A=B. *By Marko Petkovšek, Herbert S. Wilf, Doron Zeilberger*. A. K. Peters, Ltd., Wellesley, 1996. \$39.00. xii + 212 pp., cloth. ISBN 1-56881-063-6.

In their recent research, the authors of the book under review have given important contributions towards computer proofs of hypergeometric identities. Hypergeometric identities are identites about hypergeometric sums, i.e., definite sums

$$S_n := \sum_{k\in\mathbb{Z}} F(n,k) \tag{1}$$

where the summand is a hypergeometric term w.r.t. both n and k, i.e., the term ratios

$$\frac{F(n+1,k)}{F(n,k)} \quad \text{and} \quad \frac{F(n,k+1)}{F(n,k)}$$

are rational functions in n and k.

In the book under review this knowledge is collected, and a nice introduction to the topic is given.

The main idea behind these computerized proofs is to detect a *holonomic recurrence equation* for the sum S_n under consideration, i.e., a linear recurrence equation with polynomial coefficients. Zeilberger was the one having the idea how to adjust Gosper's algorithm on indefinite hypergeometric summation to the definite case.

Although many of these ideas can be generalized, e.g. towards the consideration of multiple sums, integrals, q-sums, the generation of differential rather than recurrence equations, etc., the authors are mainly concerned with the above mentioned setting.

This is the contents of the book:

Foreword: The foreword is written by Donald Knuth. He gives some examples of sums which he was investigating, and for which the new methods are great tools. The funny thing is that his main example

$$S_n = \sum_{k\in\mathbb{Z}} \binom{2n-2k}{n-k}^2 \binom{2n}{k}^2$$

is slightly corrupted by a typographical error. This one has a recurrence equation whose printout covers a whole page, which shows the power and the pitfalls of Zeilberger's method at the same time! The sum Knuth really meant is the much more well-behaved

$$S_n = \sum_{k\in\mathbb{Z}} \binom{2n-2k}{n-k}^2 \binom{2k}{k}^2$$

satisfying the simple recurrence equation

$$\begin{aligned} 0 \;=\; & (n+2)^3 S_{n+2} - 8\,(3+2\,n)\left(2\,n^2+6\,n+5\right) S_{n+1} \\ & + 256\,(n+1)^3 S_n\,. \end{aligned}$$

Note that an errata sheet can be found at the URL http://www.cis.upenn.edu/~wilf/AeqBErrata.html.

A Quick Start ...: Here by a short example it is shown how to download software in Maple and Mathematica from the World Wide Web, and how to deal with this software.

I Background

1. Proof Machines: Canonical and normal forms are discussed, and it is shown how proofs can be given "by examples", using recurrence equations as normal forms. Polynomial, trigonometric, and other types of identities are discussed.

2. Tightening the Target: Here the main topic of the book, the *hypergeometric identities*, are introduced. It is shown how Mathematica and Maple deal with hypergeometric sums, and WZ proof certificates (see Chapter 7) are introduced.

3. The Hypergeometric Database: A database of hypergeometric identities can be used to identify sums as soon as such sums are converted into hypergeometric notation. Here this conversion is considered.

II The Five Basic Algorithms

4. Sister Celine's Method: The method of Celine Fasenmyer to find a recurrence equation w.r.t. n for a sum S_n given by (1) is presented. Celine Fasenmyer uses linear algebra to detect a k-free recurrence equation w.r.t. both n and k for the *summand*, which afterwards is summed resulting in the recurrence equation searched for.

5. Gosper's Algorithm: Gosper's algorithm finds a hypergeometric term antidifference s_k for a_k, i.e., $s_{k+1} - s_k = a_k$, whenever such an antidifference exists. As a result, indefinite summation of hypergeometric terms can be treated algorithmically.

6. Zeilberger's Algorithm: Zeilberger's algorithm uses a variant of Gosper's algorithm to determine holonomic recurrence equations for definite sums, given by (1). In most cases this recurrence equation is of lowest order. If it is of first order, then one can read off the hypergeometric term solution; if not, Petkovšek's algorithm, described in Chapter 8, can be used to determine such solutions if applicable.

Note that Zeilberger's algorithm in general is much faster than Celine Fasenmyer's method since its linear algebra part deals with mainly $J+1$ rather than with $(J+1)^2$ variables if J denotes the order of the recurrence equation searched for.

7. The WZ Phenomenon: In the cases in which Zeilberger's algorithm determines a first order recurrence equation, the WZ phenomenon occurs: such a hypergeometric identity can be proved by bringing it into the form

$$S_n := \sum_{k\in\mathbb{Z}} F(n,k) = 1\,, \tag{2}$$

and by using Gosper's algorithm to find a rational multiple $G(n,k) = R(n,k)\, F(n,k)$ of $F(n,k)$ for which

$$(3)\ F(n+1,k) - F(n,k) = G(n,k+1) - G(n,k)\ .$$

Hence by summation $S_{n+1} - S_n = 0$ proving (2) (modulo one initial value). The rational function $R(n,k)$ is called the *WZ proof certificate*. Its knowledge makes a proof of (2) available by verifying a single rational identity.

8. Algorithm Hyper: Petkovšek's algorithm is a decision procedure to determine all hypergeometric term solutions of a given holonomic recurrence equation. It uses a representation lemma for rational functions initially due to Gosper, the *Gosper-Petkovšek representation*, in a clever way.

III Epilogue

9. An Operator Algebra Viewpoint: The main theme of the book are holonomic recurrence equations. Using the shift operator $N\, a_n := a_{n+1}$, these can also be understood as operator equations, and one can deal with them in a non-commutative algebra where the commutator rule $Nn - nN = N$ is valid. In the given chapter this approach is considered in more detail.

In the Appendix the WWW sites and the software are discussed in more detail.

All algorithms that are discussed in the book under review are accompanied by examples, and a few exercises for the reader some of which come with solutions. Furthermore the authors give examples for the use of Mathematica and Maple to do the computations. It is assumed that the reader has access to the World Wide Web or to other file transfer services, as well as to either Maple or Mathematica since the use of implementations of the algorithms considered seems to be a must.

The authors refer to Maple software available from Zeilberger's WWW site, and to Mathematica software due to Krattenthaler (hypergeometric database), Paule/Schorn (Gosper's and Zeilberger's algorithms) and Petkovšek (Petkovšek's algorithm). Implementational details are not discussed. Note that the Maple package `sumtools` written by the reviewer [2] comes with Maple V.4 and does also contain an implementation of both Gosper's and Zeilberger's algorithms.

The presentation of the book is charming, and gives an excellent introduction to this modern topic. I would like to mention two minor inconveniences, though. First, the fact that the rational certificate of an application of Zeilberger's algorithm might contain poles with some obvious defects is not addressed. Second, I find it a little inconvenient that in some instances the authors use different notations at different places of the book. This might be influenced by the fact that the book forms essentially a collection of previously published material [1], [4], [5], [6], [7].

There is no need, e.g., for new notations for rising and falling factorials different from the ones given on pages 39 and 149, respectively, in the proof of the "Fundamental Theorem" on p. 66. In my opinion, this causes confusion. Similarly the footnote on p. 157 about the rising factorial notation is unnecessary since this definition is given on p. 39. Even worse, the mentioned footnote contains a *wrong* notation.

The authors mention the continuous analogues of the algorithms presented, without giving the details. A forthcoming book by the reviewer [3] with the emphasis on the use of Maple for orthogonal polynomials and special function will cover these topics.

One of the highlights of the presentation is the consideration of finite sums of hypergeometric terms. The authors show how Gosper's algorithm can be extended to this case. This previously unnoticed fact is rather important since summation is a linear operation, but Gosper's original algorithm is not.

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