Two Finite Classes of Orthogonal Functions

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Abstract. By using Fourier transforms of two symmetric sequences of finite orthogonal polynomials, we introduce two new classes of finite orthogonal functions and obtain their orthogonality relations via Parseval's identity.

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1. Introduction. There exist ten sequences of real polynomials [6, 10] that are orthogonal with respect to the Pearson distributions family

$$W\left(\begin{matrix} d, & e \\ a, & b, & c \end{matrix}\middle| x\right) = \exp\left(\int \frac{dx+e}{ax^2+bx+c}\,dx\right) \qquad (a,b,c,d,e\in \mathbf{R})\,, \tag{1}$$

and its symmetric analogue [10]

$$W^*\left(\begin{matrix} r, & s \\ p, & q \end{matrix}\middle| x\right) = \exp\left(\int \frac{r\,x^2+s}{x(px^2+q)}\,dx\right) \qquad (p,q,r,s\in \mathbf{R})\,. \tag{2}$$

Five of them are infinitely orthogonal with respect to special cases of the two above-mentioned distribution (weight) functions and five other ones are finitely orthogonal [6,10] which are limited to some parametric constraints. The following table shows the main properties of these ten sequences.

Table 1: *Characteristics of ten sequences of orthogonal polynomials*

Symbol	Weight Function	Kind, Interval & Parameters constraint
$P_n^{(u,v)}(x)$	$W\left(\begin{matrix} -u-v, & -u+v \\ -1, & 0, & 1 \end{matrix}\middle\vert x\right) = (1-x)^u(1+x)^v$	Infinite, $[-1,1]$, $\forall n,\ u>-1, v>-1$
$L_n^{(u)}(x)$	$W\left(\begin{matrix} -1, & u \\ 0, & 1, & 0 \end{matrix}\middle\vert x\right) = x^u \exp(-x)$	Infinite, $[0,\infty)$, $\forall n,\ u>-1$
$H_n(x)$	$W\left(\begin{matrix} -2, & 0 \\ 0, & 0, & 1 \end{matrix}\middle\vert x\right) = \exp(-x^2)$	Infinite, $(-\infty,\infty)$
$J_n^{(u,v)}(x;a,b,c,d)$	$\begin{aligned}&((ax+b)^2+(cx+d)^2)^{-u} \\ &\times \exp(v\arctan\frac{ax+b}{cx+d})\end{aligned}$	Finite, $(-\infty,\infty)$, $\max n < u-1/2$, $\det(a,b,c,d)\neq 0$
$M_n^{(u,v)}(x)$	$W\left(\begin{matrix} -u, & v \\ 1, & 1, & 0 \end{matrix}\middle\vert x\right) = x^v(x+1)^{-(u+v)}$	Finite, $[0,\infty)$, $\max n < (u-1)/2$, $v>-1$
$N_n^{(u)}(x)$	$W\left(\begin{matrix} -u, & 1 \\ 1, & 0, & 0 \end{matrix}\middle\vert x\right) = x^{-u}\exp(-1/x)$	Finite, $[0,\infty)$, $\max n < (u-1)/2$
$S_n\left(\begin{matrix} -2u-2v-2, & 2u \\ -1, & 1 \end{matrix}\middle\vert x\right)$	$W^*\left(\begin{matrix} -2u-2v, & 2u \\ -1, & 1 \end{matrix}\middle\vert x\right) = x^{2u}(1-x^2)^v$	Infinite, $[-1,1]$, $u>-1/2,\ v>-1$
$S_n\left(\begin{matrix} -2, & 2u \\ 0, & 1 \end{matrix}\middle\vert x\right)$	$W^*\left(\begin{matrix} -2, & 2u \\ 0, & 1 \end{matrix}\middle\vert x\right) = x^{2u}\exp(-x^2)$	Infinite, $(-\infty,\infty)$, $u>-1/2$
$S_n\left(\begin{matrix} -2u-2v+2, & -2u \\ 1, & 1 \end{matrix}\middle\vert x\right)$	$W^*\left(\begin{matrix} -2u-2v, & -2u \\ 1, & 1 \end{matrix}\middle\vert x\right) = x^{-2u}(1+x^2)^{-v}$	Finite, $(-\infty,\infty)$, $\max n < u+v-1/2$, $u<1/2,\ v>0$
$S_n\left(\begin{matrix} -2u+2, & 2 \\ 1, & 0 \end{matrix}\middle\vert x\right)$	$W^*\left(\begin{matrix} -2a, & 2 \\ 1, & 0 \end{matrix}\middle\vert x\right) = x^{-2a}e\,\mathrm{xp}(-1/x^2)$	Finite, $(-\infty,\infty)$, $\max n < u-1/2$

where the sequence

$$\Phi_n(x) = S_n\left(\begin{matrix} r, & s \\ p, & q \end{matrix}\middle\vert x\right) = \sum_{k=0}^{[n/2]} \binom{[n/2]}{k}\left(\prod_{i=0}^{[n/2]-(k+1)} \frac{\left(2i+(-1)^{n+1}+2[n/2]\right)p+r}{\left(2i+(-1)^{n+1}+2\right)q+s}\right) x^{n-2k}, \quad (3)$$

is a basic class of symmetric orthogonal polynomials [10] satisfying the equation

$$x^2(px^2+q)\Phi_n''(x) + x(rx^2+s)\Phi_n'(x) - \left(n(r+(n-1)p)x^2+(1-(-1)^n)s/2\right)\Phi_n(x) = 0.$$

$$(4)$$

Except the two finite orthogonal polynomial sequences $S_n\left(\begin{matrix} -2u-2v+2, & -2u \\ 1, & 1 \end{matrix} \middle| x \right)$ and

$S_n\left(\begin{matrix} -2u+2, & 2 \\ 1, & 0 \end{matrix} \middle| x \right)$ in table 1, Fourier transforms of all ten aforesaid sequences have been found. Indeed, in [8] Fourier transforms of the generalized ultraspherical polynomials $S_n\left(\begin{matrix} -2u-2v-2, & 2u \\ -1, & 1 \end{matrix} \middle| x \right)$ and generalized Hermite polynomials $S_n\left(\begin{matrix} -2, & 2u \\ 0, & 1 \end{matrix} \middle| x \right)$ have been derived. In [9] the Fourier transform of Routh-Romanovski polynomials $J_n^{(u,v)}(x;a,b,c,d)$ has been obtained, and in [5] the Fourier transforms of finite orthogonal polynomials $M_n^{(u,v)}(x)$ and $N_n^{(u)}(x)$ have been calculated. In this sense, note that the Fourier transforms of classical Jacobi, Laguerre and Hermite polynomials are already known, see e.g. [3,7]. Hence, to complete the analysis of the families of orthogonal polynomials of table 1, only Fourier transforms of the two above-mentioned finite sequences remain, which should be determined. To reach this purpose, we need the general properties of these two sequences.

1.1. Finite orthogonal polynomials with weight $x^{-2a}(1+x^2)^{-b}$ on $(-\infty,\infty)$

If $(p,q,r,s)=(1,1,-2a-2b+2,-2a)$ is substituted in (4), then the equation

$$x^2(x^2+1)\Phi_n''(x)-2x((a+b-1)x^2+a)\Phi_n'(x)+\left(n(2a+2b-(n+1))x^2+(1-(-1)^n)a\right)\Phi_n(x)=0,$$
(5)

has the explicit solution

$$\Phi_n(x)=S_n\left(\begin{matrix} -2a-2b+2, & -2a \\ 1, & 1 \end{matrix} \middle| x \right)=\sum_{k=0}^{[n/2]}\binom{[n/2]}{k}\left(\prod_{i=0}^{[\frac{n}{2}]-(k+1)}\frac{2i+2[n/2]+(-1)^{n+1}+2-2a-2b}{2i+(-1)^{n+1}+2-2a}\right)x^{n-2k},$$
(6)

whose monic form is equivalent to the hypergeometric representation

$$A_n^{(a,b)}(x)=\bar{S}_n\left(\begin{matrix} -2a-2b+2, & -2a \\ 1, & 1 \end{matrix} \middle| x \right)=x^n\,{}_2F_1\left(\begin{matrix} -[n/2], & a+1/2-[(n+1)/2] \\ a+b-n+1/2 \end{matrix} \middle| -\frac{1}{x^2} \right).$$
(7)

${}_2F_1(.)$ in (7) is a special case of the generalized hypergeometric functions [1,4]

$$_pF_q\left(\begin{matrix} a_1 & a_2 & ... & a_p \\ b_1 & b_2 & ... & b_q \end{matrix} \middle| x \right)=\sum_{k=0}^{\infty}\frac{(a_1)_k(a_2)_k\,...(a_p)_k}{(b_1)_k(b_2)_k\,...(b_q)_k}\frac{x^k}{k!},$$
(8)

where $(r)_k=r(r+1)...(r+k-1)$.

The monic polynomials (7) satisfy the orthogonality relation

$$\int_{-\infty}^{\infty} \frac{x^{-2a}}{(1+x^2)^b} A_n^{(a,b)}(x) A_m^{(a,b)}(x)\, dx = (-1)^n \prod_{j=1}^{n} C_j \begin{pmatrix} -2a-2b+2, & -2a \\ 1, & 1 \end{pmatrix}$$
$$\times \frac{\Gamma(b+a-1/2)\Gamma(-a+1/2)}{\Gamma(b)}\, \delta_{n,m}, \tag{9}$$

in which

$$C_j \begin{pmatrix} -2a-2b+2, & -2a \\ 1, & 1 \end{pmatrix} = \frac{(j-(1-(-1)^j)a)(j-(1-(-1)^j)a-2b)}{(2j-2a-2b+1)(2j-2a-2b-1)}, \tag{10}$$

and

$$\int_{-\infty}^{\infty} \frac{x^{-2a}}{(1+x^2)^b}\, dx = \int_0^{\infty} \frac{t^{-a-1/2}}{(1+t)^b}\, dt = B(-a+\frac{1}{2};b+a-\frac{1}{2}) = \frac{\Gamma(b+a-1/2)\Gamma(-a+1/2)}{\Gamma(b)}. \tag{11}$$

According to [10], relation (9) is valid only if $m,n = 0,1,...,N \le a+b-1/2$ where $N = \max\{m,n\}$; $a < 1/2$; $(-1)^{2a} = 1$ and $b > 0$. Moreover, $B(\lambda_1,\lambda_2)$ in (11) denotes the Beta integral [1] having various definitions as

$$B(\lambda_1;\lambda_2) = \int_0^1 x^{\lambda_1-1}(1-x)^{\lambda_2-1}\, dx = \int_{-1}^1 x^{2\lambda_1-1}(1-x^2)^{\lambda_2-1}\, dx = \int_0^{\infty} \frac{x^{\lambda_1-1}}{(1+x)^{\lambda_1+\lambda_2}}\, dx$$
$$= 2\int_0^{\pi/2} \sin^{(2\lambda_1-1)} x \cos^{(2\lambda_2-1)} x\, dx = \frac{\Gamma(\lambda_1)\Gamma(\lambda_2)}{\Gamma(\lambda_1+\lambda_2)} = B(\lambda_2;\lambda_1), \tag{12}$$

in which

$$\Gamma(z) = \int_0^{\infty} x^{z-1} e^{-x} dx \qquad \mathrm{Re}(z) > 0, \tag{13}$$

denotes the well-known Gamma function satisfying the equation $\Gamma(z+1) = z\Gamma(z)$.

1.2. Finite orthogonal polynomials with weight $x^{-2a} e^{-1/x^2}$ on $(-\infty,\infty)$

Similarly, if $(p,q,r,s) = (1,0,-2a+2,2)$ is substituted in (4), then the equation

$$x^4 \Phi_n''(x) + 2x((1-a)x^2+1)\Phi_n'(x) - \big(n(n+1-2a)x^2+1-(-1)^n\big)\Phi_n(x) = 0, \tag{14}$$

has the explicit solution

$$\Phi_n(x) = S_n \begin{pmatrix} -2a+2 & 2 \\ 1 & 0 \end{pmatrix} x = \sum_{k=0}^{[n/2]} \binom{[n/2]}{k} \left(\prod_{i=0}^{[\frac{n}{2}]-(k+1)} \frac{2i+2[n/2]+(-1)^{n+1}+2-2a}{2} \right) x^{n-2k}, \tag{15}$$

whose monic form is equivalent to the hypergeometric form

$$B_n^{(a)}(x) = \bar{S}_n \left(\begin{matrix} -2a+2 & 2 \\ 1 & 0 \end{matrix} \middle| x \right) = x^n \,_1F_1 \left(\begin{matrix} -[n/2] \\ a+(-1)^n/2 \end{matrix} \middle| \frac{1}{x^2} \right). \tag{16}$$

Moreover, the orthogonality relation corresponding to these polynomials takes the form

$$\int_{-\infty}^{\infty} x^{-2a} e^{-\frac{1}{x^2}} B_n^{(a)}(x) B_m^{(a)}(x)\, dx = \left((-1)^n \prod_{j=1}^{n} C_j \left(\begin{matrix} -2a+2 & 2 \\ 1 & 0 \end{matrix} \right) \right) \Gamma\left(a-\frac{1}{2}\right) \delta_{n,m}, \tag{17}$$

where [10]

$$C_j \left(\begin{matrix} -2a+2 & 2 \\ 1 & 0 \end{matrix} \right) = \frac{-2(-1)^j(j-a)-2a}{(2j-2a+1)(2j-2a-1)}, \tag{18}$$

and $m,n = 0,1,...,N \le a-1/2$ with $N = \max\{m,n\}$ and $(-1)^{2a} = 1$.

It is known that some orthogonal polynomials are mapped onto each other by the Fourier transform. In this paper, we follow this approach for the two finite orthogonal polynomials (described in sections 1.1 and 1.2) to obtain two new classes of finite orthogonal functions via Parseval's identity.

2. Fourier transform of monic polynomials $A_n^{(a,b)}(x)$ and $B_n^{(a)}(x)$ and their orthogonality relations

The Fourier transform of a function, say $g(x)$, is defined by [2]

$$\mathbf{F}(s) = \mathbf{F}(g(x)) = \int_{-\infty}^{\infty} e^{-isx} g(x)\, dx, \tag{19}$$

and for the inverse transform we have

$$g(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{isx} \mathbf{F}(s)\, ds. \tag{20}$$

For $g, h \in L^2(\mathbb{R})$, the Parseval identity related to a Fourier transform is given by [2]

$$\int_{-\infty}^{\infty} g(x) \overline{h(x)}\, dx = \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathbf{F}(g(x)) \overline{\mathbf{F}(h(x))}\, ds. \tag{21}$$

By noting the relations (9) and (21), let us define the functions

$$\begin{cases} g(x) = x^{-2\alpha}(1+x^2)^{-\beta} A_n^{(c,d)}(x) & \text{s.t. } (-1)^{2\alpha} = 1, \\ h(x) = x^{-2l}(1+x^2)^{-u} A_m^{(v,w)}(x) & \text{s.t. } (-1)^{2l} = 1, \end{cases} \tag{22}$$

in terms of the monic polynomials (7) to which we shall apply the Fourier transform.

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Notice that for both above functions the Fourier transform exists. For instance, for $g(x)$ defined in (22) we have

$$\mathbf{F}(g(x)) = \int_{-\infty}^{\infty} e^{-isx}(1+x^2)^{-\beta} x^{-2\alpha} A_n^{(c,d)}(x)\, dx$$

$$= \int_{-\infty}^{\infty} e^{-isx}(1+x^2)^{-\beta} x^{-2\alpha+n}\left(\sum_{k=0}^{[n/2]} \frac{(-[n/2])_k (c+1/2-[(n+1)/2])_k}{(c+d-n+1/2)_k k!} \frac{(-1)^k}{x^{2k}}\right) dx \qquad (23)$$

$$= \sum_{k=0}^{[n/2]} \frac{(-[n/2])_k (c+1/2-[(n+1)/2])_k (-1)^k}{(c+d-n+1/2)_k k!}\left(\int_{-\infty}^{\infty} e^{-isx}(1+x^2)^{-\beta} x^{-2\alpha+n-2k}\, dx\right).$$

Now it remains in (23) to evaluate the definite integral

$$I_{n,k}(s;\alpha,\beta) = \int_{-\infty}^{\infty} e^{-isx}(1+x^2)^{-\beta} x^{-2\alpha+n-2k}\, dx. \qquad (24)$$

There are two ways to compute the integral (24). In the first way, by noting that $(-1)^{2\alpha}=1$, we can directly compute $I_{n,k}(s;\alpha,\beta)$ for $n=2m$ as follows

$$I_{2m,k}(s;\alpha,\beta) = \int_{-\infty}^{\infty}\left(\sum_{j=0}^{\infty}\frac{(-isx)^j}{j!}\right) x^{-2\alpha+2m-2k}(1+x^2)^{-\beta}\, dx = \sum_{j=0}^{\infty}\frac{(-1)^j i^j s^j}{j!}\left(\int_{-\infty}^{\infty} x^{j-2\alpha+2m-2k}(1+x^2)^{-\beta}\, dx\right) =$$

$$\sum_{r=0}^{\infty}\frac{(-1)^r s^{2r}}{(2r)!}\left(2\int_{0}^{\infty} x^{2r-2\alpha+2m-2k}(1+x^2)^{-\beta}\, dx\right) = \sum_{r=0}^{\infty}\frac{(-1)^r s^{2r}}{(2r)!} B(r-\alpha+m-k+\tfrac{1}{2};\beta-r+\alpha-m+k-\tfrac{1}{2}),$$
$$(25)$$

where we have used the third kind of beta integral in (12).
The last sum in (25) can be represented in terms of a hypergeometric function, so

$$I_{2m,k}(s;\alpha,\beta) = \frac{\Gamma(-\alpha+m-k+1/2)\Gamma(\beta+\alpha-m+k-1/2)}{\Gamma(\beta)}\, {}_1F_2\left(\begin{matrix} -\alpha+m-k+1/2 \\ 1/2, \quad -\beta-\alpha+m-k+3/2\end{matrix}\middle|\frac{s^2}{4}\right).$$
$$(26)$$

By knowing that

$$\int_{-\infty}^{\infty} x^{j-2\alpha+2m+1-2k}(1+x^2)^{-\beta}\, dx = 0 \text{ for any } j = 0,2,4,\dots,$$

this method can be similarly applied to $I_{2m+1,k}(s;\alpha,\beta)$ so that after some computations we obtain

$$I_{2m+1,k}(s;\alpha,\beta) = (-is)\frac{\Gamma(-\alpha+m-k+3/2)\Gamma(\beta+\alpha-m+k-3/2)}{\Gamma(\beta)}\, {}_1F_2\left(\begin{matrix} -\alpha+m-k+3/2 \\ 3/2, \quad -\beta-\alpha+m-k+5/2\end{matrix}\middle|\frac{s^2}{4}\right).$$
$$(27)$$

Hence, combining both relations (26) and (27) and using the identity

$$[\frac{n+1}{2}]-[\frac{n}{2}]=\frac{1-(-1)^n}{2}, \qquad (28)$$

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gives the final form of (24) as

$$I_{n,k}(s;\alpha,\beta) = \Gamma(-\alpha - k + \frac{1}{2} + [\frac{n+1}{2}])\,\Gamma(\beta + \alpha + k - \frac{1}{2} - [\frac{n+1}{2}])$$

$$\times \frac{(-is)^{\frac{1-(-1)^n}{2}}}{\Gamma(\beta)}\;{}_1F_2\left(\begin{array}{c} 1/2 - \alpha - k + [(n+1)/2] \\ 1 - \frac{(-1)^n}{2}, \quad -\beta - \alpha - k + 3/2 + [(n+1)/2] \end{array}\middle|\; \frac{s^2}{4}\right). \tag{29}$$

The second way of computing $I_{n,k}(s;\alpha,\beta)$ is that we respectively suppose $n = 2m$ and $n = 2m+1$ and then directly apply the cosine and sine Fourier transforms [2] to the function $(1+x^2)^{-\beta} x^{-2\alpha+n-2k}$. For example, by noting that $(-1)^{2\alpha} = 1$ we have

$$I_{2m,k}(s;\alpha,\beta) = \int_{-\infty}^{\infty} \cos(sx)(1+x^2)^{-\beta} x^{-2\alpha+2m-2k}\,dx - i\int_{-\infty}^{\infty} \sin(sx)(1+x^2)^{-\beta} x^{-2\alpha+2m-2k}\,dx$$

$$= 2\int_{0}^{\infty} \cos(sx)(1+x^2)^{-\beta} x^{-2\alpha+2m-2k}\,dx$$

$$= \frac{\Gamma(-\alpha + m - k + 1/2)\,\Gamma(\beta + \alpha - m + k - 1/2)}{\Gamma(\beta)}\;{}_1F_2\left(\begin{array}{c} -\alpha + m - k + 1/2 \\ 1/2, \quad -\beta - \alpha + m - k + 3/2 \end{array}\middle|\; \frac{s^2}{4}\right). \tag{30}$$

Remark 1. To prove the last equality of (30), one can use dominated convergence theorem (DCT) [11] so that define the sequence

$$\Phi_n(x) = (1+x^2)^{-\beta} x^{-2\lambda} \sum_{k=0}^{n} \frac{(-1)^k (sx)^{2k}}{(2k)!} \quad \text{for} \quad (-1)^\lambda = 1, \tag{31}$$

and then conclude

$$\left|\Phi_n(x)\right| \le \cosh(sx)(1+x^2)^{-\beta} x^{-2\lambda} = \Phi(x). \tag{32}$$

Inequality (32) allows us to explicitly compute $I_{2m,k}(s;\alpha,\beta)$ in (30) as the same form as we have done in (25).

By referring to remark 1, we can similarly conclude that

$$I_{2m+1,k}(s;\alpha,\beta) = \int_{-\infty}^{\infty} \cos(sx)(1+x^2)^{-\beta} x^{-2\alpha+2m+1-2k}\,dx - i\int_{-\infty}^{\infty} \sin(sx)(1+x^2)^{-\beta} x^{-2\alpha+2m+1-2k}\,dx$$

$$= (-2i)\int_{0}^{\infty} \sin(sx)(1+x^2)^{-\beta} x^{-2\alpha+2m+1-2k}\,dx$$

$$= (-is)\Gamma(-\alpha + m - k + \frac{3}{2})\Gamma(\beta + \alpha - m + k - \frac{3}{2})\;{}_1F_2\left(\begin{array}{c} -\alpha + m - k + 3/2 \\ 3/2, \quad -\beta - \alpha + m - k + 5/2 \end{array}\middle|\; \frac{s^2}{4}\right). \tag{33}$$

Therefore, the result (29) would simplify (23) as

$$\mathbf{F}(g(x)) = \frac{1}{\Gamma(\beta)}\Gamma(-\alpha + \frac{1}{2} + [\frac{n+1}{2}])\Gamma(\beta + \alpha - \frac{1}{2} - [\frac{n+1}{2}])(-is)^{\frac{1-(-1)^n}{2}} \times$$

$$\sum_{k=0}^{[n/2]} \frac{(-[n/2])_k(c+1/2-[(n+1)/2])_k(\beta+\alpha-1/2-[(n+1)/2])_k}{(c+d-n+1/2)_k(1/2+\alpha-[(n+1)/2])_k k!} \tag{34}$$

$$\times {}_1F_2\left(\begin{array}{c} 1/2-\alpha-k+[(n+1)/2] \\ 1-\frac{(-1)^n}{2}, \quad -\beta-\alpha-k+3/2+[(n+1)/2] \end{array}\middle| \frac{s^2}{4}\right).$$

If for simplicity in (34) we define the symmetric function

$$A_n(x; p_1, p_2, p_3, p_4) = x^{\frac{1-(-1)^n}{2}}\sum_{k=0}^{[n/2]} \frac{(-[n/2])_k(p_3+1/2-[(n+1)/2])_k(p_1+p_2-1/2-[(n+1)/2])_k}{(p_3+p_4-n+1/2)_k(1/2+p_1-[(n+1)/2])_k k!}$$

$$\times {}_1F_2\left(\begin{array}{c} 1/2-p_1-k+[(n+1)/2] \\ 1-\frac{(-1)^n}{2}, \quad -p_1-p_2-k+3/2+[(n+1)/2] \end{array}\middle| \frac{x^2}{4}\right), \tag{35}$$

then clearly

$$\mathbf{F}(g(x)) = (-i)^{\frac{1-(-1)^n}{2}}\frac{1}{\Gamma(\beta)}\Gamma(-\alpha + \frac{1}{2} + [\frac{n+1}{2}])\Gamma(\beta + \alpha - \frac{1}{2} - [\frac{n+1}{2}]) A_n(s; \alpha, \beta, c, d). \tag{36}$$

By substituting (36) in the Parseval identity (21) and noting (22) one gets

$$2\pi \int_{-\infty}^{\infty} x^{-2(\alpha+l)}(1+x^2)^{-(\beta+u)} A_n^{(c,d)}(x) A_m^{(v,w)}(x)\,dx = i^{\frac{(-1)^n-(-1)^m}{2}}\Gamma(-\alpha+\frac{1}{2}+[\frac{n+1}{2}])$$

$$\times\frac{1}{\Gamma(\beta)\,\Gamma(u)}\Gamma(\beta+\alpha-\frac{1}{2}-[\frac{n+1}{2}])\Gamma(-l+\frac{1}{2}+[\frac{m+1}{2}])\Gamma(u+l-\frac{1}{2}-[\frac{m+1}{2}]) \tag{37}$$

$$\times\int_{-\infty}^{\infty} A_n(s; \alpha, \beta, c, d) A_m(s; l, u, v, w)\,ds.$$

Now, if in the left hand side of (37)

$$c = v = \alpha + l \quad \text{and} \quad d = w = \beta + u, \tag{38}$$

then according to the orthogonality relation (9) we have,

Theorem 1. *The special function $A_n(x; p_1, p_2, p_3, p_4)$ defined in (35) satisfies the finite orthogonality relation*

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} A_n\left(x;\alpha,\beta,p,q\right) A_m\left(x;p-\alpha,q-\beta,p,q\right)dx = \prod_{j=1}^{n}\frac{\left(-j+(1-(-1)^j)p\right)\left(j-(1-(-1)^j)p-2q\right)}{(2j-2p-2q+1)(2j-2p-2q-1)}\times$$

$$\frac{\Gamma(\beta)\Gamma(q-\beta)\Gamma(p+q-1/2)\Gamma(-p+1/2)\quad \delta_{n,m}}{\Gamma(q)\Gamma\left(-\alpha+\frac{1}{2}+[\frac{n+1}{2}]\right)\Gamma\left(\alpha+\beta-\frac{1}{2}-[\frac{n+1}{2}]\right)\Gamma\left(\alpha-p+\frac{1}{2}+[\frac{n+1}{2}]\right)\Gamma\left(p+q-\alpha-\beta-\frac{1}{2}-[\frac{n+1}{2}]\right)},$$

(39)

for $m,n=0,1,...,N=\max\{m,n\}\le p+q-1/2$, $p<1/2$, $(-1)^{2p}=1$, $q>\beta>0$, $0<\alpha<1/2$ *and* $\alpha+\beta>1/2$.

This approach can similarly be applied for the monic polynomials $B_n^{(a)}(x)$ in (16). First, let us define the functions

$$u(x)=x^{-2a}e^{\frac{-1}{2x^2}}B_n^{(b)}(x) \quad \text{and} \quad v(x)=x^{-2c}e^{\frac{-1}{2x^2}}B_m^{(d)}(x) \quad \text{for} \quad (-1)^{2a}=(-1)^{2c}=1. \quad (40)$$

The Fourier transform of e.g. $u(x)$ is computed as

$$\mathbf{F}(u(x))=\int_{-\infty}^{\infty}e^{-isx}x^{-2a}e^{\frac{-1}{2x^2}}B_n^{(b)}(x)\,dx = \int_{-\infty}^{\infty}e^{-isx}e^{\frac{-1}{2x^2}}x^{-2a+n}\left(\sum_{k=0}^{[n/2]}\frac{(-[n/2])_k}{(b+(-1)^n/2)_k}\frac{x^{-2k}}{k!}\right)dx$$

$$= \sum_{k=0}^{[n/2]}\frac{(-[n/2])_k}{(b+(-1)^n/2)_k\,k!}\left(\int_{-\infty}^{\infty}e^{-isx}e^{\frac{-1}{2x^2}}x^{-2a+n-2k}\,dx\right).$$

(41)

Again, the following definite integral should be evaluated

$$R_{n,k}(s;a)=\int_{-\infty}^{\infty}e^{-isx}e^{\frac{-1}{2x^2}}x^{-2a+n-2k}\,dx. \quad (42)$$

To do this, we can use the same method as we applied to $I_{n,k}(s;\alpha,\beta)$, i.e.

$$R_{2m,k}(s;a)=\int_{-\infty}^{\infty}\left(\sum_{j=0}^{\infty}\frac{(-isx)^j}{j!}\right)e^{\frac{-1}{2x^2}}x^{-2a+2m-2k}\,dx = \sum_{j=0}^{\infty}\frac{(-1)^j i^j s^j}{j!}\left(\int_{-\infty}^{\infty}e^{\frac{-1}{2x^2}}x^{j-2a+2m-2k}\,dx\right)$$

$$=\sum_{r=0}^{\infty}\frac{(-1)^r s^{2r}}{(2r)!}\left(2\int_{0}^{\infty}x^{2r-2a+2m-2k}e^{\frac{-1}{2x^2}}\,dx\right)=\sum_{r=0}^{\infty}\frac{(-1)^r s^{2r}}{(2r)!}2^{-r+a-m+k-\frac{1}{2}}\Gamma(-r+a-m+k-\frac{1}{2})$$

$$=2^{a-m+k-\frac{1}{2}}\Gamma(a-m+k-\frac{1}{2})\,_0F_2\left(\begin{array}{c}-\\ \frac{1}{2},\quad \frac{3}{2}-a+m-k\end{array}\middle|\frac{s^2}{8}\right),$$

(43)

as well as

$$R_{2m+1,k}(s;a) = \sum_{j=0}^{\infty} \frac{(-1)^j i^j s^j}{j!} \left(\int_{-\infty}^{\infty} e^{\frac{-1}{2x^2}} x^{j-2a+2m+1-2k} dx \right) = (-is) \sum_{r=0}^{\infty} \frac{(-1)^r s^{2r}}{(2r+1)!} \left(2 \int_0^{\infty} x^{2r-2a+2m-2k+2} e^{\frac{-1}{2x^2}} dx \right)$$

$$= (-is) 2^{a-m+k-\frac{3}{2}} \Gamma(a-m+k-\frac{3}{2}) \, _0F_2 \left(\begin{array}{cc} 3 & 5 \\ \frac{3}{2}, & \frac{5}{2} - a + m - k \end{array} \middle| \frac{-}{\frac{s^2}{8}} \right).$$

(44)

Consequently we have

$$R_{n,k}(s;a) = 2^{a+k-\frac{1}{2}-[\frac{n+1}{2}]} \Gamma\left(a+k-\frac{1}{2}-[\frac{n+1}{2}]\right)(-is)^{\frac{1-(-1)^n}{2}} \, _0F_2 \left(\begin{array}{cc} - \\ 1-\frac{(-1)^n}{2}, & -a-k+\frac{3}{2}+[\frac{n+1}{2}] \end{array} \middle| \frac{s^2}{8} \right),$$

(45)

and therefore

$$\mathbf{F}(u(x)) = \Gamma(a-\frac{1}{2}-[\frac{n+1}{2}]) 2^{a-\frac{1}{2}-[\frac{n+1}{2}]}(-is)^{\frac{1-(-1)^n}{2}} \times$$

(46)

$$\sum_{k=0}^{[n/2]} \frac{(-[\frac{n}{2}])_k (a-\frac{1}{2}-[\frac{n+1}{2}])_k}{(b+(-1)^n/2)_k} \frac{2^k}{k!} \, _0F_2 \left(\begin{array}{cc} - \\ 1-\frac{(-1)^n}{2}, & -a-k+\frac{3}{2}+[\frac{n+1}{2}] \end{array} \middle| \frac{s^2}{8} \right).$$

Now if for simplicity in (46) we define the symmetric function

$$B_n(x;q_1,q_2) = x^{\frac{1-(-1)^n}{2}} \sum_{k=0}^{[n/2]} \frac{(-[\frac{n}{2}])_k (q_1-\frac{1}{2}-[\frac{n+1}{2}])_k}{(q_2+(-1)^n/2)_k} \frac{2^k}{k!} \, _0F_2 \left(\begin{array}{cc} - \\ 1-\frac{(-1)^n}{2}, & -q_1-k+\frac{3}{2}+[\frac{n+1}{2}] \end{array} \middle| \frac{x^2}{8} \right),$$

(47)

then by referring to definitions (40) and applying Parseval identity we get

$$2\pi \int_{-\infty}^{\infty} x^{-2(a+c)} e^{\frac{-1}{x^2}} B_n^{(b)}(x) B_m^{(d)}(x) dx$$

(48)

$$= i^{\frac{(-1)^n-(-1)^m}{2}} \frac{\Gamma(a-\frac{1}{2}-[\frac{n+1}{2}])\Gamma(c-\frac{1}{2}-[\frac{m+1}{2}])}{2^{-a+\frac{1}{2}+[\frac{n+1}{2}]} 2^{-c+\frac{1}{2}+[\frac{m+1}{2}]}} \int_{-\infty}^{\infty} B_n(s;a,b) B_m(s;c,d) ds.$$

It is sufficient in (48) to assume that $b = d = a + c$ and then refer to the finite orthogonality relation (17) to reach,

Theorem 2. *The special function $B_n(x;q_1,q_2)$ defined in (47) satisfies the orthogonality relation*

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}B_n(x;a,b)B_m(x;b-a,b)\,dx = 2^{-b+1+2\,[\frac{n+1}{2}]}\prod_{j=1}^{n}\frac{2(-1)^j(j-b)+2b}{(2j-2b+1)(2j-2b-1)}$$

$$\times\frac{\Gamma(b-1/2)}{\Gamma(a-\frac{1}{2}-[\frac{n+1}{2}])\Gamma(b-a-\frac{1}{2}-[\frac{n+1}{2}])}\,\delta_{n,m}, \tag{49}$$

for $m,n = 0,1,...,N = \max\{m,n\} \le b-\dfrac{1}{2},\ (-1)^{2b}=1\ and\ \dfrac{1}{2} < a < b-\dfrac{1}{2}.$

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