Computer Algebra Algorithms for Orthogonal Polynomials and Special Functions

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Online Demonstrations with Computer Algebra

- I will use the computer algebra system *Maple* to demonstrate and program the algorithms presented.
- Of course, we could also easily use any other system like *Mathematica* or MuPAD.
- We first give a short introduction about the capabilities of *Maple*.

Computation of Power Series

- Assume, given an expression f depending of the variable x, we would like to compute a formula for the coefficient a_k of the power series

$$f(x) = \sum_{k=0}^{\infty} a_k x^k$$

representing $f(x)$.

Algorithm

- Input: expression $f(x)$
- Determine a holonomic differential equation DE (homogeneous and linear with polynomial coefficients) by computing the derivatives of $f(x)$ iteratively.
- Convert DE to a holonomic recurrence equation RE for a_k.
- Solve RE for a_k.
- Output: a_k resp. $\Sigma\, a_k\, x^k$

Computation of Holonomic Differential Equations

- Input: expression $f(x)$
- Compute $c_0 f(x) + c_1 f\,'(x) + \cdots + c_J f^{(J)}(x)$ with still undetermined coefficients c_j.
- Sort this w. r. t. linearly independent functions $\in \mathbb{Q}(x)$ and determine their coefficients.
- Set these zero, and solve the corresponding linear system for the unknowns $c_0, c_1, \ldots, c_J$.
- Output: DE:=$c_0 f(x) + c_1 f\,'(x) + \cdots + c_J f^{(J)}(x) = 0$.

Algebra of Holonomic Functions

- We call a function that satisfies a holonomic differential equation a holonomic function.
- Sum and product of holonomic functions turn out to be holonomic.
- We call a sequence that satisfies a holonomic recurrence equation a holonomic sequence.
- Sum and product of holonomic sequences are holonomic.
- A function is holonomic iff it is the generating function of a holonomic sequence.

Hypergeometric Functions

- The power series

$$ {}_pF_q\left(\begin{matrix} a_1, \cdots, a_p \\ b_1, \cdots, b_q \end{matrix} \middle| x \right) = \sum_{k=0}^{\infty} A_k x^k, $$

whose coefficients A_k have rational term ratio

$$ \frac{A_{k+1}x^{k+1}}{A_k x^k} = \frac{(k+a_1)\cdots(k+a_p)}{(k+b_1)\cdots(k+b_q)} \cdot \frac{x}{k+1} $$

is called the generalized hypergeometric function.

Coefficients of Hypergeometric Functions

- For the coefficients of the hypergeometric function we get the formula

$$ {}_pF_q\left(\left.\begin{matrix} a_1, \cdots, a_p \\ b_1, \cdots, b_q \end{matrix}\right| x\right) = \sum_{k=0}^{\infty} \frac{(a_1)_k \cdots (a_p)_k}{(b_1)_k \cdots (b_q)_k} \frac{x^k}{k!}, $$

where $(a)_k = a(a+1)\cdots(a+k\text{-}1)$ is called the Pochhammer symbol (or shifted factorial).

Examples of Hypergeometric Functions

$$e^{x} = {}_0F_0(x)$$

$$\sin x = x \cdot {}_0F_1\left(\begin{matrix} - \\ 3/2 \end{matrix} \middle| -\frac{x^2}{4} \right)$$

Further examples: cos(x), arcsin(x), arctan(x), ln(1+x), erf(x), $L_n^{(\alpha)}(x)$, ..., but for example not tan(x), ...

Identification of Hypergeometric Functions

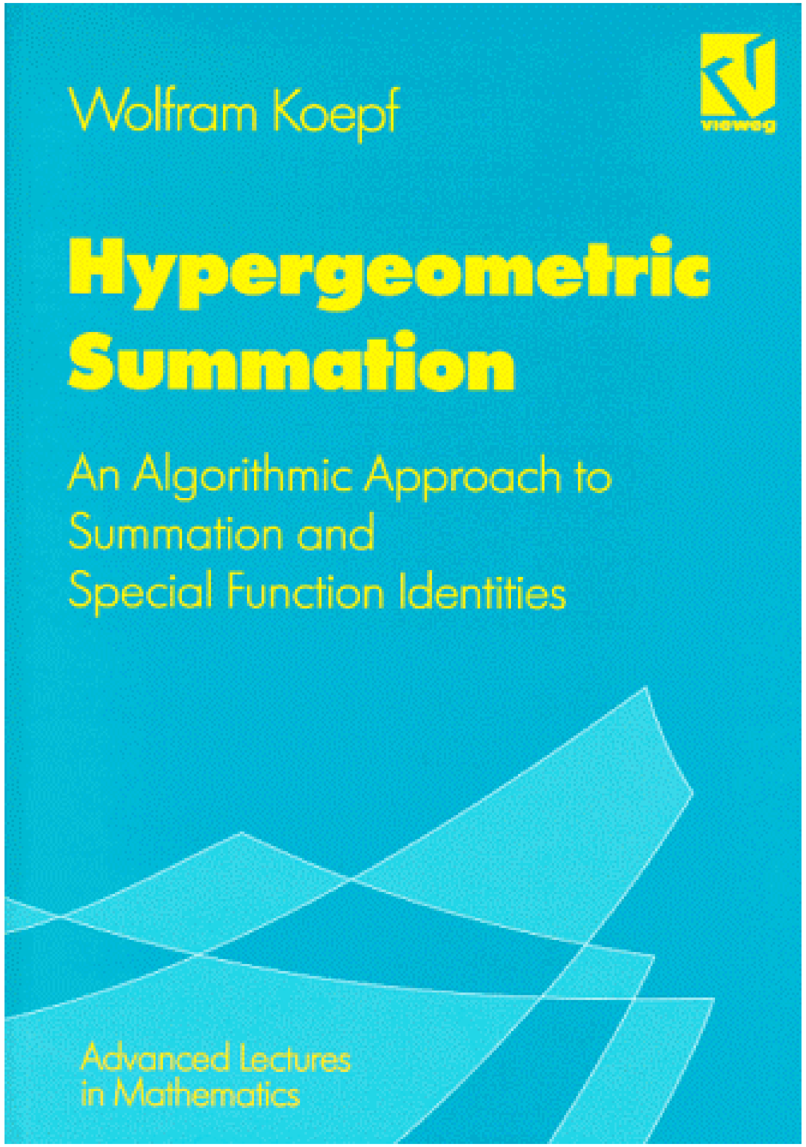
- Assume we have

$$s = \sum_{k=0}^{\infty} a_k \; .$$

- How do we find out which ${}_pF_q(x)$ this is?

Identification Algorithm

- Input: a_k
- Compute
$$r_k := \frac{a_{k+1}}{a_k}$$
and check whether the term ratio r_k is rational.
- Factorize r_k.
- Output: read off the upper and lower parameters and an initial value.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Hypergeometric Summation
An Algorithmic Approach to Summation and Special Function Identities
Advanced Lectures in Mathematics

Recurrence Equations for Hypergeometric Functions

- Given a sequence s_n, as hypergeometric sum

$$s_n = \sum_{k=-\infty}^{\infty} F(n,k)$$

how do we find a recurrence equation for s_n?

Celine Fasenmyer's Algorithm

- Input: summand $F(n,k)$
- Compute

$$\text{ansatz} \coloneqq \sum_{\substack{i=0,\ldots,I \\ j=0,\ldots,J}} \frac{F(n+j,k+i)}{F(n,k)} \in \mathbb{Q}(n,k)$$

- Bring this into rational normal form, and set the numerator coefficient list w.r.t. k zero.
- Output: Sum the resulting recurrence equation for $F(n,k)$ w.r.t. k.

Drawbacks of Fasenmyer's Algorithm

- In easy cases this algorithm succeeds, but:
  - In many cases the algorithm generates a recurrence equation of too high order.
  - The algorithm is slow. If, e.g., $I = 2$ and $J = 2$, then already 9 linear equations have to be solved.
  - Therefore the algorithm might fail.

Indefinite Summation

- Given a sequence a_k, find a sequence s_k which satisfies

$$a_k = s_{k+1} - s_k = \Delta\, s_k \ .$$

- Having found s_k makes definite summation easy since by telescoping for arbitrary *m*, *n*

$$\sum_{k=m}^{n} a_k = s_{n+1} - s_m \ .$$

- Indefinite summation is the inverse of Δ .

Gosper's Algorithm

- Input: a_k , a hypergeometric term.
- Compute p_k, q_k, $r_k \in \mathbb{Q}[k]$ with

$$\frac{a_{k+1}}{a_k} = \frac{p_{k+1}}{p_k} \frac{q_{k+1}}{r_{k+1}} \quad \text{and} \quad \gcd(q_k, r_{k+j}) = 1 \text{ for all } j \geq 0 .$$

- Find a polynomial solution f_k of the recurrence equation $q_{k+1} f_k - r_{k+1} f_{k-1} = p_k$.
- Output: the hyperg. term $s_k = \frac{r_k}{p_k} f_{k-1}\, a_k$.

Definite Summation: Zeilberger's Algorithm

- Zeilberger had the brilliant idea to use a modified version of Gosper's algorithm to compute definite hypergeometric sums

$$s_n = \sum_{k=-\infty}^{\infty} F(n,k).$$

- Note however that, whenever s_n is itself a hypergeometric term, then Gosper's algorithm, applied to $F(n,k)$, fails!

Zeilberger's Algorithm

- Input: summand $F(n,k)$
- For suitable $J \in \mathbb{N}$ set

$$a_k := F(n,k) + \sigma_1 F(n+1,k) + \cdots + \sigma_J F(n+J,k) .$$

- Apply the following modified version of Gosper's algorithm to a_k:
  - In the last step, solve at the same time for the coefficients of f_k and the unknowns $\sigma_j \in \mathbb{Q}(n)$.
- Output by summation: The recurrence equation

$$\mathrm{RE} := s_n + \sigma_1 s_{n+1} + \cdots + \sigma_J s_{n+J} = 0 .$$

The output of Zeilberger's Algorithm

- We apply Zeilberger's algorithm iteratively for $J = 1,2, \dots$ until it succeeds.
- If $J = 1$ is successful, then the resulting recurrence equation for s_n is of first order, hence s_n is a hypergeometric term.
- If $J > 1$, then the result is a holonomic recurrence equation for s_n.
- One can prove that Zeilberger's algorithm terminates for suitable input.
- Zeilberger's algorithm is much faster than Fasenmyer's.

Representations of Legendre Polynomials

$$P_n(x) = \sum_{k=0}^{n} \binom{n}{k} \binom{-n-1}{k} \left(\frac{1-x}{2} \right)^k = {}_2F_1\left(\left. \begin{matrix} -n, n+1 \\ 1 \end{matrix} \right| \frac{1-x}{2} \right)$$

$$= \frac{1}{2^n} \sum_{k=0}^{n} \binom{n}{k}^2 (x-1)^{n-k} (x+1)^k = \left(\frac{x-1}{2} \right)^n {}_2F_1\left(\left. \begin{matrix} -n, -n \\ 1 \end{matrix} \right| \frac{x+1}{x-1} \right)$$

$$= \frac{1}{2^n} \sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k \binom{n}{k} \binom{2n-2k}{n} x^{n-2k} = \left(\frac{x}{2} \right)^n \binom{2n}{n} {}_2F_1\left(\left. \begin{matrix} -n/2, (-n+1)/2 \\ -n+1/2 \end{matrix} \right| \frac{1}{x^2} \right)$$

$$= x^n \, {}_2F_1\left(\left. \begin{matrix} -n/2, (1-n)/2 \\ 1 \end{matrix} \right| 1 - \frac{1}{x^2} \right)$$

Dougall's Identity

- Dougall (1907) found the following identity

$$
{}_7F_6\left(\left.\begin{matrix} a, 1+\frac{a}{2}, b, c, d, 1+2a-b-c-d+n, -n \\ \frac{a}{2}, 1+a-b, 1+a-c, 1+a-d, b+c+d-a-n, 1+a+n \end{matrix}\right| 1\right)
$$

$$
= \frac{(1+a)_n (a+1-b-c)_n (a+1-b-d)_n (a+1-c-d)_n}{(1+a-b)_n (1+a-c)_n (1+a-d)_n (1+a-b-c-d)_n}.
$$

Clausen's Formula

- Clausen's formula gives the cases when a Clausen ${}_3F_2$ function is the square of a Gauss ${}_2F_1$ function:

$$ {}_2F_1\left(\begin{matrix} a,b \\ a+b+\frac{1}{2} \end{matrix}\middle| x\right)^2 = {}_3F_2\left(\begin{matrix} 2a,2b,a+b \\ 2a+2b,a+b+\frac{1}{2} \end{matrix}\middle| x\right). $$

- The right hand side can be detected from the left hand side by Zeilberger's algorithm.

A Generating Function Problem

- Recently Folkmar Bornemann showed me a newly developed generating function of the Legendre polynomials and asked me to generate it automatically.
- Here is the question:

  Write
  $$G(x,z,\alpha) := \sum_{n=0}^{\infty} \binom{\alpha+n-1}{n} P_n(x)\, z^n$$
  as a hypergeometric function!

Generating Function as a Double Sum

- We can take any of the four given hyper-geometric representations of the Legendre polynomials to write $G(x,z,\alpha)$ as double sum.
- Then the trick is to change the order of summation

$$\sum_{n=0}^{\infty}\binom{\alpha+n-1}{n}\sum_{k=0}^{\infty}p_k(n,x)\,z^n=\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\binom{\alpha+n-1}{n}p_k(n,x)z^n\,.$$

Automatic Computation of Infinite Sums

- Whereas Zeilberger's algorithm finds Chu-Vandermonde's formula for $n \in \mathbb{N}_{\geq 0}$

$$ {}_2F_1\left(\begin{matrix}-n, b\\ c\end{matrix}\,\middle|\, 1\right) = \frac{(c-b)_n}{(c)_n}, $$

the question arises to detect Gauss' identity

$$ {}_2F_1\left(\begin{matrix}a, b\\ c\end{matrix}\,\middle|\, 1\right) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} $$

for $a,b,c \in \mathbb{C}$ in case of convergence.

Solution

- The idea is to detect automatically

$$ {}_2F_1\left(\begin{matrix} a,b \\ c+m \end{matrix}\middle| 1\right) = \frac{(c-a)_m (c-b)_m}{(c)_m (c-a-b)_m}\, {}_2F_1\left(\begin{matrix} a,b \\ c \end{matrix}\middle| 1\right) $$

  and then to consider the limit as $m \to \infty$.
- Using appropriate limits for the Γ function, this and similar questions can be handled automatically by a Maple package of Vidunas and Koornwinder.

The WZ Method

- Assume we want to prove an identity

$$\sum_{k=-\infty}^{\infty} f(n,k) = \tilde{s}_n$$

with hypergeometric terms $f(n,k)$ and $\tilde{s}_n$.

- Dividing by $\tilde{s}_n$, we may put the identity into the form

$$s_n := \sum_{k=-\infty}^{\infty} F(n,k) = 1.$$

Rational Certificate

- If Gosper's algorithm, applied to $F(n+1,k)$-$F(n,k)$, is successful, then it generates a rational multiple $G(n,k)$ of $F(n,k)$, i.e. $G(n,k) = R(n,k)\, F(n,k)$, such that

$$F(n+1,k) - F(n,k) = G(n,k+1) - G(n,k)$$

- By telescoping, this proves $s_{n+1} - s_n = 0$, hence the identity.
- Second proof: Dividing by $F(n,k)$, we may prove

$$\frac{F(n+1,k)}{F(n,k)} - 1 = R(n,k+1)\frac{F(n,k+1)}{F(n,k)} - R(n,k),$$

a purely rational identity.

Differential Equations

- Zeilberger's algorithm can easily be adapted to generate holonomic differential equations for hyperexponential sums

$$s(x) = \sum_{k=-\infty}^{\infty} F(x,k).$$

- For this purpose, the summand $F(x,k)$ must be a hyperexponential term, i.e.

$$\frac{F'(x,k)}{F(x,k)} \in \mathbb{Q}(x,k).$$

Petkovsek's Algorithm

- Petkovsek's algorithm is an adaption of Gosper's.
- Given a holonomic recurrence equation, it determines all hypergeometric term solutions.
- Petkovsek's algorithm is slow, especially if the leading and trailing terms have many factors. Maple 9 will contain a much more efficient algorithm due to Mark van Hoeij.

Combining Zeilberger’s and Petkovsek’s Algorithm

- Zeilberger’s algorithm may not give a recurrence of first order, even if the sum is a hypergeometric term. This rarely happens, though.
- Therefore the combination of Zeilberger’s algorithm with Petkovsek’s guarantees to find out whether a given sum can be written as a hypergeometric term.
- Exercise 9.3 of my book gives 9 examples for this situation, all from p. 556 of
  - Prudnikov, Brychkov, Marichev: Integrals and Series, Vol. 3: More Special Functions. Gordon Breach, 1990.

INTEGRALS AND SERIES

VOLUME 3: MORE SPECIAL FUNCTIONS

A.P. Prudnikov, Yu.A. Brychkov, O.I. Marichev

Translated from the Russian by G.G. Gould

Gordon and Breach Science Publishers

26. $${}_4F_3\left(\begin{matrix}-n,\ a,\ a/2+1,\ b;\ 1\\ a/2,\ 1+a-b,\ c\end{matrix}\right)=\frac{(c-2b-1)_n}{(c)_n}\,{}_4F_3\left(\begin{matrix}-n,\ a-2b-1,\ (a+1)/2-b,\ -b-1;\ 1\\ (a-1)/2-b,\ 1+a-b,\ c-2b-1\end{matrix}\right)=$$

27. $$=\frac{c+n}{c}\,{}_3F_2\left(\begin{matrix}-n,\ a+1,\ b+1;\ 1\\ c+1,\ 1+a-b\end{matrix}\right).$$

28. $${}_4F_3\left(\begin{matrix}-n,\ a,\ 1-a,\ b;\ 1\\ 1-b-n,\ c,\ 1+2b-c\end{matrix}\right)=\frac{((a+c-1)/2)_n\,((c-a)/2)_n\,(2b)_n}{(b)_n\,(b+1/2)_n\,(c)_n}\times$$
$$\times{}_4F_3\left(\begin{matrix}-n,\ 1+b-(a+c)/2,\ b+(1+a-c)/2,\ 1-c-n;\ 1\\ (3-a-c)/2-n,\ 1+(a-c)/2-n,\ 1+2b-c\end{matrix}\right).$$

29. $${}_4F_3\left(\begin{matrix}-n,\ a,\ a/2+1,\ b;\ 1\\ a/2,\ 1+a+n,\ 1+a-b\end{matrix}\right)=\frac{(1+a)_n\,((1+a)/2-b)_n}{((1+a)/2)_n\,(1+a-b)_n}.$$

30. $${}_4F_3\left(\begin{matrix}-n,\ a,\ a/2+1,\ b;\ 1\\ a/2,\ 1+a-b,\ 2+2b-n\end{matrix}\right)=$$
$$=\frac{(a-2b-1)_n}{(-2b-1)_n}\,{}_3F_2\left(\begin{matrix}-n,\ (a+1)/2,\ a-2b+n-1;\ 1\\ 1+a-b,\ (a-1)/2-b\end{matrix}\right)=$$

31. $$=\frac{(a-2b-1)_n\,(-b-1)_n\,(a-2b+2n-1)}{(1+a-b)_n\,(-2b-1)_n\,(a-2b-1)}.$$

32. $${}_4F_3\left(\begin{matrix}-n,\ a,\ a+1/2,\ b;\ 1\\ 2a,\ (b-n+1)/2,\ (b-n)/2+1\end{matrix}\right)=\frac{(2a-b)_n\,(b-n)}{(1-b)_n\,(b+n)}.$$

33. $${}_4F_3\left(\begin{matrix}-n,\ a,\ a+1/2,\ b;\ 1\\ 2a+1,\ (b-n)/2,\ (b-n+1)/2\end{matrix}\right)=\frac{(1+2a-b)_n}{(1-b)_n}.$$

34. $${}_4F_3\left(\begin{matrix}-n,\ a,\ a+1/2,\ b;\ 1\\ 2a+1,\ (b-n+1)/2,\ (b-n)/2+1\end{matrix}\right)=\frac{(1+2a-b)_n\,(2a-b-n)\,(b-n)}{(1-b)_n\,(2a-b+n)\,(b+n)}.$$

35. $${}_4F_3\left(\begin{matrix}-n,\ a,\ b,\ -1/2-a-b-n;\ 1\\ -a-n,\ -b-n,\ a+b+1/2\end{matrix}\right)=\frac{(2a+1)_n\,(2b+1)_n\,(a+b+1)_n}{(a+1)_n\,(b+1)_n\,(2a+2b+1)_n}.$$

36. $${}_4F_3\left(\begin{matrix}-n,\ a,\ b,\ 1/2-a-b-n;\ 1\\ -a-n,\ 1-b-n,\ a+b+1/2\end{matrix}\right)=\frac{(2a+1)_n\,(2b)_n\,(a+b)_n}{(a+1)_n\,(b)_n\,(2a+2b)_n}.$$

37. $${}_4F_3\left(\begin{matrix}-n,\ a,\ b,\ 1/2-a-b-n;\ 1\\ 1-a-n,\ 1-b-n,\ a+b\pm1/2\end{matrix}\right)=\frac{(2a)_n\,(2b)_n\,(a+b)_n}{(a)_n\,(b)_n\,(2a+2b-(1\mp1)/2)_n}.$$

38. $${}_4F_3\left(\begin{matrix}-n,\ a,\ b,\ 3/2-a-b-n;\ 1\\ 1-a-n,\ 1-b-n,\ a+b+1/2\end{matrix}\right)=\frac{(2a)_n\,(2b)_n\,(a+b)_n\,(2a+2b-1)}{(a)_n\,(b)_n\,(2a+2b-1)_n\,(2a+2b+2n-1)}.$$

39. $${}_4F_3\left(\begin{matrix}-n,\ a,\ b,\ 3/2-a-b-n;\ 1\\ 1-a-n,\ 2-b-n,\ a+b-1/2\end{matrix}\right)=\frac{(2a)_n\,(2b-1)_n\,(a+b-1)_n}{(a)_n\,(b-1)_n\,(2a+2b-2)_n}.$$

40. $${}_4F_3\left(\begin{matrix}-n,\ a,\ b,\ 5/2-a-b-n;\ 1\\ 2-a-n,\ 2-b-n,\ a+b-1/2\end{matrix}\right)=$$
$$=\frac{(2a-1)_n\,(2b-1)_n\,(a+b-1)_n\,(2a+2b-3)}{(a-1)_n\,(b-1)_n\,(2a+2b-3)_n\,(2a+2b+2n-3)}.$$

41. $${}_4F_3\left(\begin{matrix}-n,\ 1+n,\ a,\ a+1/2;\ 1\\ 1/2,\ b,\ 2a-b+2\end{matrix}\right)=\frac{1}{2(a-b+1)}\left[\frac{(1-b)_{n+1}}{(2a-b+2)_n}-\frac{(b-2a-1)_{n+1}}{(b)_n}\right].$$

42. $${}_4F_3\left(\begin{matrix}-n,\ 2+n,\ a,\ a+1/2;\ 1\\ 3/2,\ b,\ 2a-b+2\end{matrix}\right)=$$
$$=\frac{1}{2(n+1)(a-b+1)(1-2a)}\left[\frac{(1-b)_{n+2}}{(2a-b+2)_n}-\frac{(b-2a-1)_{n+2}}{(b)_n}\right].$$

43. $${}_4F_3\left(\begin{matrix}-n,\ 1,\ 1,\ a;\ 1\\ 2,\ b,\ 1+a-b-n\end{matrix}\right)=$$
$$=\frac{(b-1)(a-b-n)}{(n+1)(a-1)}\,[\psi(n+b)+\psi(1+a-b)-\psi(b-1)-\psi(a-b-n)].$$

Examples

- As an example, we take

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\binom{ck}{n}=(-c)^n\,, \qquad (c=2,3,\ldots)$$

- and Exercise 9.3 (a), resp. PBM (7.5.3.32):

$${}_4F_3\left(\begin{matrix}-n,a,a+\frac{1}{2},b\\ 2a,\frac{b-n+1}{2},\frac{b-n}{2}+1\end{matrix}\middle|1\right)=\frac{(2a-b)_n(b-n)}{(1-b)_n(b+n)}\,.$$

Indefinite Integration

- To find recurrence and differential equations for hypergeometric and hyperexponential integrals, one needs a continuous version of Gosper’s algorithm.
- Almkvist and Zeilberger gave such an algorithm. It finds hyperexponential antiderivatives if those exist.

Recurrence and Differential Equations for Integrals

- Applying the continuous Gosper algorithm, one can easily adapt the discrete versions of Zeilberger’s algorithm to the continuous case.
- The resulting algorithms find holonomic recurrence and differential equations for hypergeometric and hyperexponential integrals.

Example 1

- As example, we would like to find holonomic equations for

$$S(x,y) := \int_0^1 t^x (1-t)^y \, dt$$

Resulting recurrence equations:

$$-(x+y+2)S(x+1,y)+(x+1)S(x)=0$$

$$-(x+y+2)S(x,y+1)+(y+1)S(y)=0$$

Example ctd.

- Solving both recurrence equations shows that $S(x,y)$ must be a multiple of

$$S(x,y) \sim \frac{\Gamma(x+1)\Gamma(y+1)}{\Gamma(x+y+2)}$$

- Computing the initial value

$$S(0,0) := \int_0^1 dt = 1$$

proves that the above is an identity.

Example 2

- The integral

$$I(x)=\int_0^{\infty}\frac{x^2}{(x^4+t^2)(1+t^2)}\,dt$$

satisfies the differential equation

$$x(x-1)(x+1)(x^2+1)I''(x)+$$
$$(1+7x^4)I'(x)+8x^3I(x)=0$$

from which it can be derived that

$$I(x)=\frac{\pi}{2(x^2+1)}\,.$$

Rodrigues Formulas

- Using Cauchy's integral formula

$$h^{(n)}(x) = \frac{n!}{2\pi i} \oint_{\gamma} \frac{h(t)}{(t-x)^{n+1}}\, dt$$

for the *n*th derivative makes the integration algorithm accessible for Rodrigues type expressions

$$f_n(x) = g_n(x) \frac{d^n}{dx^n} h_n(x) .$$

Orthogonal Polynomials

- Hence we can easily show that the functions

$$P_n(x) = \frac{(-1)^n}{2^n n!} \frac{d^n}{dx^n}(1-x^2)^n$$

are the Legendre polynomials, and

$$L_n^{(\alpha)}(x) = \frac{e^x}{n! x^\alpha} \frac{d^n}{dx^n}(e^{-x} x^{\alpha+n})$$

are the generalized Laguerre polynomials.

Generating Functions

- If $F(z)$ is the generating function of the sequence $a_n f_n(x)$

$$F(z) = \sum_{n=0}^{\infty} a_n f_n(x) z^n \,,$$

then by Cauchy's formula and Taylor's theorem

$$f_n(x) = \frac{1}{a_n} \frac{F^{(n)}(0)}{n!} = \frac{1}{a_n} \frac{1}{2\pi i} \int_{\gamma} \frac{F(t)}{t^{n+1}} dt \,.$$

Laguerre Polynomials

- Hence we can easily prove the following generating function identity

$$(1-z)^{-\alpha-1}e^{\frac{xz}{z-1}} = \sum_{n=0}^{\infty} L_n^{(\alpha)}(x)z^n$$

for the generalized Laguerre polynomials.

Basic Hypergeometric Series

- Instead of considering series whose coefficients A_k have rational term ratio $A_{k+1}/A_k \in \mathbb{Q}(k)$, we can also consider such series whose coefficients A_k have term ratio $A_{k+1}/A_k \in \mathbb{Q}(q^k)$.
- This leads to the q-hypergeometric series

$$ {}_r\varphi_s\left(\left.\begin{matrix} a_1, \cdots, a_r \\ b_1, \cdots, b_s \end{matrix}\right| q; x \right) = \sum_{k=0}^{\infty} A_k x^k . $$

Coefficients of the Basic Hypergeometric Series

- Here the coefficients are given by

$$A_k = \frac{(a_1;q)_k \cdots (a_r;q)_k}{(b_1;q)_k \cdots (b_s;q)_k} \frac{x^k}{(q;q)_k} \left((-1)^k q^{\binom{k}{2}} \right)^{1+s-r},$$

where

$$(a;q)_k = \prod_{j=0}^{k-1} \left(1 - aq^j\right)$$

denotes the *q*-Pochhammer symbol.

Further q-Expressions

- q-Pochhammer symbol: $(a;q)_\infty = \lim_{n\to\infty}(a;q)_n$
- q-factorial: $[k]_q! = \dfrac{(q;q)_k}{(1-q)^k}$
- q-Gamma function: $\Gamma_q(z) = \dfrac{(q;q)_\infty}{(q^z;q)_\infty}(1-q)^{1-z}$
- q-binomial coefficient: $\begin{bmatrix} n \\ k \end{bmatrix}_q = \dfrac{(q;q)_n}{(q;q)_k (q;q)_{n-k}}$
- q-brackets:

$$[k]_q = \frac{1-q^k}{1-q} = 1 + q + \cdots + q^{k-1}\,.$$

q-Chu-Vandermonde Theorem

- For all classical hypergeometric theorems corresponding q-versions exist.
- For example, the q- Chu-Vandermonde theorem states that

$$
{}_2\varphi_1\left(\left.\begin{matrix} q^{-n}, b \\ c \end{matrix}\right| q; \frac{cq^n}{b}\right) = \frac{(c/b;q)_n}{(c;q)_n}
$$

and can be proved by a q-version of Zeilberger's algorithm.

q-Hypergeometric Orthogonal Polynomials

- All classical orthogonal systems have (several) q-hypergeometric equivalents.
- The Little and the Big q-Legendre Polynomials, respectively, are given by

$$p_n(x \mid q) = {}_2\varphi_1\left(\left.\begin{matrix} q^{-n}, q^{n+1} \\ q \end{matrix}\right| q; qx\right),$$

$$P_n(x; c; q) = {}_3\varphi_2\left(\left.\begin{matrix} q^{-n}, q^{n+1}, x \\ q, cq \end{matrix}\right| q; q\right).$$

Operator Equations

- q-orthogonal polynomials satisfy q-holonomic recurrence equations with respect to n and – in the classical Hahn case – holonomic q-difference equations.
- For the latter one uses Hahn's q-difference operator

$$D_q f(x) = \frac{f(x) - f(qx)}{(1-q)x}.$$

Scalar Products

- Given: a scalar product

$$\langle f, g \rangle := \int_a^b f(x) g(x) d\mu(x)$$

  with non-negative measure μ supported in the interval $[a,b]$.

- Particular cases:
  - absolutely continuous measure $d\mu(x) = \rho(x)dx$,
  - discrete measure $\rho(x)$ supported by $\mathbb{Z}$,
  - discrete measure $\rho(x)$ supported by $q^{\mathbb{Z}}$.

Orthogonal Polynomials

- A family $P_n(x)$ of polynomials

$$P_n(x) = k_n x^n + k_n' x^{n-1} + \cdots, \quad k_n \neq 0$$

is orthogonal w. r. t. the measure $\mu(x)$ if

$$\langle P_n, P_m \rangle = \begin{cases} 0 & \text{if } m \neq n \\ d_n^2 \neq 0 & \text{if } m = n \end{cases}.$$

Classical Families

- The classical orthogonal polynomials can be alternatively defined as the polynomial solutions of the differential equation

$$\sigma(x)P_n''(x)+\tau(x)P_n'(x)+\lambda_n P_n(x)=0\,.$$

- Conclusions:
  - $n = 1$ implies $\tau(x) = d\,x + e$, $d \neq 0$
  - $n = 2$ implies $\sigma(x) = a\,x^2 + b\,x + c$
  - coefficient of x^n implies $\lambda_n = -\,n(a(n\text{-}1)+d)$

Classification

- The classical systems can be classified according to the scheme
- $\sigma(x) = 0$ powers x^n
- $\sigma(x) = 1$ Hermite polynomials
- $\sigma(x) = x$ Laguerre polynomials
- $\sigma(x) = x^2$ Bessel polynomials
- $\sigma(x) = x^2 - 1$ Jacobi polynomials

Weight function

- The weight function ρ(*x*) corresponding to the differential equation satisfies Pearson's differential equation

$$\frac{d}{dx}\left(\sigma(x)\rho(x)\right)=\tau(x)\rho(x)$$

- Hence it is given as

$$\rho(x)=\frac{C}{\sigma(x)}e^{\int\frac{\tau(x)}{\sigma(x)}dx}\quad .$$

Classical Discrete Families

- The classical discrete orthogonal polynomials can be defined as the polynomial solutions of the difference equation

$$\sigma(x)\Delta\nabla P_n(x) + \tau(x)\Delta P_n(x) + \lambda_n P_n(x) = 0\,.$$

- Conclusions:
  - $n = 1$ implies $\tau(x) = d\,x + e,\ d \neq 0$
  - $n = 2$ implies $\sigma(x) = a\,x^2 + b\,x + c$
  - coefficient of x^n implies $\lambda_n = -\,n(a(n\text{-}1)+d)$

Classification

- The classical discrete systems can be classified according to the scheme
- $\sigma(x) = 1$ translated Charlier pols.
- $\sigma(x) = x$ falling factorials
- $\sigma(x) = x$ Charlier, Meixner, Krawtchouk pols.
- $\sigma(x) = x\,(N+\alpha-x)$ Hahn polynomials

Weight function

- The weight function $\rho(x)$ corresponding to the difference equation satisfies Pearson's difference equation

$$\Delta\left(\sigma(x)\rho(x)\right)=\tau(x)\rho(x)$$

- Hence it is given as

$$\frac{\rho(x+1)}{\rho(x)}=\frac{\sigma(x)+\tau(x)}{\sigma(x+1)}.$$

Classical q-Families

- The q-orthogonal polynomials of the Hahn class can be defined as the polynomial solutions of the q-difference equation

$$\sigma(x) D_q D_{1/q} P_n(x) + \tau(x) D_q P_n(x) + \lambda_n P_n(x) = 0 .$$

- Conclusions:
  - $n = 1$ implies $\tau(x) = d\,x + e,\ d \neq 0$
  - $n = 2$ implies $\sigma(x) = a\,x^2 + b\,x + c$
  - coefficient of x^n implies $\lambda_n = -a[n]_{1/q}[n-1]_q - d[n]_q$

Classification

- The classical q-systems can be classified according to the scheme
- $\sigma(x) = 0$ powers and q-Pochhammers
- $\sigma(x) = 1$ discrete q-Hermite polynomials II
- $\sigma(x) = x$ q-Charlier, q-Laguerre pols.
- $\sigma(x) = (x-a\,q)(x-b\,q)$ Big q- Jacobi pols.

Weight function

- The weight function ρ(x) corresponding to the q-difference equation satisfies the q-Pearson differential equation

$$D_q\big(\sigma(x)\rho(x)\big) = \tau(x)\rho(x)$$

- Hence it is given as

$$\frac{\rho(qx)}{\rho(x)} = \frac{\sigma(x) + (q-1)x\tau(x)}{\sigma(qx)} \quad .$$

Computing Differential Equation from a Recurrence Equation

- From the differential or (q)-difference equation one can determine the three-term recurrence equation for $P_n(x)$ in terms of the coefficients of $\sigma(x)$ and $\tau(x)$.
- Using this information in the opposite direction, one can find the corresponding differential or (q)-difference equation from a given three-term recurrence equation.

Example 1

- Given the recurrence equation

$$P_{n+2}(x) - (x - n - 1)P_{n+1}(x) + \alpha(n+1)^2 P_n(x) = 0$$

one finds that for $\alpha = 1/4$ translated Laguerre polynomials, and for $\alpha < 1/4$, Meixner and Krawtchouk polynomials are solutions.

Example 2

- Given the recurrence equation

$$P_{n+2}(x) - xP_{n+1}(x) + \alpha q^{n}(q^{n+1} - 1)P_{n}(x) = 0$$

one finds that for every α there are *q*-orthogonal polynomial solutions.

Epilogue

- Software development is a time consuming activity!
- Software developers love when their software is used.
- But they need your support.
- Hence my suggestion: If you use one of the packages mentioned for your research, please cite ist use!