# Spaces of Functions Satisfying Simple Differential Equations

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**Abstract:**

In [6]–[9] the first author published an algorithm for the conversion of analytic functions for which derivative rules are given into their representing power series $\sum_{k=0}^{\infty} a_k z^k$ at the origin and vice versa, implementations of which exist in Mathematica [19], (s. [9]), Maple [12] (s. [4]) and Reduce [5] (s. [13]).
One main part of this procedure is an algorithm to derive a homogeneous linear differential equation with polynomial coefficients for the given function. We call this type of ordinary differential equations *simple*.
Whereas the opposite question to find functions satisfying given differential equations is studied in great detail, our question to find differential equations that are satisfied by given functions seems to be rarely posed.
In this paper we consider the family $F$ of functions satisfying a simple differential equation generated by the rational, the algebraic, and certain transcendental functions. It turns out that $F$ forms a linear space of transcendental functions. Further $F$ is closed under multiplication and under the composition with rational functions and rational powers. These results had been published by Stanley who had proved them by theoretical algebraic considerations.
In contrast our treatment is purely algorithmically oriented. We present algorithms that generate simple differential equation for $f+g$, $f \cdot g$, $f \circ r$ ($r$ rational), and $f \circ x^{p/q}$ ($p, q \in \mathbb{N}_0$), given simple differential equations for $f$, and $g$, and give a priori estimates for the order of the resulting differential equations. We show that all order estimates are sharp.
After finishing this article we realized that in independent work Salvy and Zimmermann published similar algorithms. Our treatment gives a detailed description of those algorithms and their validity.

## 1 Simple functions

Many mathematical functions satisfy a homogeneous linear differential equation with polynomial coefficients. We call such an ordinary differential equation *simple*. Also a function $f$ that satisfies a simple differential equation is called simple. The least order of such a simple differential equation fulfilled by $f$ is called the order of $f$.

Technically there is no difference between a differential equation with polynomial coefficients, and one with rational coefficients, as we can multiply such a differential equation by its common denominator, so we may consider the case that the coefficients are members of the field $K[x]$ where $K[x]$ is one of $\mathbb{Q}[x]$, $\mathbb{R}[x]$, or $\mathbb{C}[x]$.

Examples of simple functions are

1. all rational functions $p/q$ ($p, q$ polynomials) are simple of order 1: $p\,q\,f' + (p\,q' - q\,p')\,f = 0$,
2. all algebraic functions $f$ are simple (see e. g. [2]–[3],[18], and [10]) of the order of $f$,
3. the power function $x^\alpha$ satisfies the simple differential equation $x\,f' - \alpha\,f = 0$,
4. the exponential function $e^x$ satisfies the simple differential equation $f' - f = 0$,
5. the logarithm function $\ln x$ satisfies the simple differential equation $x\,f'' + f' = 0$,
6. the sine function $\sin x$ and the cosine function $\cos x$ satisfy the simple differential equation $f'' + f = 0$,
7. the inverse sine function $\arcsin x$ and the inverse cosine function $\arccos x$ satisfy the simple differential equation $(x^2 - 1)\,f'' + x\,f' = 0$,
8. the inverse tangent function $\arctan x$ and the inverse cotangent functions $\mathrm{arccot}\,x$ satisfy the simple differential equation $(1 + x^2)\,f'' + 2\,x\,f' = 0$,
9. the inverse secant function $\mathrm{arcsec}\,x$ and the inverse cosecant function $\mathrm{arccsc}\,x$ satisfy the simple differential equation $(x^3 - x)\,f'' + (2x^2 - 1)\,f' = 0$,
10. the error function $\mathrm{erf}\,x$ satisfies the simple differential equation $f'' + 2\,x\,f' = 0$.

Note that for rational functions with rational coefficients, algebraic functions given by a rational coefficient equation, the power function for $\alpha \in \mathbb{Q}$, and in all other examples we have $K[x] = \mathbb{Q}[x]$.

Also many special functions are simple with $K[x] = \mathbb{Q}[x]$: Airy functions (see e. g. [1], § 10.4), Bessel functions (see e. g. [1], Ch. 9–11), all kinds of orthogonal polynomials (see e. g. [16], [17]), and functions of hypergeometric type (see [6]).

Not all elementary transcendental functions, however, are simple, the simplest example of which probably is the tangent function $\tan x$, compare [15], Example 2.5.

**Theorem 1** The functions $\tan x$ and $\sec x$ do not satisfy simple differential equations.

*Proof:* It is easily seen that the tangent function $f(x) = \tan x$ satisfies the nonlinear differential equation

$$f' = 1 + f^2\,. \tag{1}$$

Differentiation of (1) yields after further substitution of (1)

$$f'' = (1 + f^2)' = 2\,f\,f' = 2\,f\,(1 + f^2)\,,$$

and inductively we get representations of $f^{(k)} = P_k(f)$ ($k \in \mathbb{N}$) by polynomial expressions $P_k$ in $f$.

Assume now, the tangent function would satisfy the simple differential equation

$$\sum_{k=0}^{n} p_k \, f^{(k)} = 0 \qquad (p_k \ \text{polynomials}) \ , \tag{2}$$

then we can substitute $f^{(k)}$ by $P_k$ arriving at an algebraic identity for the tangent function. This obviously is a contradiction to the fact that the tangent function is transcendental, and therefore a differential equation of type (2) cannot be satisfied.

Let us now consider $f(x) = \sec x$. In this case we may establish the nonlinear differential equation

$$f'^2 = f^4 - f^2$$

from which, by differentiation, the second order differential equation

$$f'' = 2f^3 - f \tag{3}$$

follows. If now a simple differential equation (2) would be valid for $f$, then also

$$p_1 \, f' = - \sum_{\substack{k=0 \\ k \neq 1}}^{n} p_k \, f^{(k)} \ ,$$

and taking the square we get

$$p_1^2 \, f'^2 = p_1^2 \, (f^4 - f^2) = \left( \sum_{\substack{k=0 \\ k \neq 1}}^{n} p_k \, f^{(k)} \right)^2 \ .$$

If we substitute (3), and the corresponding representations for $f^{(k)}$ $(k = 3, \ldots, n)$ which we get by differentiating (3), we arrive at an algebraic identity, and the conclusion follows by the transcendency of $f$, again. □

By similar means we see that the elementary functions $\cot x$, and $\csc x$ as well as the corresponding hyperbolic functions are not simple. Moreover, the generating functions

$$\frac{x}{e^x - 1} = \sum_{k=0}^{\infty} B_k \, \frac{x^k}{k!} \ , \qquad \text{and} \qquad \frac{e^{x/2}}{e^x + 1} = \sum_{k=0}^{\infty} \frac{E_k}{2^{k+1}} \, \frac{x^k}{k!}$$

of the Bernoulli and Euler numbers $B_k$, and $E_k$ $(k \in \mathbb{N}_0)$ (see e. g. [1], (23.1)) satisfy the nonlinear differential equations

$$f' = \frac{1}{x} \left( (1 - x) \, f - f^2 \right) , \qquad \text{and} \qquad f'^2 = \frac{f^2}{4} - f^4 \ ,$$

respectively, and so by a similar procedure are realized not to be simple. As exactly the simple differential equations correspond to simple recurrence equations for the Taylor series coefficients (see e. g. [6]), this implies the following

**Corollary 1** Neither the Bernoulli nor the Euler numbers satisfy a finite homogeneous linear recurrence equation with polynomial coefficients.

## 2 Construction of simple functions

If simple functions are given, then we may construct further examples of simple functions by the following procedures: integration, differentiation, addition, multiplication, and the composition with certain functions, namely rational functions and rational powers. Before we prove this main result, we consider an important particular case. We will prove that if $f$ is simple, and $q \in \mathbb{N}$, then $f \circ x^{1/q}$ is simple, too. To get this result, we need the following

**Lemma 1** Let $f : D \to \mathbb{R}$ be infinitely often differentiable, let be $D \subset \mathbb{R}^+ \neq \emptyset$, and let $q \in \mathbb{N}$ with $q > 2$ be given. Consider the function $h : D \to \mathbb{R}$ with $h(x) := f(x^{1/q})$.

(a) Then we have for $r \in \mathbb{N}$ with $1 \le r \le q-1$

$$x^{r/q}\, f^{(r)}(x^{1/q}) = \sum_{k=1}^{r} C^0_{qrk}\, x^k\, h^{(k)}(x) \tag{4}$$

with constants $C^0_{qrk}$, in particular

$$C^0_{q11} = q\ , \qquad C^0_{qr1} = q\,(q-1)\cdots(q-r+1)\ , \qquad \text{and} \qquad C^0_{qrr} = q^r\ ,$$

and

$$f^{(q)}(x^{1/q}) = \sum_{k=1}^{q} C^1_{q0k}\, x^{k-1}\, h^{(k)}(x) \tag{5}$$

with constants $C^1_{q0k}$, in particular

$$C^1_{q01} = q!\ , \qquad \text{and} \qquad C^1_{q0q} = q^q\ .$$

(b) Further for $s \in \mathbb{N}$ we have

$$f^{(sq)}(x^{1/q}) = \sum_{k=s}^{sq} C^s_{q0k}\, x^{k-s}\, h^{(k)}(x)$$

with constants $C^s_{q0k}$, in particular

$$C^s_{q,0,sq} = q^{sq}\ , \tag{6}$$

and for $r \in \mathbb{N}$ with $1 \le r \le q-1$

$$x^{r/q}\, f^{(sq+r)}(x^{1/q}) = \sum_{k=s+1}^{sq+r} C^s_{qrk}\, x^{k-s}\, h^{(k)}(x)$$

with constants $C^s_{qrk}$, in particular

$$C^s_{q,r,sq+r} = q^{sq+r}\ . \tag{7}$$

*Proof:* Differentiating the identity $f(x^{1/q}) = h(x)$, we get

$$x^{1/q}\, f'(x^{1/q}) = q\, x\, h'(x)\ ,$$

a further differentiation yields

$$x^{2/q}\, f''(x^{1/q}) + x^{1/q}\, f'(x^{1/q}) = q^2\, x\, h'(x) + q^2\, x^2\, h''(x)\ ,$$

and therefore

$$x^{2/q}\, f''(x^{1/q}) = q\,(q-1)\, x\, h'(x) + q^2\, x^2\, h''(x)\ .$$

Differentiating once more and rearranging similarly, gives

$$x^{3/q}\, f'''(x^{1/q}) = q\,(q-1)\,(q-2)\, x\, h'(x) + 3\, q^2\,(q-1)\, x^2\, h''(x) + q^3\, x^3\, h'''(x)\ .$$

Assume now, for $r \in \mathbb{N}$ with $1 \le r \le q-1$ the relation

$$x^{r/q}\, f^{(r)}(x^{1/q}) = \sum_{k=1}^{r} C^0_{qrk}\, x^k\, h^{(k)}(x)$$

with

$$C^0_{qr1} = q\,(q-1)\cdots(q-r+1)\ , \qquad \text{and} \qquad C^0_{qrr} = q^r$$

is valid, then by another differentiation we get

$$\begin{aligned} x^{(r+1)/q}\, f^{(r+1)}(x^{1/q}) &= (q-r)\, C^0_{qr1}\, x\, h'(x) \\ &= +\sum_{k=2}^{r} \Big( (k\,q - r)\, C^0_{qrk} + q\, C^0_{q,r,k-1} \Big)\, x^k\, h^{(k)}(x) + q\, C^0_{qrr}\, x^{r+1}\, h^{(r+1)}(x)\ . \end{aligned}$$

For $r \le q-1$, we set

$$C^0_{q,r+1,1} := (q-r)\, C^0_{qr1} = q\,(q-1)\cdots(q-r)\ ,$$
$$C^0_{q,r+1,r+1} := q\, C^0_{qrr} = q\, q^r = q^{r+1}\ ,$$

and

$$C^0_{q,r+1,k} := (k\,q - r)\, C^0_{qrk} + q\, C^0_{q,r,k-1} \qquad (2 \le k \le r)\ .$$

For $r = q$ the same method applies, if we set

$$C^1_{q01} := C^0_{q,q-1,1} = q!\ ,$$

$$C^1_{q0k} := q\, C^0_{q,q-1,q-1} = q\, q^{q-1} = q^q\ ,$$

and

$$C^1_{q0k} := ((k-1)\, q + 1)\, C^0_{q,q-1,k} + q\, C^0_{q,q-1,k-1} \qquad (2 \le k \le q-1)\ ,$$

which proves (a).

To prove (b) we assume that for some $s \in \mathbb{N}$

$$f^{(sq)}(x^{1/q}) = \sum_{k=s}^{sq} C^s_{q0k}\, x^{k-s}\, h^{(k)}(x)$$

with

$$C^s_{q,0,sq} = q^{sq} \ .$$

If we define the functions $f_s$ and $h_s$ by

$$f_s(x) := f^{(sq)}(x) \ , \qquad \text{and} \qquad h_s(x) := \sum_{k=s}^{sq} C^s_{q0k}\, x^{k-s}\, h^{(k)}(x) \ ,$$

then we may apply (a), and get for $1 \le r \le q-1$ that

$$x^{r/q}\, f_s^{(r)}(x^{1/q}) = \sum_{k=1}^{r} C^0_{qrk}\, x^k\, h_s^{(k)}(x) \ ,$$

and further

$$f_s^{(q)}(x^{1/q}) = \sum_{k=1}^{q} C^1_{q0k}\, x^{k-1}\, h_s^{(k)}(x) \ ,$$

hence

$$x^{r/q}\, f^{(sq+r)}(x^{1/q}) = \sum_{k=1}^{r} C^0_{qrk}\, x^k\, h_s^{(k)}(x) \ , \tag{8}$$

and

$$f^{((s+1)q)}(x^{1/q}) = \sum_{k=1}^{q} C^1_{q0k}\, x^{k-1}\, h_s^{(k)}(x) \ . \tag{9}$$

To finish our proof, we will calculate the derivatives $h_s^{(k)}$ $(k = 1, \ldots, q)$ algorithmically. We have

$$h_s(x) = \sum_{j=s}^{sq} C^s_{q0j}\, x^{j-s}\, h^{(j)}(x) \ ,$$

and therefore

$$\begin{aligned} h_s'(x) &= \sum_{j=s}^{sq} C^s_{q0j}\, x^{j-s}\, h^{(j+1)}(x) + \sum_{j=s+1}^{sq} (j-s)\, C^s_{q0j}\, x^{j-s-1}\, h^{(j)}(x) \\ &= \sum_{j=s+1}^{sq+1} C^s_{q,0,j-1}\, x^{j-s-1}\, h^{(j)}(x) + \sum_{j=s+1}^{sq} (j-s)\, C^s_{q0j}\, x^{j-s-1}\, h^{(j)}(x) \\ &= \sum_{j=s+1}^{sq} \Big( (j-s)\, C^s_{q0j} + C^s_{q,0,j-1} \Big)\, x^{j-s-1}\, h^{(j)}(x) + C^s_{q,0,sq}\, x^{sq-q}\, h^{(sq+1)}(x) \ . \end{aligned}$$

We may write this as

$$h_s'(x) = \sum_{j=s+1}^{sq+1} D^1_j\, x^{j-s-1}\, h^{(j)}(x)$$

with

$$D^1_j := \begin{cases} (j-s)\, C^s_{q0j} + C^s_{q,0,j-1} & \text{if } s+1 \le j \le s\,q \\ C^s_{q,0,sq} = q^{sq} & \text{if } j = s\,q+1 \end{cases} \ .$$

Continuing iteratively in the same fashion, we get representations

$$h_s^{(k)}(x) = \sum_{j=s+k}^{sq+k} D_j^k \, x^{j-s-k} \, h^{(j)}(x)$$

with

$$D_j^k = \begin{cases} D_{j-1}^{k-1} + (j-s-k+1)\, D_j^{k-1} & \text{if } s+k \le j \le s\,q+k-1 \\ q^{sq} & \text{if } j = s\,q+k \end{cases} . \tag{10}$$

We substitute these equations in (8) to get

$$x^{r/q} \, f^{(sq+r)}(x^{1/q}) = \sum_{k=1}^{r} \sum_{j=s+k}^{sq+k} C_{qrk}^0 \, D_j^k \, x^{j-s} \, h^{(j)}(x) \; .$$

For $j$ with $s+1 \le j \le s\,q+r$ we use the notation

$$M_j := \{ k \in \mathbb{N} \mid 1 \le k \le r \quad \text{and} \quad s+k \le j \le s\,q+k \} \; .$$

Then $M_{sq+r} = \{r\}$, and

$$x^{r/q} \, f^{(sq+r)}(x^{1/q}) = \sum_{j=s+1}^{sq+r} C_{qrj}^s \, x^{j-s} \, h^{(j)}(x)$$

with

$$C_{qrj}^s := \sum_{k \in M_j} C_{qrk}^0 \, D_j^k$$

for $s+1 \le j \le s\,q+r$, in particular

$$C_{q,r,sq+r}^s = C_{qrr}^0 \, D_{sq+r}^r = q^r \, q^{sq} = q^{sq+r}$$

proving (7).

To prove (6), we substitute Equations (10) in (9) to get

$$f^{((s+1)q)}(x^{1/q}) = \sum_{j=s+1}^{(s+1)q} C_{q0j}^{s+1} \, x^{j-s-1} \, h^{(j)}(x)$$

with

$$C_{q0j}^{s+1} := \sum_{k \in M_j} C_{q0k}^1 \, D_j^k$$

for $s+1 \le j \le (s+1)\,q$, in particular

$$C_{q,0,(s+1)q}^{s+1} = C_{q0q}^1 \, D_{sq+q}^q = q \, q^{sq} = q^{(s+1)q}$$

finishing our proof. □

**Remark 1** We remark that one easily can establish that all constants $C^s_{qrk}$ are integers. Further, we note that using the differential operator $\theta_q := q\,x\,\frac{d}{dx}$ (compare with the proof of [6], Theorem 8.1) Equation (4) gets the simple form

$$x^{r/q}\,f^{(r)}(x^{1/q}) = \theta_q\,(\theta_q - 1)\,(\theta_q - 2)\cdots(\theta_q - (r-1))\,h(x) \qquad (1 \le r \le q)\ .$$

Next we give some applications of the lemma. First assume that $f' = f$, a solution of which is given by the exponential function $f(x) = e^x$. Then $f^{(q)} = f$, and so $f^{(q)}(x^{1/q}) = f(x^{1/q})$, and therefore we get for $h(x) := e^{x^{1/q}}$ the differential equation

$$\sum_{k=1}^{q} C^1_{q0k}\,x^{k-1}\,h^{(k)} = h\ ,$$

e. g. for $q = 3$ we have the differential equation for $e^{x^{1/3}}$

$$27\,x^2\,h''' + 54\,x\,h'' + 6\,h' = h\ .$$

Further we consider the homogeneous Euler differential equation

$$\sum_{k=0}^{n} a_k\,x^k\,f^{(k)} = 0 \tag{11}$$

with constants $a_k$ $(k = 0, \ldots, n)$, $a_n \neq 0$. As by the Lemma we have

$$\begin{aligned}
t^r\,f^{(r)}(t)\Big|_{t=x^{1/q}} &= \sum_{k=1}^{r} C^0_{qrk}\,x^k\,h^{(k)}(x) \quad (1 \le r \le q-1)\ ,\\
t^q\,f^{(q)}(t)\Big|_{t=x^{1/q}} &= \sum_{k=1}^{q} C^1_{q0k}\,x^k\,h^{(k)}(x)\ ,\\
t^{sq}\,f^{(sq)}(t)\Big|_{t=x^{1/q}} &= \sum_{k=s}^{sq} C^s_{q0k}\,x^k\,h^{(k)}(x) \quad (s \in \mathbb{N})\ ,
\end{aligned}$$

and

$$t^{sq+r}\,f^{(sq+r)}(t)\Big|_{t=x^{1/q}} = \sum_{k=s+1}^{sq+r} C^s_{qrk}\,x^k\,h^{(k)}(x) \quad (1 \le r \le q-1)\ ,$$

we immediately conclude the following

**Corollary 2** Assume, the function $f$ satisfies a homogeneous Euler differential equation (11) of order $n$. Then the function $f \circ x^{1/q}$ satisfies a homogeneous Euler differential equation of the same order $n$. □

As an application of Corollary 2 we get

**Corollary 3** Assume $P$ is a polynomial of degree $n$, then $P \circ x^{1/q}$ satisfies a homogeneous Euler differential equation of order $\le n + 1$.

*Proof:* $P$ satisfies the differential equation $x^{n+1}\, f^{(n+1)} = 0$. □

Our next step is to prove another lemma which we will need to prove that if $f$ is simple, and $q \in \mathbb{N}$, then so is $f \circ x^{1/q}$.

**Lemma 2** Let $f : D \to \mathbb{R}$ be an infinitely often differentiable function that is simple of order $n$, let be $D \subset \mathbb{R}^+ \neq \emptyset$, and let $q \in \mathbb{N}$ with $q > 2$ be given. Consider the function $h : D \to \mathbb{R}$ with $h(x) := f(x^{1/q})$. Then the linear space $L_f$ over $K[x]$ (i.e. with rational function coefficients) generated by the functions $x^{r/q}\, f^{(k)}(x^{1/q})$ $(r, k \in \mathbb{N}_0)$ has dimension $\leq n\, q$.

*Proof:* Given simple $f$ of order $n$, the linear space $L_f$ is generated by the functions of the $\mathbb{N}_0 \times \mathbb{N}_0$ matrix

$$\begin{array}{cccccc}
f(y), & y\,f(y), & y^2\,f(y), & \ldots, & y^r\,f(y), & \ldots \\
f'(y), & y\,f'(y), & y^2\,f'(y), & \ldots, & y^r\,f'(y), & \ldots \\
f''(y), & y\,f''(y), & y^2\,f''(y), & \ldots, & y^r\,f''(y), & \ldots \\
\vdots & \vdots & \vdots & & \vdots & \\
f^{(k)}(y), & y\,f^{(k)}(y), & y^2\,f^{(k)}(y), & \ldots, & y^r\,f^{(k)}(y), & \ldots \\
\vdots & \vdots & \vdots & & \vdots &
\end{array}$$

using the abbreviation $y := x^{1/q}$.

At most the first $q$ expressions $f(x^{1/q}), x^{1/q}\, f(x^{1/q}), \ldots, x^{(q-1)/q}\, f(x^{1/q})$ of the first row are linearly independent (over $K[x]$) as the expressions for $r = sq$ $(s \in \mathbb{N})$ depend linearly on the first expression $f(y)$ $(y^{sq} = x^s$ is a polynomial), further the expressions for $r = sq + 1$ $(s \in \mathbb{N})$ depend linearly on the second expression $y\, f(y)$, and so on. The same conclusion follows for each other row so that we obtain that $L_f$ is generated by the $q \times \mathbb{N}_0$ matrix $M$

$$\begin{array}{ccccc}
f(y), & y\,f(y), & y^2\,f(y), & \ldots, & y^{q-1}\,f(y), \\
f'(y), & y\,f'(y), & y^2\,f'(y), & \ldots, & y^{q-1}\,f'(y), \\
f''(y), & y\,f''(y), & y^2\,f''(y), & \ldots, & y^{q-1}\,f''(y), \\
\vdots & \vdots & \vdots & & \vdots \\
f^{(k)}(y), & y\,f^{(k)}(y), & y^2\,f^{(k)}(y), & \ldots, & y^{q-1}\,f^{(k)}(y), \\
\vdots & \vdots & \vdots & & \vdots
\end{array} .$$

Now, we consider the first column. We will show that at most the first $n$ expressions of this column are linearly independent. By the given simple differential equation for $f$ there are polynomials $p$, and $p_k$ $(k = 0, \ldots, n-1)$ such that

$$p(y)\, f^{(n)}(y) = \sum_{k=0}^{n-1} p_k(y)\, f^{(k)}(y) \;,$$

or

$$p(x^{1/q})\, f^{(n)}(x^{1/q}) = \sum_{k=0}^{n-1} p_k(x^{1/q})\, f^{(k)}(x^{1/q}) \;. \tag{12}$$

If $p(x^{1/q})$ turns out to be a polynomial in $x$, then this equation tells that $f^{(n)}(x^{1/q})$ depends linearly (over $K[x]$) on the elements $x^{r/q}\, f^{(k)}(x^{1/q})$ $(r \in \mathbb{N}_0,\ 0 \le k \le n-1)$ of the rows above in our scheme. As this, however, is not necessarily the case, we construct a polynomial $r$ such that $p(x^{1/q})\, r(x^{1/q})$ is a polynomial in $x$.

Therefore we use the complex factorization $p(y) = c\,(y-y_1)\,(y-y_2)\cdots(y-y_m)$ of $p$, and by the identity

$$y^q - y_l^q = (y - y_l) \sum_{j=0}^{q-1} y^j\, y_l^{q-1-j}$$

we see that multiplication of $p$ by the polynomial

$$r(y) := \prod_{l=1}^{m} \left( \sum_{j=0}^{q-1} y^j\, y_l^{q-1-j} \right)$$

yields a polynomial $s$ in the variable $x^q$. Even though, for technical reasons, we use a complex factorization, it turns out that $r \in \mathbb{R}[x]$ $(\mathbb{Q}[x])$ if $p \in \mathbb{R}[x]$ $(\mathbb{Q}[x])$. Hence $P(x) := s(x^{1/q}) = p(x^{1/q})\, r(x^{1/q})$ is a polynomial in $x$. Thus we see that multiplying (12) by $r(y)$ generates the representation

$$P(x)\, f^{(n)}(x^{1/q}) = \sum_{k=0}^{n-1} P_k(x^{1/q})\, f^{(k)}(x^{1/q})$$

with polynomials $P$, and $P_k$ $(k = 0, \ldots, n-1)$.

This equation tells that $f^{(n)}(x^{1/q})$ depends linearly on the functions of the rows above. Similarly, by an induction argument, $f^{(j)}(x^{1/q})$ $(j > n)$ depend linearly on those functions $x^{r/q}\, f^{(k)}(x^{1/q})$ $(0 \le r \le q-1, 0 \le k \le n-1)$ of the first $n$ rows of $M$.

Considering the other columns, the same argument can be applied to show that $y^r\, f^{(j)}(y)$ depend linearly on the functions of the first $n$ rows of $M$. This shows finally that $L_f$ is generated by the $n\,q$ functions $x^{r/q}\, f^{(k)}(x^{1/q})$ $(0 \le r \le q-1, 0 \le k \le n-1)$, and therefore has dimension $\le n\,q$. □

Lemma 1 shows immediately that for $h := f \circ x^{1/q}$ the linear space $L_h$ over $K[x]$ generated by $h, h', h'', \ldots$ is a subset of $L_f$ (declared in Lemma 2), and so by Lemma 2 is of dimension $\le n\,q$. Thus we have proved the following Theorem (compare [15], Theorem 2.7, [14], MAPLE function `algebraicsubs`).

**Theorem 2** Let $f$ be simple of order $n$, and let $q \in \mathbb{N}$. Then $f \circ x^{1/q}$ is simple of order $\le n\,q$. □

Whereas Theorem 2 gives a complete answer with regard to the existence of a simple differential equation, in [15], Theorem 2.7, it is shown that this is true for the composition with each algebraic function $\varphi(x)$ with $\varphi(0) =)$), it does not tell anything how such a differential equation may be obtained. Therefore we continue to describe an algorithm that generates the differential equation of lowest order for $h$, which gives a second proof for Theorem 2.

**Algorithm to Theorem 2**: Given the differential equation

$$\sum_{k=0}^{n} p_k(x)\, f^{(k)}(x) = 0 \qquad (p_k \text{ polynomials } (k = 0, \ldots, n), \quad p_n \not\equiv 0) \tag{13}$$

for $f$ we have to construct a simple differential equation for $h(x) := f(x^{1/q})$ $(q \in \mathbb{N}, q > 1)$. In this construction Lemma 1 will play a key role as it did in the proof of Corollary 2. Therefore we prepare the given polynomials $p_k$ in the following way: If $P(x) = \sum_{j=0}^{N} a_j\, x^j$ is any polynomial with $a_N \neq 0$, then we decompose $P$ by the $q$ polynomials

$$P_m(x) := \sum_{\substack{j=0 \\ j\equiv m \pmod q}}^{N} a_j\, x^j \qquad (m = 0, 1, \ldots, q-1)$$

in the form

$$P(x) = \sum_{m=0}^{q-1} P_m(x)\ ,$$

and $P$ is the zero polynomial if and only if $P_m$ is the zero polynomial for all $m = 0, 1, \ldots, q-1$. If $m \in \{0, 1, \ldots, q-1\}$ then $j \equiv m \pmod q$ if and only if $j = s_j\, q + m$ with $s_j \in N_0$. Therefore we have

$$P_m(x) = x^m \sum_{\substack{j=0 \\ j\equiv m \pmod q}}^{N} a_j\, x^{s_j q}\ .$$

If we set

$$P_m^*(y) := \sum_{\substack{j=0 \\ j\equiv m \pmod q}}^{N} a_j\, y^{s_j}$$

we have $P_m(x) = x^m\, P_m^*(x^q)$, and therefore we arrive at the decomposition $P(x) = \sum_{m=0}^{q-1} x^m\, P_m^*(x^q)$ with the polynomials $P_m^*$. For the degrees of the polynomials $P_m^*(y)$ we obtain the relations

$$\deg P_m^*(y) \begin{cases} \leq & s_N \qquad (0 \leq m < r_N) \\ = & s_N \qquad (m = r_N) \\ \leq & s_N - 1 \quad (r_N < m \leq q-1) \end{cases}, \tag{14}$$

and $P_{r_N}^*(y) \not\equiv 0$, where we use the representation $N = s_N\, q + r_N$ $(s_N \in \mathbb{N}_0, r_N \in \mathbb{N}_0, 0 \leq r_N \leq q-1)$.

Now we continue with (13). We differentiate (13) iteratively $l^* := n(q-1)$ times to get the equations

$$\sum_{k=0}^{n+l} p_{kl}(x)\, f^{(k)}(x) = 0 \tag{15}$$

with polynomials $p_{kl}$ $(l = 0, 1, \ldots, l^*)$, and we note that $p_{n+l,l} = p_n \not\equiv 0$, and that $p_{k0} = p_k$. These form a set of $l^* + 1 = n(q-1) + 1$ equations.

We decompose each polynomial $p_{kl}$ $(l = 0, 1, \ldots, l^*)$ in the way introduced above to get

$$p_{kl}(x) = \sum_{m=0}^{q-1} x^m\, Q_{klm}(x^q)$$

with polynomials $Q_{klm}$. This brings our $l^*+1$ equations (15) into the form

$$\sum_{k=0}^{n+l}\sum_{m=0}^{q-1} Q_{klm}(x^q)\,\left(x^m\,f^{(k)}(x)\right)=0 \qquad (l=0,1,\ldots,l^*)\ . \tag{16}$$

If we write $k=s_k\,q+r_k$ ($s_k\in\mathbb{N}_0,\ r_k=0,1,\ldots,q-1$), a brief look in Lemma 1 shows that for the purpose to arrive at a differential equation for $h$ the "variables" to be eliminated in the above equations are the terms $\left(x^m\,f^{(k)}(x)\right)$ where $m\in\{0,1,\ldots,q-1\}\setminus\{r_k\}$. Every derivative $f^{(k)}$ so generates $q-1$ unknowns, and as we have $nq+1$ derivatives, these are $(nq+1)(q-1)=q(l^*+1)-1$ unknowns. Therefore we need to generate $q$ equations out of each of our $l^*+1$ equations (16) to arrive at a set of $q(l^*+1)$ equations, i. e. enough equations to yield the differential equation searched for.

This is done by multiplying each of the equations (16) by the factors $x^p$ ($p=0,\ldots,q-1$) generating the $q(l^*+1)$ equations

$$\sum_{k=0}^{n+l}\sum_{m=p}^{q+p-1} Q_{k,l,m-p}(x^q)\,\left(x^m\,f^{(k)}(x)\right)=\sum_{k=0}^{n+l}\sum_{j=0}^{q-1} Q_{klpj}(x^q)\,\left(x^j\,f^{(k)}(x)\right)=0\ , \tag{17}$$

$$(l=0,1,\ldots,l^*,\ k=s_k\,q+r_k,\ s_k\in\mathbb{N}_0,\ r_k=0,1,\ldots,q-1,\ p=0,1,\ldots,q-1)$$

of the $q(l^*+1)-1$ unknowns $\left(x^j\,f^{(k)}(x)\right)$, where we use the abbreviations

$$Q_{klpj}(y):=\begin{cases} y\,Q_{k,l,q+j-p}(y) & \text{if } j=0,\ldots,p-1\\ Q_{k,l,j-p}(y) & \text{if } j=p,\ldots,q-1\end{cases} \tag{18}$$

and

$$Q_{kl0j}(y):=Q_{klj}(y) \qquad (j=0,1,\ldots,q-1)\ . \tag{19}$$

We will show that these unknowns always can be eliminated, therefore arriving at a differential equation of order $nq$ for $h$.

We note, that if we differentiate (13) iteratively, generating $q$ new equations at each step by multiplication with $x^p$ ($p=0,\ldots,q-1$), and checking at each step if the unknowns can be eliminated, this algorithm obviously results in the differential equation of lowest order valid for $h$.

Now we show that in the last step the algorithm always succeeds. Therefore we rewrite the equations system (17) using the $q\times q$ matrices

$$A_{kl}(y):=\left(Q_{klpj}(y)\right)_{pj}\ .$$

With the aid of these matrices each block of equations with fixed $l$ can be written as the matrix equation

$$\sum_{k=0}^{n+l} A_{kl}(x^q)\begin{pmatrix} f^{(k)}(x)\\ x\,f^{(k)}(x)\\ x^2 f^{(k)}(x)\\ \vdots\\ x^{q-1}f^{(k)}(x)\end{pmatrix}=0\ .$$

Because of (18) and (19) the matrices $A_{kl}(y)$ are of the following special form

$$A_{kl}(y) = \begin{pmatrix} Q_{kl0}(y) & Q_{kl1}(y) & Q_{kl2}(y) & \cdots & Q_{k,l,q-2}(y) & Q_{k,l,q-1}(y) \\ y\,Q_{k,l,q-1}(y) & Q_{kl0}(y) & Q_{kl1}(y) & \cdots & Q_{k,l,q-3}(y) & Q_{k,l,q-2}(y) \\ y\,Q_{k,l,q-2}(y) & y\,Q_{k,l,q-1}(y) & Q_{kl0}(y) & \cdots & Q_{k,l,q-4}(y) & Q_{k,l,q-3}(y) \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ y\,Q_{kl2}(y) & y\,Q_{kl3}(y) & y\,Q_{kl4}(y) & \cdots & Q_{kl0}(y) & Q_{kl1}(y) \\ y\,Q_{kl1}(y) & y\,Q_{kl2}(y) & y\,Q_{kl3}(y) & \cdots & y\,Q_{k,l,q-1}(y) & Q_{kl0}(y) \end{pmatrix} .$$

From the relations $\sum\limits_{k=0}^{n+l} p_{kl}(x)\,f^{(k)}(x) = 0$ with $p_{kl}(x) = x^m \sum\limits_{m=0}^{q-1} Q_{klm}(x^q)$, $p_{n+l,l}(x) = p_n(x) \not\equiv 0$, and $p_{k0}(x) = p_k(x)$ we see that the matrix $A_{kl}(y)$ is completely determined by the polynomial $p_{kl}(x)$, and we have

$$A_{n+l,l}(y) = A_{n0}(y) =: A_n(y) .$$

In particular, the matrix $A_n(y)$ is completely determined by the nonvanishing coefficient polynomial $p_n(x)$ of $f^{(n)}$ in the original equation (13). We will show now that the matrix $A_n(y)$ is invertible. Let $p_n(x) = \sum\limits_{j=0}^{N} a_j\,x^j$ with $a_N \neq 0$, $N = s_N\,q + r_N$ ($s_N, r_N \in \mathbb{N}_0$, $0 \leq r_N \leq q-1$). We have $p_n(x) = p_{n0}(x) = x^m \sum\limits_{m=0}^{q-1} Q_{n0m}(x^q)$. If we use the notation $Q_m(y) := Q_{n0m}(y)$, then for $Q_m$ the degree relations (14) read

$$\deg Q_m(y) \begin{cases} \leq & s_N \qquad (0 \leq m < r_N) \\ = & s_N \qquad (m = r_N) \\ \leq & s_N - 1 \quad (r_N < m \leq q-1) \end{cases} ,$$

and we have $Q_{r_N} \not\equiv 0$. This information leads to the following observations about the degrees of the polynomial entries of the matrix $A_n$: The entries of $A_n$ for which the difference of colomn index and row index $\Delta$ is $\geq (r_N + 1)$ (region I) have degrees $\leq s_N - 1$, the entries for which $0 \leq \Delta \leq r_N$ (region II) have degrees $\leq s_N$, further if $-r_N + 1 \leq \Delta \leq -1$ (region III), then the degrees again are $\leq s_N$, and finally if $\Delta \leq -r_N$ (region IV), then the degrees are $\leq s_N + 1$.

Assume next that $p_n(0) \neq 0$, then we get $|A_n(0)| = Q_0^q(0) = p_n^q(0) \neq 0$, and therefore $A_n(y)$ is invertible. To obtain the same result if $p_n(0) = 0$, we use our degree observations. By the Weierstraß representation of the determinant $D = |d_{jk}|$ of a $q \times q$ matrix $(d_{jk})$

$$D = \sum_{\pi \in S_q} \text{sign}\,\pi\; d_{1,\pi(1)}\,d_{2,\pi(2)} \cdots d_{q,\pi(q)}$$

where the sum is to be taken over all permutations $\pi \in S_q$, we observe that in the Weierstraß representation of the determinant of our matrix $A_n(y)$ the summands

$$Q_0^q(y), \pm y\,Q_1^q(y), \ldots, \pm y^{r_N}\,Q_{r_N}^q(y), \ldots, y^{q-1}\,Q_{q-1}^q(y)$$

occur. Now it follows that

$$\deg\,\left(y^{r_N}\,Q_{r_N}^q(y)\right) = q\,s_N + r_N = N = \deg p_n(x) > 0 ,$$

and for all other summands $P(y)$ using our degree observations we get the relation

$$\deg P(y) < q\, s_N + r_N = N\ .$$

Hence the degree of $|A_n(y)|$ equals $N$, thus $|A_n(y)|$ cannot be the zero polynomial, and $A_n(y)$ therefore is invertible.

Using the inverse $A_n^{-1}$, we can now write

$$\begin{pmatrix} f^{(n+l)}(x) \\ x\,f^{(n+l)}(x) \\ x^2 f^{(n+l)}(x) \\ \vdots \\ x^{q-1} f^{(n+l)}(x) \end{pmatrix} = -\sum_{k=0}^{n+l-1} A_n^{-1}(x^q)\cdot A_{kl}(x^q) \begin{pmatrix} f^{(k)}(x) \\ x\,f^{(k)}(x) \\ x^2 f^{(k)}(x) \\ \vdots \\ x^{q-1} f^{(k)}(x) \end{pmatrix} \qquad (l = 0,\ldots,l^*)\ .$$

Using the fact that the inhomogeneous parts of these equations do not vanish, it is now easily established that the unknowns $\left(x^j\, f^{(k)}(x)\right)$ can be eliminated, which finishes the algorithm. □

After these preparations we are ready to state our main theorem (compare [15], [14]).

**Theorem 3** Let the two functions $f$ and $g$ be simple of order $n$ and $m$, respectively, and let $r$ be rational. Then

(a) $\int f(x)\, dx$ is simple of order $\leq n+1$ ,

(b) $f'$ is simple of order $\leq n$ ,

(c) $f+g$ is simple of order $\leq n+m$ ,

(d) $f\cdot g$ is simple of order $\leq n\, m$ ,

(e) $f\circ r$ is simple of order $\leq n$ ,

(f) $f\circ x^{p/q}$ $(p,q\in \mathbb{Z})$ is simple of order $\leq n\, q$ .

*Proof:* (a): Let $h = \int f(x)\, dx$, and let $f$ satisfy the simple differential equation (2), then obviously

$$\sum_{k=0}^{n} p_k\, h^{(k+1)} = 0\ ,$$

and so $h$ is simple of order $\leq n+1$.

(b): Let $h = f'$, and (2) be valid. If $p_0 \equiv 0$ then (2) is a simple differential equation valid for $h$, and so $h$ is simple of order $n$. If $p_0 \not\equiv 0$, however, then, dividing by $p_0$, we write (2) as

$$f = \sum_{k=1}^{n} r_k\, f^{(k)} \tag{20}$$

with rational functions $r_k$. If we differentiate the original differential equation, we get a further differential equation

$$\sum_{k=0}^{n+1} P_k\, f^{(k)} = 0$$

with polynomials $P_k$ $(k = 0, \ldots, n+1)$. We substitute $f$ by (20), and multiply by the common denominator to get a simple differential equation of order $\leq n$ for $h$.
(c): First we give an algebraic argument for the existence statement. Let $f$ and $g$ satisfy simple differential equations of order $n$ and $m$, respectively. We consider the linear space $L_f$ of functions with rational coefficients generated by $f, f', f'', \ldots, f^{(k)}, \ldots$. As $f, f', \ldots, f^{(n)}$ are linearly dependent by (2), and as the same conclusion follows for $f', f'', \ldots, f^{(n+1)}$ by differentiation, and so on inductively, the dimension of $L_f$ is $\leq n$. Similarly $L_g$ has dimension $\leq m$. We now build the sum $L_f + L_g$ which is of dimension $\leq n+m$. As $f+g, (f+g)', \ldots, (f+g)^{(k)}, \ldots$ are elements of $L_f + L_g$, arbitrary $n+m+1$ many of them are linearly dependent. In particular, $f+g$ is simple of order $\leq n+m$.

Now we like to present an algorithm which generates the differential equation for the sum: We may bring the given differential equations for $f$ and $g$ into the form

$$f^{(n)} = \sum_{j=0}^{n-1} p_j \, f^{(j)} \;, \tag{21}$$

and

$$g^{(m)} = \sum_{k=0}^{m-1} q_k \, g^{(k)} \;, \tag{22}$$

with rational functions $p_j$, and $q_k$ respectively. By iterative differentiation and recursive substitution of (21), and (22), we generate sets of rules

$$f^{(l)} = \sum_{j=0}^{n-1} p_j^l \, f^{(j)} \qquad (l = n, \ldots, n+m) \;, \tag{23}$$

and

$$g^{(l)} = \sum_{k=0}^{m-1} q_k^l \, g^{(k)} \qquad (l = m, \ldots, n+m) \;, \tag{24}$$

with rational functions $p_j^l$, and $q_k^l$, respectively. Now, by iterative differentiation of the defining equation $h := f+g$, we generate the set of equations

$$\begin{array}{rcl} h & = & f+g \\ h' & = & f'+g' \\ h'' & = & f''+g'' \\ \vdots & & \vdots \\ h^{(n+m)} & = & f^{(n+m)} + g^{(n+m)} \end{array} \quad . \tag{25}$$

Next, we take the first $\max\{n,m\}$ of these equations, and use the rules (23) and (24) to eliminate all occurences of $f^{(l)}$ $(l \geq n)$, and $g^{(l)}$ $(l \geq m)$. On the right hand sides remain the $n+m$ variables $f^{(l)}$ $(l = 0, \ldots, n-1)$ and $g^{(l)}$ $(l = 0, \ldots, m-1)$. We solve the remaining linear system of equations for the variables $h^{(l)}$ $(l = 0, \ldots, \max\{n,m\})$ trying to eliminate the variables $f^{(l)}$ $(l = 0, \ldots, n-1)$ and $g^{(l)}$ $(l = 0, \ldots, m-1)$. If this procedure is successful, it generates the simple differential equation for $h$, searched for.

If the procedure fails, however, we go on taking the first $\max\{n,m\}+1$ of equations (25), doing the same manipulations, and so on, until we take the whole set of equations (25), where the procedure must stop.

(d): This situation is handled in exactly the same way as case (c). The only difference is that to differentiate the product $h := fg$ we use the Leibniz rule, and the variables to be eliminated are the $n\,m$ products $f^{(j)}\,g^{(k)}$ $(j = 0, \ldots, n-1,\ k = 0, \ldots, m-1)$. It is easily seen that this procedure generates a differential equation for $h$ of order $\leq n\,m$.

That the algorithm stops is seen by the fact that finally we arrive at a set of $n\,m + 1$ equations with $n\,m$ variables to be eliminated, where all equations actually possess a nonvanishing inhomogeneous part $h^{(l)}$.

(e): Let $h := f \circ r$ for some rational $r$, and let $f$ satisfy (2). We compose (2) with $r$, and get

$$0 = \sum_{k=0}^{n} (p_k \circ r) \cdot (f^{(k)} \circ r) = \sum_{k=0}^{n} q_k \cdot \left(f^{(k)} \circ r\right) \qquad (q_k \text{ rational functions}) \ . \tag{26}$$

Differentiating the identity $h = f \circ r$ leads to

$$h' = f' \circ r \cdot r' \qquad \text{or} \qquad f' \circ r = h'/r' \ ,$$
$$h'' = f' \circ r \cdot r'' + f'' \circ r \cdot r'^2 \qquad \text{or} \qquad f'' \circ r = \frac{1}{r'^2} \left(h'' - f' \circ r \cdot r''\right) = \frac{1}{r'^2} \left(h'' - \frac{r''}{r'} h'\right) \ ,$$

and inductively it follows

$$h^{(k)} = \sum_{j=1}^{k} r_j \cdot \left(f^{(j)} \circ r\right) \qquad (r_j \text{ rational functions}) \ .$$

Solving for $f^{(k)} \circ r$ and substituting the previous results we are lead to a representation

$$f^{(k)} \circ r = \sum_{j=1}^{k} R_j \cdot h^{(j)}$$

with rational functions $R_j$.

Substituting these results in (26) and multiplying by the common denominator yields a simple differential equation for $h$, and we see that the order of $h$ is $\leq n$.

(f): The statement follows immediately by an application of Theorem 2 and (e). □

**Remark 2** We remark that again our proofs provide algorithms to generate the differential equations searched for.

Further we note that by Theorem 1 the functions $1/g$ and $f/g$ in general are not simple, if $f$ and $g$ are, as $\sec x = 1/\cos x$ and $\tan x = \sin x/\cos x$ show.

Note that Theorem 3 (f) generalizes Theorem 8.1 in [6].

The following consideration strengthens Theorem 3 (c), and brings it in connection with the fundamental systems of simple differential equations.

**Corollary 4** Assume, $f$ satisfies a simple differential equation $F$ of order $n$, $g$ satisfies a simple differential equation $G$ of order $m$, and $F$ and $G$ do not have any *common nontrivial* solution. Then the order of any simple differential equation satisfied by the sum $h := f + g$ is $\geq n + m$. In particular, $h$ satisfies no simple differential equation of order $< n + m$, and exactly one differential equation $H$ of order $n + m$ which is generated by the algorithm given in Theorem 3 (c).

*Proof:* Let $f_1, f_2, \ldots, f_n$ be a fundamental system of the differential equation $F$, and $g_1, g_2, \ldots, g_m$ be a fundamental system of the differential equation $G$. Then $f_1, f_2, \ldots, f_n$ are linearly independent (in the usual sense), and $g_1, g_2, \ldots, g_m$ are linearly independent, as well. Assume now, the $n+m$ functions $f_1, f_2, \ldots, f_n, g_1, g_2, \ldots, g_m$ are linearly dependent. Then there are constants $a_j$ $(k = j, \ldots, n)$ and $b_k$ $(k = 1, \ldots, m)$, not all equal to zero, such that

$$\sum_{j=1}^{n} a_j\, f_j + \sum_{k=1}^{m} b_k\, g_k = 0\ . \tag{27}$$

From the linear independence of the subsets $f_1, f_2, \ldots, f_n$, and $g_1, g_2, \ldots, g_m$ it follows that at least one of the numbers $a_j$ $(j = 1, \ldots, n)$, and one of the numbers $b_k$ $(k = 1, \ldots, m)$ does not vanish. Therefore by the linearity of $F$ and $G$ the functions $f := \sum\limits_{j=1}^{n} a_j\, f_j$ and $g := -\sum\limits_{k=1}^{m} b_k\, g_k$ are solutions of $F$, and $G$, respectively, which by (27) agree, and we have a contradiction.

Thus we have proved that $f_1, f_2, \ldots, f_n, g_1, g_2, \ldots, g_m$ are linearly independent. As the zero function is a solution of both $F$, and $G$, it turns out that $f_1, f_2, \ldots, f_n, g_1, g_2, \ldots, g_m$ must satisfy any linear differential equation $H$ for the sum, and so form a fundamental system for $H$. Therefore the order of $H$ is $\geq n+m$. □

We mention that, on the other hand, if the algorithm presented in Theorem 3 (c) generates a differential equation of order $< m+n$ for $h := f+g$, then the simple differential equations $F$ for $f$ and $G$ for $g$ must have a common nonzero solution, i. e. the fundamental systems $F$ and $G$ are linearly dependent. We gain this insight without solving any of the differential equations!

From Theorem 3 (d) a similar statement can be obtained for the product $h := fg$: If our algorithm generates a differential equation of order $< n\,m$ for $h$, then the product of the fundamental systems $S_F$ and $S_G$ of $F$ and $G$, i. e. the set $\{f\,g\,|f \in S_F,\ g \in S_G\}$, is linearly dependent. Again, we gain this insight without solving any of the differential equations.

In connection with Theorem 3 (d) for the product $h := fg$ we note that for the special case of the square $h := f^2 = f \cdot f$ the bound for the order of $h$ can be considerably strengthened.

**Corollary 5** Let $f$ be simple of order $n$, then $h := f^2$ is simple of order $\leq \frac{n(n+1)}{2}$.

*Proof:* A careful study of the algorithm given in the proof of Theorem 3 (d) shows that the order cannot exceed $\frac{n(n+1)}{2}$. This depends on the fact that the variables to be eliminated are the $\frac{n(n+1)}{2}$ different products $f^{(j)}\,f^{(k)}$ $(j = 0, \ldots, n-1,\ k = 0, \ldots, n-1)$. □

# 3 Algorithmic calculation of simple differential equations

Theorem 3 enables us to define the linear space $F$ of functions generated by the algebraic functions, $\exp x$, $\ln x$, $\sin x$, $\cos x$, $\arcsin x$, $\arctan x$, (and, if we wish, further special functions satisfying simple differential equations,) and the functions that are constructed by an application of finitely many of the procedures (a)–(f) of Theorem 3. The theorem then states in particular that $F$ forms a differential ring.

The algorithm which generates the simple differential equation of lowest order for $f$ given in [6]–[9] is strengthened by Theorem 3.

**Algorithm 1 (Find a simple differential equation)** Let $f \in F$. Then the following procedure generates the simple differential equation of lowest order valid for $f$:

(a) Find out whether there exists a simple differential equation for $f$ of order $N := 1$. Therefore differentiate $f$, and solve the linear equation

$$f'(x) + A_0 f(x) = 0$$

for $A_0$; i. e. set $A_0 := -\frac{f'(x)}{f(x)}$. Is $A_0$ rational in $x$, then you are done after multiplication with its denominator.

(b) Increase the order $N$ of the differential equation searched for by one. Expand the expression

$$f^{(N)}(x) + A_{N-1} f^{(N-1)}(x) + \cdots + A_0 f(x) \,,$$

and check, if the summands contain exactly $N$ rationally independent expressions (i. e.: linearly independent over $K[x]$) considering the numbers $A_0, A_1, \ldots, A_{N-1}$ as constants. Just in that case there exists a solution as follows: Sort with respect to the rationally independent terms and create a system of linear equations by setting their coefficients to zero. Solve this system for the numbers $A_0, A_1, \ldots, A_{N-1}$. Those are rational functions in $x$, and there exists a unique solution. After multiplication by the common denominator of $A_0, A_1, \ldots, A_{N-1}$ you get the differential equation searched for. Finally cancel common factors of the polynomial coefficients.

(c) If part (b) was not successful, repeat step (b).

*Proof:* The proof given in [6] shows that the given algorithm indeed generates the simple differential equation of lowest order valid for $f$ whenever such a differential equation exists. Theorem 3 now guarantees the existence of such a differential equation, and therefore the algorithm does not end in an infinite loop. □

**Remark 3** Obviously $f \in F$ can be checked by a pattern matching mechanism applied to the given expression $f$, and Theorem 3 then moreover gives a priori estimates for the possible order of the solution differential equation.

We further point out that an actual implementation of the algorithm should be able to decide the rational dependency of a set of functions. Otherwise, it may fail to find the simple differential equation of lowest order. An example of this type is given by $f(x) := \sin(2x) - 2 \sin x \cos x$, for which the algorithm yields the differential equation $f'' + f = 0$ rather than $f' = 0$ if the rational dependency of $\sin(2x)$, and $\sin x \cos x$ is not discovered.

In a forthcoming paper [11] we discuss the more general situation of functions $f$ that depend on other special functions not contained in $F$, like Airy functions, Bessel functions and orthogonal polynomials.

We mention that the algorithms presented in Theorem 3 can be combined to get the following different

**Algorithm 2 (Find a simple differential equation)** Let be $f \in F$. Then recursive application of the algorithms presented in Theorem 3 generates a simple differential equation for $f$.

**Remark 4** We note that, in general, we cannot control the order of the resulting differential equation as the algorithm uses recursive descent through the expression tree of the given $f$. Given a sum of two functions or order $n$ and $m$, however, it is easily seen that the order of the resulting differential equation is the lowest possible order which is $\geq \max\{n, m\}$. So with the given algorithm, it is principally impossible to derive any differential equation of order lower than two for the example function $f(x) := \sin(2x) - 2 \sin x \cos\ x$ as the lowest order differential equations for the summand functions $\sin(2x)$, and $\sin x \cos\ x$ are of order two.

This can be interpreted as follows: By construction, for a sum $f := f_1 + f_2$ a differential equation is obtained that is valid not only for $f_1 + f_2$, but for any linear combination $\lambda_1 f_1 + \lambda_2 f_2$ or $f_1$ and $f_2$. In other words, this subalgorithm generates a differential equation for the linear hull of $f_1$ and $f_2$.

# 4 Sharpness of the orders

In this section we give examples that show that the bounds for the least orders that are given in Theorems 2–3, and in Corollary 5 in fact may be assumed.

First we consider the statement of Corollary 5. We get

**Theorem 4** For each $n \in \mathbb{N}$ there is a function $f_n$ that is simple of order $n$, such that $h := f_n^2$ is simple of order $\frac{n(n+1)}{2}$. Indeed, an example function of that type is

$$f_n(x) := \sum_{k=1}^{n} e^{x^k} . \tag{28}$$

*Proof:* We show that $f_n$, given by (28), is simple of order $n$, and that $f_n^2$ is simple of order $\frac{n(n+1)}{2}$.

As each function $e^{x^k}$ $(k = 1, \ldots, n)$ is simple of order 1 (satisfying the differential equation $f' - k\, x^{k-1} f = 0$), by an inductive application of Theorem 3 (c), we see that $f_n$ is simple of order $\leq n$. To show that $f_n$ is simple of order $\geq n$, we first show that $e^{x^k}$ $(k = 1, \ldots, n)$ are rationally independent. Suppose, any linear combination

$$r_1\, e^x + r_2\, e^{x^2} + \cdots + r_n\, e^{x^n} = 0 \tag{29}$$

with rational functions $r_k$ $(k = 1, \ldots, n)$ representing zero is given. From (29) we get, assuming $r_k \not\equiv 0$

$$r_k\, e^{x^k} = - \sum_{\substack{j=1 \\ j \neq k}}^{n} r_j\, e^{x^j} ,$$

and dividing by $r_k\, e^{x^k}$, we get

$$1 = - \sum_{\substack{j=1 \\ j \neq k}}^{n} R_j\, e^{x^j - x^k}$$

with rational functions $R_j$ $(j = 1, \ldots, n)$. For $k = n$, we get by taking the limit $x \to \infty$, along the positive real axis

$$1 = - \lim_{x \to \infty} \sum_{j=1}^{n-1} R_j\, e^{x^j - x^n} = 0 ,$$

a contradiction to the assumption $r_n \not\equiv 0$. For $k < n$, the right hand limit

$$- \lim_{x\to\infty} \sum_{\substack{j=1 \\ j\neq k}}^{n} R_j \, e^{x^j - x^k}$$

does not exist, whereas the left hand side equals 1, a contradiction to the assumption $r_k \not\equiv 0$. Therefore $r_k \equiv 0$ $(k = 1, \ldots, n)$, and $e^{x^k}$ $(k = 1, \ldots, n)$ are rationally independent.

From this it follows that applying Algorithm 1 to $f_n$, in each step $n$ rationally independent are involved (the derivative of $e^{x^k}$ is of the same type), so that the algorithm can be successful not earlier than in the $n$th step. As this algorithm always finds the simple differential equaton of least order, the order of $f_n$ is $\geq n$.

Next we consider the function $f_n^2$. Expanding $f_n^2$ we get the representation

$$f_n^2(x) = e^{2x} + e^{2x^2} + \cdots + e^{2x^n} + 2\, e^{x}\, e^{x^2} + \cdots + 2\, e^{x^{n-1}}\, e^{x^n}$$

consisting of $n + \binom{n}{2} = \frac{n(n+1)}{2}$ summands. An application of Theorem 3 (c), again, shows that $f_n^2$ is simple of order $\leq \frac{n(n+1)}{2}$, as each summand $f := e^{x^j}\, e^{x^k}$ satisfies the simple differential equation

$$f' - 2\,(j\, x^j + k\, x^k)\, f = 0$$

of order 1. Similarly as above one realizes that $e^{x^j}\, e^{x^k}$ $(j, k = 1, \ldots, n$ $(j \neq k))$ are rationally independent, and therefore by an application of Algorithm 1 we conclude that $f_n^2$ is simple of order $\geq \frac{n(n+1)}{2}$, finishing the proof. $\square$

Next we consider the statement of Theorem 2. We get

**Theorem 5** For each $n, q \in \mathbb{N}$ there is a function $f_{nq}$ that is simple of order $n$, such that $h := f \circ x^{1/q}$ is simple of order $n\, q$. Indeed, an example function of that type is

$$f_{nq}(x) := \sum_{k=1}^{n} e^{x^{kq+1}} \; . \tag{30}$$

*Proof:* Everything follows as in Theorem 4, if we show that

$$f(x) := e^{x^{kq+1}} \qquad (k = 1, \ldots, n, \; q \in \mathbb{N})$$

is simple of order 1, whereas

$$h(x) := e^{x^{k+1/q}} \qquad (k = 1, \ldots, n, \; q \in \mathbb{N})$$

is simple of order $q$. The first statement was already proven so that it remains to prove the second statement. This, however, turns out to be a consequence, of Lemma 1 (a). We start with one rationally independent function, namely $h$ itself. As $h(x) = f(x^{1/q})$, Lemma 1 (a) shows that in the $r$th differentiation step $(r \leq q)$ for $h$ the new summand

$$x^{r/q}\, f^{(r)}(x^{1/q})$$

is introduced. By a similar argument as in Theorem 4 one sees that for each $r \leq q-1$ the functions involved

$$\left\{h(x), x^{1/q} f'(x^{1/q}), \ldots, x^{r/q} f^{(r)}(x^{1/q})\right\}$$

are rationally independent.

In the $q$th step, however, by (5), $h^{(q)}$ rationally depends on $h, h', \ldots, h^{(q-1)}$, so that an arbitrary linear combination of $h, h', \ldots h^{(q)}$ contains exactly $q$ rationally independent terms, showing that $h$ is of order $q$. □

Finally we consider the statements of Theorem 3. We get

**Theorem 6** The statements of Theorem 3 are sharp which is seen by the following example functions:

(a) For each $n$ the function $f_n$ given by (28) is simple of order $n$, and $\int f_n$ is simple of order $n+1$.

(b) For each $n$ the function $f_n$ given by (28) is simple of order $n$, and $f_n'$ is simple of order $n$, too.

(c) For each $n, m \in \mathbb{N}$ there are functions $f$ and $g$ that are simple of order $n$, and $m$, respectively, such that $h := f + g$ is simple of order $n + m$. Indeed, example functions of that type are

$$f(x) := \sum_{k=1}^{n} e^{x^k} \qquad \text{and} \qquad g(x) := \sum_{k=n+1}^{n+m} e^{x^k} . \tag{31}$$

(d) For each $n, m \in \mathbb{N}$ there are functions $f$ and $g$ that are simple of order $n$, and $m$, respectively, such that $h := f\,g$ is simple of order $n\,m$. Indeed, example functions of that type are given by (31).

(e) For each $n$ the function $f_n$ given by (28) is simple of order $n$, and for each rational function $r$ the function $f \circ r$ is simple of order $n$, too.

(f) For each $n$ the function $f_{nq}$ given by (30) is simple of order $n$, and for each $p \in \mathbb{N}$ the function $f \circ x^{p/q}$ is simple of order $n\,q$.

*Proof:* (a) By Theorem 4 $f_n$ is simple of order $n$. As the antiderivative of $f_n$ is not an elementary function, it is rationally independent, and so its order is $\geq n + 1$.
(b) As the derivatives of each summand of $f_n$ depend rationally on $f_n$, by the linearity the order of $f_n'$ equals $n$, too.
(c) In Theorem 4 we already proved that $f$ is simple of order $n$, and that $f + g$ is simple of order $n + m$. A similar argument shows that $g$ is simple of order $m$, and we are done.
(d) If we expand the function $f\,g$, we get

$$f(x)\,g(x) = \sum_{k=1}^{n} e^{x^k} \sum_{k=n+1}^{n+m} e^{x^k} = \sum_{k=1}^{n} \sum_{j=1}^{m} e^{x^k}\, e^{x^{j+n}}$$

with $n\,m$ rationally independent expressions. Thus the order of $f\,g$ is $\geq n\,m$.
(e) By the argumentation of Theorem 4 one sees that $f_n \circ r$ has order $n$.
(f) This follows from (e) and Theorem 5. □

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