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Divided-difference equation, inversion, connection, multiplication and linearization formulae of the continuous Hahn and the Meixner-Pollaczek polynomials

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Abstract From the study of various properties of some difference operators, we prove in the first part of this work that the continuous Hahn and the Meixner-Pollaczek polynomials are solutions of a second order divided-difference equation of hypergeometric type. Next, using some algorithmic tools, we solve the inversion, connection, multiplication and linearization problem for the continuous Hahn and the Meixner-Pollaczek polynomials.

Keywords Meixner-Pollaczek polynomials · Continuous Hahn polynomials · Divided-difference equations · Inversion formula · Connection formula · Multiplication formula · Linearization formula

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1 Introduction

Classical orthogonal polynomials on a quadratic and q-quadratic lattice [12], [18], [19] are known to satisfy a divided-difference equation of the type ([3], [5], [19], [28])

$$\left\{\phi(x(s))\frac{\Delta}{\nabla x_1(s)}\frac{\nabla}{\nabla x(s)}+\frac{\psi(x(s))}{2}\left[\frac{\Delta}{\Delta x(s)}+\frac{\nabla}{\nabla x(s)}\right]+\lambda_n\right\}p_n(x(s))=0,\ n\geq 0, \quad (1)$$

where $\phi(x(s))=\phi_2 x^2(s)+\phi_1 x(s)+\phi_0$ and $\psi(x(s))=\psi_1 x(s)+\psi_0$ are polynomials of maximal degree two and one respectively, λ_n is a constant depending on the integer n and

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the leading coefficients ϕ_2 and ψ_1 of ϕ and ψ respectively, and $x(s)$ is a quadratic or q-quadratic lattice defined by [18]

$$x(s)=\begin{cases} c_1q^s+c_2q^{-s}+c_3 & \text{if } \; q\neq 1,\\ c_4s^2+c_5s+c_6 & \text{if } \; q=1,\end{cases}$$

with

$$x_\mu(s)=x\left(s+\frac{\mu}{2}\right),\quad \mu,\ c_1,\ \ldots,\ c_6\in\mathbb{C}.$$

Classical orthogonal polynomials on a quadratic and q-quadratic lattice are represented usually in terms of generalized or basic hypergeometric series.

Definition 1 1. The *generalized hypergeometric series* is defined by

$${}_pF_q\left(\begin{matrix}a_1,\ldots,a_p\\ b_1,\ldots,b_q\end{matrix}\middle|\,x\right):=\sum_{m=0}^{\infty}A_mx^m=\sum_{m=0}^{\infty}\frac{(a_1,\cdots,a_p)_m}{(b_1,\cdots,b_q)_m}\frac{x^m}{m!},$$

where $(a_1,\cdots,a_p)_m$ denotes the *Pochhammer symbol (or shifted factorial)* defined by

$$(a_1,\cdots,a_p)_m=(a_1)_m\cdots(a_p)_m \text{ with } (a_i)_m=\begin{cases}1 & \text{if } m=0\\ a_i(a_i+1)\cdots(a_i+m-1)=\frac{\Gamma(a_i+m)}{\Gamma(a_i)} & \text{if } m\in\mathbb{N}.\end{cases}$$

We say that a term A_m is a *hypergeometric term* with respect to m if $\frac{A_{m+1}}{A_m}$ is a rational function in the variable m.

2. The *basic hypergeometric series* ${}_r\phi_s$ is defined by

$${}_r\phi_s\left(\begin{matrix}a_1,\ldots,a_r\\ b_1,\ldots,b_s\end{matrix}\middle|\,q;z\right)=\sum_{k=0}^{\infty}\frac{(a_1,\ldots,a_r;q)_k}{(b_1,\ldots,b_s;q)_k}\left((-1)^kq^{\frac{k(k-1)}{2}}\right)^{1+s-r}\frac{z^k}{(q;q)_k},$$

where the q-Pochhammer symbol $(a_1,a_2,\ldots,a_r;q)_k$ is defined by

$$(a_1,\ldots,a_r;q)_k:=(a_1;q)_k\cdots(a_r;q)_k,\ \text{ with }\ (a_i;q)_k=\begin{cases}\prod_{j=0}^{k-1}(1-a_iq^j) & \text{if }\ k=1,2,\ldots\\ 1 & \text{if }\ k=0.\end{cases}$$

Foupouagnigni showed in [5] that for some classical orthogonal polynomials on a quadratic or q-quadratic lattice, Equation (1) is equivalent to a second-order divided-difference equation of the form

$$\phi(x(s))\mathbb{D}_x^2p_n(x(s))+\psi(x(s))\mathbb{S}_x\mathbb{D}_xp_n(x(s))+\lambda_np_n(x(s))=0, \tag{2}$$

where the operators $\mathbb{D}_x$ and $\mathbb{S}_x$ are defined by

$$\mathbb{D}_xf(x(s))=\frac{f(x(s+\frac{1}{2}))-f(x(s-\frac{1}{2}))}{x(s+\frac{1}{2})-x(s-\frac{1}{2})},\quad \mathbb{S}_xf(x(s))=\frac{f(x(s+\frac{1}{2}))+f(x(s-\frac{1}{2}))}{2}.$$

Note that (2) is equivalent to a difference or q-difference equation of the form (see [12, chapters 9, 14])

$$\lambda_ny(x(s))=B(s)y(x(s+1))-(B(s)+D(s))y(x(s))+D(s)y(x(s-1)), \tag{3}$$

with

$$B(s)=\frac{\phi(x(s))}{(x(s+1/2)-x(s-1/2))(x(s+1)-x(s))}+\frac{\psi(x(s))}{2(x(s+1)-x(s))},$$

$$D(s)=\frac{\phi(x(s))}{(x(s+1/2)-x(s-1/2))(x(s)-x(s-1))}-\frac{\psi(x(s))}{2(x(s)-x(s-1))}.$$

Following the work by Foupouagnigni [5], Njionou Sadjang et al. [22] using the same approach proved that the Wilson and the continuous dual Hahn polynomials are solutions of the divided-difference equation of the form

$$\phi(x)\mathbf{D}^2p_n(x)+\psi(x)\mathbf{S}\mathbf{D}p_n(x)+\lambda_np_n(x)=0, \tag{4}$$

where the operators **S** and the Wilson operator **D** (see [4], [11]) are defined by

$$\mathbf{D}f(x)=\frac{f\left(x+\frac{i}{2}\right)-f\left(x-\frac{i}{2}\right)}{2ix},\quad \mathbf{S}f(x)=\frac{f\left(x+\frac{i}{2}\right)+f\left(x-\frac{i}{2}\right)}{2}.$$

The divided-difference equations given in the form (2) or (4) are very useful: their coefficients can be used for instance to compute the structure formula, the connection and the inversion coefficients of classical orthogonal polynomials on a quadratic and q-quadratic lattice (see e. g. [7], [22], [29] and references therein).

Let us set the following notations:

$$B_n(a,x)=(aq^s;q)_n(aq^{-s};q)_n=\prod_{k=0}^{n-1}(1-2axq^k+a^2q^{2k}),\ n\geq 1,\ B_0(a,x)\equiv 1, \tag{5}$$

where $x=x(s)=\cos\theta=\dfrac{q^s+q^{-s}}{2}$, $q^s=e^{i\theta}$;

$$\vartheta_n(a,x)=(a+ix)_n(a-ix)_n; \tag{6}$$

$$\begin{cases}\xi_n(\gamma,\delta,\mu(x))=(q^{-x};q)_n(\gamma\delta q^{x+1};q)_n=\prod\limits_{k=0}^{n-1}(1+\gamma\delta q^{2k+1}-\mu(x)q^k),\ n\geq 1,\\ \xi_0(\gamma,\delta,\mu(x))\equiv 1,\end{cases} \tag{7}$$

with $\mu(x)=q^{-x}+\gamma\delta q^{x+1}$;

$$\begin{cases}\chi_n(\gamma,\delta,\lambda(x))=(-x)_n(x+\gamma+\delta+1)_n=\prod\limits_{k=0}^{n-1}\Big(k(\gamma+\delta+k+1)-\lambda(x)\Big),\ n\geq 1,\\ \chi_0(\gamma,\delta,\lambda(x))\equiv 1,\end{cases} \tag{8}$$

for $\lambda(x)=x(x+\gamma+\delta+1)$.
The hypergeometric and the basic hypergeometric representations of the classical orthogonal polynomials on a quadratic or q-quadratic lattice suggest to use the natural bases $\{B_n(a,x)\}$, $\{(a+ix)_n\}$, $\{\xi_n(\gamma,\delta,\mu(x))\}$ or $\{\chi_n(\gamma,\delta,\lambda(x))\}$ whose elements are polynomials of degree n in the variables x, x, $\mu(x)$ or $\lambda(x)$, respectively, and the basis $\{\vartheta_n(a,x)\}$ whose elements are polynomials of degree n in the variable x^2. The operator $\mathbb{D}_x$ is appropriate for $B_n(a,x)$, $\xi_n(\gamma,\delta,\mu(x))$ and $\chi_n(\gamma,\delta,\lambda(x))$ whereas the corresponding operator for the basis $\{\vartheta_n(a,x)\}$ is **D**.

In the second section of this work, using the operator δ_x defined in [23, p. 436] (which is appropriate for the basis $\{(a+ix)_n\}$) and following the same approach as in [5] and [22], we derive the divided-difference equation satisfied by the continuous Hahn and the Meixner-Pollaczek polynomials taking advantage that these polynomials are expanded in the basis $\{(a+ix)_n\}$.

The third section is devoted to the solution of the inversion, connection, multiplication and linearization problem for the Meixner-Pollaczek and the Continuous Hahn polynomials, proceeding as in [29, chapter 4] or ([7], [22]). The literature on the inversion, connection, multiplication and linearization problems is vast, and a variety of methods and approaches for computing the coefficients have been developed for classical continuous, discrete, q-discrete orthogonal polynomials and also for orthogonal polynomials on a nonuniform lattice (see e. g. [1], [2], [7], [10], [13], [15], [16], [17], [24], [25], [27], [29], ...). It should be noted that the *connection problem* is the problem of finding the coefficients $C_m(n)$ in the *connection formula*

$$P_n(x) = \sum_{m=0}^{n} C_m(n)Q_m(x),$$

where P_n are Q_n are polynomial sequences with $\deg(P_n) = \deg(Q_n) = n$, $\forall n \geq 0$. When $P_n(x) = \upsilon_n(x)$, where $Q_n(x)$ is expanded in the basis $\upsilon_n(x)$ that is

$$Q_n(x) = \sum_{m=0}^{n} A_m(n)\upsilon_m(x),$$

we are faced with the so-called *inversion problem* for the family $Q_m(x)$:

$$\upsilon_n(x) = \sum_{m=0}^{n} I_m(n)Q_m(x). \tag{9}$$

If we substitute $P_n(x)$ and $Q_m(x)$ in the connection formula, respectively, by $P_n(ax)$ and $P_m(x)$, we have the *multiplication formula* for the polynomials $P_n(x)$ and the *multiplication problem* is the problem of finding the coefficients $D_m(n;a)$ in the multiplication formula

$$P_n(ax) = \sum_{m=0}^{n} D_m(n;a)P_m(x),$$

where a designates a non-zero complex number.
The *linearization problem* is the problem of finding the coefficients $C_k(m,n)$ in the expansion of the product $P_n(x)Q_m(x)$ of two polynomial systems in terms of a third sequence of polynomials $R_k(x)$,

$$P_n(x)Q_m(x) = \sum_{k=0}^{n+m} L_k(m,n)R_k(x).$$

We find our results by an application of the Maple computer algebra systems. The main algorithmic tools for our development are Zeilberger's algorithm which searches for a homogeneous linear recurrence equation with polynomial coefficients for $S_n = \sum_{m=-\infty}^{\infty} A(n,m)$ (see [14, Chapter 7] and references therein), the Petkovšek-van-Hoeij algorithm which finds all hypergeometric term solutions of a homogeneous linear recurrence equation with polynomial coefficients (see [14, Chapter 9] and references therein) and is implemented in Maple by the procedure `LREtools[hypergeomsols]`, the Maple procedure `Sumtohyper` which is an implementation of Algorithm 2.8, p. 22 of [14] which converts a sum into hypergeometric notation. For all computation, we use the `hsum` package accompanying [14].

2 Divided-difference equation of the Continuous Hahn and the Meixner-Pollaczek polynomials

The Continuous Hahn and the Meixner-Pollaczek polynomials are defined, respectively, by (see [12, pages 200, 213])

$$p_n(x;a,b,c,d) = i^n \frac{(a+c)_n(a+d)_n}{n!}\,{}_3F_2\left(\left.\begin{array}{c} -n, n+a+b+c+d-1, a+ix \\ a+c, a+d \end{array}\right| 1\right),$$

$$P_n^{(\lambda)}(x;\theta) = \frac{(2\lambda)_n}{n!} e^{in\theta}\,{}_2F_1\left(\left.\begin{array}{c} -n, \lambda+ix \\ 2\lambda \end{array}\right| 1-e^{-2i\theta}\right).$$

According to these definitions, they are expanded in the basis $\{(a+ix)_n\}$. Let us define the difference operator δ_x (see [23, p. 436], compare [12, p. 201 and 214], [20], [21], [30, Equation (1.15)]) and its companion operator $\mathscr{S}$ as follows:

$$\delta_x f(x) = \frac{f\left(x+\frac{i}{2}\right) - f\left(x-\frac{i}{2}\right)}{i}, \quad \mathscr{S} f(x) = \frac{f\left(x+\frac{i}{2}\right) + f\left(x-\frac{i}{2}\right)}{2}.$$

Note that the operator $\mathscr{D}$ given in ([20], [21]) and the operator δ given in [12, p. 201 and 214] and [30, Equation(1.15)] are equal to $i\delta_x$. The operator δ_x transforms a polynomial of degree n in the variable x into a polynomial of degree $n-1$ (since $\delta_x x = 1$) and is appropriate for the basis $(a+ix)_n$ as shown in

Proposition 2 *The action of the operators δ_x and $\mathscr{S}$ on the basis $(a+ix)_n$ is given by*

$$\begin{aligned}
\delta_x(a+ix)_n &= ni(a+\frac{1}{2}+ix)_{n-1};\\
\delta_x^k(a+ix)_n &:= \delta_x^{k-1}\left(\delta_x(a+ix)_n\right)\\
&= i^k \frac{n!}{(n-k)!}(a+\frac{k}{2}+ix)_{n-k},\ k=1,2,\ldots; \qquad (10)\\
\mathscr{S}(a+ix)_n &= (a+\frac{1}{2}+ix)_n - \frac{n}{2}(a+\frac{1}{2}+ix)_{n-1};\\
(a+ix)\delta_x^2(a+ix)_n &= -n(n-1)(a+ix)_{n-1}; \qquad (11)\\
(a+ix)\mathscr{S}\delta_x(a+ix)_n &= ni(a+ix)_n - \frac{n(n-1)}{2} i(a+ix)_{n-1};\\
x(a+ix)_n &= -i(a+ix)_{n+1} + i(n+a)(a+ix)_n; \qquad (12)\\
(a+ix)(a+ix)_n &= (a+ix)_{n+1} - n(a+ix)_n;\\
(a+ix)(a+1+ix)_n &= (a+ix)_{n+1}.
\end{aligned}$$

Proof The results are obtained by direct computation.

Proposition 3 *The operators δ_x and $\mathscr{S}$ satisfy the following product rules*

$$\begin{aligned}
\delta_x(fg) &= \delta_x f \mathscr{S} g + \mathscr{S} f \delta_x g, \qquad (13)\\
\mathscr{S}(fg) &= -\frac{1}{4}\delta_x f \delta_x g + \mathscr{S} f \mathscr{S} g,\\
\delta_x \mathscr{S} &= \mathscr{S}\delta_x, \qquad (14)\\
\mathscr{S}^2 &= -\frac{1}{4}\delta_x^2 + \boldsymbol{I}, \qquad (15)
\end{aligned}$$

where $\boldsymbol{I}f = f$.

Proof The proof follows from the definition of the operators δ_x and $\mathscr{S}$.

Starting from the equations (see [12, pages 202, 214])

$$\delta_x[w(x;a,b,c,d)p_n(x;a,b,c,d)] = -(n+1)w\left(x;a-\frac{1}{2},b-\frac{1}{2},c-\frac{1}{2},d-\frac{1}{2}\right) \times p_{n+1}\left(x;a-\frac{1}{2},b-\frac{1}{2},c-\frac{1}{2},d-\frac{1}{2}\right), \quad (16)$$

where

$$w(x;a,b,c,d) = \Gamma(a+ix)\Gamma(b+ix)\Gamma(c-ix)\Gamma(d-ix),$$

and

$$\delta_x[\omega(x;\lambda,\theta)P_n^{(\lambda)}(x;\theta)] = -(n+1)\omega\left(x;\lambda-\frac{1}{2},\theta\right)P_{n+1}^{(\lambda+\frac{1}{2})}(x;\theta), \quad (17)$$

where

$$\omega(x;\lambda,\theta) = \Gamma(\lambda+ix)\Gamma(\lambda-ix)e^{(2\theta-\pi)x},$$

and using the relations (see [12, pages 201, 214])

$$\delta_x p_n(x;a,b,c,d) = (n+a+b+c+d-1)p_{n-1}\left(x,a+\frac{1}{2},b+\frac{1}{2},c+\frac{1}{2},d+\frac{1}{2}\right), \quad (18)$$

$$\delta_x P_n^{(\lambda)}(x;\theta) = 2\sin\theta P_{n-1}^{(\lambda+\frac{1}{2})}(x;\theta), \quad (19)$$

and (13), we show that

Proposition 4 *The continuous Hahn and the Meixner-Pollaczek polynomials are, respectively, solution of the divided-difference equations*

$$(2x^2-i(a+b-c-d)x-cd-ab)\delta_x^2 y(x) + (2(a+b+c+d)x-2i(ab-cd))\mathscr{S}\delta_x y(x) - 2n(n+a+b+c+d-1)y(x) = 0, \quad (20)$$

$$(x\cos(\theta)-\lambda\sin(\theta))\delta_x^2 y(x) + 2(x\sin(\theta)+\lambda\cos(\theta))\mathscr{S}\delta_x y(x) - 2n\sin(\theta)\,y(x) = 0. \quad (21)$$

Proof First combine (16) and (18) to get the relation

$$\begin{aligned}
&\delta_x\left[w\left(x;a+\frac{1}{2},b+\frac{1}{2},c+\frac{1}{2},d+\frac{1}{2}\right)\delta_x p_n(x;a,b,c,d)\right]\\
&\quad = (n+a+b+c+d-1)\\
&\qquad \times\delta_x\left[w\left(x;a+\frac{1}{2},b+\frac{1}{2},c+\frac{1}{2},d+\frac{1}{2}\right)p_{n-1}\left(x;a+\frac{1}{2},b+\frac{1}{2},c+\frac{1}{2},d+\frac{1}{2}\right)\right]\\
&\quad = -n(n+a+b+c+d-1)w(x;a,b,c,d)p_n(x;a,b,c,d).
\end{aligned}$$

Next, use the product rule (13) to write the left-hand side as

$$\begin{aligned}
&\delta_x\left[w\left(x;a+\frac{1}{2},b+\frac{1}{2},c+\frac{1}{2},d+\frac{1}{2}\right)\delta_x p_n(x;a,b,c,d)\right]\\
&\qquad = \mathscr{S}w\left(x;a+\frac{1}{2},b+\frac{1}{2},c+\frac{1}{2},d+\frac{1}{2}\right)\delta_x^2 p_n(x;a,b,c,d)\\
&\qquad\qquad +\delta_x w\left(x;a+\frac{1}{2},b+\frac{1}{2},c+\frac{1}{2},d+\frac{1}{2}\right)\mathscr{S}\delta_x p_n(x;a,b,c,d).
\end{aligned}$$

We therefore have by identification

$$\phi(x)\delta_x^2 p_n(x;a,b,c,d)+\psi(x)\mathscr{S}\delta_x p_n(x;a,b,c,d)=-n(n+a+b+c+d-1)p_n(x;a,b,c,d)$$

with

$$\phi(x)=\frac{\mathscr{S}w\left(x;a+\frac{1}{2},b+\frac{1}{2},c+\frac{1}{2},d+\frac{1}{2}\right)}{w(x;a,b,c,d)},\ \psi(x)=\frac{\delta_x w\left(x;a+\frac{1}{2},b+\frac{1}{2},c+\frac{1}{2},d+\frac{1}{2}\right)}{w(x;a,b,c,d)}$$

which are simplified into polynomials to get (20).
(21) follows by the same procedure using (17) and (19).

Remark 5 From the representations of $\delta_x^2 y(x)$, $\mathscr{S}\delta_x y(x)$, we get $y(x+i)$ and $y(x-i)$ in terms of $\delta_x^2 y(x)$, $\mathscr{S}\delta_x y(x)$ and $y(x)$. If we substitute $y(x+i)$ and $y(x-i)$ in (see [12, Eqs. (9.4.5) and (9.7.5)])

$$\lambda_n y(x)=B(x)y(x+i)-(B(x)+D(x))y(x)+D(x)y(x-i), \tag{22}$$

we see that the divided-difference equation

$$\phi(x)\delta_x^2 y(x)+\psi(x)\mathscr{S}\delta_x y(x)-2\lambda_n y(x)=0,$$

is equivalent to the well-known difference equation (22), with

$$\phi(x)=-(B(x)+D(x)),\ \psi(x)=2i(B(x)-D(x)).$$

For the continuous Hahn polynomials (see [12, page 201])

$$B(x)=(c-ix)(d-ix),\ D(x)=(a+ix)(b+ix),\ \lambda_n=n(n+a+b+c+d-1);$$

and for the Meixner-Pollaczek polynomials (see [12, page 214])

$$B(x)=e^{i\theta}(\lambda-ix),\ D(x)=-e^{-i\theta}(\lambda+ix),\ \lambda_n=2in\sin(\theta).$$

It is well known that all the derivatives of functions of hypergeometric type, i. e., which are solution of differential equations of the form $\phi(x)y''(x)+\psi(x)y'(x)+\lambda_n y(x)=0$, are also of hypergeometric type (see e. g. [19, p. 6]). In the following proposition, we want to prove a similar result.

Proposition 6 *If f is a function satisfying*

$$\phi(x)\delta_x^2 y(x)+\psi(x)\mathscr{S}\delta_x y(x)-\lambda_n y(x)=0, \tag{23}$$

then $\delta_x^m f$ is solution of the equation

$$\phi^m(x)\delta_x^2 y(x)+\psi^m(x)\mathscr{S}\delta_x y(x)-\lambda_n^m y(x)=0,$$

where $\phi^{m+1}(x)=\mathscr{S}\phi^m(x)-\frac{1}{4}\delta_x\psi^m(x)$, $\psi^{m+1}(x)=\delta_x\phi^m(x)+\mathscr{S}\psi^m(x)$, $\lambda_n^{m+1}=\delta_x\psi^m(x)+\lambda_n^m$, for $m=1,2,\ldots$, with $\phi^0(x)=\phi(x)$, $\psi^0(x)=\psi(x)$ and $\lambda_n^0=\lambda_n$.

Proof We apply the difference operator δ_x to the divided-difference equation (23) and use the relations (13)–(15) to obtain the result (see e. g. [6, Equation (54)]).

3 Inversion, connection, multiplication and linearization formulae of the Continuous Hahn and the Meixner-Pollaczek polynomials

In this section, proceeding as in [29, chapter 4], [7], [22], we solve the inversion, connection, multiplication and linearization problem for the Continuous Hahn and the Meixner-Pollaczek polynomials.

3.1 Inversion formulae of the Continuous Hahn and the Meixner-Pollaczek polynomials

To derive the inversion formulae of the Continuous Hahn and the Meixner-Pollaczek polynomials, we use the recurrence relations given below.

Proposition 7 *The following recurrence relations are valid:*
1. for the basis $(a+ix)_n$

$$x\delta_x^2(a+ix)_n = i(n+a-1)\delta_x^2(a+ix)_n - i\frac{n-1}{n+1}\delta_x^2(a+ix)_{n+1}, \tag{24}$$

2. for the continuous Hahn polynomials

$$\begin{aligned} xp_n(x;a,b,c,d) &= \frac{(n+a+b+c+d-1)(n+1)}{(2n+a+b+c+d-1)(2n+a+b+c+d)}p_{n+1}(x;a,b,c,d) + \\ & i\Big(\frac{n(n-1+a+d)(n-1+a+c)}{2n+a+b+c+d-2} - \frac{(n+1)(n+a+d)(n+a+c)}{2n+a+b+c+d} + n+a\Big)p_n(x;a,b,c,d) \\ &+ \frac{(n-1+d+b)(n-1+c+b)(a+d+n-1)(a+c+n-1)}{(2n+a+b+c+d-2)(2n+a+b+c+d-1)}p_{n-1}(x;a,b,c,d), \text{ (see [12, p. 201])} \end{aligned} \tag{25}$$

$$\begin{aligned} x\delta_x^2 p_n(x;a,b,c,d) &= \frac{(n-1)(n+a+b+c+d-1)}{(2n+a+b+c+d-1)(2n+a+b+c+d)}\delta_x^2 p_{n+1}(x;a,b,c,d) + \\ & i\Big(\frac{(n-1+a+c)(n-1+a+d)(n-2)}{2n+a+b+c+d-2} - \frac{(n+a+c)(n+a+d)(n-1)}{2n+a+b+c+d} + n+a-1\Big)\delta_x^2 p_n(x;a,b,c,d) \\ &+ \frac{(a+c+n-1)(a+d+n-1)(n-1+d+b)(n-1+c+b)(n+a+b+c+d)}{(2n+a+b+c+d-2)(2n+a+b+c+d-1)(n-2+a+b+c+d)}\delta_x^2 p_{n-1}(x;a,b,c,d), \end{aligned} \tag{26}$$

3. for the Meixner-Pollaczeck polynomials

$$\begin{aligned} xP_n^{(\lambda)}(x;\theta) &= \frac{(n+1)ie^{i\theta}}{e^{2i\theta}-1}P_{n+1}^{(\lambda)}(x;\theta) - \frac{(n+\lambda)i(e^{2i\theta}+1)}{e^{2i\theta}-1}P_n^{(\lambda)}(x;\theta) \\ &+ \frac{(2\lambda+n-1)ie^{i\theta}}{e^{2i\theta}-1}P_{n-1}^{(\lambda)}(x;\theta), \text{ (see [12, Eq. (9.7.3), p. 213])} \end{aligned} \tag{27}$$

$$\begin{aligned} x\delta_x^2 P_n^{(\lambda)}(x;\theta) &= \frac{(n-1)ie^{i\theta}}{e^{2i\theta}-1}\delta_x^2 P_{n+1}^{(\lambda)}(x;\theta) - \frac{(n+\lambda-1)i(e^{2i\theta}+1)}{e^{2i\theta}-1}\delta_x^2 P_n^{(\lambda)}(x;\theta) \\ &+ \frac{(2\lambda+n-1)ie^{i\theta}}{e^{2i\theta}-1}\delta_x^2 P_{n-1}^{(\lambda)}(x;\theta). \end{aligned} \tag{28}$$

Proof From (11) we obtain $(a+ix)_n$ in terms of $\delta_x^2(a+ix)_n$. If we substitute this in the recurrence equation (12), (24) follows.

To get (25) and (27), we substitute

$$p_n(x) = \sum_{m=0}^{n} A_m(n)(a+ix)_m \tag{29}$$

in the recurrence equation

$$xp_n = \alpha_n p_{n+1} + \beta_n p_n + \gamma_n p_{n-1}, \tag{30}$$

and use (12). By equating the coefficients of $(a+ix)_{n+1}$, one gets α_n. Equating the coefficients of $(a+ix)_n$ and $(a+ix)_{n-1}$ yields respectively β_n and γ_n.

We derive (26) and (28) as follows: we substitute the expression of p_n given by (29) in the recurrence equation

$$x\delta_x^2 p_n = \alpha_n^* \delta_x^2 p_{n+1} + \beta_n^* \delta_x^2 p_n + \gamma_n^* \delta_x^2 p_{n-1}, \tag{31}$$

and then multiply the equation obtained by $(a+ix)$. Next we use (11) and (12) respectively to eliminate $(a+ix)\delta_x^2(a+ix)_n$ and $x(a+ix)_n$. Equating the coefficients of $(a+ix)_n$, $(a+ix)_{n-1}$, $(a+ix)_{n-2}$ yields respectively α_n^*, β_n^* and γ_n^*.

Remark 8 We note that the recurrence relation of the continuous Hahn polynomials given by Equations (9.4.3) and (9.4.4) in [12, p. 201]) are equivalent to the recurrence relation (25). Indeed Equation (9.4.4) of [12, p. 201]) is

$$xp_n(x) = p_{n+1}(x) + i(A_n + C_n + a)p_n(x) - A_{n-1}C_n p_{n-1}(x),$$

where

$$p_n(x) = \frac{n!}{(n+a+b+c+d-1)_n} p_n(x;a,b,c,d),$$
$$A_n = -\frac{(n+a+b+c+d-1)(n+a+c)(n+a+d)}{(2n+a+b+c+d-1)(2n+a+b+c+d)},$$
$$C_n = \frac{n(n+b+c-1)(n+b+d-1)}{(2n+a+b+c+d-2)(2n+a+b+c+d-1)}.$$

The above recurrence equation is equivalent to

$$xp_n(x;a,b,c,d) = \frac{(n+a+b+c+d-1)(n+1)}{(2n+a+b+c+d-1)(2n+a+b+c+d)} p_{n+1}(x;a,b,c,d) + i(A_n + C_n + a) \times$$
$$p_n(x;a,b,c,d) + \frac{(n-1+d+b)(n-1+c+b)(a+d+n-1)(a+c+n-1)}{(2n+a+b+c+d-2)(2n+a+b+c+d-1)} p_{n-1}(x;a,b,c,d).$$

Since

$$A_n + C_n + a = \frac{n(n-1+a+d)(n-1+a+c)}{2n+a+b+c+d-2} - \frac{(n+1)(n+a+d)(n+a+c)}{2n+a+b+c+d} + n + a,$$

we have the equivalence between Equation (9.4.4) in [12, p. 201]) and Equation (25).

Using the above recurrence relations, we prove that

Proposition 9 *For the continuous Hahn polynomials $p_n(x;a,b,c,d)$ and the Meixner-Pollaczek polynomials $P_m^{(\lambda)}(x;\theta)$, the following inversion formulae are valid:*

$$(a+ix)_n = \sum_{m=0}^{n} \frac{i^m n!(m+a+c,m+a+d)_{n-m}}{(n-m)!(m+a+b+c+d-1)_m(2m+a+b+c+d)_{n-m}} p_m(x;a,b,c,d),\ \text{(see [20], [21])},$$

$$(\lambda+ix)_n = \sum_{m=0}^{n} \frac{(-1)^m n!(m+2\lambda)_{n-m}}{(n-m)!e^{im\theta}(1-e^{-2i\theta})^n} P_m^{(\lambda)}(x;\theta).$$

Proof Substituting the expression of $v_n(a,x)=(a+ix)_n$ given by (9) in (12) and in (24) (but with Q_n replaced by $p_n(x;a,b,c,d)$ or $P_n^{(\lambda)}(x;\theta)$), and using the three-term recurrence relations (30) and (31), we get by an appropriate shift of indices the following recurrence relations in n and m

$$-iI_m(n+1)+i(n+a)I_m(n)=\alpha_{m-1}I_{m-1}(n)+\beta_m I_m(n)+\gamma_{m+1}I_{m+1},$$
$$-i\tfrac{n-1}{n+1}I_m(n+1)+i(n+a-1)I_m(n)=\alpha^*_{m-1}I_{m-1}(n)+\beta^*_m I_m(n)+\gamma^*_{m+1}I_{m+1}.$$

By linear algebra, we eliminate the term $I_m(n+1)$ to obtain a pure recurrence equation with respect to m. By the Petkovšek-van-Hoeij algorithm, we solve the recurrence equation obtained for each family by replacing α_n, β_n, γ_n, α^*_n, β^*_n, γ^*_n by their expressions in (25) and (26) for the continuous Hahn polynomials or (27) and (28) for the Meixner-Pollaczek polynomials, then we get the solution up to a multiplicative constant. Identification of the coefficient of $(a+ix)_n$ on both sides of the inversion formula gives the desired constant.

3.2 Connection and linearization formulae

The following relations are necessary to solve the connection and linearization problems of the Continuous Hahn and the Meixner-Pollaczek polynomials.

Proposition 10 *The following linearization and connection formulae of the basis $(a+ix)_n$ are valid:*

$$(a+ix)_n(b+ix)_m = \sum_{k=0}^{m} \tfrac{m!}{k!(m-k)!}(b-a-n)_{m-k}(a+ix)_{n+k},\ n,m=0,1,\ldots, \tag{32}$$
$$(a+ix)_n = \sum_{m=0}^{n} \tfrac{n!}{m!(n-m)!}(a-b)_{n-m}(b+ix)_m,\ n=0,1,\ldots. \tag{33}$$

Proof We first remark that

$$v_n(a,x) := (a+ix)_n = \prod_{j=0}^{n-1}(a+ix+j).$$

Hence, for $x=\xi_j(a)=i(a+j)$, we have

$$v_n(a,\xi_j(a))=0,\quad j=0,1,\ldots,n-1,\quad \text{and}\quad v_n(a,\xi_n(a))\neq 0,\ n\geq 1.$$

We now expand the product $v_n(a,x)v_m(b,x)$ in the basis $v_k(a,x)$

$$v_n(a,x)v_m(b,x)=\sum_{k=0}^{n+m} J_k(m,n)v_k(a,x).$$

Clearly, we have

$$0 = v_n(a,\xi_0(a))v_m(b,\xi_0(a)) = J_0(m,n) + \sum_{k=1}^{n+m} J_k(m,n)v_k(a,\xi_0(a)) = J_0(m,n).$$

Hence, we can write

$$v_n(a,x)v_m(b,x) = \sum_{k=1}^{n+m} J_k(m,n)v_k(a,x).$$

By the same procedure, we get

$$J_1(m,n)v_1(a,\xi_1(a)) = v_n(a,\xi_1(a))v_m(b,\xi_1(a)) = 0,$$

and hence $J_1(m,n) = 0$ since $v_1(a,\xi_1(a)) \neq 0$. Progressively, we prove that

$$J_0(m,n) = J_1(m,n) = \ldots = J_j(m,n) = 0, \quad j \leq n-1.$$

Therefore, we can actually write

$$v_n(a,x)v_m(b,x) = \sum_{k=n}^{n+m} J_k(m,n)v_{n+k}(a,x) = \sum_{k=0}^{m} J_{n+k}(m,n)v_{n+k}(a,x). \tag{34}$$

Next, we have

$$v_n(a,\xi_n(a))v_m(b,\xi_n(a)) = J_n(m,n)v_n(a,\xi_n(a)),$$

and hence

$$J_n(m,n) = v_m(b,\xi_n(a)) = (b-a-n)_m.$$

Using (34), we can write

$$v_m(b,x) = \sum_{k=n}^{m} J_{n+k}(m,n)\frac{v_{n+k}(a,x)}{v_n(a,x)} = \sum_{k=0}^{m} J_{n+k}(m,n)v_k(a+n,x).$$

The use of Relation (10) yields

$$(i)^l \frac{m!}{(m-l)!}v_{m-l}\left(b+\frac{l}{2},x\right) = \sum_{k=l}^{m} J_{n+k}(m,n)(i)^l \frac{k!}{(k-l)!}v_{k-l}\left(a+n+\frac{l}{2},x\right).$$

Taking $k = l$ and $x = \xi_0\left(a+n+\frac{l}{2}\right)$, it follows that

$$J_{n+l}(m,n) = \frac{m!}{l!(m-l)!}v_{m-l}\left(b+\frac{l}{2},\xi_0\left(a+n+\frac{l}{2}\right)\right) = \frac{m!}{l!(m-l)!}(b-a-n)_{m-l},$$

therefore, the required result (32) is proved.
(33) follows by taking $n = 0$ in (32).

From the hypergeometric representation and the inversion problem of the continuous Hahn and the Meixner-Pollaczek polynomials, we get:

Proposition 11 *For the continuous Hahn polynomials $p_n(x;a,b,c,d)$ and the Meixner-Pollaczek polynomials $P_m^{(\lambda)}(x;\theta)$, the following connection and linearization formulae are valid:*
1. the continuous Hahn polynomials
Connection formula

$$p_n(x;a,b,c,d) = \sum_{m=0}^{n} \frac{(-1)^n i^{n-m}(m+a+c,m+a+d)_{n-m}(b-b_1)_n}{(n-m)!(m+a+b_1+c+d-1,n+a+b_1+c+d,b_1+1-n-b)_m} \frac{(n+a+b+c+d-1)_m(a+b_1+c+d)_{2m}}{(a+b_1+c+d)_n} p_m(x;a,b_1,c,d), \quad (35)$$

$$p_n(x;a,b,c,d) = \sum_{m=0}^{n} \frac{(-1)^m i^{n-m}(m+a+d,m+b+d)_{n-m}(c-c_1)_n}{(n-m)!(m+a+b+c_1+d-1,n+a+b+c_1+d,c_1+1-n-c)_m} \frac{(n+a+b+c+d-1)_m(a+b+c_1+d)_{2m}}{(a+b+c_1+d)_n} p_m(x;a,b,c_1,d), \quad (36)$$

$$p_n(x;a,b,c,d) = \sum_{m=0}^{n} \frac{(-1)^m i^{n-m}(m+a+c,m+b+c)_{n-m}(d-d_1)_n}{(n-m)!(m+a+b+c+d_1-1,n+a+b+c+d_1,d_1+1-n-d)_m} \frac{(n+a+b+c+d-1)_m(a+b+c+d_1)_{2m}}{(a+b+c+d_1)_n} p_m(x;a,b,c,d_1), \quad (37)$$

$$p_n(x;a,b,c,d) = \sum_{m=0}^{n} \frac{i^{n-m}(n+a+b+c+d-1)_m(m+a+c,m+a+d)_{n-m}}{(n-m)!(m+a+\beta+\gamma+\delta-1)_m} \times {}_4F_3\left(\left.\begin{matrix} m-n, m+n+a+b+c+d-1, m+a+\gamma, m+a+\delta \\ 2m+a+\beta+\gamma+\delta, m+a+c, m+a+d \end{matrix}\right| 1\right) p_m(x;a,\beta,\gamma,\delta), \quad (38)$$

Linearization formula

$$p_n(x;a,b,c,d)p_m(x;a,b,c,d) = \sum_{r=0}^{n+m} L_r(m,n)p_r(x;a,b,c,d) \text{ with}$$

$$L_r(m,n) = \sum_{l=0}^{n+m-r} \sum_{j=\max(0,l+r-m)}^{min(n,l+r)} \sum_{k=0}^{min(m+j-l-r,j)} \frac{i^{m+n+r}(a+c,a+d)_n(-n)_j(n+a+b+c+d-1)_j}{n!(a+c,a+d)_j j!m!(a+c,a+d)_{l+r-j+k}} \frac{(l+r)!}{(l+r-j)!k!l!} \frac{(a+c,a+d)_m(-m,m+a+b+c+d-1)_{l+r-j+k}(-j)_k(r+a+c,r+a+d)_l}{(r+a+b+c+d-1)_r(2r+a+b+c+d)_l},$$

2. the Meixner-Pollaczek polynomials
Connection formula

$$P_n^{(\lambda)}(x;\theta) = \sum_{m=0}^{n} \frac{(-n)_m(2\lambda)_n}{n!(2\lambda)_m} \left(\frac{e^{2i\theta_1}-e^{2i\theta}}{e^{i\theta}(e^{2i\theta_1}-1)}\right)^n \left(\frac{e^{i\theta_1}(e^{2i\theta}-1)}{e^{2i\theta}-e^{2i\theta_1}}\right)^m P_m^{(\lambda)}(x;\theta_1), \quad (39)$$

Linearization formula

$$P_n^{(\lambda)}(x;\theta)P_m^{(\lambda)}(x;\theta) = \sum_{r=0}^{n+m} \sum_{l=0}^{n+m-r} \sum_{j=\max(0,l+r-m)}^{min(n,l+r)} \sum_{k=0}^{min(m+j-l-r,j)} (-1)^r(2\lambda)_n e^{i\theta(m+n-r)}(-n)_j \frac{(l+r)!}{(l+r-j)!k!l!} \frac{(1-e^{-2i\theta})^k(2\lambda)_m(-m)_{l+r-j+k}(-j)_k(r+2\lambda)_l}{n!m!j!(2\lambda)_j(2\lambda)_{l+r-j+k}} P_r^{(\lambda)}(x;\theta).$$

Proof We combine the hypergeometric representation

$$p_n(x) = \sum_{j=0}^{n} A_j(n)(a+ix)_j$$

and the inversion formula

$$(a+ix)_j = \sum_{m=0}^{j} I_m(j)p_m(x) \tag{40}$$

to obtain the the connection formula with the connection coefficient

$$C_m(n) = \sum_{j=0}^{n-m} A_{j+m}(n)I_m(j+m).$$

To get (38) we apply the Sumtohyper algorithm. The Zeilberger algorithm combined with the Petkovšek-van-Hoeij algorithm yield the specialized cases (35)-(37) and (39).

By combining the hypergeometric representations

$$p_n(x) = \sum_{j=0}^{n} A_j(n)(a+ix)_j,\ p_m(x) = \sum_{k=0}^{m} A_k(m)(a+ix)_k,$$

the linearization of the basis (32)

$$(a+ix)_j(a+ix)_k = \sum_{l=0}^{k} J_{j+l}(k,j)(a+ix)_{j+k},$$

and the inversion formula

$$(a+ix)_l = \sum_{r=0}^{l} I_r(l)p_r(x),$$

we get the linearization formula

$$p_n(x)p_m(x) = \sum_{r=0}^{n+m} L_r(m,n)p_r(x)$$

with

$$L_r(m,n) = \sum_{l=0}^{n+m-r} \sum_{j=\max(0,l+r-m)}^{\min(n,l+r)} \sum_{k=0}^{\min(m-l-r+j,j)} A_j(n)A_{l+r-j+k}(m)J_{l+r}(l+r-j+k,j)I_r(l+r).$$

We remark that in the connection formula of the continuous Hahn polynomials of Proposition 11, the parameter a is kept identical on both sides of the formula. We would now like to get a similar formula for different a. For this purpose, we need the connection formula (33). We use the connection formula (33) to derive the representation of the continuous Hahn and the Meixner-Pollaczek polynomials in the basis $(\alpha+ix)_n$. In fact, from

$$p_n(x) = \sum_{j=0}^{n} A_j(n)(a+ix)_j \ \text{ and } \ (a+ix)_j = \sum_{m=0}^{j} F_m(j)(\alpha+ix)_m,$$

we get

$$p_n(x) = \sum_{m=0}^{n} G_m(n)(\alpha+ix)_m,$$

with

$$G_m(n) = \sum_{j=0}^{n-m} A_{j+m}(n)F_m(j+m).$$

Using the Sumtohyper algorithm, one gets the following representations.

Proposition 12 *The elements $p_n(x;a,b,c,d)$ of the continuous Hahn polynomials and $P_n^{(\lambda)}(x;\theta)$ of the Meixner-Pollaczek polynomials have the following representations in the basis $((\alpha+ix)_n)_n$*

$$p_n(x;a,b,c,d) = \sum_{m=0}^{n} i^n \frac{(a+c,a+d)_n(-n,n+a+b+c+d-1)_m}{n!(a+c,a+d)_m m!} \times {}_3F_2\left(\begin{matrix} m-n, a-\alpha, n+m+a+b+c+d-1 \\ m+a+c, m+a+d \end{matrix}\middle| 1\right)(\alpha+ix)_m, \quad (41)$$

$$P_n^{(\lambda)}(x;\theta) = \sum_{m=0}^{n} e^{in\theta}\left(\frac{e^{2i\theta}-1}{e^{2i\theta}}\right)^m \frac{(2\lambda)_n(-n)_m}{(2\lambda)_m n! m!} {}_2F_1\left(\begin{matrix} m-n, \lambda-\alpha \\ 2\lambda+m \end{matrix}\middle| \frac{e^{2i\theta}-1}{e^{2i\theta}}\right)(\alpha+ix)_m. \quad (42)$$

Remark 13 For $\alpha = b$, the representation (41) of $p_n(x;a,b,c,d)$ reduces using Zeilberger's algorithm to

$$p_n(x;a,b,c,d) = p_n(x;b,a,c,d)$$

from which we derive the following inversion and connection formulae for the continuous Hahn polynomials:

$$(b+ix)_n = \sum_{m=0}^{n} \frac{i^m n!(m+b+c,m+b+d)_{n-m}}{(n-m)!(m+a+b+c+d-1)_m(2m+a+b+c+d)_{n-m}} p_m(x;a,b,c,d),$$

$$p_n(x;a,b,c,d) = \sum_{m=0}^{n} \frac{(-1)^n i^{n-m}(m+b+c,m+b+d)_{n-m}(a-a_1)_n}{(n-m)!(m+a_1+b+c+d-1,n+a_1+b+c+d,a_1+1-n-a)_m} \frac{(n+a+b+c+d-1)_m(a_1+b+c+d)_{2m}}{(a_1+b+c+d)_n} p_m(x;a_1,b,c,d).$$

From the representation of $p_n(x;a,b,c,d)$ and $P_n^{(\lambda)}(x;\theta)$ of Proposition 12 and the inversion formula, we have

$$p_n(x;a,b,c,d) = \sum_{j=0}^{n} G_j(n)(\alpha+ix)_j \text{ and } (\alpha+ix)_j = \sum_{m=0}^{j} I_m(j) p_m(x;\alpha,\beta,\gamma,\delta),$$

from which we get

$$p_n(x;a,b,c,d) = \sum_{m=0}^{n} C_m(n) p_m(x;\alpha,\beta,\gamma,\delta),$$

with

$$C_m(n) = \sum_{j=0}^{n-m} G_{j+m}(n) I_m(j+m).$$

Using once more the Sumtohyper algorithm, one gets:

Proposition 14 *The continuous Hahn polynomials $p_n(x;a,b,c,d)$ and the Meixner-Pollaczek polynomials $P_m^{(\lambda)}(x;\theta)$ satisfy the following connection formulae:*
1. the continuous Hahn polynomials

$$p_n(x;a,b,c,d) = \sum_{m=0}^{n} \frac{i^{n-m}(n+a+b+c+d-1)_m(m+a+c,m+a+d)_{n-m}}{(n-m)!(m+\alpha+\beta+\gamma+\delta-1)_m} \times \sum_{k=0}^{n-m} \frac{(m-n,m+\alpha+\gamma,m+\alpha+\delta,m+n+a+b+c+d-1)_k}{k!(m+a+c,m+a+d,2m+\alpha+\beta+\gamma+\delta)_k} \times {}_3F_2\left(\begin{matrix} k+m-n, m+n+k+a+b+c+d-1, a-\alpha \\ m+k+a+c, m+k+a+d \end{matrix}\middle| 1\right) p_m(x;\alpha,\beta,\gamma,\delta),$$

2. the Meixner-Pollaczek polynomials

$$P_n^{(\lambda)}(x,\theta) = \sum_{k=0}^{n}\sum_{m=0}^{n-k}(-1)^{m+k}\frac{e^{i(n-k)\theta}(2\lambda)_n(2\alpha+k)_m(-n+m)_k}{m!(n-m)!(2\lambda)_{m+k}} \times$$
$${}_2F_1\left(\begin{array}{c} k+m-n,\lambda-\alpha \\ 2\lambda+m+k \end{array}\middle| 1-e^{-2i\theta}\right)P_k^{(\alpha)}(x,\theta).$$

For some applications, it is important to know the rate of change in the direction of the parameters of the orthogonal systems, given in terms of the system itself called the parameter derivative (see [9], [13], [26]).

Corollary 15 *The following parameter derivatives are valid:*
1. for the continuous Hahn polynomials

$$\frac{\partial}{\partial a}p_n(x;a,b,c,d) = \sum_{m=0}^{n-1}\Big(\frac{p_n(x;a,b,c,d)}{m+n+a+b+c+d-1} + \frac{(-1)^n i^{n-m}(n-1)!(a+b+c+d)_{2m}}{(n-m)!(a+b+c+d)_n} \times$$
$$\frac{(n+a+b+c+d-1)_m(m+b+c,m+b+d)_{n-m}}{(m+a+b+c+d-1,n+a+b+c+d,1-n)_m}p_m(x;a,b,c,d)\Big),$$

$$\frac{\partial}{\partial b}p_n(x;a,b,c,d) = \sum_{m=0}^{n-1}\Big(\frac{p_n(x;a,b,c,d)}{m+n+a+b+c+d-1} + \frac{(-1)^n i^{n-m}(n-1)!(a+b+c+d)_{2m}}{(n-m)!(a+b+c+d)_n} \times$$
$$\frac{(n+a+b+c+d-1)_m(m+a+c,m+a+d)_{n-m}}{(m+a+b+c+d-1,n+a+b+c+d,1-n)_m}p_m(x;a,b,c,d)\Big),$$

$$\frac{\partial}{\partial c}p_n(x;a,b,c,d) = \sum_{m=0}^{n-1}\Big(\frac{p_n(x;a,b,c,d)}{m+n+a+b+c+d-1} + \frac{(-1)^m i^{n-m}(n-1)!(a+b+c+d)_{2m}}{(n-m)!(a+b+c+d)_n} \times$$
$$\frac{(n+a+b+c+d-1)_m(m+a+d,m+b+d)_{n-m}}{(m+a+b+c+d-1,n+a+b+c+d,1-n)_m}p_m(x;a,b,c,d)\Big),$$

$$\frac{\partial}{\partial d}p_n(x;a,b,c,d) = \sum_{m=0}^{n-1}\Big(\frac{p_n(x;a,b,c,d)}{m+n+a+b+c+d-1} + \frac{(-1)^m i^{n-m}(n-1)!(a+b+c+d)_{2m}}{(n-m)!(a+b+c+d)_n} \times$$
$$\frac{(n+a+b+c+d-1)_m(m+a+c,m+b+c)_{n-m}}{(m+a+b+c+d-1,n+a+b+c+d,1-n)_m}p_m(x;a,b,c,d)\Big),$$

2. for the Meixner-Pollaczek polynomials

$$\frac{\partial}{\partial\theta}P_n^{(\lambda)}(x;\theta) = in\frac{e^{2i\theta}+1}{e^{2i\theta}-1}P_n^{(\lambda)}(x;\theta) - \frac{2ie^{i\theta}(2\lambda+n-1)}{e^{2i\theta}-1}P_{n-1}^{(\lambda)}(x;\theta).$$

Proof (compare [13]) Given the connection relation

$$p_n^{\alpha}(x) = \sum_{m=0}^{n}C_m(n;\alpha,\beta)p_m^{\beta}(x),$$

we build the difference quotient

$$\begin{aligned}\frac{p_n^{\alpha}(x)-p_n^{\beta}(x)}{\alpha-\beta} &= \sum_{m=0}^{n}\frac{C_m(n;\alpha,\beta)}{\alpha-\beta}p_m^{\beta}(x) - \frac{p_n^{\beta}(x)}{\alpha-\beta}\\ &= \frac{C_n(n;\alpha,\beta)-1}{\alpha-\beta}p_n^{\beta}(x) + \sum_{m=0}^{n-1}\frac{C_m(n;\alpha,\beta)}{\alpha-\beta}p_m^{\beta}(x)\end{aligned}$$

so that with $\beta \to \alpha$

$$\frac{\partial}{\partial\alpha}p_n^{\alpha}(x) = \lim_{\beta\to\alpha}\frac{C_n(n;\alpha,\beta)-1}{\alpha-\beta}p_n^{\beta}(x) + \sum_{m=0}^{n-1}\lim_{\beta\to\alpha}\frac{C_m(n;\alpha,\beta)}{\alpha-\beta}p_m^{\beta}(x)$$

since the systems $p_n^{\alpha}(x)$ are continuous with respect to α. This gives the results.

3.3 Multiplication formulae of the Continuous Hahn and the Meixner-Pollaczek polynomials

In this section, we solve the multiplication problem for the continuous Hahn and the Meixner-Pollaczek polynomials. We first present the following results for the operator δ_x.

Theorem 16 (see [20]) *Assume $f(x)$ is a polynomial of degree n in the variable x. Then*

$$f(x) = \sum_{k=0}^{n} f_k(a+ix)_k, \quad with \;\; f_k = \frac{(-i)^k}{k!}(\delta_x^k f)\Big(i(a+\frac{k}{2})\Big).$$

Proof Let $j = 0,1,\ldots,k$. We apply δ_x^j to both sides of $f(x) = \sum_{k=0}^{n} f_k(a+ix)_k$ and use (10) to get

$$\delta_x^j f(x) = f_j i^j j! + \sum_{k=j+1}^{n} f_k(i)^j\frac{k!}{(k-j)!}(a+ix+\frac{j}{2})_{k-j}.$$

For $x = i\left(a+\frac{j}{2}\right)$, we obtain

$$\delta_x^j f\left(i\left(a+\frac{j}{2}\right)\right) = i^j f_j j!,$$

and the proof is completed.

Proposition 17 (see [20], [30]) *Let k be a nonnegative integer, then the following relation holds*

$$\delta_x^k f(x) = \sum_{l=0}^{k}\frac{(-1)^l k!}{i^k l!(k-l)!} f\left(x+\frac{k-2l}{2}i\right). \tag{43}$$

Proof The proof is done by induction w.r.t. k.

It follows from Propositions 16 and 17 that

Proposition 18 *The following duplication formula is valid:*

$$(a+i\alpha x)_n = \sum_{k=0}^{n}\sum_{l=0}^{k}\frac{(-1)^l}{l!(k-l)!}(a-\alpha(a+l))_n(a+ix)_k. \tag{44}$$

Proof First we apply Proposition 16 with $f(x) = (a+i\alpha x)_n$ to get

$$(a+i\alpha x)_n = \sum_{k=0}^{n}\frac{(-i)^k}{k!}\delta_x^k(a+i\alpha x)_n\Big|_{x=i(a+\frac{k}{2})}(a+ix)_k.$$

Next, using Proposition 17, we have

$$\delta_x^k(a+i\alpha x)_n\Big|_{x=i(a+\frac{k}{2})} = \sum_{l=0}^{k}\frac{(-1)^l k!}{i^k l!(k-l)!}(a-\alpha(a+k-l))_n.$$

We substitute $l \to k-l$ to complete the proof.

It follows from the above result that

Proposition 19 *The continuous Hahn polynomials $p_n(x;a,b,c,d)$ and the Meixner-Pollaczek polynomials $P_m^{(\lambda)}(x;\theta)$ satisfied the following multiplication formulae:*

$$p_n(\alpha x;a,b,c,d)=\sum_{m=0}^{n}\frac{i^{m+n}(a+c,a+d)_n}{n!(a+b+c+d-1+m)_m}\sum_{s=0}^{n-m}\frac{(n-s)!}{(n-s-m)!}\frac{(a+c+m,a+d+m)_{n-m-s}}{(a+b+c+d+2m)_{n-m-s}}$$
$$\times\sum_{l=0}^{n-s}\sum_{j=0}^{s}\frac{1}{l!(n-s-l)!}\frac{(-1)^l(-n,n+a+b+c+d-1,a-\alpha(a+l))_{j+n-s}}{(j+n-s)!(a+c,a+d)_{j+n-s}}p_m(x;a,b,c,d).$$

$$P_n^{(\lambda)}(\alpha x,\theta)=\sum_{m=0}^{n}\frac{(2\lambda)_n e^{i\theta(n-m)}}{n!}\sum_{s=0}^{n-m}\frac{(2\lambda+m)_{n-s-m}}{(n-s-m)!}$$
$$\times\sum_{l=0}^{n-s}\sum_{j=0}^{s}\frac{(n-s)!}{l!(n-s-l)!}\frac{(-1)^{m+l}(-n,\lambda-\alpha(\lambda+l))_{j+n-s}(1-e^{-2i\theta})^j}{(2\lambda)_{j+n-s}(j+n-s)!}P_m^{(\lambda)}(x,\theta).$$

Proof Combining $p_n(\alpha x)=\sum_{k=0}^{n}A_k(n)(a+i\alpha x)_k$, $(a+i\alpha x)_k=\sum_{i=0}^{k}E_i(k,a,\alpha)(a+ix)_i$ with $E_i(k,a,\alpha)=\sum_{l=0}^{i}F_l(i,k,a,\alpha)$, $(a+ix)_i=\sum_{m=0}^{i}I_m(i)p_m(x)$, interchanging the order of summation and substituting i by $n-m-j$ yields the duplication relation $p_n(\alpha x)=\sum_{m=0}^{n}D_m(n,\alpha)p_m(x)$ with

$$D_m(n,\alpha)=\sum_{j=0}^{n-m}\sum_{k=0}^{j}\sum_{l=0}^{n-j}A_{k+n-j}(n)F_l(n-j,k+n-j,a,\alpha)I_m(n-j).$$

Remark 20 By setting (see [12, p. 215]) $x\to x+t$, $a\to\lambda-it$, $c\to\lambda+it$ and $b=d=t\tan\theta$ in the definition of the continuous Hahn polynomials and taking the limit $t\to\infty$ we obtain the Meixner-Pollaczek polynomials $P_n^{(\lambda)}(x;\theta)$:

$$P_n^{(\lambda)}(x;\theta)=(\cos\theta)^n\lim_{t\to\infty}\frac{p_n(x+t;\lambda-it,t\tan\theta,\lambda+it,t\tan\theta)}{t^n}.$$

Note that in [12, p. 215], there is a misprint in this limit relation ($n!$ must not be in the denominator). Using this limit relation, the inversion, connection, linearization, multiplication and parameter derivatives relations for the Meixner-Pollaczek polynomials can be derived from those of the continuous Hahn polynomials.

4 Conclusion

In this work, we find the divided-difference equation of the continuous Hahn and the Meixner-Pollaczek polynomials and then we solve the inversion, connection, multiplication and linearization problem for these two polynomial families. To the best of our knowledge, the results obtained in this work are completely new. We find our results by an application of the Maple computer algebra systems. The main algorithmic tools for our development are Zeilberger's algorithm, the Petkovšek-van-Hoeij algorithm, the Maple procedure `Sumtohyper` which is an implementation of Algorithm 2.8, p. 22 of [14].

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