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Irrationality of certain infinite series

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Summary: In this paper a new direct proof for the irrationality of Euler's number

$$e = \sum_{k=0}^{\infty} \frac{1}{k!}$$

is presented. Furthermore, formulas for the base b digits are given which, however, are not computably effective. Finally we generalize our method and give a simple criterium for some fast converging series representing irrational numbers.

1 Introduction

Let

$$e = \sum_{k=0}^{\infty} \frac{1}{k!}$$

be Euler's number. It is well-known that e is transcendental. However, whereas transcendency proofs typically are quite hard, it is mostly much easier to show irrationality. In this paper we will give a new direct irrationality proof for e which can be generalized to many other constants given by a similar type of series.

Of course $e = \lim\limits_{n\to\infty} s_n$ for the partial sums

$$s_n := \sum_{k=0}^{n} \frac{1}{k!} \; . \tag{1.1}$$

Our direct proof of irrationality of e will use the identity

$$\lfloor n\, s_n \rfloor = \lfloor n\, e \rfloor \quad (n \in \mathbb{N} = \{0, 1, 2, 3, \ldots\}) \tag{1.2}$$

which is interesting in its own. Here

$$\lfloor x \rfloor := \max\{n \in \mathbb{N} \mid n \leqq x\}$$

denotes the floor function (Gauss bracket). From (1.2) we furthermore deduce an explicit formula for the base b digits of e, before we consider our method in a more general setting.

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2 Irrationality of *e*

To show the irrationality of e, we proceed with several lemmas.

Lemma 2.1 *Let $c \in \mathbb{R}_{>0}$ be arbitrary, and let the remainder $0 \leqq R_n < 1$ be defined by the division algorithm as*

$$n\,c = \lfloor n\,c \rfloor + R_n \ .$$

Then c is irrational if and only if $R_n > 0$ for all $n \in \mathbb{N}$.

Proof: The proof of this lemma is obvious. □

Lemma 2.2 (see e. g. [1, p. 198]) *Let s_n be the partial sum given by (1.1). Then*

$$s_n < e < s_n + \frac{1}{n \cdot n!} \ . \tag{2.1}$$

Proof: The left-hand inequality is trivial, and the right inequality follows from the computations

$$\begin{aligned} e &= s_n + \sum_{k=n+1}^{\infty} \frac{1}{k!} = s_n + \frac{1}{n!} \sum_{k=n+1}^{\infty} \frac{n!}{k!} \\ &= s_n + \frac{1}{n!} \sum_{k=1}^{\infty} \frac{n!}{(n+k)!} < s_n + \frac{1}{n!} \sum_{k=1}^{\infty} \frac{1}{(n+1)^k} = s_n + \frac{1}{n \cdot n!} \end{aligned}$$

by evaluating the latter geometric series. □

For the next lemma we consider the representations

$$n\,s_n = M_n + R_n$$

with $M_n = \lfloor n\,s_n \rfloor$ and remainder $0 \leqq R_n < 1$ and

$$n\,e = \widetilde{M}_n + \widetilde{R}_n$$

with $\widetilde{M}_n = \lfloor n\,e \rfloor$ and remainder $0 \leqq \widetilde{R}_n < 1$, both given by the division algorithm.

Lemma 2.3 *For all $n \in \mathbb{N}$ the number $(n-1)!\,R_n \in \mathbb{N}$.*

Proof: If we multiply the equation

$$R_n = n\,s_n - M_n$$

by $(n-1)!$, we get

$$(n-1)!\,R_n = n!\,s_n - (n-1)!\,M_n \ .$$

Since

$$n!\,s_n = \sum_{k=0}^{n} \frac{n!}{k!} \in \mathbb{Z}\,,$$

the conclusion follows from $R_n \geqq 0$. □

We remark that $R_n > 0$ therefore implies the stronger relation $R_n \geqq \frac{1}{(n-1)!}$.
The above lemmas result in the following

Theorem 2.4 *For all $n \in \mathbb{N}$ it follows that*

(a) $M_n = \widetilde{M}_n$*, hence (1.2),*

(b) and $\widetilde{R}_n > 0$ *for all* $n \in \mathbb{N}$.

(c) Therefore, by Lemma 2.1, e is irrational.

Proof: From Lemma 2.2 we get

$$0 < n!\,e - n!\,s_n < \frac{1}{n}\,.$$

From the definitions of R_n and $\widetilde{R}_n$ it follows furthermore that

$$\begin{aligned} n!\,e &= (n-1)!\,\widetilde{M}_n + (n-1)!\,\widetilde{R}_n \\ n!\,s_n &= (n-1)!\,M_n + (n-1)!\,R_n \end{aligned}$$

and therefore we get for the difference

$$0 < n!\,(e - s_n) = (n-1)!\,\big(\widetilde{M}_n - M_n\big) + (n-1)!\,\big(\widetilde{R}_n - R_n\big) < \frac{1}{n}\,.$$

Since $\widetilde{R}_n < 1$, this gives

$$-(n-1)! < (n-1)!\,\big(\widetilde{M}_n - M_n\big) - (n-1)!\,R_n < \frac{1}{n}\,.$$

From Lemma 2.3 we know that $(n-1)!\,R_n \in \mathbb{N}$. Therefore, we deduce that

$$(n-1)!\,\big(\widetilde{M}_n - M_n\big) - (n-1)!\,R_n \in \mathbb{Z}$$

and since

$$(n-1)!\,\big(\widetilde{M}_n - M_n\big) - (n-1)!\,R_n < \frac{1}{n}$$

we conclude

$$(n-1)!\,\big(\widetilde{M}_n - M_n\big) - (n-1)!\,R_n \leqq 0\,.$$

From $0 \leqq R_n < 1$ we therefore deduce that

$$-(n-1)! < (n-1)!\,\big(\widetilde{M}_n - M_n\big) \leqq (n-1)!\,R_n < (n-1)!$$

and finally through division by $(n-1)!$ we deduce

$$-1 < \widetilde{M}_n - M_n < 1 \; .$$

Since $\widetilde{M}_n - M_n \in \mathbb{Z}$, this is equivalent to (a).

From $s_n < e$ it follows that

$$M_n + R_n < \widetilde{M}_n + \widetilde{R}_n \; ,$$

and using $\widetilde{M}_n = M_n$ we get for all $n \in \mathbb{N}$

$$0 \leqq R_n < \widetilde{R}_n \; .$$

Therefore the second conclusion (b) follows. Finally, statement (c) is an immediate consequence of Lemma 2.1 applied to the constant $c = e$. □

We would like to mention that a simple computation gives the following extension of (b):

$$0 < \widetilde{R}_n < R_n + \frac{1}{n!}$$

connecting the two remainder sequences considered.

In the next section, we will utilize Equation (1.2) in more detail and give explicit representations for the base b digits of e.

3 Base *b* digits

Let $b \in \mathbb{N}_{\geqq 2}$ be an arbitrary base, and

$$e = 2 + \sum_{j=1}^{\infty} c_j(b)\, b^{-j} \quad (c_j(b) \in \{0, 1, \ldots, b-1\}) \tag{3.1}$$

be the base b representation of Euler's number e. For $b = 10$ this is the usual decimal representation. Since

$$e = 2.71828182845904523 53\ldots,$$

we have for example $c_1(10) = 7$, $c_2(10) = 1$, $c_3(10) = 8$, ... We would like to find explicit representations for the digits $c_j(b)$ in (3.1). We get the following relation between this representation and the partial sums s_n.

Theorem 3.1 *For the truncated series in (3.1) the identity*

$$\frac{\lfloor b^k\, s_{b^k} \rfloor}{b^k} = 2 + \sum_{j=1}^{k} c_j(b)\, b^{-j} \tag{3.2}$$

is valid. Therefore, by telescoping, the explicit representation

$$c_k(b) = \lfloor b^k\, s_{b^k} \rfloor - b \cdot \lfloor b^{k-1}\, s_{b^{k-1}} \rfloor \tag{3.3}$$

follows.

Proof: Let (3.1) be valid. Then fix an arbitrary $k \in \mathbb{N}_{>0}$ and consider the decomposition

$$e = 2 + \sum_{j=1}^{k} c_j(b)\, b^{-j} + \sum_{j=k+1}^{\infty} c_j(b)\, b^{-j} \ . \tag{3.4}$$

From the construction of the base b representation through iterative division by b (see e. g. [4]), it follows for the remainder part

$$\sum_{j=k+1}^{\infty} c_j(b)\, b^{-j} < \frac{1}{b^k} \ ,$$

hence

$$0 \leqq b^k \cdot \sum_{j=k+1}^{\infty} c_j(b)\, b^{-j} < 1 \ . \tag{3.5}$$

From (3.4), we conclude

$$b^k \cdot e = 2\, b^k + \sum_{j=1}^{k} c_j(b)\, b^{k-j} + b^k \cdot \sum_{j=k+1}^{\infty} c_j(b)\, b^{-j} \ .$$

Now we get using (3.5)

$$\lfloor b^k \cdot e \rfloor = 2\, b^k + \sum_{j=1}^{k} c_j(b)\, b^{k-j} \ .$$

Theorem 2.4 (a) leads to the conclusion

$$\lfloor b^k \cdot s_{b^k} \rfloor = 2\, b^k + \sum_{j=1}^{k} c_j(b)\, b^{k-j}$$

and therefore to (3.2). By telescoping formula (3.3) is generated. □

The computation

$$c_2(10) = \lfloor 100\, s_{100} \rfloor - 10 \cdot \lfloor 10\, s_{10} \rfloor = 271 - 10 \cdot \left\lfloor \frac{98641010}{3628800} \right\rfloor = 271 - 270 = 1$$

gives gives $c_2(10)$. Since

$$\begin{aligned} s_{100} = {} & 4299778907798767752801199122242037634663518280784714275131782 \\ & 8133465975238709567206600082275449499964960577581750509066713 \\ & 4768643813040977474177102242650833 9 \Big/ \\ & 1581800261761765299689817607733333906622304546853925787603270 \\ & 5744952135592072867052362959999595873191292435557980122436580 5 \\ & 2856289689600000000000000000000000, \end{aligned}$$

it is obvious that the explicit formula (3.3) clearly cannot be used to compute the base b digits in an efficient way. For the computation of the tenth decimal digit $c_{10}(10)$, e. g., one has to compute the partial sum $s_{10.000.000.000}$, a clearly impractical approach. With rational arithmetic, this is not feasible, and even with robust decimal arithmetic this computation is slow. Although not computably efficient, our formula (3.3) seems to be interesting from a theoretical point of view.

4 Irrationality of series of exponential type

Although e and therefore e^{-1} are irrational, it is not immediately clear that

$$\cosh 1 = \frac{e+e^{-1}}{2} \quad \text{and} \quad \sinh 1 = \frac{e-e^{-1}}{2}$$

are also irrational. Nevertheless, our method yields this result, too. This will follow in a more general context from the following considerations.

Let a sequence $(d_k)_{k\in\mathbb{N}}$ be given which has the following properties:

(a) $d_k \in \mathbb{N}$ for all $k \in \mathbb{N}$,

(b) $d_k > 0$ for infinitely many $k \in \mathbb{N}$,

(c) $d_k \leqq K$ for all $k \in \mathbb{N}$ and some constant $K \in \mathbb{R}$.

Now assume

$$a = \sum_{k=0}^{\infty} \frac{d_k}{k!} \ ,$$

and by

$$\widehat{s}_n = \sum_{k=0}^{n} \frac{d_k}{k!}$$

let us denote the corresponding partial sums. Then we get

Lemma 4.1 *For all $n \in \mathbb{N}$ the inequality*

$$\widehat{s}_n < a < \widehat{s}_n + \frac{K}{n\,n!}$$

is valid.

Proof: The left-hand inequality follows directly from property (b), and the right-hand inequality is proved with the aid of property (c) in a similar way as Lemma 2.2. □

Next we use again the decompositions

$$n\,\widehat{s}_n = \widehat{M}_n + \widehat{R}_n$$

with $\widehat{M}_n = \lfloor n\,\widehat{s}_n \rfloor$ and remainder $0 \leqq \widehat{R}_n < 1$ and

$$n\,a = \widetilde{M}_n + \widetilde{R}_n$$

with $\widetilde{M}_n = \lfloor n\,a \rfloor$ and remainder $0 \leqq \widetilde{R}_n < 1$, both given by the division algorithm. We get

Lemma 4.2 *For all $n \in \mathbb{N}$ the number $(n-1)!\,\widehat{R}_n \in \mathbb{N}$.*

Proof: The proof mimics the proof of Lemma 2.3. □

This gives us the ingredients to prove

Theorem 4.3 *For all $n \in \mathbb{N}$ with $n \geqq K$ it follows that*

(a) $\widehat{M}_n = \widetilde{M}_n$, *hence (1.2),*

(b) *and* $\widetilde{R}_n > 0$.

Proof: As in the proof of Theorem 2.4, initially we arrive at the inequality

$$-(n-1)! < (n-1)!\,\widetilde{M}_n - (n-1)!\,\widehat{M}_n - (n-1)!\,\widehat{R}_n < \frac{K}{n}$$

for all $n \in \mathbb{N}$. Now, if $n \geqq K$, then $\frac{K}{n} \le 1$, and therefore the rest of the proof continues in the same way as in Theorem 2.4. □

To deduce irrationality from Theorem 4.3, we need a refinement of Lemma 2.1.

Lemma 4.4 *Let $c \in \mathbb{R}_{>0}$ be arbitrary, and let the remainder $0 \leqq R_n < 1$ be defined by the division algorithm as*

$$n\,c = \lfloor n\,c \rfloor + R_n \ .$$

If $R_n > 0$ for almost all $n \in \mathbb{N}$, i. e. for all but finitely many $n \in \mathbb{N}$, then c is irrational.

Proof: Assume that $R_n > 0$ for almost all $n \in \mathbb{N}$ and c is rational. Then $c = \frac{p}{q}$ with $p \in \mathbb{Z}$ and $q \in \mathbb{N}_{>0}$. We get $q\,c = p$ and therefore

$$\lfloor q\,c \rfloor = p = q\,c \ ,$$

hence $R_q = 0$. However, for arbitrary $m \in \mathbb{N}_{>0}$ we have $c = \frac{m\,p}{m\,q}$ implying $R_{mq} = 0$ as well. This contradicts the assumption $R_n > 0$ for almost all $n \in \mathbb{N}$. □

Combining Lemma 2.1 and Lemma 4.4 yields

Lemma 4.5 *Under the same conditions of Lemma 4.4 we have: If $R_n > 0$ for almost all $n \in \mathbb{N}$ then $R_n > 0$ for all $n \in \mathbb{N}$.*

Now we are in the position to prove the essential

Theorem 4.6 *Assume*

$$a = \sum_{k=0}^{\infty} \frac{d_k}{k!} \ ,$$

and let d_k have properties (a)–(c). Then a is irrational.

Proof: This is an immmediate consequence of Theorem 4.3 and Lemma 4.4. □

Theorem 4.6 should be compared to the irrationality result given in [3, Satz 8.4].

Example 4.7 As an example, we show the irrationality of cosh 1 and sinh 1 as announced. For this purpose we set

$$d_k = \begin{cases} 1 & \text{for even } k, \\ 0 & \text{for odd } k. \end{cases}$$

This sequence obviously has properties (a)–(c) with $K = 1$. Therefore

$$\cosh 1 = \sum_{k=0}^{\infty} \frac{d_k}{k!}$$

is irrational. In a similar way, the irrationality of

$$\sinh 1 = \sum_{j=0}^{\infty} \frac{1}{(2j+1)!}$$

follows. We would like to mention that this leads to similar representations for the base b representations of cosh 1 and sinh 1 as in Theorem 3.1.

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