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Weinstein, Lenard
The Bieberbach conjecture. (English)
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In this excellent paper the author gives a new proof of the Bieberbach, Robertson and Milin conjectures. The Milin conjecture is now known as the de Branges theorem because *L. de Branges* [Acta Math. 154, 137-152 (1985; Zbl. 573.30014)] was the first to prove it in 1984. Let S denote the family of analytic and univalent functions $f(z) = z + a_2z^2 + \dots$ of the unit disk . S is compact with respect to the topology of locally uniform convergence so that $k_n := \max_{f\in S} |a_n(f)|$ exists. In 1916 *L. Bieberbach* [S.-B. Preuss. Akad. Wiss. 38, 940-955 (1916; FdM 46, 552)] proved that $k_2 = 2$, with equality if and only if f is a rotation of the Koebe function

$$k(z) := \frac{z}{(1-z)^2} = \frac{1}{4}\left(\left(\frac{1+z}{1-z}\right)^2 - 1\right) = \sum_{n=1}^{\infty} nz^n,$$

and in a footnote he mentioned “Vielleicht ist ueberhaupt $k_n = n$.”. This statement is known as the Bieberbach conjecture. In 1923 *K. Loewner* [Math. Ann. 89, 103-121 (1923; FdM 49, 714)] was able to prove the Bieberbach conjecture for $n = 3$. His method was to embed a univalent function $f(z)$ into a Loewner chain, i.e. a family $\{f(z,t)|t \geq 0\}$ of univalent functions of the form

$$f(z,t) = e^t z + \sum_{n=2}^{\infty} a_n(t)z^n, \quad (z \in, t \geq 0, a_n(t) \in (n \geq 2))$$

which start with f

$$f(z,0) = f(z),$$

and for which the relation

$$Rep(z,t) = Re\left(\frac{\dot{f}(z,t)}{zf'(z,t)}\right) > 0 \quad (z \in) \tag{1}$$

is satisfied. Here $'$ and $\dot{}$ denote the partial derivatives with respect to z and t, respectively. (1) is referred to as the Loewner differential equation, and geometrically it states that the image domains of f_t expand as t increases. A function $f(z) = \sum_{n=1}^{\infty} a_nz^n \in S$ if and only if the square root transform $h(z) := z\sqrt{\frac{f(z^2)}{z^2}}$ is an odd univalent function. *M. S. Robertson* [Ann. Math., II. Ser. 37, 374-408 (1936; Zbl. 14, 165)] conjectured that for an odd univalent function h the relation $\sum_{k=1}^{n} |c_{2k-1}|^2 \leq n$ is satisfied which by $a_n = \sum_{k=1}^{n} c_{2(n-k)+1} \cdot c_{2k-1}$ implies the Bieberbach conjecture. The history of the Bieberbach conjecture showed that it was easier to obtain results about the logarithmic coefficients of a univalent function f, i.e. the coefficients d_n of the expansion

$$\varphi(z) = \ln\frac{f(z)}{z} =: \sum_{n=1}^{\infty} d_nz^n$$

rather than for the coefficients a_n of f itself. So, in the mid-sixties, *N. A. Lebedev* and *I. M. Milin* [Vestn. Leningr. Univ. 20, No. 19, 157-158 (1965; Zbl. 144, 333)] developed methods to exponentiate such information. The so-called second Lebedev-Milin inequality states that if $\psi(z) = \sum_{k=0}^{\infty} \beta_kz^k (\beta_0 = 1)$ and $\varphi(z) := \ln\psi(z) = \sum_{k=1}^{\infty} \alpha_kz^k$, then

$$\frac{1}{n+1}\sum_{k=0}^{n} |\beta_k|^2 \leq \exp\left(\frac{1}{n+1}\sum_{k=1}^{n}(n+1-k)\left(k|\alpha_k|^2 - \frac{1}{k}\right)\right).$$

Now assume, for $f \in S$ the Milin conjecture

$$(2) \qquad \sum_{k=1}^{n}(n+1-k)\left(k|d_k|^2-\frac{4}{k}\right)\leq 0$$

on its logarithmic coefficients is satisfied for some $n \in$, then as for the square root transform h we have $\ln\frac{h(z)}{z}=\frac{1}{2}\varphi(z^2)$, by the second Lebedev-Milin inequality $\frac{1}{n+1}\sum_{k=1}^{n+1}|c_{2k-1}|^2\leq 1$, i.e. the Robertson conjecture, and so the Bieberbach conjecture, for the index $n+1$ follows. In 1984 de Branges verified the Milin conjecture. De Branges' idea is the following: to prove the Milin conjecture, set

$$\psi(t):=\sum_{k=1}^{n}\tau_k(t)\left(k|d_k(t)|^2-\frac{4}{k}\right),$$

where the system $(\tau_k)_{k=1,\ldots,n+1}$ of functions $\tau_k :^+\to$ will be defined later. Here $d_k(t)$ denotes the logarithmic coefficients of $e^{-t}f(z,t)$ where $f(z,t)$ is a Loewner chain $f(z,t)$ of f. If (3) $\tau_k(0)=n+1-k$ $(k=1,\ldots,n)$, then the relation $\psi(0)\leq 0$ is the statement of the Milin conjecture. L. de Branges succeeded in showing that the Milin conjecture for the index $n\geq 2$ is true if the functions $(\tau_k)_{k=1,\ldots,n-1}$ have the properties

$$(4) \quad (\tau_k(t)-\tau_{k+1}(t)=-\frac{\dot\tau_k(t)}{k}-\frac{\dot\tau_{k+1}(t)}{k+1}(k=1,\ldots,n)\ \tau_{n+1}=0,$$

$$(5) \quad \lim_{t\to\infty}\tau_k(t)=0\ (k=1,\ldots,n), \quad (6) \quad \dot\tau_k(t)<0\ (t\in^+),$$

as then after a lengthy calculation using the Loewner differential equation the relation $\dot\psi(t)\geq 0$, and therefore $\psi(0)=-\int_0^\infty\dot\psi(t)dt\leq 0$ follows.
Since the system of differential equations (4) with the initial conditions (3) has a unique solution, the relations (5) and (6) are additional. Here L. de Branges had the luck of the ingenious researcher! Whereas (5) is easily shown, (6) is a deep theorem due to *R. Askey* and *G. Gasper* [Am. J. Math. 98, 709-737 (1976; Zbl. 355.33005)]. In the present paper Weinstein proves the Milin conjecture without using the Askey-Gasper result. He uses the following representation of the partial derivatives with respect to t of the logarithmic coefficients $d_k(t)$ of the Loewner chain $(\zeta:=re^{i\theta}, r\in(0,1))$

$$\dot d_k(t)(Cauchy formula)\to=\frac{1}{2\pi}\int_0^{2\pi}\frac{\dot f(\zeta,t)}{f(\zeta,t)}\frac{d\theta}{\zeta^k}(1)\to=\frac{1}{2\pi}\int_0^{2\pi}p(\zeta,t)\frac{\zeta f'(\zeta,t)}{f(\zeta,t)}\frac{d\theta}{\zeta^k}=\frac{1}{2\pi}\int_0^{2\pi}p(\zeta,t)\left(1+\sum_{j=1}^{\infty}\right.$$

Further the uses the following special Loewner chain of bounded univalent functions $\varphi(z,t):=k^{-1}(e^{-t}k(z))$ $(z\in,\ t\geq 0)$ for which

$$(8) \qquad \frac{z}{(1-z)^2}=e^t\frac{\varphi}{(1-\varphi)^2}\quad (t\geq 0).$$

$\varphi(,t)$ is the unit disk with a radial slit increasing with t. If we take the derivative of the identity $k(\varphi(z,t))=e^{-t}k(z)$, we get the relation

$$(9) \qquad \dot\varphi=-\varphi\frac{1-\varphi}{1+\varphi}.$$

Rather than proving the Milin conjecture for each $n\in$ separately like L. de Branges (in fact, for each n his special function system is a different one), L. Weinstein wants to prove them altogether using a generating function. He observes that for the generating function of the Milin expressions (2) by a rearrangement we automatically get

$$\omega(z) := \sum_{n=1}^{\infty} \left(\sum_{k=1}^{n} (n+1-k) \left(\frac{4}{k} - k|d_k(0)|^2 \right) \right) z^{n+1} = \frac{z}{(1-z)^2} \sum_{k=1}^{\infty} \left(\frac{4}{k} - k|d_k(0)|^2 \right) z^k$$

the Koebe function as a factor. Using (8) and the main theorem of calculus, by $\phi(z,\infty) = 0$ and $\varphi(z,0) = z$

$$\omega(z) = \int_0^\infty - \frac{e^t \varphi}{(1-\varphi)^2} \frac{d}{dt} \left(\sum_{k=1}^{\infty} \left(\frac{4}{k} - k|d_k(t)|^2 \right) \varphi^k \right) dt.$$

If one now uses relations (7) and (9), then after a lengthy (but elementary) calculation ($\zeta := re^{i\theta}$, $r \in (0,1)$)

$$\omega(z) = \int_0^\infty \frac{e^t \varphi}{1-\varphi^2} \sum_{k=1}^{\infty} A_k(t) \varphi^k dt,$$

with

$$A_k(t) := \lim_{r \to 1} \frac{1}{2\pi} \int_0^{2\pi} Rep(\zeta,t) \left| 1 + 2 \sum_{j=1}^{k} j d_j(t) \zeta^j - k d_k(t) \zeta^k \right|^2 d\theta.$$

By the Loewner theory $Rep(\zeta,t) > 0(1)$ so that $A_k(t) \geq 0 (t \geq 0)$, and the Milin conjecture follows if

$$B_{kn}(t) \geq 0 \quad (t \geq 0, k, n \in) \tag{10}$$

in the expansion

$$\frac{e^t \varphi^{k+1}}{1-\varphi^2} =: \sum_{n=1}^{\infty} B_{kn}(t) z^{n+1}.$$

This, again, is a statement about special functions which, however, are in direct connection with the theory of univalent functions. As $\varphi(,t)$ is the unit disk with a radial slit, and as the function ($\gamma \in$)

$$h_\gamma(z) := \frac{z}{1 - 2\cos\gamma \cdot z + z^2}$$

for $\gamma \neq 0 \pmod{\pi}$ maps the unit disk onto the plane with two slits on the real axis, φ can be interpreted as $\varphi = h_\theta^{-1}(e^{-t} h_\gamma)$ for a suitable pair (θ, γ), and a calculation shows that (11) $\cos\gamma = (1 - e^{-t}) + e^{-t}\cos\theta$. So we get finally the representation

$$h_\gamma(z) = e^t \cdot h_\theta(\varphi) = \frac{e^t \varphi}{1-\varphi^2} \left(\frac{1-\varphi^2}{1 - 2\cos\theta \cdot \varphi + \varphi^2} \right) = \frac{e^t \varphi}{1-\varphi^2} Re \frac{1 + e^{i\theta}\varphi}{1 - e^{i\theta}\varphi} = \frac{e^t \varphi}{1-\varphi^2} \left(1 + 2 \sum_{k=1}^{\infty} \varphi^k \cos k\theta \right) =$$

As

$$\eta(z) := \frac{\sqrt{h_\gamma(z)}}{z} = \frac{1}{\sqrt{1 - 2\cos\gamma \cdot z + z^2}}$$

is the generating function of the Legendre polynomials, Weinstein uses now the addition theorem for Legendre polynomials to finish the proof of (10), and thus of the Milin conjecture. It is remarkable that L. Bieberbach easily would have understood Weinstein's proof! It was, however, not a single researcher to be able to collect all the necessary ideas for the proof. In fact, the history of the

Bieberbach conjecture is a particularly interesting example how mathematical knowledge is established. Weinstein's proof is so short that it fits on four pages. This may be compared with the fact that the first proof of the Bieberbach conjecture for the fourth coefficient was a paper of 38 pages *R. P. Garabedian*, and *M. M. Schiffer* [J. Rat. Mech. Anal. 4, 427-465, (1955; Zbl. 65, 69)]. On the other hand, some of Weinstein's nice ideas are covered in this short presentation, and the work of elaboration is due to the reader.

W.Koepf (Berlin)

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- 30C50 Coefficient problems for univalent and multivalent functions