}-q)\,(\alpha q^{2n}+q)\,q^{2n-3}x^{2} + (q+1)\,\alpha\,(q^{n}-q)\,q^{3n-4}x + q^{3n-4}\right) S(n, \alpha, 0) + \left(-(\alpha q^{2n}-\alpha q^{n+1}-\alpha q^{n}-1)\,q^{2n-3}x - q^{3n-4}\right) S(n-1, \alpha, 0) \quad \textbf{(19.5)}$$

Bound (13) of the manuscript

```
> bound10:=combine(solve(op([2,2],recAQC11),x),power)
```

$$bound10 := \frac{q^{n-1}}{\alpha q^{n}-\alpha q^{2n}+\alpha q^{n+1}+1} \quad \textbf{(19.6)}$$

Mixed recurrence equation involving $S(n-3, \alpha q^{6}, 0)$ giving an upper bound of x_(n,1)

```
> recAQC2:=qMixRec(Faqc,q,k,S(n),3,alpha,6,0,0):
> recAQC21:=combine(denom(rhs(recAQC2))*lhs(recAQC2)  =collect
  (numer(rhs(recAQC2)),[S(n, alpha, 0),S(n-1, alpha, 0),x],
  qsimpcomb),power)
```

$$recAQC21 := \alpha^{2}\,(1+\alpha q^{2+n})\,(1+\alpha q^{n})\,(-q^{2}+q^{n})\,(q^{n}-q)\,(1+\alpha q^{n+1})\,(\alpha q^{2n}+q)\,x^{6}\,S(n-3, q^{6}\alpha, 0) = \left(-\alpha^{2}\,(-q^{2}+q^{n})\,(q^{n}-q)\,(\alpha q^{2n}+q)\,q^{3n-6}x^{3} - \alpha^{2}\,(-q^{2}+q^{n})\,(q^{n}-q)\,q^{4n-8}x^{2}\,(q^{2}+q+1) - (-\alpha q^{n+4}+\alpha q^{2n+2}-\alpha q^{n+3}+\alpha q^{2n+1}-\alpha q^{2+n}+\alpha q^{2n}-\alpha q^{n+1}-q^{2})\,q^{3n-9}x - q^{4n-9}\right) S(n, \alpha, 0) + \left((-\alpha^{2} q^{3n+2}+\alpha^{2} q^{2n+3}+\alpha^{2} q^{4n}-\alpha^{2} q^{3n+1}+\alpha^{2} q^{2n+2}-q^{3n}\alpha^{2}+\alpha^{2} q^{2n+1}+\alpha q^{2+n}-\alpha q^{2n}+\alpha q^{n+1}+\alpha q^{n}+1)\,q^{2n-6}x^{2} + (q+1)\,(-\alpha q^{2+n}+\alpha q^{2n}-\alpha q^{n}-1)\,q^{3n-8}x + q^{4n-9}\right) S(n-1, \alpha, 0) \quad \textbf{(19.7)}$$

```
> GAQC[3,6]:=op([2,2],recAQC21)/S(n-1, alpha, 0)
```

$$GAQC_{3,6} := (-\alpha^{2} q^{3n+2}+\alpha^{2} q^{2n+3}+\alpha^{2} q^{4n}-\alpha^{2} q^{3n+1}+\alpha^{2} q^{2n+2}-q^{3n}\alpha^{2}+\alpha^{2} q^{2n+1}+\alpha q^{2+n}-\alpha q^{2n}+\alpha q^{n+1}+\alpha q^{n}+1)\,q^{2n-6}x^{2} + (q+1)\,(-\alpha q^{2+n}+\alpha q^{2n}-\alpha q^{n}-1)\,q^{3n-8}x + q^{4n-9} \quad \textbf{(19.8)}$$

The upper bound (14) of the manuscript for x_{n,1}

```
> boundLQJC2:=combine((-coeff(GAQC[3,6],x,1)-sqrt(((coeff(GAQC[3,
  6],x,1)^2-4*coeff(GAQC[3,6],x,0)*coeff(GAQC[3,6],x,2)))))/(2*
  coeff(GAQC[3,6],x,2)),power)
```

$$boundLQJC2 := \frac{1}{2}\left(\left(-(q+1)\left(-\alpha q^{2+n}+\alpha q^{2n}-\alpha q^{n}-1\right)q^{3n-8} - \left((q+1)^2\left(-\alpha q^{2+n}+\alpha q^{2n}-\alpha q^{n}-1\right)^2 q^{6n-16} - 4\left(-\alpha^2 q^{3n+2} + \alpha^2 q^{2n+3} + \alpha^2 q^{4n} - \alpha^2 q^{3n+1} + \alpha^2 q^{2n+2} - q^{3n}\alpha^2 + \alpha^2 q^{2n+1} + \alpha q^{2+n} - \alpha q^{2n} + \alpha q^{n+1} + \alpha q^{n} + 1\right) q^{6n-15}\right)^{1/2}\right) q^{-2n+6}\right) \Big/ \left(-\alpha^2 q^{3n+2} + \alpha^2 q^{2n+3} + \alpha^2 q^{4n} - \alpha^2 q^{3n+1} + \alpha^2 q^{2n+2} - q^{3n}\alpha^2 + \alpha^2 q^{2n+1} + \alpha q^{2+n} - \alpha q^{2n} + \alpha q^{n+1} + \alpha q^{n} + 1\right) \quad \textbf{(19.9)}$$

Comparison between the bounds and the extreme zeros

```
> xnAQC:= sort([solve(expand(subs({alpha=100,q=0.8},(AQC(70,
  alpha,x,q)))),x)]):
> extzeroxnAQC:=evalf[15]([min(xnAQC) ,max(xnAQC)]);
```

$$extzeroxnAQC := [2.05671150501128\ 10^{-7}, 1.00000000000000] \quad \textbf{(19.10)}$$

```
> AQC240:=unapply(bound10 ,[n,alpha,q]):
> evalf[15](AQC240(70,100,0.8))
```

$$2.05681977554343\ 10^{-7} \quad \textbf{(19.11)}$$

```
> AQC360:=unapply(boundLQJC2 ,[n,alpha,q]):
> evalf[15](AQC360(70,100,0.8))
```

$$2.05671151311302\ 10^{-7} \quad \textbf{(19.12)}$$

the q-Charlier polynomials

```
> QC:=(n,alpha,x,q)->add(qphihyperterm([q^(-n),x],[0],q,-q^(n+1)
  /alpha,k),k=0..n);
```

$$QC := (n, \alpha, x, q) \rightarrow add\left(qphihyperterm\left([q^{-n}, x], [0], q, -\frac{q^{n+1}}{\alpha}, k\right), k=0..n\right) \quad \textbf{(20.1)}$$

```
> Fqc:=(qphihyperterm([q^(-n),x],[0],q,-q^(n+1)/alpha,k));
```

The weight function

```
> rhoQC:=alpha->alpha^x*q^binomial(x,2)/qpochhammer(q,q,x)
```

$$rhoQC := \alpha \rightarrow \frac{\alpha^x q^{\mathrm{binomial}(x,2)}}{qpochhammer(q,q,x)} \quad \textbf{(20.2)}$$

```
> cQC:=qsimplify(rhoQC(alpha*q^s)/rhoQC(alpha))
```

$$cQC := q^{sx} \quad \textbf{(20.3)}$$

cQC is a ploynomial of degree s of the variable q^x.

Mixed recurrence equation involving $S(n-3, \alpha, 0)$ giving a lower bound of x_(n,n)

```
> recQC1:=qMixRec(Fqc,q,k,S(n),3,alpha,0,0,0):
> recQC11:=combine(denom(rhs(recQC1))*lhs(recQC1)   =collect(numer
```

```
(rhs(recQC1)),[S(n, alpha, 0),S(n-1, alpha, 0),x],qsimpcomb),
power)
```

$$recQC11 := (q^n - q)(-q^2 + q^n)(\alpha q + q^n)(q^2 \alpha + q^n) S(n-3, \alpha, 0) = \big(\alpha q^{2n+1} x + q^3 \alpha (-q^2 \alpha + \alpha q^n - \alpha q - q^n)\big) S(n, \alpha, 0) + \big(q^{4n} x^2 + (q+1) q^{2n+1} x (-q^2 \alpha + \alpha q^n - q^n - \alpha) + q^3 (-\alpha^2 q^{n+2} + \alpha^2 q^3 + \alpha^2 q^{2n} - \alpha^2 q^{n+1} + \alpha q^{n+2} + q^2 \alpha^2 - q^{2n} \alpha - q^n \alpha^2 + \alpha q^{n+1} + \alpha^2 q + q^{2n} + \alpha q^n)\big) S(n-1, \alpha, 0) \quad (20.4)$$

```
> GQC[3,0]:=op([2,2],recQC11)/S(n-1, alpha, 0):
> boundQC1:=(-coeff(GQC[3,0],x,1)+sqrt(((coeff(GQC[3,0],x,1)^2-4*
  coeff(GQC[3,0],x,0)*coeff(GQC[3,0],x,2)))))/(2*coeff(GQC[3,0],
  x,2))
```

$$boundQC1 := \frac{1}{2} \frac{1}{q^{4n}} \Big(-(q+1) q^{2n+1} (-q^2 \alpha + \alpha q^n - q^n - \alpha) + \big((q+1)^2 (q^{2n+1})^2 (-q^2 \alpha + \alpha q^n - q^n - \alpha)^2 - 4 q^3 (-\alpha^2 q^{n+2} + \alpha^2 q^3 + \alpha^2 q^{2n} - \alpha^2 q^{n+1} + \alpha q^{n+2} + q^2 \alpha^2 - q^{2n} \alpha - q^n \alpha^2 + \alpha q^{n+1} + \alpha^2 q + q^{2n} + \alpha q^n) q^{4n}\big)^{1/2}\Big) \quad (20.5)$$

Mixed recurrence equation involving $S(n-3, \alpha q^{-6}, 0)$ giving an upper bound of x_(n,1)

```
> recQC2:=qMixRec(Fqc,q,k,S(n),3,alpha,-6,0,0):
> recQC21:=combine(denom(rhs(recQC2))*lhs(recQC2)   =collect(numer
  (rhs(recQC2)),[S(n, alpha, 0),S(n-1, alpha, 0),x],qsimpcomb),
  power)
```

$$recQC21 := (-q^2 + q^n)(q^n - q) x^6 S\left(n-3, \frac{\alpha}{q^6}, 0\right) = \Big(-\frac{\alpha^3 (q^n - q)(-q^2 + q^n) x^3}{q^9} - (-q^2 + q^n)(q^n - q)(q^3 + \alpha) \alpha^3 (q^2 + q + 1) q^{-11-n} x^2 - (q^3 + \alpha) \alpha (q^4 + \alpha)(q^2 + \alpha) q^{-n-12} x (-q^{n+4} - q^4 \alpha + \alpha q^{n+2} - q^3 \alpha + \alpha q^{n+1} - q^2 \alpha + \alpha q^n - \alpha q) - \alpha (q^3 + \alpha)(q^2 + \alpha)(q + \alpha)(q^4 + \alpha)(q^5 + \alpha) q^{-n-12}\Big) S(n, \alpha, 0) + \Big((q^3 + \alpha)(q^n + \alpha)(q^{2n+6} + \alpha q^{n+5} - \alpha q^{2n+3} + \alpha q^{n+4} - \alpha^2 q^{n+2} + \alpha q^{3+n} + \alpha^2 q^3 + \alpha^2 q^{2n} - \alpha^2 q^{n+1} + q^2 \alpha^2 - q^n \alpha^2 + \alpha^2 q) q^{-9-n} x^2 + (q^3 + \alpha)(q^4 + \alpha)(q^n + \alpha)(-q^{3+n} - q^2 \alpha + \alpha q^n - \alpha)(q^2 + \alpha)(q + 1) q^{-11-n} x + (q^5 + \alpha)(q^3 + \alpha)(q^4 + \alpha)(q^n + \alpha)(q + \alpha)(q^2 + \alpha) q^{-n-12}\Big) S(n-1, \alpha, 0) \quad (20.6)$$

```
> GQC[3,6]:=collect((op([2,2],recQC21)/S(n-1, alpha, 0)/((q^3+
  alpha)*(q^n+alpha))),x,qsimpcomb):
> boundQC2:=combine((-coeff(GQC[3,6],x,1)-sqrt(((coeff(GQC[3,6],
  x,1)^2-4*coeff(GQC[3,6],x,0)*coeff(GQC[3,6],x,2)))))/(2*coeff
  (GQC[3,6],x,2)),power)
```

$$boundQC2 := \frac{1}{2} \Big(\big(-(q^4 + \alpha)(-q^{3+n} - q^2 \alpha + \alpha q^n - \alpha)(q^2 + \alpha)(q + 1) q^{-11-n} - \big((q^4 + \alpha)^2 (-q^{3+n} - q^2 \alpha + \alpha q^n - \alpha)^2 (q^2 + \alpha)^2 (q + 1)^2 q^{-22-2n} \quad (20.7)$$

$$-4\left(q^5+\alpha\right)\left(q^4+\alpha\right)\left(q+\alpha\right)\left(q^2+\alpha\right)\left(q^{2n+6}+\alpha q^{n+5}-\alpha q^{2n+3}+\alpha q^{n+4}-\alpha^2 q^{n+2}+\alpha q^{3+n}+\alpha^2 q^3+\alpha^2 q^{2n}-\alpha^2 q^{n+1}+q^2\alpha^2-q^n\alpha^2+\alpha^2 q\right) q^{-2n-21}\Big)^{1/2}\Big) q^{n+9}\Big) \Big/ \left(q^{2n+6}+\alpha q^{n+5}-\alpha q^{2n+3}+\alpha q^{n+4}-\alpha^2 q^{n+2}+\alpha q^{3+n}+\alpha^2 q^3+\alpha^2 q^{2n}-\alpha^2 q^{n+1}+q^2\alpha^2-q^n\alpha^2+\alpha^2 q\right)$$

Comparison between the bounds and the extreme zeros

```
> xnQC:= sort([solve(expand(subs({alpha=15,q=0.9},QC(40,alpha,x,
  q))),x)]):
> extzeroxnQC:=evalf[15]([min(xnQC) ,max(xnQC)]);
```

$$extzeroxnQC := \left[6.68351291879645,\ 1.68895659963783\ 10^5\right] \quad \textbf{(20.8)}$$

```
> QC300:=unapply(boundQC1 ,[n,alpha,q]):
> evalf[15](QC300(40,15,0.9))
```

$$1.58960382328626\ 10^5 \quad \textbf{(20.9)}$$

```
> QC3m60:=unapply(boundQC2 ,[n,alpha,q]):
> evalf[15](QC3m60(40,15,0.9))
```

$$7.14402718821895 \quad \textbf{(20.10)}$$

the Stieltjes-Wigert polynomials

```
> SW:=(n,x,q)->1/qpochhammer(q,q,n) *add(qphihyperterm([q^(-n)],
  [0],q,-q^(n+1)*x,k),k=0..n);
```

$$SW := (n, x, q) \rightarrow \frac{add\left(qphihyperterm\left(\left[q^{-n}\right], [0], q, -q^{n+1} x, k\right), k=0..n\right)}{qpochhammer(q, q, n)} \quad \textbf{(21.1)}$$

```
> Fsw:=1/qpochhammer(q,q,n) *(qphihyperterm([q^(-n)],[0],q,-q^
  (n+1)*x,k));
```

Equation (16) of the manuscript involving $S(n-3, q)$ giving a good lower bound for x_{n,n}

```
> recSW1:=qMixRec(Fsw,q,k,S(n),3,0,0,0,0):
> recSW11:=combine(denom(rhs(recSW1))*lhs(recSW1)  =collect(numer
  (rhs(recSW1)),[S(n, 0, 0),S(n-1, 0, 0),x],qsimpcomb),power)
```

$$recSW11 := q^6 S(n-3,0,0) = \left(-\left(q^n-1\right) q^{2n+1} x - q^3\left(q^n-1\right)\left(-q^2+q^n-q\right)\right) S(n, 0, 0) + \left(q^{4n} x^2 + \left(-q^2+q^n-1\right)(q+1) q^{2n+1} x + q^3\left(q^3+q^2-q^n-q^{n+2}-q^{n+1}+q^{2n}+q\right)\right) S(n-1,0,0) \quad \textbf{(21.2)}$$

```
> GSW[3]:=op([2,2],recSW11)/S(n-1, 0, 0)
```

$$GSW_3 := q^{4n} x^2 + \left(-q^2+q^n-1\right)(q+1) q^{2n+1} x + q^3\left(q^3+q^2-q^n-q^{n+2}-q^{n+1}+q^{2n}+q\right) \quad \textbf{(21.3)}$$

```
> bound17:=(-coeff(GSW[3],x,1)+sqrt(((coeff(GSW[3],x,1)^2-4*coeff
  (GSW[3],x,0)*coeff(GSW[3],x,2)))))/(2*coeff(GSW[3],x,2))
```

$$bound14 := \frac{1}{2}\,\frac{1}{q^{4n}}\Big(-\left(-q^2+q^n-1\right)(q+1)\,q^{2n+1} \quad \textbf{(21.4)}$$

$$+\left(\left((q+1)^2 \left(q^{2n+1}\right)^2 \left(-q^2+q^n-1\right)^2-4q^3\left(q^3+q^2-q^n-q^{n+2}-q^{n+1}+q^{2n}+q\right)q^{4n}\right)^{1/2}\right)$$

Equation involving $S(n-4,q)$ giving a good lower bound for x_{n,n}

```
> recSW2:=qMixRec(Fsw,q,k,S(n),4,0,0,0,0):
> recSW21:=combine(denom(rhs(recSW2))*lhs(recSW2)   =collect(numer
  (rhs(recSW2)),[S(n, 0, 0),S(n-1, 0, 0),x],qsimpcomb),power)
```

$$recSW21 := q^{12}\, S(n-4,0,0) = \Big(\big((q^n-1)\,q^{4n+1}x^2+(q^n-1)\,(q+1)\,q^{2n+3}x\,(-q^3+q^n-q)+q^6\,(q^n-1)\,(q^5-q^{3+n}+q^4-q^{n+2}+q^3+q^{2n}-q^{n+1})\big)\,S(n,0,0)+\big(-q^{6n}x^3-(q^2+q+1)\,q^{4n+1}x^2\,(-q^3+q^n-1)-(q^7-q^{n+5}+q^6-2\,q^{n+4}+2\,q^5+q^{2n+2}-2\,q^{3+n}+2\,q^4+q^{2n+1}-2\,q^{n+2}+2\,q^3+q^{2n}-2\,q^{n+1}+q^2-q^n+q)\,q^{2n+3}x-q^6\,(q^{n+5}-q^6-q^{2n+3}+q^{n+4}-q^5-q^{2n+2}+2\,q^{3+n}-q^4+q^{3n}-q^{2n+1}+q^{n+2}-q^3-q^{2n}+q^{n+1})\big)\,S(n-1,0,0)\Big) \tag{21.5}$$

The polynomial -A_3(x) given by Equation (17) in the manuscript

```
> GSW[4]:=op([2,2],recSW21)/S(n-1, 0, 0)
```

$$GSW_4 := -q^{6n}x^3-(q^2+q+1)\,q^{4n+1}x^2\,(-q^3+q^n-1)-(q^7-q^{n+5}+q^6-2\,q^{n+4}+2\,q^5+q^{2n+2}-2\,q^{3+n}+2\,q^4+q^{2n+1}-2\,q^{n+2}+2\,q^3+q^{2n}-2\,q^{n+1}+q^2-q^n+q)\,q^{2n+3}x-q^6\,(q^{n+5}-q^6-q^{2n+3}+q^{n+4}-q^5-q^{2n+2}+2\,q^{3+n}-q^4+q^{3n}-q^{2n+1}+q^{n+2}-q^3-q^{2n}+q^{n+1}) \tag{21.6}$$

Comparison between the bounds and the extreme zeros

```
> xnSW:= sort([solve(expand(subs(q=0.98,SW(30,x,q))),x)]):
> extzeroSW:=evalf[15]([min(xnSW)  ,max(xnSW)])
```

$$extzeroSW := [0.429268662682186,\ 7.82899963873761] \tag{21.7}$$

```
> SW30:=unapply(bound17,[n,q]):
> evalf[15](SW30(30,0.98))
```

$$6.74396892396525 \tag{21.8}$$

```
> SW40:=unapply(GSW[4],[n,q,x]):
> evalf[15](max(solve(SW40(30,0.98,x),x)))
```

$$7.45024245429932 \tag{21.9}$$

the discrete q-Hermite I polynomials

```
> DQHI:=(n,x,q)->q^binomial(n,2)*add(qphihyperterm([q^(-n),1/x],
  [0],q,-q*x,k),k=0..n);
```

$$DQHI := (n,x,q)\rightarrow q^{\mathrm{binomial}(n,2)}\ add\left(qphihyperterm\left(\left[q^{-n},\frac{1}{x}\right],[0],q,-q\,x,k\right),k=0..n\right) \tag{22.1}$$

```
> Fdqh1:=q^binomial(n,2)*(qphihyperterm([q^(-n),1/x],[0],q,-q*x,
  k));

> recDQHI1:=qMixRec(Fdqh1,q,k,S(n),6,0,0,0,0):
> recDQHI11:=combine(denom(rhs(recDQHI1))*lhs(recDQHI1)   =collect
```

```
(numer(rhs(recDQHI1)),[S(n, 0, 0),S(n-1, 0, 0),x],qsimpcomb),
power)
```

$$recDQHI11 := (-q^5+q^n)(-q^4+q^n)(q^n-q)(-q^2+q^n)(-q^3+q^n)S(n-6,0,0) = (q^{35-5n}x^4+(q^2+q+1)q^{26-4n}x^2(-q^4+q^{n+2}-q^{n+1}+q^n)+(-q^2+q^n)(-q^4+q^n)q^{21-3n})S(n,0,0)+(-q^{35-5n}x^5-(q^2+1)q^{26-4n}x^3(q^{n+4}-q^5-q^4+q^n)-(-q^{n+5}+q^6+q^{2n+2}-q^{2n+1}-q^{n+2}+q^{2n})(q^2+q+1)q^{21-3n}x)S(n-1,0,0) \quad (22.2)$$

```
> GDQHI[3,6]:=op([2,2],recDQHI11)/S(n-1, 0, 0):
> boundDQHI:=combine(sqrt((-coeff(GDQHI[3,6],x,3)+sqrt(((coeff
(GDQHI[3,6],x,3)^2-4*coeff(GDQHI[3,6],x,1)*coeff(GDQHI[3,6],x,
5)))))/(2*coeff(GDQHI[3,6],x,5))),power)
```

$$boundDQHI := \frac{1}{2}\left(-2\left((q^2+1)q^{26-4n}(q^{n+4}-q^5-q^4+q^n)+\left((q^2+1)^2q^{52-8n}(q^{n+4}-q^5-q^4+q^n)^2-4(-q^{n+5}+q^6+q^{2n+2}-q^{2n+1}-q^{n+2}+q^{2n})(q^2+q+1)q^{56-8n}\right)^{1/2}\right)q^{-35+5n}\right)^{1/2} \quad (22.3)$$

Comparison between the bounds and the extreme zeros

```
> xnDQHI:= sort([solve(expand(subs(q=0.98,DQHI(50,x,q))),x)]):
> extzeroDQHI:=evalf[15](max(xnDQHI))
```

$$extzeroDQHI := 0.999787629884629 \quad (22.4)$$

```
> DQI60:=unapply(boundDQHI,[n,q]):
> evalf[15](DQI60(50,0.98))
```

$$0.490937892965192 \quad (22.5)$$

the discrete q-Hermite II polynomials

```
> DQHII:=(n,x,q)->(I)^(-n)*q^(-binomial(n,2))*add(qphihyperterm(
[q^(-n),I*x],[],q,-q^(n),k),k=0..n);
```

$$DQHII := (n,x,q) \to I^{-n}q^{-\mathrm{binomial}(n,2)}\,add\left(qphihyperterm\left([q^{-n},Ix],[\,],q,-q^n,k\right),k=0..n\right) \quad (23.1)$$

```
> Fdqh2:=(I)^(-n)*q^(-binomial(n,2))*(qphihyperterm([q^(-n),I*x],
[],q,-q^(n),k));
```

$$Fdqh2 := \frac{I^{-n}q^{-\mathrm{binomial}(n,2)}\,qpochhammer(q^{-n},q,k)\,qpochhammer(Ix,q,k)\,(-q^n)^k}{(-1)^k q^{\frac{1}{2}k(k-1)}\,qpochhammer(q,q,k)} \quad (23.2)$$

Equation involving $S(n-5,q)$ giving a good lower bound for x_{n,n}

```
> recDQHII1:=qMixRec(Fdqh2,q,k,S(n),5,0,0,0,0):
> recDQHII11:=combine(denom(rhs(recDQHII1))*lhs(recDQHII1)  =
collect(numer(rhs(recDQHII1)),[S(n, 0, 0),S(n-1, 0, 0),x],
qsimpcomb),power)
```

$$recDQHII11 := (q^n-q)(-q^2+q^n)(-q^3+q^n)(-q^4+q^n)S(n-5,0,0) = (\quad (23.3)$$

$-q^{8n-14}x^3 - (-q^4 + q^{n+1} - q^2 + q^n)\, q^{6n-11}x)\, S(n,0,0) + (q^{8n-14}x^4 + (-q^3 + q^2 + q^n - q)\,(q^2+q+1)\, q^{6n-12}x^2 + (q^n - q)\,(-q^3 + q^n)\, q^{4n-8})\, S(n-1,0,0)$

```
> GDQHII[5]:=op([2,2],recDQHII11)/S(n-1, 0, 0)
```

$$GDQHII_5 := q^{8n-14}x^4 + (-q^3 + q^2 + q^n - q)\,(q^2+q+1)\, q^{6n-12}x^2 + (q^n - q)\,(-q^3 + q^n)\, q^{4n-8} \quad \textbf{(23.4)}$$

The bound (18) of the manuscript

```
> bound19:=sqrt(combine((-coeff(GDQHII[5],x,2)+sqrt(((coeff
  (GDQHII[5],x,2)^2-4*coeff(GDQHII[5],x,0)*coeff(GDQHII[5],x,4)))
  ))/(2*coeff(GDQHII[5],x,4)),power))
```

$$bound17 := \frac{1}{2}\sqrt{2}\left(\left(-(-q^3+q^2+q^n-q)\,(q^2+q+1)\,q^{6n-12} + \sqrt{(-q^3+q^2+q^n-q)^2\,(q^2+q+1)^2\,q^{12n-24} - 4\,(q^n-q)\,(-q^3+q^n)\,q^{12n-22}}\right) q^{-8n+14}\right)^{1/2} \quad \textbf{(23.5)}$$

Equation involving $S(n-7,q)$ giving a good lower bound for x_{n,n}

```
> recDQHII2:=qMixRec(Fdqh2,q,k,S(n),7,0,0,0,0):
> recDQHII21:=combine(denom(rhs(recDQHII2))*lhs(recDQHII2)  =
  collect(numer(rhs(recDQHII2)),[S(n, 0, 0),S(n-1, 0, 0),x],
  qsimpcomb),power)
```

$$recDQHII21 := (q^n-q)\,(-q^2+q^n)\,(-q^3+q^n)\,(-q^4+q^n)\,(-q^5+q^n)\,(-q^6+q^n)\,S(n-7,0,0) = (-q^{12n-27}x^5 - (q^2+1)\,(-q^6+q^{n+1}-q^2+q^n)\,q^{10n-24}x^3 - (q^2+q+1)\,(q^8-q^7-q^{n+5}+q^6-q^{n+2}+q^{2n})\,q^{8n-19}x)\,S(n,0,0) + (q^{12n-27}x^6 + (-q^5+q^4-q^3+q^2+q^n-q)\,q^{10n-25}x^4\,(q^4+q^3+q^2+q+1) + (q^2+q+1)\,(q^{10}-q^9-q^{n+7}+q^8-q^{n+5}+q^6-q^5+q^{2n+2}-q^{n+3}+q^4+q^{2n}-q^{n+1})\,q^{8n-21}x^2 + (-q^5+q^n)\,(q^n-q)\,(-q^3+q^n)\,q^{6n-15})\,S(n-1,0,0) \quad \textbf{(23.6)}$$

```
> GDQHII[7]:=op([2,2],recDQHII21)/S(n-1, 0, 0)
```

$$GDQHII_7 := q^{12n-27}x^6 + (-q^5+q^4-q^3+q^2+q^n-q)\,q^{10n-25}x^4\,(q^4+q^3+q^2+q+1) + (q^2+q+1)\,(q^{10}-q^9-q^{n+7}+q^8-q^{n+5}+q^6-q^5+q^{2n+2}-q^{n+3}+q^4+q^{2n}-q^{n+1})\,q^{8n-21}x^2 + (-q^5+q^n)\,(q^n-q)\,(-q^3+q^n)\,q^{6n-15} \quad \textbf{(23.7)}$$

Comparison between the bounds and the extreme zeros

```
> xnDQHII:= sort([solve(expand(subs(q=0.98,qsimpcomb(DQHII(10,x,
  q)))),x)]):
> extzeroDQHII:=evalf[15](max(xnDQHII))
```

$$extzeroDQHII := 0.766549224725163 \quad \textbf{(23.8)}$$

```
> DQHII50:=unapply(bound19,[n,q]):
> evalf[15](DQHII50(10,0.98))
```

$$0.729684868073074 \quad \textbf{(23.9)}$$

```
> DQHII70:=unapply(GDQHII[7],[n,q,x]):
> evalf[15](max(solve(DQHII70(10,0.98,x),x)))
```

$$0.764180845761768 \quad (23.10)$$

the Al-Salam-Carlitz I polynomials

```
> ASCI:=(n,alpha,x,q)->(-alpha)^n*q^(binomial(n,2))*add
  (qphihyperterm([q^(-n),1/x],[0],q,q*x/alpha,k),k=0..n);
```

$$ASCI := (n, \alpha, x, q) \rightarrow (-\alpha)^n q^{\mathrm{binomial}(n, 2)}\, add\left(qphihyperterm\left(\left[q^{-n}, \frac{1}{x}\right], [0], q, \frac{q\,x}{\alpha}, k\right), k = 0..n\right) \quad (24.1)$$

```
> Fasc1:=(-alpha)^n*q^(binomial(n,2))*(qphihyperterm([q^(-n),
  1/x],[0],q,q*x/alpha,k));
```

$$Fasc1 := \frac{(-\alpha)^n q^{\mathrm{binomial}(n, 2)}\, qpochhammer(q^{-n}, q, k)\, qpochhammer\left(\frac{1}{x}, q, k\right) \left(\frac{q\,x}{\alpha}\right)^k}{qpochhammer(q, q, k)} \quad (24.2)$$

The weight function

```
> rhoASCI:=alpha->qpochhammer(q*x,q,infinity)*qpochhammer(q*
  x/apha,q,infinity)
```

$$rhoASCI := \alpha \rightarrow qpochhammer(q\,x, q, \infty)\, qpochhammer\left(\frac{q\,x}{apha}, q, \infty\right) \quad (24.3)$$

```
> cASCI:=qpochhammer(q*x/(a*q^s), q, s);
```

$$cASCI := qpochhammer\left(\frac{q\,x}{a\,q^s}, q, s\right) \quad (24.4)$$

since $\alpha < x < 1$, a shift on α will change the interval of orthogonality and we can not apply Theorem 1. So we just consider the bound from Equation (2)

```
> recASCI1:=qMixRec(Fasc1,q,k,S(n),3,alpha,0,0,0):
> recASCI11:=combine(denom(rhs(recASCI1))*lhs(recASCI1)   =collect
  (numer(rhs(recASCI1)),[S(n, alpha, 0),S(n-1, alpha, 0),x],
  qsimpcomb),power)
```

$$recASCI11 := (q^n - q)\, \alpha^2 (-q^2 + q^n)\, S(n-3, \alpha, 0) = \left(-q^{8-2n} x + (\alpha + 1)\, q^{6-n}\right) S(n, \alpha, 0) + \left(q^{8-2n} x^2 - (q+1)(\alpha+1)\, q^{6-n} x + \left(q^n \alpha^2 + q^n \alpha + \alpha\,q + q^n\right) q^{5-n}\right) S(n-1, \alpha, 0) \quad (24.5)$$

```
> GASCI[3,0]:=op([2,2],recASCI11)/S(n-1, alpha, 0)
```

$$GASCI_{3,0} := q^{8-2n} x^2 - (q+1)(\alpha+1)\, q^{6-n} x + \left(q^n \alpha^2 + q^n \alpha + \alpha\,q + q^n\right) q^{5-n} \quad (24.6)$$

```
> boundASCI:=[solve(GASCI[3,0],x)]
```

$$boundASCI := \left[\frac{1}{2}\,\frac{1}{q^{8-2n}}\left(\alpha\,q^{7-n} + \alpha\,q^{6-n} + q^{7-n} + q^{6-n}\right.\right. \quad (24.7)$$

$$+ \left(-4\,q^{8-2n} \alpha^2 q^5 - 4\,q^{8-2n} \alpha\,q^5 - 4\,q^{8-2n} q^5 + \left(q^{7-n}\right)^2 \alpha^2\right.$$

$$+2\,q^{7-n}q^{6-n}\alpha^2+\left(q^{6-n}\right)^2\alpha^2+2\left(q^{7-n}\right)^2\alpha+4\,q^{7-n}q^{6-n}\alpha-4\,q^{8-2n}q^{6-n}\alpha$$

$$+2\left(q^{6-n}\right)^2\alpha+\left(q^{7-n}\right)^2+2\,q^{7-n}q^{6-n}+\left(q^{6-n}\right)^2\Big)^{1/2}\Big),\ \frac{1}{2}\,\frac{1}{q^{8-2n}}\Big(\alpha\,q^{7-n}$$

$$+\alpha\,q^{6-n}+q^{7-n}+q^{6-n}$$

$$-\Big(-4\,q^{8-2n}\alpha^2q^5-4\,q^{8-2n}\alpha\,q^5-4\,q^{8-2n}q^5+\left(q^{7-n}\right)^2\alpha^2$$

$$+2\,q^{7-n}q^{6-n}\alpha^2+\left(q^{6-n}\right)^2\alpha^2+2\left(q^{7-n}\right)^2\alpha+4\,q^{7-n}q^{6-n}\alpha-4\,q^{8-2n}q^{6-n}\alpha$$

$$+2\left(q^{6-n}\right)^2\alpha+\left(q^{7-n}\right)^2+2\,q^{7-n}q^{6-n}+\left(q^{6-n}\right)^2\Big)^{1/2}\Big)\Big]$$

```
> combine((-coeff(GASCI[3,0],x,1)+sqrt(((coeff(GASCI[3,0],x,1)^2
  -4*coeff(GASCI[3,0],x,0)*coeff(GASCI[3,0],x,2)))))/(2*coeff
  (GASCI[3,0],x,2)),power)
```

$$\frac{1}{2}\Big((q+1)\,(\alpha+1)\,q^{6-n}+\sqrt{(q+1)^2\,(\alpha+1)^2\,q^{12-2n}-4\left(q^n\alpha^2+q^n\alpha+\alpha\,q+q^n\right)q^{13-3n}}\Big)\,q^{-8+2n} \quad \textbf{(24.8)}$$

Comparison of the bounds and the extreme zeros

```
> hypasc1:=(alpha)->alpha<0
```

$$hypasc1 := \alpha \to \alpha < 0 \quad \textbf{(24.9)}$$

```
> xnboundASCI:= sort([solve(expand(subs({alpha=-50,q=0.9},
  qsimpcomb(ASCI(30,alpha,x,q)))),x)]):
> extzeroboundASCI:=evalf[15]([min(xnboundASCI),max(xnboundASCI)]
  )
```

$$extzeroboundASCI := [-50.0000000000000,\ 0.209239543368237] \quad \textbf{(24.10)}$$

```
> ASCI30:=unapply(boundASCI,[n,alpha,q]):
> sort(evalf[15](ASCI30(30,-50,0.9)))
```

$$[-4.02045809486024,\ -0.851908368624455] \quad \textbf{(24.11)}$$

the Al-Salam-Carlitz II polynomials

```
> ASCII:=(n,alpha,x,q)->(-alpha)^n*q^(-binomial(n,2))*add
  (qphihyperterm([q^(-n),x],[],q,q^(n)/alpha,k),k=0..n);
```

$$ASCII := (n,\alpha,x,q) \to (-\alpha)^n\,q^{-\mathrm{binomial}(n,\,2)}\,add\Big(qphihyperterm\Big([q^{-n},x],[\,],q,\frac{q^n}{\alpha},k\Big),k=0..n\Big) \quad \textbf{(25.1)}$$

```
> Fasc2:=(-alpha)^n*q^(-binomial(n,2))*(qphihyperterm([q^(-n),x],
  [],q,q^(n)/alpha,k));
```

$$Fasc2 := \frac{(-\alpha)^n q^{-\mathrm{binomial}(n,2)}\, qpochhammer(q^{-n}, q, k)\, qpochhammer(x, q, k) \left(\frac{q^n}{\alpha}\right)^k}{(-1)^k q^{\frac{1}{2}k(k-1)}\, qpochhammer(q, q, k)} \tag{25.2}$$

The weight function

```
> rhoASCII:=alpha->q^(k^2)*alpha^k/qpochhammer(q,q,k)/qpochhammer
  (alpha*q,q,k)
```

$$rhoASCII := \alpha \rightarrow \frac{q^{k^2} \alpha^k}{qpochhammer(q, q, k)\, qpochhammer(\alpha q, q, k)} \tag{25.3}$$

```
> qsimplify(rhoASCII(alpha*q^(-s))/rhoASCII(alpha))
```

$$\frac{qpochhammer(\alpha q, q, k)}{q^{s k}\, qpochhammer\left(\frac{\alpha q}{q^s}, q, k\right)} \tag{25.4}$$

```
> qsimpcomb(subs(s=4,qsimplify(rhoASCII(alpha*q^(-s))/rhoASCII
  (alpha))))
```

$$\frac{(-q^3 + \alpha q^k)(\alpha q^k - q^2)(\alpha q^k - q)(\alpha q^k - 1)}{(q^k)^4 (-q^3 + \alpha)(-q^2 + \alpha)(-q + \alpha)(-1 + \alpha)} \tag{25.5}$$

cASCII is a polynomial of degree s in q^{-k}.

Equation (4) is valid only for negative shifts k=0,-1,-2,...... For example

```
> qMixRec(Fasc2,q,k,S(n),2,alpha,-2,0,0)
```

$$S\left(n-2, \frac{\alpha}{q^2}, 0\right) = \frac{(q^n \alpha - \alpha q + q^2)\, q\, S(n, \alpha, 0)}{\alpha (\alpha - x)(-q x + \alpha)(q^n - q)} + \frac{S(n-1, \alpha, 0)\, q^{-n+2} (\alpha q^{n+1} - q^{n+1} x + q^n \alpha - \alpha q + q^2)}{\alpha (\alpha - x)(-q x + \alpha)(q^n - q)} \tag{25.6}$$

But with the substitution $\alpha \rightarrow \alpha q^{-s}$, the orthogonality condition 0<αq<1 is not more valid and Theorem 1 can not be applied. Therefore we use Equation (2)

```
> recASCII1:=qMixRec(Fasc2,q,k,S(n),3,alpha,0,0,0):
> recASCII11:=combine(denom(rhs(recASCII1))*lhs(recASCII1)  =
  collect(numer(rhs(recASCII1)),[S(n, alpha, 0),S(n-1, alpha, 0),
  x],qsimpcomb),power)
```

$$recASCII11 := (q^n - q)\, \alpha^2 (-q^2 + q^n)\, S(n-3, \alpha, 0) = \left(-q^{4n-5} x + (\alpha + 1) q^{3n-3}\right) S(n, \alpha, 0) + \left(q^{4n-5} x^2 - (q+1)(\alpha+1) q^{3n-4} x + (\alpha^2 q + q^n \alpha + \alpha q + q) q^{2n-3}\right) S(n-1, \alpha, 0) \tag{25.7}$$

```
> GASCII[3,0]:=op([2,2],recASCII11)/S(n-1, alpha, 0)
```

$$GASCII_{3,0} := q^{4n-5} x^2 - (q+1)(\alpha+1) q^{3n-4} x + (\alpha^2 q + q^n \alpha + \alpha q + q) q^{2n-3} \tag{25.8}$$

```
> boundASCII:=[solve(GASCII[3,0],x)]
```

$$boundASCII := \left[\frac{1}{2} \frac{1}{q^{4n-5}} \left(\alpha q^{3n-3} + \alpha q^{3n-4} + q^{3n-3} + q^{3n-4} \right.\right. \tag{25.9}$$

$$+\left(\left(q^{3n-4}\right)^2\alpha^2+2\,q^{3n-4}\,q^{3n-3}\,\alpha^2-4\,q^{4n-5}\,q^{2n-2}\,\alpha^2+\left(q^{3n-3}\right)^2\alpha^2\right.$$

$$+2\left(q^{3n-4}\right)^2\alpha+4\,q^{3n-4}\,q^{3n-3}\,\alpha-4\,q^{4n-5}\,q^{2n-2}\,\alpha-4\,q^{4n-5}\,q^{3n-3}\,\alpha$$

$$\left.\left.+2\left(q^{3n-3}\right)^2\alpha+\left(q^{3n-4}\right)^2+2\,q^{3n-4}\,q^{3n-3}-4\,q^{4n-5}\,q^{2n-2}+\left(q^{3n-3}\right)^2\right)^{1/2}\right),$$

$$\frac{1}{2}\,\frac{1}{q^{4n-5}}\left(\alpha\,q^{3n-3}+\alpha\,q^{3n-4}+q^{3n-3}+q^{3n-4}\right.$$

$$-\left(\left(q^{3n-4}\right)^2\alpha^2+2\,q^{3n-4}\,q^{3n-3}\,\alpha^2-4\,q^{4n-5}\,q^{2n-2}\,\alpha^2+\left(q^{3n-3}\right)^2\alpha^2\right.$$

$$+2\left(q^{3n-4}\right)^2\alpha+4\,q^{3n-4}\,q^{3n-3}\,\alpha-4\,q^{4n-5}\,q^{2n-2}\,\alpha-4\,q^{4n-5}\,q^{3n-3}\,\alpha$$

$$\left.\left.+2\left(q^{3n-3}\right)^2\alpha+\left(q^{3n-4}\right)^2+2\,q^{3n-4}\,q^{3n-3}-4\,q^{4n-5}\,q^{2n-2}+\left(q^{3n-3}\right)^2\right)^{1/2}\right)\Big]$$

```
> combine((-coeff(GASCII[3,0],x,1)+sqrt(((coeff(GASCII[3,0],x,1)
  ^2-4*coeff(GASCII[3,0],x,0)*coeff(GASCII[3,0],x,2)))))/(2*coeff
  (GASCII[3,0],x,2)),power)
```

$$\frac{1}{2}\left((q+1)\,(\alpha+1)\,q^{3n-4}\right.$$

$$\left.+\sqrt{(q+1)^2\,(\alpha+1)^2\,q^{6n-8}-4\left(\alpha^2\,q+q^n\,\alpha+\alpha\,q+q\right)q^{6n-8}}\right)q^{-4n+5} \qquad \textbf{(25.10)}$$

Comparison of the bounds and the extreme zeros

```
> hypasc:=(alpha,q)->0<alpha*q and alpha*q<1
```

$$hypasc := (\alpha, q) \rightarrow 0 < q\,\alpha \textbf{ and } q\,\alpha < 1 \qquad \textbf{(25.11)}$$

```
> xnboundASCII:= sort([solve(expand(subs({alpha=1.05,q=0.7},
  qsimpcomb(ASCII(20,alpha,x,q)))),x)]):
> extzeroboundASCII:=evalf[15]([min(xnboundASCII),max
  (xnboundASCII)])
```

$$extzeroboundASCII := [1.09202100143873, 2396.35575452363] \qquad \textbf{(25.12)}$$

```
> ASCII30:=unapply(boundASCII,[n,alpha,q]):
> sort(evalf[15](ASCII30(20,1.05,0.7)))
```

$$[730.035896040315, 2327.27794299901] \qquad \textbf{(25.13)}$$