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TWO FINITE SEQUENCES OF SYMMETRIC Q-ORTHOGONAL POLYNOMIALS GENERATED BY TWO Q-STURM-LIOUVILLE PROBLEMS

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By using a symmetric generalization of Sturm-Liouville problems in q-difference spaces, we introduce two finite sequences of symmetric q-orthogonal polynomials and obtain their basic properties such as a second order q-difference equations, the explicit form of the polynomials in terms of basic hypergeometric series, three term recurrence relations and norm square values based on a Ramanujan identity. We also show that one of the introduced sequences is connected with the Little q-Jacobi polynomials.

Keywords: q-Sturm-Liouville problems; symmetric finite q-orthogonal polynomials; Ramanujan's identity; Little q-Jacobi polynomials; norm square value.

1. Introduction

Let us start with the following identity discovered by Ramanujan [10]. For $0 < q < 1$, $|a| > q$, $|b| < 1$ and $|\frac{b}{a}| < |x| < 1$, we have

$$\Psi(a, b; q; x) = \sum_{n=-\infty}^{\infty} \frac{(a;\, q)_n}{(b;\, q)_n} x^n \tag{1}$$
$$= \prod_{n=0}^{\infty} \frac{(1 - \frac{bq^n}{a})(1 - q^{n+1})(1 - \frac{q^{n+1}}{ax})(1 - axq^n)}{(1 - bq^n)(1 - \frac{q^{n+1}}{a})(1 - \frac{bq^n}{ax})(1 - xq^n)},$$

where

$$(a;\, q)_n = \prod_{j=0}^{n-1} (1 - aq^j).$$

We apply this identity (1) for explicitly computing the norm square values of two new finite classes of symmetric q-orthogonal polynomials. In general, classical q-orthogonal polynomials are solutions of a q-Sturm-Liouville problem of the form [9, 10]

$$Ly(x; q) + \lambda_q \varrho(x; q) y(x; q) = 0, \tag{2}$$

where

$$Ly(x; q) = (D_q(r\, D_{q^{-1}} y))(x; q) \qquad (r(x; q) > 0,\ \varrho(x; q) > 0), \tag{3}$$

with the boundary conditions

$$\alpha_1 y(a; q) + \beta_1 D_q y(a; q) = 0, \qquad \alpha_2 y(b; q) + \beta_2 D_q y(b; q) = 0, \tag{4}$$

and D_q is the q-difference operator defined by [1, 2, 6]

$$D_q f(x) = \frac{f(qx) - f(x)}{(q-1)x} \qquad (x \neq 0,\ q \neq 1),$$

with $D_q f(0) := f'(0)$, provided that $f'(0)$ exists. In the q-Sturm-Liouville problem (2)-(4), there is an orthogonality property for eigenfunctions of equation (3) on (a, b) with respect to the weight function $\varrho(x; q)$. In other words, if $y_m(x; q)$ and $y_n(x; q)$ are two solutions of the problem (2)-(4), then by referring to the boundary conditions (4) at $x = a$ we have

$$\alpha_1 y_m(a; q) + \beta_1 D_q y_m(a; q) = 0, \quad \textit{and} \quad \alpha_2 y_n(a; q) + \beta_2 D_q y_n(a; q) = 0,$$

which is also valid for $x = b$. In the sequel, if the q-analogue of integration by parts [22, 10] is applied for $y_m(x; q)$ and $y_n(x; q)$ we find

$$\int_a^b (y_m L y_n - y_n L y_m)(x; q) d_q x = 0.$$

This means that if $y_n(x;q)$ and $y_m(x;q)$ are two eigenfunctions of the q-difference equation (2), they are orthogonal with respect to the weight function $\varrho(x;q)$ and

$$\int_a^b y_m(x;q)\, y_n(x;q)\, \varrho(x;q) d_q x = 0, \quad (\lambda_m \neq \lambda_n), \tag{5}$$

in which the q-integral operator [8] is defined by

$$\int_0^x f(t) d_q t = (1-q)x \sum_{j=0}^{\infty} q^j f(q^j x), \qquad (x \in A), \tag{6}$$

where A is a μ-geometric set for fixed $\mu \in \mathbb{C}$ [4] and the right hand series is convergent. Note that for any arbitrary interval $[a,b]$ we have from (6) that

$$\int_a^b f(x) d_q x = \int_0^b f(x) d_q x - \int_0^a f(x) d_q x \qquad (a, b \in A). \tag{7}$$

Also, from (6) and (7) one can conclude that

$$\int_{-b}^b f(t) d_q t = b(1-q) \sum_{n=0}^{\infty} q^n \left(f(bq^n) + f(-bq^n) \right), \qquad (b \in A). \tag{8}$$

Finally, if $b \to \infty$, (8) changes to [10]

$$\int_{-\infty}^{\infty} f(t) d_q t = (1-q) \sum_{n=-\infty}^{\infty} q^n \left(f(q^n) + f(-q^n) \right).$$

Recently many symmetric special functions of continuous type have been generalized in [12, 13, 14, 17, 18] and a discrete analogue of the main theorem 1 given in this paper on the linear lattice $x(s) = s$ has been proved in [16]. Also, a basic class of symmetric orthogonal polynomials of a discrete variable with four free parameters has been introduced in [15]. In [4], the authors have presented a theorem by which one can generalize q-Sturm-Liouville problems with symmetric solutions.

Theorem 1. *Let $\phi_n(x;q) = (-1)^n \phi_n(-x;q)$ be a sequence of symmetric functions that satisfies the q-difference equation*

$$\varphi(x) D_q D_{q^{-1}} \phi_n(x;q) + \tau(x) D_q \phi_n(x;q) + \left(\lambda_{n,q} \theta(x) + \pi(x) + \sigma_n \eta(x) \right) \phi_n(x;q) = 0, \tag{9}$$

where $\varphi(x)$, $\tau(x)$, $\theta(x)$, $\pi(x)$ and $\eta(x)$ are real functions, σ_n is defined as

$$\sigma_n = \frac{1-(-1)^n}{2} = \begin{cases} 0 & n \text{ even}, \\ 1 & n \text{ odd}, \end{cases}$$

and $\lambda_{n,q}$ is a sequence of constants. If $\varphi(x)$, $(\theta(x)>0)$, $\pi(x)$ and $\eta(x)$ are even functions and $\tau(x)$ is odd, then

$$\int_{-b}^{b}\varrho^*(x;q)\phi_n(x;q)\phi_m(x;q)d_qx=\left(\int_{-b}^{b}\varrho^*(x;q)\phi_n^2(x;q)d_qx\right)\delta_{n,m},\quad \delta_{n,m}=\begin{cases}0 & (n\neq m),\\ 1 & (n=m),\end{cases}$$

where

$$\varrho^*(x;q)=\theta(x)\varrho(x;q), \tag{10}$$

and $\varrho(x;q)$ is a solution of the Pearson q-difference equation

$$D_q\left(\varphi(x)\varrho(x;q)\right)=\tau(x)\varrho(x;q),$$

which is equivalent to

$$\frac{\varrho(qx;q)}{\varrho(x;q)}=\frac{(q-1)x\tau(x)+\varphi(x)}{\varphi(qx)}.$$

Of course, the weight function defined in (10) must be positive and even, and the function $\varphi(x)\varrho(x;q)$ must vanish at $x=b$ see [4].

Using the above theorem, we can introduce two finite sequences of symmetric q-orthogonal polynomials and obtain their general properties in detail. For this purpose, we should first refer to basic hypergeometric series

$${}_r\phi_s\left(\begin{matrix}a_1,a_2,...,a_r\\ b_1,b_2,...,b_s\end{matrix};q;z\right)=\sum_{k=0}^{\infty}\frac{(a_1;\,q)_k\dots(a_r;\,q)_k}{(q;\,q)_k(b_1;\,q)_k\dots(b_s;\,q)_k}\left((-1)^kq^{\binom{k}{2}}\right)^{1+s-r}z^k, \tag{11}$$

where $r,\,s\in\mathbb{Z}_+$ and $a_1,a_2,...,a_r,\,b_1,b_2,...,b_s,\,z\in\mathbb{C}$. Note in (11) that

$$(a;\,q)_\infty=\prod_{j=0}^{\infty}(1-aq^j),\quad \text{for}\quad 0<|q|<1,$$

and in order to have a well–defined series in (11), the condition $b_1,b_2,...,b_s\neq q^{-k}$ for $k=0,1,\dots$ is necessary. The base of definition of such q-hypergeometric series, from historical point of view, is q-numbers defined by

$$[z]_q=\frac{q^z-1}{q-1},\quad z\in\mathbb{C}. \tag{12}$$

The classical orthogonal q-polynomials are known in the literature as Askey-Schem of hypergeometric q-orthogonal polynomials [11], see also [3, 7]. Since we need some of them in order to compare with two polynomials introduced in this paper, here we recall some of them whose orders are respectively $(1,1)$, $(2,1)$ and $(2,0)$. For instance

$$P_n(x;a,b;q)=\frac{1}{(b^{-1}q^{-n};q)_n}\,{}_2\varphi_1\left(\begin{matrix}q^{-n},aqx^{-1}\\ aq\end{matrix};q,\frac{x}{b}\right),$$

are Big q-Laguerre polynomials that satisfy the orthogonality property

$$\int_{bq}^{aq} \frac{(a^{-1}x, b^{-1}x;q)_\infty}{(x;q)_\infty} P_m(x;a,b;q)P_n(x;a,b;q)d_qx$$
$$= aq(1-q)\frac{(q, a^{-1}b, ab^{-1}q;q)_\infty}{(aq,bq;q)_\infty}\frac{(q;q)_n}{(aq,bq;q)_n}(-abq^2)^n q^{\binom{n}{2}}\delta_{n,m},$$

where $0 < aq < 1$ and $b < 0$. Also

$$p_n(x;a;q) = {}_2\varphi_1\left(\begin{matrix} q^{-n}, 0 \\ aq \end{matrix} ;q, qx\right), \tag{13}$$

are known as the Little q-Laguerre polynomials with the orthogonality property

$$\sum_{k=0}^{\infty} \frac{(aq)^k}{(q;q)_k} p_m(q^k;a;q)p_n(q^k;a;q) = \frac{(aq)^n}{(aq;q)_\infty}\frac{(q;q)_n}{(aq;q)_n}\delta_{n,m}, \quad (0 < aq < 1),$$

and

$$L_n^{(\alpha)}(x;q) = \frac{(q^{\alpha+1};q)_n}{(q;q)_n}{}_1\varphi_1\left(\begin{matrix} q^{-n} \\ q^{\alpha+1} \end{matrix} ;q, -q^{n+\alpha+1}x\right)$$
$$= \frac{1}{(q;q)_n}{}_2\varphi_1\left(\begin{matrix} q^{-n}, -x \\ 0 \end{matrix} ;q, -q^{n+\alpha+1}\right),$$

are q-Laguerre polynomials with two orthogonality properties

$$\int_0^\infty \frac{x^\alpha}{(x-;q)_\infty} L_m^{(\alpha)}(x;q)L_n^{(\alpha)}(x;q)dx$$
$$= \frac{(q^{-\alpha};q)_\infty}{(q;q)_\infty}\frac{(q^{\alpha+1};q)_n}{(q;q)_n q^n}\Gamma(-\alpha)\Gamma(\alpha+1)\delta_{n,m}, \quad (\alpha > -1),$$

and

$$\sum_{-\infty}^{\infty} \frac{q^{k\alpha+k}}{(-cq^k;q)_\infty} L_m^{(\alpha)}(cq^k;q)L_n^{(\alpha)}(cq^k;q)$$
$$= \frac{(q, -cq^{\alpha+1}, -c^{-1}q^{-\alpha};q)_\infty}{(q^{\alpha+1}, -c, -c^{-1}q;q)_\infty}\frac{(q^{\alpha+1};q)_n}{(q;q)_n q^n}\delta_{n,m} \quad (\alpha > -1, \quad c > 0).$$

The Little q-Jacobi polynomials are defined by

$$J_n(x;a,b;q) = {}_2\varphi_1\left(\begin{matrix} q^{-n}, abq^{n+1} \\ aq \end{matrix} ;q, qx\right), \tag{14}$$

which satisfy the orthogonality property

$$\sum_{k=0}^{\infty} \frac{(bq;q)_k}{(q;q)_k}(aq)^k J_m(q^k;a,b;q) J_n(q^k;a,b;q)$$
$$= \frac{(abq^2;q)_\infty}{(aq;q)_\infty} \frac{(1-abq)(aq)^n}{(1-abq^{2n+1})} \frac{(q,bq;q)_n}{(aq,abq;q)_n} \delta_{n,m},$$

where $0 < aq < 1$ and $bq < 1$ and

$$M_n(q^{-x};b,c;q) = {}_2\varphi_1\left(\begin{matrix} q^{-n}, q^{-x} \\ bq \end{matrix} ; q, -\frac{q^{n+1}}{c} \right),$$

are known as q-Meixner polynomials with the orthogonality property

$$\sum_{x=0}^{\infty} \frac{(bq;q)_x}{(q,-bcq;q)_x} c^x q^{\binom{x}{2}} M_m(q^{-x};b,c;q) M_n(q^{-x};b,c;q)$$
$$= \frac{(-c;q)_\infty}{(-bcq;q)_\infty} \frac{(q,-c^{-1}q;q)_n}{(bq;q)_n} q^{-n} \delta_{n,m} \quad (0 \le bq < 1, \quad c > 0).$$

Finally

$$h_n(x;q) = q^{\binom{n}{2}} {}_2\varphi_1\left(\begin{matrix} q^{-n}, x^{-1} \\ 0 \end{matrix} ; q, -qx \right)$$
$$= x^n {}_2\varphi_0\left(\begin{matrix} q^{-n}, q^{-n+1} \\ - \end{matrix} ; q^2, \frac{q^{2n-1}}{x^2} \right),$$

are Discrete q-Hermite polynomials with the orthogonality property

$$\int_{-1}^{1} (qx,-qx;q)_\infty h_m(x;q) h_n(x;q) d_q x$$
$$= (1-q)(q;q)_n (q,-1,-q;q)_\infty q^{\binom{n}{2}} \delta_{n,m},$$

and

$$V_n^{(a)}(x;q) = (-a)^n q^{-\binom{n}{2}} {}_2\varphi_0\left(\begin{matrix} q^{-n}, x \\ - \end{matrix} ; q, \frac{q^n}{a} \right),$$

are Al-Salam-Carlitz II polynomials that satisfy the orthogonality property

$$\int_{a}^{1} (qx, a^{-1}qx;q)_\infty V_m^{(a)}(x;q) V_n^{(a)}(x;q) d_q x$$
$$= (-a)^n (1-q)(q;q)_n (q,a,a^{-1}q;q)_\infty q^{\binom{n}{2}} \delta_{n,m} \quad (a < 0).$$

Here Let us add that the q-Bessel polynomials are also important in certain problems of mathematical physics; for example, they appear in the study of electrical networks and when the wave equation is considered in spherical coordinates, see e.g. [20, 21].

2. Two finite sequences of symmetric q-orthogonal polynomials

In this section, we introduce two finite classes of symmetric orthogonal q-polynomials which are particular solutions of q-difference equation (9) and have not been considered in [4]. It is straightforward to check [4, 12] that if $\varphi(x)$ is a polynomial of degree at most four, $\tau(x)$ an odd polynomial of degree at most three, $\theta(x)$ a symmetric quadratic polynomial and $\pi(x)$ and $\eta(x)$ are two constants, one can find symmetric polynomial solutions for equation (9). By noting these comments, recently in [4], a q-difference equation of type (9) has been introduced as

$$x^2\left(ax^2+b\right)D_qD_{q^{-1}}\phi_n(x;q)+x\left(cx^2+d\right)D_q\phi_n(x;q) \\ -\left([n]_q\left(c-[1-n]_qa\right)x^2+\sigma_nd\right)\phi_n(x;q)=0, \quad (15)$$

whose explicit q-polynomial solution is

$$\phi_n(x;q)=\sum_{k=0}^{\left[\frac{n}{2}\right]}q^{(k-1)k}x^{n-2k}\begin{bmatrix}\left[\frac{n}{2}\right]\\ k\end{bmatrix}_{q^2}\prod_{j=0}^{\left[\frac{n}{2}\right]-k-1}\frac{a[2j+\sigma_n+n-1]_q+cq^{2j+\sigma_n+n-1}}{b[\left(2j+(-1)^{n+1}+2\right)]_q+dq^{2j+(-1)^{n+1}+2}}, \quad (16)$$

where

$$\begin{bmatrix}n\\ m\end{bmatrix}_q=\frac{(q;\,q)_n}{(q;\,q)_m(q;\,q)_{n-m}},$$

denotes the q-binomial coefficient and $[z]_q$ is the q-number defined in (12).

Also, it is shown in [4] that the monic form of these polynomials satisfies a three term recurrence relation as

$$\bar{\phi}_{n+1}(x;q)=x\bar{\phi}_n(x;q)-C_{n,q}\bar{\phi}_{n-1}(x;q), \quad \text{with} \quad \bar{\phi}_0(x;q)=1, \quad \bar{\phi}_1(x;q)=x, \quad (17)$$

where

$$C_{n,q}=[q^{n+1}(q^{2n}(a+c(q-1))((d-dq)\sigma_n-b)+q^n(ab(q^2+1)+ad(q-1)q^2+bc(q-1)) \\ -aq^2(b+d(q-1)\sigma_{n-1}))]/[a^2q^4+q^{4n}(a+c(q-1))^2-a(q^3+q)q^{2n}(a+c(q-1))]. \quad (18)$$

There are two special cases of equation (15) whose polynomial solutions are finitely orthogonal on $(-\infty,\infty)$.

2.1. First sequence

For $u,v\in\mathbb{R}$, consider the equation

$$x^2\left(x^2+1\right)D_qD_{q^{-1}}\phi_n(x;q)-2x\left((u+v-1)x^2+u\right)D_q\phi_n(x;q) \\ +\left([n]_q\left(2u+2v-2+[1-n]_q\right)x^2+2u\sigma_n\right)\phi_n(x;q)=0, \quad (19)$$

whose monic polynomial solution can be represented as

$$\bar{\phi}_n(x;q,u,v) = K_1 x^{\sigma_n}\,{}_2\phi_1\left(\begin{matrix} q^{-n+\sigma_n}, (1-2(q-1)(u+v-1))q^{n+\sigma_n-1} \\ (1-2u(q-1))q^{2\sigma_n+1} \end{matrix}; q^2; -q^2x^2\right), \tag{20}$$

where

$$K_1 = \frac{q^{[n/2]([n/2]-1)}(q^{n+\sigma_n-1}(1-2(u+v-1)(q-1));\, q^2)_{[n/2]}}{(q^{(-1)^n+2}(1-2u(q-1));\, q^2)_{[n/2]}}.$$

It is not difficult to verify that these polynomials (20) are connected with the Little q-Jacobi polynomials (14) as follows

$$\phi_n(x;q,u,v) = \\ x^{\sigma_n} J_{[\frac{n}{2}]}\left(-x^2; (1-2u(q-1))\frac{1+q^2+(-1)^n(1-q^2)}{2q}, \frac{1-2(q-1)(u+v-1)}{q^2(1-2u(q-1))}; q^2\right). \tag{21}$$

Moreover, as a special case of polynomials (21) for $u+v=1$, one can derive the Little q-Laguerre polynomials (13) as

$$\phi_n(x;q,u,1-u) = x^{\sigma_n} p_{[\frac{n}{2}]}(-x^2; (1-2u(q-1))(\frac{1+q^2+(-1)^n(1-q^2)}{2q}); q^2).$$

In order to prove the orthogonality of the finite set $\{\bar{\phi}_n(x;q,u,v)\}_{n=0}^N$ on $(-\infty,\infty)$, it is necessary to impose a specific condition, which indeed leads to a finite orthogonality [22, 19] as $N < \frac{1-\log_q(1-2(q-1)(u+v-1))}{2}$, because if equation (19) is written in a self-adjoint form, then

$$D_q\left(x^2(x^2+1)\varrho_1(x;q,u,v)D_{q^{-1}}\phi_n(x;q)\right) + \left(\lambda_{n,q}x^2 + 2u\sigma_n\right)\varrho_1(x;q,u,v)\phi_n(x;q) = 0, \tag{22}$$

and

$$D_q\left(x^2(x^2+1)\varrho_1(x;q,u,v)D_{q^{-1}}\phi_m(x;q)\right) + \left(\lambda_{m,q}x^2 + 2u\sigma_m\right)\varrho_1(x;q,u,v)\phi_m(x;q) = 0, \tag{23}$$

where

$$\varrho_1(x;q,u,v) = x^{\log_q\left(\frac{1-2(q-1)(u+v-1)}{q^4}\right)} \frac{\left(-\frac{1}{q^2x^2}; q^2\right)_\infty}{\left(-\frac{1-2u(q-1)}{(1-2(q-1)(u+v-1))x^2}; q^2\right)_\infty}.$$

Now, by multiplying (22) by $\phi_m(x;q)$ and (23) by $\phi_n(x;q)$ and subtracting each other we get

$$\begin{aligned} &\phi_m(x;q)D_q\left(x^2(x^2+1)\varrho_1(x;q,u,v)D_{q^{-1}}\phi_n(x;q)\right) \\ &\quad - \phi_n(x;q)D_q\left(x^2(x^2+1)\varrho_1(x;q,u,v)D_{q^{-1}}\phi_m(x;q)\right) \\ &\quad + (\lambda_{n,q}-\lambda_{m,q})\,x^2\varrho_1(x;q,u,v)\phi_n(x;q)\phi_m(x;q) \\ &\quad + ((-1)^m-(-1)^n)u\varrho_1(x;q,u,v)\phi_n(x;q)\phi_m(x;q) = 0. \end{aligned} \tag{24}$$

Since the q-integration of any odd integrand over a symmetric interval is equal to zero and $\varrho_1(x;q,u,v)$ is an even function, q-integrating on both sides of (24) over $\mathbb{R}$ yields

$$\begin{aligned}
&\int_{-\infty}^{\infty} \phi_m(x;q) D_q \left(x^2(x^2+1)\varrho_1(x;q,u,v) D_{q^{-1}}\phi_n(x;q)\right) d_qx \\
&\quad - \int_{-\infty}^{\infty} \phi_n(x;q) D_q \left(x^2(x^2+1)\varrho_1(x;q,u,v) D_{q^{-1}}\phi_m(x;q)\right) d_qx \\
&\quad + (\lambda_{n,q}-\lambda_{m,q}) \int_{-\infty}^{\infty} x^2 \varrho_1(x;q,u,v)\phi_n(x;q)\phi_m(x;q)\, d_qx \\
&\quad + u((-1)^m-(-1)^n) \int_{-\infty}^{\infty} \varrho_1(x;q,u,v)\phi_n(x;q)\phi_m(x;q)\, d_qx = 0, \qquad (25)
\end{aligned}$$

which can be transformed, by using the rule of q-integration by parts, to

$$\begin{aligned}
&\left[x^2(x^2+1)\varrho_1(x;q,u,v)\phi_m(x;q)D_{q^{-1}}\phi_n(x;q)\right]_{-\infty}^{\infty} \\
&\quad - \left[x^2(x^2+1)\varrho_1(x;q,u,v)\phi_n(x;q)D_{q^{-1}}\phi_m(x;q)\right]_{-\infty}^{\infty} \\
&\quad + (\lambda_{n,q}-\lambda_{m,q}) \int_{-\infty}^{\infty} x^2 \varrho_1(x;q,u,v)\phi_n(x;q)\phi_m(x;q) d_qx \\
&\quad + u((-1)^m-(-1)^n) \int_{-\infty}^{\infty} \varrho_1(x;q,u,v)\phi_n(x;q)\phi_m(x;q) d_qx = 0. \qquad (26)
\end{aligned}$$

In other words, (26) is simplified as

$$\begin{aligned}
&\left[(x^2+1)\varrho_1^*(x;q,u,v)\left(\phi_m(x;q)D_{q^{-1}}\phi_n(x;q) - \phi_n(x;q)D_{q^{-1}}\phi_m(x;q)\right)\right]_{-\infty}^{\infty} \\
&\quad = (\lambda_{m,q}-\lambda_{n,q}) \int_{-\infty}^{\infty} \varrho_1^*(x;q,u,v)\phi_n(x;q)\phi_m(x;q) d_qx, \qquad (27)
\end{aligned}$$

in which

$$\varrho_1^*(x;q,u,v) = x^2\, \varrho_1(x;q,u,v) = \varrho_1^*(-x;q,u,v),$$

provided that $(-1)^{\log_q\left(\frac{1-2(q-1)(u+v-1)}{q^4}\right)} = 1$.

Now since

$$deg\left(\phi_m(x;q)D_{q^{-1}}\phi_n(x;q) - \phi_n(x;q)D_{q^{-1}}\phi_m(x;q)\right) = m+n-1,$$

the left hand side of (27) is zero if

$$\lim_{x\to\pm\infty} x^{m+n+1} \varrho_1^*(x;q,u,v) = 0. \qquad (28)$$

By taking $\max\{m,n\} = N$, relation (28) would be equivalent to

$$\lim_{x\to\pm\infty} x^{2N+1+\log_q\left(\frac{1-2(q-1)(u+v-1)}{q^2}\right)} \frac{\left(-\frac{1}{q^2x^2};q^2\right)_\infty}{\left(-\frac{1-2u(q-1)}{(1-2(q-1)(u+v-1))x^2};q^2\right)_\infty} = 0. \qquad (29)$$

And (29) is valid if and only if

$$2N-1+\log_q\left(1-2(q-1)(u+v-1)\right)<0 \quad \text{or} \quad N<\frac{1-\log_q(1-2(q-1)(u+v-1))}{2}.$$

Now, by noting Favard's theorem [5], the orthogonality relation of q-polynomials (20) can be represented as

$$\int_{-\infty}^{\infty} \varrho_1^*(x;q,u,v)\bar{\phi}_n(x;q,u,v)\bar{\phi}_m(x;q,u,v)d_qx = \left(\prod_{j=1}^{n} C_{j,q}^{(u,v)} \int_{-\infty}^{\infty} \varrho_1^*(x;q,u,v)d_qx\right)\delta_{n,m}, \tag{30}$$

where $\{C_{j,q}^{(u,v)}\}$ are directly derived from (18) as

$$C_{j,q}^{(u,v)} = [q^{j+1}(q^{2j}(1-2(u+v-1)(q-1))(2u(q-1)\sigma_j-1)+q^j((q^2+1)-2u(q-1)q^2-2(u+v-1)(q-1)) -q^2(1-2u(q-1)\sigma_{j-1}))]/[q^4+q^{4j}(1-2(u+v-1)(q-1))^2-(q^3+q)q^{2j}(1-2(u+v-1)(q-1))].$$

Hence, in order to obtain the norm square value, it just remains to compute the q-integral

$$\int_{-\infty}^{\infty} \varrho_1^*(x;q,u,v)d_qx = \int_{-\infty}^{\infty} x^{\log_q\left(\frac{1-2(q-1)(u+v-1)}{q^2}\right)} \frac{\left(-\frac{1}{q^2x^2};q^2\right)_\infty}{\left(-\frac{1-2u(q-1)}{(1-2(q-1)(u+v-1))x^2};q^2\right)_\infty} d_qx. \tag{31}$$

For this purpose, we can directly use the Ramanujan identity (1) for computing the q-integral (31) as follows

$$\begin{aligned}
\int_{-\infty}^{\infty} \varrho_1^*(x;q,u,v)d_qx &= 2(1-q)\sum_{n=-\infty}^{\infty} q^{n\left(\log_q\left(\frac{1-2(q-1)(u+v-1)}{q^2}\right)+1\right)} \frac{\left(-q^{-2}q^{-2n};q^2\right)_\infty}{\left(-\frac{1-2u(q-1)}{1-2(q-1)(u+v-1)}q^{-2n};q^2\right)_\infty} \\
&= 2(1-q)\sum_{n=-\infty}^{\infty} q^{n\left(\log_q\left(\frac{1-2(q-1)(u+v-1)}{q^2}\right)+1\right)} \frac{\left(-\frac{1-2u(q-1)}{1-2(q-1)(u+v-1)};q^2\right)_{-n}\left(-q^{-2};q^2\right)_\infty}{\left(-\frac{1-2u(q-1)}{1-2(q-1)(u+v-1)};q^2\right)_\infty\left(-q^{-2};q^2\right)_{-n}} \\
&= h_1\sum_{n=-\infty}^{\infty} q^{n\left(\log_q\left(\frac{1-2(q-1)(u+v-1)}{q^2}\right)+1\right)} \frac{\left(-\frac{1-2u(q-1)}{1-2(q-1)(u+v-1)};q^2\right)_{-n}}{\left(-q^{-2};q^2\right)_{-n}} \\
&= h_1\sum_{n=-\infty}^{\infty} q^{n\left(-\log_q\left(\frac{1-2(q-1)(u+v-1)}{q^2}\right)-1\right)} \frac{\left(-\frac{1-2u(q-1)}{1-2(q-1)(u+v-1)};q^2\right)_{n}}{\left(-q^{-2};q^2\right)_{n}} \\
&= h_1\Psi\left(-\frac{1-2u(q-1)}{1-2(q-1)(u+v-1)}, -q^{-2}; q^2; q^{\left(-\log_q\left(\frac{1-2(q-1)(u+v-1)}{q^2}\right)-1\right)}\right),
\end{aligned} \tag{32}$$

where

$$h_1 = \frac{2(1-q)\left(-q^{-2};q^2\right)_\infty}{\left(-\frac{1-2u(q-1)}{1-2(q-1)(u+v-1)};q^2\right)_\infty}.$$

For instance, the polynomial set $\{\bar{\phi}_k(x;0.5,128,896)\}_{k=0}^{N=5}$ is finitely orthogonal with respect to the weight function $\frac{x^8(-4x^{-2};\frac{1}{4})_\infty}{(-\frac{129}{1024}x^{-2};\frac{1}{4})_\infty}$ on $(-\infty,\infty)$.

2.2. Second sequence

For $u\in\mathbb{R}$, consider the equation

$$x^4 D_q D_{q^{-1}}\phi_n(x;q) + 2x\left((1-u)x^2+1\right)D_q\phi_n(x;q) \\ + \left([n]_q\left(2u-2+[1-n]_q\right)x^2+2\sigma_n\right)\phi_n(x;q)=0, \quad (33)$$

whose monic polynomial solution can be represented as

$$\bar{\phi}_n(x;q,u) = K_2 x^{\sigma_n}\,{}_2\phi_0\left(\begin{array}{c} q^{-n+\sigma_n},(1+(2-2u)(q-1))q^{n+\sigma_n-1} \\ - \end{array};q^2;\frac{q^{1-2\sigma_n}x^2}{2(1-q)}\right) \quad (34)$$

$$= x^n\,{}_2\phi_1\left(\begin{array}{c} q^{-n+\sigma_n},0 \\ q^{3-2n}(1+(2-2u)(q-1))^{-1} \end{array};q^2;\frac{2q^2(1-q)}{(1+(2-2u)(q-1))x^2}\right), \quad (35)$$

where

$$K_2 = \frac{(q^{n+\sigma_n-1}(1-2(u-1)(q-1));\,q^2)_{[n/2]}}{(2-2q)^{[n/2]}\;q^{[n/2](2+(-1)^{n+1})}}.$$

Once again, it is necessary for orthogonality of the finite set $\{\bar{\phi}_n(x;q,u)\}_{n=0}^N$ to impose a specific condition as $N<\frac{1-\log_q(1+2(q-1)(1-u))}{2}$, because if we write equation (33) in a self-adjoint form, then

$$D_q\left(x^4\varrho_2(x;q,u)D_{q^{-1}}\phi_n(x;q)\right)+\left(\lambda_{n,q}x^2+2\sigma_n\right)\varrho_2(x;q,u)\phi_n(x;q)=0, \quad (36)$$

and

$$D_q\left(x^4\varrho_2(x;q,u)D_{q^{-1}}\phi_m(x;q)\right)+\left(\lambda_{m,q}x^2+2\sigma_m\right)\varrho_2(x;q,u)\phi_m(x;q)=0, \quad (37)$$

where

$$\varrho_2(x;q,u)=\frac{x^{\log_q\left(\frac{1+2(q-1)(1-u)}{q^4}\right)}}{(-\frac{2(q-1)}{(1+2(q-1)(1-u))x^2};q^2)_\infty}.$$

By multiplying (36) by $\phi_m(x;q)$ and (37) by $\phi_n(x;q)$ and subtracting each other we get

$$\phi_m(x;q)D_q\left(x^4\varrho_2(x;q,u)D_{q^{-1}}\phi_n(x;q)\right) \\ -\phi_n(x;q)D_q\left(x^4\varrho_2(x;q,u)D_{q^{-1}}\phi_m(x;q)\right) \\ +\left(\lambda_{n,q}-\lambda_{m,q}\right)x^2\varrho_2(x;q,u)\phi_n(x;q)\phi_m(x;q) \\ +((-1)^m-(-1)^n)\varrho_2(x;q,u)\phi_n(x;q)\phi_m(x;q)=0. \quad (38)$$

Since $\varrho_2(x;q,u)$ is an even function, q-integrating on both sides of (38) over $\mathbb{R}$ yields

$$\begin{aligned}\int_{-\infty}^{\infty}\phi_m(x;q)D_q\left(x^4\varrho_2(x;q,u)D_{q^{-1}}\phi_n(x;q)\right)d_qx\\ -\int_{-\infty}^{\infty}\phi_n(x;q)D_q\left(x^4\varrho_2(x;q,u)D_{q^{-1}}\phi_m(x;q)\right)d_qx\\ +(\lambda_{n,q}-\lambda_{m,q})\int_{-\infty}^{\infty}x^2\varrho_2(x;q,u)\phi_n(x;q)\phi_m(x;q)\,d_qx\\ +((-1)^m-(-1)^n)\int_{-\infty}^{\infty}\varrho_2(x;q,u)\phi_n(x;q)\phi_m(x;q)\,d_qx=0,\end{aligned}\tag{39}$$

which is transformed to

$$\begin{aligned}\left[x^4\varrho_2(x;q,u)\phi_m(x;q)D_{q^{-1}}\phi_n(x;q)\right]_{-\infty}^{\infty}\\ -\left[x^4\varrho_2(x;q,u)\phi_n(x;q)D_{q^{-1}}\phi_m(x;q)\right]_{-\infty}^{\infty}\\ +(\lambda_{n,q}-\lambda_{m,q})\int_{-\infty}^{\infty}x^2\varrho_2(x;q,u)\phi_n(x;q)\phi_m(x;q)d_qx\\ +((-1)^m-(-1)^n)\int_{-\infty}^{\infty}\varrho_2(x;q,u)\phi_n(x;q)\phi_m(x;q)d_qx=0.\end{aligned}\tag{40}$$

On the other hand, (40) can be simplified as

$$\begin{aligned}\left[x^2\varrho_2^*(x;q,u)\left(\phi_m(x;q)D_{q^{-1}}\phi_n(x;q)-\phi_n(x;q)D_{q^{-1}}\phi_m(x;q)\right)\right]_{-\infty}^{\infty}\\ =(\lambda_{m,q}-\lambda_{n,q})\int_{-\infty}^{\infty}\varrho_2^*(x;q,u)\phi_n(x;q)\phi_m(x;q)d_qx,\end{aligned}\tag{41}$$

in which

$$\varrho_2^*(x;q,u)=x^2\,\varrho_2(x;q,u)=\varrho_2^*(-x;q,u),$$

provided that $(-1)^{\log_q\left(\frac{1+2(q-1)(1-u)}{q^4}\right)}=1$. Now, since

$$deg\left(\phi_m(x;q)D_{q^{-1}}\phi_n(x;q)-\phi_n(x;q)D_{q^{-1}}\phi_m(x;q)\right)=m+n-1,$$

the left hand side of (41) is equal to zero if and only if

$$\lim_{x\to\pm\infty}x^{m+n+1}\varrho_2^*(x;q,u)=0.\tag{42}$$

Again if $\max\{m,n\}=N$, relation (42) is equivalent to

$$\lim_{x\to\pm\infty}\frac{x^{2N+1+\log_q\left(\frac{1+2(q-1)(1-u)}{q^2}\right)}}{(-\frac{2(q-1)}{(1+2(q-1)(1-u))x^2};q^2)_\infty}=0,\tag{43}$$

and in the sequal (43) is valid if and only if

$$2N-1+\log_q\left(1+2(q-1)(1-u)\right)<0 \quad \text{or} \quad N<\frac{1-\log_q(1+2(q-1)(1-u))}{2}.$$

Now, by noting Favard's theorem [5], the orthogonality relation of q-polynomials (34) can be represented as

$$\int_{-\infty}^{\infty}\varrho_2^*(x;q,u)\bar{\phi}_n(x;q,u)\bar{\phi}_m(x;q,u)d_qx=\left(\prod_{j=1}^{n}C_{j,q}^{(u)}\int_{-\infty}^{\infty}\varrho_2^*(x;q,u)d_qx\right)\delta_{n,m}, \quad (44)$$

where $\{C_{j,q}^{(u)}\}$ are derived from (18) as

$$C_{j,q}^{(u)}=[q^{j+1}(q^{2j}(1+2(1-u)(q-1))(2-2q)\sigma_j+2q^{j+2}(q-1)-2q^2(q-1)\sigma_{j-1}))]$$
$$/[q^4+q^{4j}(1+2(1-u)(q-1))^2-(q^3+q)q^{2j}(1+2(1-u)(q-1))].$$

Therefore, to obtain the norm square value, it just remains to compute the q-integral

$$\int_{-\infty}^{\infty}\varrho_2^*(x;q,u)d_qx=\int_{-\infty}^{\infty}\frac{x^{\log_q\left(\frac{1+2(q-1)(1-u)}{q^2}\right)}}{(-\frac{2(q-1)}{(1+2(q-1)(1-u))x^2};q^2)_\infty}d_qx. \quad (45)$$

Here we can again use the Ramanujan identity for computing (45) to directly obtain

$$\begin{aligned}\int_{-\infty}^{\infty}\varrho_2^*(x;q,u)d_qx&=2(1-q)\sum_{n=-\infty}^{\infty}\frac{q^{n\left(\log_q\left(\frac{1+2(q-1)(1-u)}{q^2}\right)+1\right)}}{\left(-\frac{2(q-1)}{1+2(q-1)(1-u)}q^{-2n};q^2\right)_\infty}\\&=2(1-q)\sum_{n=-\infty}^{\infty}q^{n\left(\log_q\left(\frac{1+2(q-1)(1-u)}{q^2}\right)+1\right)}\frac{\left(-\frac{2(q-1)}{1+2(q-1)(1-u)};q^2\right)_{-n}}{\left(-\frac{2(q-1)}{1+2(q-1)(1-u)};q^2\right)_\infty}\\&=h_2\sum_{n=-\infty}^{\infty}q^{n\left(\log_q\left(\frac{1+2(q-1)(1-u)}{q^2}\right)+1\right)}\left(-\frac{2(q-1)}{1+2(q-1)(1-u)};q^2\right)_{-n}\\&=h_2\sum_{n=-\infty}^{\infty}q^{n\left(-\log_q\left(\frac{1+2(q-1)(1-u)}{q^2}\right)-1\right)}\left(-\frac{2(q-1)}{1+2(q-1)(1-u)};q^2\right)_{n}\\&=h_2\Psi\left(-\frac{2(q-1)}{1+2(q-1)(1-u)},0;q^2;q^{\left(-\log_q\left(\frac{1+2(q-1)(1-u)}{q^2}\right)-1\right)}\right),\end{aligned}$$

where

$$h_2=\frac{2(1-q)}{\left(-\frac{2(q-1)}{1+2(q-1)(1-u)};q^2\right)_\infty}.$$

For instance, the polynomial set $\{\bar{\phi}_k(x;0.5,256)\}_{k=0}^{N=4}$ is finitely orthogonal with respect to the weight function $\frac{x^6}{(\frac{x^{-2}}{256};\frac{1}{4})_\infty}$ on $(-\infty,\infty)$.

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