Wolfram Koepf: Introduction to Computer Algebra with Maple 9

```
> restart;
```

Numbers

Calculator

```
> (1.23 + 2.25)/3.67;
```

0.9482288828

```
> evalf(Pi);
```

3.141592654

```
> evalf(Pi,500);
```

3.14159265358979323846264338327950288419716939937510582097494459230781640\
628620899862803482534211706798214808651328230664709384460955058223172535\
940812848111745028410270193852110555964462294895493038196442881097566593\
344612847564823378678316527120190914564856692346034861045432664821339360\
726024914127372458700660631558817488152092096282925409171536436789259036\
001133053054882046652138414695194151160943305727036575959195309218611738\
193261179310511854807446237996274956735188575272489122793818301194912

Long Integers Arithmetic

```
> 100!;
```

933262154439441526816992388562667004907159682643816214685929638952175999\
932299156089414639761565182862536979208272237582511852109168640000000000\
00000000000000

```
> 2^32-1;
```

4294967295

```
> evalf(%);
```

$0.4294967295\ 10^{10}$

```
> 100!/2^100;
```

588971222367687651371627846346807888288472382883312574253249804256440585\
603406374176100610302040933304083276457607746124267578125 / 8

```
> binomial(123,45);
```

896647319101861715891695497019268 4

```
> product(123-k,k=0..44)/45!;
```

8966473191018617158916954970192684

```
> sum(1/k,k=1..100);
```

$$\frac{14466636279520351160221518043104131447711}{2788815009188499086581352357412492142272}$$

Prime Decomposition of Integers

```
> ifactor(100!);
```

$(2)^{97}\ (3)^{48}\ (5)^{24}\ (7)^{16}\ (11)^{9}\ (13)^{7}\ (17)^{5}\ (19)^{5}\ (23)^{4}\ (29)^{3}\ (31)^{3}\ (37)^{2}\ (41)^{2}\ (43)^{2}\ (47)^{2}\ (53)\ (59)\ (61)\ (67)\ (71)\ (73)\ (79)\ (83)\ (89)\ (97)$

```
> isprime(1234567);
```

$$false$$

```
> ifactor(1234567);
```

$$(127)\ (9721)$$

```
> nextprime(1234567);
```

$$1234577$$

```
> isprime(%);
```

$$true$$

Integer Division and Euclidean Algorithm

```
> iquo(100,3);
```

$$33$$

```
> irem(100,3);
```

$$1$$

```
> modp(100,3);
```

$$1$$

```
> gcd(1234567,1604137);
```

$$127$$

Algebraic Numbers

```
> x:=sqrt(2)+sqrt(3);
```

$$x := \sqrt{2} + \sqrt{3}$$

```
> 1/x;
```

$$\frac{1}{\sqrt{2} + \sqrt{3}}$$

```
> simplify(1/x);
```

$$\frac{1}{\sqrt{2} + \sqrt{3}}$$

```
> y:=sqrt(3)-sqrt(2);
```

$$y := \sqrt{3} - \sqrt{2}$$

```
> x*y;
```

$$(\sqrt{2} + \sqrt{3})\,(\sqrt{3} - \sqrt{2})$$

```
> simplify(x*y);
```

$$1$$

```
> sin(Pi);
```

$$0$$

```
> sin(Pi/3);
```

$$\frac{\sqrt{3}}{2}$$

```
> [seq(sin(Pi/k),k=3..5)];
```

$$\left[\frac{\sqrt{3}}{2}, \frac{\sqrt{2}}{2}, \sin\left(\frac{\pi}{5}\right)\right]$$

```
> convert(%,radical);
```

$$\left[\frac{\sqrt{3}}{2}, \frac{\sqrt{2}}{2}, \frac{\sqrt{2}\sqrt{5-\sqrt{5}}}{4}\right]$$

Complex Numbers

```
> (1+I)/(1-I);
```

$$I$$

```
> Re(2*exp(3*x+I*y));
```

$$2\,\mathbf{e}^{(3\sqrt{2}+3\sqrt{3})}\cos(\sqrt{3}-\sqrt{2})$$

```
> x:='x': y:='y':
> Re(2*exp(3*x+I*y));
```

$$2\,\Re(\mathbf{e}^{(3x+yI)})$$

```
>
```

Symbols

Polynomials

```
> p:=(x+y)^10-(x-y)^10;
```

$$p := (x+y)^{10} - (x-y)^{10}$$

```
> expand(p);
```

$$20\,x^9\,y + 240\,x^7\,y^3 + 504\,x^5\,y^5 + 240\,x^3\,y^7 + 20\,x\,y^9$$

```
> factor(p);
```

$$4\,x\,y\,(y^4 + 10\,x^2\,y^2 + 5\,x^4)\,(5\,y^4 + 10\,x^2\,y^2 + x^4)$$

```
> rat:=(1-x^10)/(1-x^4);
```

$$rat := \frac{1-x^{10}}{1-x^4}$$

```
> normal(rat);
```

$$\frac{x^8 + x^6 + x^4 + x^2 + 1}{x^2+1}$$

```
> gcd(1-x^10,1-x^4);
```

$$-1 + x^2$$

```
> factor(rat);
```

$$\frac{(x^4 + x^3 + x^2 + x + 1)\,(x^4 - x^3 + x^2 - x + 1)}{x^2+1}$$

```
> factor(x^6+x^2+1);
```

$$x^6 + x^2 + 1$$

```
> factor(x^4+1);
```

$$x^4 + 1$$

```
> factor(x^4+1,I);
```

$$(x^2 - I)(x^2 + I)$$

```
> factor(x^4+1,sqrt(2));
```

$$(x^2 - x\sqrt{2} + 1)(x^2 + x\sqrt{2} + 1)$$

```
> convert(1/(1+x^4),parfrac);
```

$$\frac{1}{x^4 + 1}$$

```
> convert(1/(1+x^4),parfrac,sqrt(2));
```

$$\frac{2 + x\sqrt{2}}{4(x^2 + x\sqrt{2} + 1)} + \frac{2 - x\sqrt{2}}{4(x^2 - x\sqrt{2} + 1)}$$

```
> int(1/(1+x^4),x);
```

$$\frac{1}{8}\sqrt{2}\ln\left(\frac{x^2 + x\sqrt{2} + 1}{x^2 - x\sqrt{2} + 1}\right) + \frac{1}{4}\sqrt{2}\arctan(x\sqrt{2} + 1) + \frac{1}{4}\sqrt{2}\arctan(x\sqrt{2} - 1)$$

Linear Algebra

```
> with(LinearAlgebra);
```

[*&x, Add, Adjoint, BackwardSubstitute, BandMatrix, Basis, BezoutMatrix, BidiagonalForm, BilinearForm, CharacteristicMatrix, CharacteristicPolynomial, Column, ColumnDimension, ColumnOperation, ColumnSpace, CompanionMatrix, ConditionNumber, ConstantMatrix, ConstantVector, Copy, CreatePermutation, CrossProduct, DeleteColumn, DeleteRow, Determinant, Diagonal, DiagonalMatrix, Dimension, Dimensions, DotProduct, EigenConditionNumbers, Eigenvalues, Eigenvectors, Equal, ForwardSubstitute, FrobeniusForm, GaussianElimination, GenerateEquations, GenerateMatrix, GetResultDataType, GetResultShape, GivensRotationMatrix, GramSchmidt, HankelMatrix, HermiteForm, HermitianTranspose, HessenbergForm, HilbertMatrix, HouseholderMatrix, IdentityMatrix, IntersectionBasis, IsDefinite, IsOrthogonal, IsSimilar, IsUnitary, JordanBlockMatrix, JordanForm, LA_Main, LUDecomposition, LeastSquares, LinearSolve, Map, Map2, MatrixAdd, MatrixExponential, MatrixFunction, MatrixInverse, MatrixMatrixMultiply, MatrixNorm, MatrixPower, MatrixScalarMultiply, MatrixVectorMultiply, MinimalPolynomial, Minor, Modular, Multiply, NoUserValue, Norm, Normalize, NullSpace, OuterProductMatrix, Permanent, Pivot, PopovForm, QRDecomposition, RandomMatrix, RandomVector, Rank, RationalCanonicalForm, ReducedRowEchelonForm, Row, RowDimension, RowOperation, RowSpace, ScalarMatrix, ScalarMultiply, ScalarVector, SchurForm, SingularValues, SmithForm, SubMatrix, SubVector, SumBasis, SylvesterMatrix, ToeplitzMatrix, Trace, Transpose, TridiagonalForm, UnitVector, VandermondeMatrix, VectorAdd, VectorAngle, VectorMatrixMultiply, VectorNorm, VectorScalarMultiply, ZeroMatrix, ZeroVector, Zip*]

```
> H:=HilbertMatrix(10);
```

$$H := \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \frac{1}{6} & \frac{1}{7} & \frac{1}{8} & \frac{1}{9} & \frac{1}{10} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \frac{1}{6} & \frac{1}{7} & \frac{1}{8} & \frac{1}{9} & \frac{1}{10} & \frac{1}{11} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \frac{1}{6} & \frac{1}{7} & \frac{1}{8} & \frac{1}{9} & \frac{1}{10} & \frac{1}{11} & \frac{1}{12} \\ \frac{1}{4} & \frac{1}{5} & \frac{1}{6} & \frac{1}{7} & \frac{1}{8} & \frac{1}{9} & \frac{1}{10} & \frac{1}{11} & \frac{1}{12} & \frac{1}{13} \\ \frac{1}{5} & \frac{1}{6} & \frac{1}{7} & \frac{1}{8} & \frac{1}{9} & \frac{1}{10} & \frac{1}{11} & \frac{1}{12} & \frac{1}{13} & \frac{1}{14} \\ \frac{1}{6} & \frac{1}{7} & \frac{1}{8} & \frac{1}{9} & \frac{1}{10} & \frac{1}{11} & \frac{1}{12} & \frac{1}{13} & \frac{1}{14} & \frac{1}{15} \\ \frac{1}{7} & \frac{1}{8} & \frac{1}{9} & \frac{1}{10} & \frac{1}{11} & \frac{1}{12} & \frac{1}{13} & \frac{1}{14} & \frac{1}{15} & \frac{1}{16} \\ \frac{1}{8} & \frac{1}{9} & \frac{1}{10} & \frac{1}{11} & \frac{1}{12} & \frac{1}{13} & \frac{1}{14} & \frac{1}{15} & \frac{1}{16} & \frac{1}{17} \\ \frac{1}{9} & \frac{1}{10} & \frac{1}{11} & \frac{1}{12} & \frac{1}{13} & \frac{1}{14} & \frac{1}{15} & \frac{1}{16} & \frac{1}{17} & \frac{1}{18} \\ \frac{1}{10} & \frac{1}{11} & \frac{1}{12} & \frac{1}{13} & \frac{1}{14} & \frac{1}{15} & \frac{1}{16} & \frac{1}{17} & \frac{1}{18} & \frac{1}{19} \end{bmatrix}$$

```
> H^(-1);
```

[100 , -4950 , 79200 , -600600 , 2522520 , -6306300 , 9609600 , -8751600 , 4375800 , -923780]

[-4950 , 326700 , -5880600 , 47567520 , -208107900 , 535134600 , -832431600 , 770140800 , -389883780 , 83140200]

[79200 , -5880600 , 112907520 , -951350400 , 4281076800 , -11237826600 , 17758540800 , -16635041280 , 8506555200 , -1829084400]

[-600600 , 47567520 , -951350400 , 8245036800 , -37875637800 , 101001700800 , -161602721280 , 152907955200 , -78843164400 , 17071454400]

[2522520 , -208107900 , 4281076800 , -37875637800 , 176752976400 , -477233036280 , 771285715200 , -735869534400 , 382086104400 , -83223340200]

[-6306300 , 535134600 , -11237826600 , 101001700800 , -477233036280 , 1301544644400 , -2121035716800 , 2037792556800 , -1064382719400 , 233025352560]

[9609600 , -832431600 , 17758540800 , -161602721280 , 771285715200 , -2121035716800 , 3480673996800 , -3363975014400 , 1766086882560 , -388375587600]

[-8751600 , 770140800 , -16635041280 , 152907955200 , -735869534400 , 2037792556800 , -3363975014400 , 3267861442560 , -1723286307600 , 380449555200]

[4375800 , -389883780 , 8506555200 , -78843164400 , 382086104400 , -1064382719400 , 1766086882560 , -1723286307600 , 912328045200 , -202113826200]

[-923780 , 83140200 , -1829084400 , 17071454400 , -83223340200 , 233025352560 ,

-388375587600 , 380449555200 , -202113826200 , 44914183600]

```
> Determinant(H);
```

$$\frac{1}{46206893947914691316295628839036278726983680000000000}$$

```
> V:=VandermondeMatrix([u,v,w,x,y,z]);
```

$$V := \begin{bmatrix} 1 & u & u^2 & u^3 & u^4 & u^5 \\ 1 & v & v^2 & v^3 & v^4 & v^5 \\ 1 & w & w^2 & w^3 & w^4 & w^5 \\ 1 & x & x^2 & x^3 & x^4 & x^5 \\ 1 & y & y^2 & y^3 & y^4 & y^5 \\ 1 & z & z^2 & z^3 & z^4 & z^5 \end{bmatrix}$$

```
> Determinant(V);
```

$$u\,w^2 y^3 z^4 v^5 - u\,w^2 z^3 x^4 y^5 + z\,v^2 w^3 y^4 u^5 - z\,u^2 v^3 x^4 w^5 + u\,z^2 x^3 y^4 w^5 + w\,z^2 x^3 y^5 u^4$$
$$+ z\,u^2 w^3 v^4 y^5 - z\,u^2 y^3 x^4 v^5 + y\,u^2 x^3 z^4 w^5 + w\,u^2 x^3 z^4 v^5 + y\,u^2 v^3 z^4 x^5 - u\,w^2 y^3 x^4 v^5$$
$$+ y\,x^2 z^3 v^4 u^5 + y\,v^2 x^3 u^5 z^4 + z\,x^2 u^3 w^4 v^5 - w\,v^2 x^3 u^5 z^4 + x\,u^2 v^3 w^4 y^5 + u\,y^2 v^3 x^4 z^5$$
$$- u\,z^2 v^3 y^4 w^5 + v\,z^2 x^3 w^4 y^5 + z\,y^2 u^3 x^4 v^5 + v\,u^2 w^3 y^4 z^5 - w\,y^2 v^3 u^4 x^5 - v\,x^2 z^3 y^4 u^5$$
$$- x\,w^2 v^3 y^4 z^5 + x\,y^2 v^3 u^4 w^5 - u\,x^2 w^3 y^5 z^4 + x\,v^2 y^3 u^4 z^5 - u\,z^2 v^3 x^4 y^5 + z\,w^2 u^3 v^4 x^5$$
$$+ x\,y^2 v^3 u^5 z^4 + y\,v^2 x^3 u^4 w^5 - z\,v^2 w^3 y^5 u^4 + u\,v^2 x^3 y^4 z^5 - u\,y^2 w^3 v^4 x^5 - u\,z^2 w^3 y^4 x^5$$
$$+ u\,w^2 z^3 y^4 x^5 - w\,z^2 x^3 v^4 y^5 + u\,x^2 v^3 w^4 z^5 - y\,z^2 w^3 v^4 x^5 + v\,z^2 y^3 w^4 u^5 - v\,u^2 w^3 y^5 z^4$$
$$+ u\,w^2 x^3 y^4 v^5 + x\,z^2 w^3 u^4 v^5 - u\,y^2 x^3 v^4 z^5 - u\,y^2 w^3 x^4 z^5 - v\,x^2 u^3 y^4 w^5 + u\,v^2 w^3 y^5 z^4$$
$$+ z\,u^2 y^3 x^4 w^5 - z\,w^2 u^3 y^4 x^5 + v\,u^2 z^3 x^4 w^5 + y\,v^2 z^3 w^4 u^5 + u\,x^2 w^3 z^4 v^5 + u\,z^2 w^3 v^4 x^5$$
$$- u\,z^2 x^3 w^4 y^5 - z\,w^2 v^3 y^4 u^5 - u\,y^2 x^3 z^4 w^5 + u\,w^2 z^3 v^4 y^5 + u\,v^2 z^3 x^4 y^5 + y\,v^2 w^3 z^4 x^5$$
$$- y\,u^2 v^3 z^4 w^5 + z\,w^2 y^3 u^4 x^5 + v\,y^2 x^3 w^4 u^5 + w\,x^2 y^3 v^4 u^5 - u\,z^2 w^3 x^4 v^5 + w\,z^2 u^3 x^4 v^5$$
$$- x\,u^2 w^3 y^4 z^5 - u\,x^2 v^3 y^4 z^5 - v\,w^2 z^3 y^4 x^5 - u\,v^2 z^3 w^4 y^5 + w\,x^2 y^3 z^4 v^5 + z\,x^2 y^3 v^4 w^5$$
$$- z\,w^2 v^3 u^4 x^5 - v\,w^2 x^3 u^4 z^5 - v\,w^2 x^3 y^5 z^4 + v\,w^2 x^3 y^5 u^4 + v\,w^2 x^3 u^5 z^4 - v\,w^2 u^3 y^4 z^5$$
$$+ v\,w^2 u^3 y^5 z^4 + v\,w^2 y^3 z^4 x^5 - v\,w^2 y^3 u^5 z^4 - v\,w^2 y^3 u^4 x^5 - v\,w^2 y^3 x^4 z^5 + v\,w^2 y^3 x^4 u^5$$
$$+ v\,w^2 y^3 u^4 z^5 - v\,w^2 u^3 z^4 x^5 + v\,w^2 u^3 x^4 z^5 + v\,w^2 z^3 x^4 y^5 - v\,w^2 z^3 x^4 u^5 - v\,w^2 z^3 y^5 u^4$$
$$+ v\,w^2 z^3 y^4 u^5 + v\,w^2 z^3 u^4 x^5 - v\,w^2 u^3 x^4 y^5 + v\,w^2 u^3 y^4 x^5 - v\,u^2 x^3 y^4 z^5 + v\,u^2 x^3 y^5 z^4$$
$$- v\,u^2 y^3 z^4 x^5 + v\,u^2 y^3 x^4 z^5 - v\,u^2 z^3 x^4 y^5 + v\,u^2 z^3 y^4 x^5 + v\,x^2 u^3 y^4 z^5 - v\,x^2 u^3 y^5 z^4$$
$$+ v\,x^2 y^3 u^5 z^4 - v\,x^2 y^3 u^4 z^5 + v\,x^2 z^3 y^5 u^4 - v\,x^2 w^3 y^4 z^5 + v\,x^2 w^3 y^4 u^5 + v\,x^2 w^3 u^4 z^5$$
$$+ v\,x^2 w^3 y^5 z^4 - v\,x^2 w^3 y^5 u^4 - v\,x^2 w^3 u^5 z^4 - v\,x^2 y^3 z^4 w^5 + v\,x^2 y^3 u^4 w^5 + v\,x^2 y^3 w^4 z^5$$
$$- v\,x^2 y^3 w^4 u^5 + v\,x^2 u^3 z^4 w^5 - v\,x^2 u^3 w^4 z^5 - v\,x^2 z^3 w^4 y^5 + v\,u^2 x^3 w^4 z^5 - v\,u^2 x^3 z^4 w^5$$
$$- v\,z^2 w^3 y^4 u^5 - v\,z^2 y^3 u^4 w^5 + v\,x^2 u^3 w^4 y^5 + v\,w^2 x^3 y^4 z^5 + v\,z^2 w^3 y^5 u^4 + v\,x^2 z^3 y^4 w^5$$
$$- v\,x^2 z^3 u^4 w^5 - v\,w^2 x^3 y^4 u^5 + v\,x^2 z^3 w^4 u^5 + v\,u^2 y^3 z^4 w^5 - v\,u^2 y^3 w^4 z^5 + v\,u^2 z^3 w^4 y^5$$
$$- v\,u^2 z^3 y^4 w^5 + v\,y^2 w^3 x^4 z^5 - v\,y^2 w^3 z^4 x^5 - v\,y^2 w^3 x^4 u^5 + v\,y^2 x^3 u^4 z^5 - v\,y^2 x^3 u^5 z^4$$
$$+ v\,y^2 u^3 z^4 x^5 - v\,y^2 u^3 x^4 z^5 + v\,y^2 z^3 x^4 u^5 - v\,y^2 z^3 u^4 x^5 - v\,y^2 w^3 u^4 z^5 + v\,y^2 w^3 u^5 z^4$$
$$- v\,y^2 u^3 z^4 w^5 + v\,y^2 u^3 w^4 z^5 - v\,y^2 z^3 w^4 u^5 + v\,y^2 z^3 u^4 w^5 + v\,y^2 w^3 u^4 x^5 + v\,y^2 x^3 z^4 w^5$$

$$- v y^{2} x^{3} u^{4} w^{5} - v y^{2} x^{3} w^{4} z^{5} + v y^{2} z^{3} w^{4} x^{5} - v y^{2} z^{3} x^{4} w^{5} - v y^{2} u^{3} w^{4} x^{5} + v y^{2} u^{3} x^{4} w^{5}$$
$$- v u^{2} w^{3} x^{4} z^{5} + v u^{2} w^{3} z^{4} x^{5} - v u^{2} z^{3} w^{4} x^{5} - v z^{2} w^{3} x^{4} y^{5} + v z^{2} w^{3} x^{4} u^{5} + v z^{2} x^{3} y^{4} u^{5}$$
$$- v z^{2} x^{3} y^{5} u^{4} + v z^{2} y^{3} u^{4} x^{5} - v z^{2} y^{3} x^{4} u^{5} + v z^{2} u^{3} x^{4} y^{5} - v z^{2} u^{3} y^{4} x^{5} - v z^{2} u^{3} w^{4} y^{5}$$
$$+ v z^{2} u^{3} y^{4} w^{5} - v z^{2} w^{3} u^{4} x^{5} + v z^{2} x^{3} u^{4} w^{5} - v z^{2} x^{3} w^{4} u^{5} + v z^{2} u^{3} w^{4} x^{5} - v z^{2} u^{3} x^{4} w^{5}$$
$$+ v z^{2} w^{3} y^{4} x^{5} - v z^{2} x^{3} y^{4} w^{5} - v z^{2} y^{3} w^{4} x^{5} + v z^{2} y^{3} x^{4} w^{5} + v u^{2} w^{3} x^{4} y^{5} - v u^{2} w^{3} y^{4} x^{5}$$
$$+ v u^{2} x^{3} y^{4} w^{5} - v u^{2} x^{3} w^{4} y^{5} + v u^{2} y^{3} w^{4} x^{5} - v u^{2} y^{3} x^{4} w^{5} + u w^{2} x^{3} y^{5} z^{4} - u w^{2} y^{3} z^{4} x^{5}$$
$$+ u w^{2} y^{3} x^{4} z^{5} + u x^{2} w^{3} y^{4} z^{5} + u x^{2} y^{3} z^{4} w^{5} - u x^{2} y^{3} w^{4} z^{5} + u x^{2} z^{3} w^{4} y^{5} - u w^{2} x^{3} y^{4} z^{5}$$
$$- u x^{2} z^{3} y^{4} w^{5} + u y^{2} w^{3} z^{4} x^{5} + u y^{2} x^{3} w^{4} z^{5} - u y^{2} z^{3} w^{4} x^{5} + u y^{2} z^{3} x^{4} w^{5} + u z^{2} w^{3} x^{4} y^{5}$$
$$+ u z^{2} y^{3} w^{4} x^{5} - u z^{2} y^{3} x^{4} w^{5} + w u^{2} x^{3} y^{4} z^{5} - w u^{2} x^{3} y^{5} z^{4} + w u^{2} y^{3} z^{4} x^{5} - w u^{2} y^{3} x^{4} z^{5}$$
$$+ w u^{2} z^{3} x^{4} y^{5} - w u^{2} z^{3} y^{4} x^{5} - w x^{2} u^{3} y^{4} z^{5} + w x^{2} u^{3} y^{5} z^{4} - w x^{2} y^{3} u^{5} z^{4} + w x^{2} y^{3} u^{4} z^{5}$$
$$- w x^{2} z^{3} y^{5} u^{4} + w x^{2} z^{3} y^{4} u^{5} - w u^{2} v^{3} x^{4} y^{5} + w u^{2} v^{3} y^{4} x^{5} - w u^{2} y^{3} z^{4} v^{5} - w u^{2} z^{3} v^{4} y^{5}$$
$$- w u^{2} x^{3} y^{4} v^{5} + w u^{2} y^{3} v^{4} z^{5} - w y^{2} x^{3} u^{4} z^{5} + w y^{2} x^{3} u^{5} z^{4} - w y^{2} u^{3} z^{4} x^{5} + w y^{2} u^{3} x^{4} z^{5}$$
$$- w y^{2} z^{3} x^{4} u^{5} + w y^{2} z^{3} u^{4} x^{5} - w z^{2} x^{3} y^{4} u^{5} - w z^{2} y^{3} u^{4} x^{5} + w z^{2} y^{3} x^{4} u^{5} - w z^{2} u^{3} x^{4} y^{5}$$
$$+ w z^{2} u^{3} y^{4} x^{5} - w v^{2} x^{3} y^{4} z^{5} + w v^{2} x^{3} y^{4} u^{5} + w v^{2} x^{3} u^{4} z^{5} + w v^{2} x^{3} y^{5} z^{4} - w v^{2} x^{3} y^{5} u^{4}$$
$$+ w v^{2} u^{3} y^{4} z^{5} - w v^{2} u^{3} y^{5} z^{4} - w v^{2} y^{3} z^{4} x^{5} + w v^{2} y^{3} u^{5} z^{4} + w v^{2} y^{3} u^{4} x^{5} + w v^{2} y^{3} x^{4} z^{5}$$
$$- w v^{2} y^{3} x^{4} u^{5} - w v^{2} y^{3} u^{4} z^{5} + w v^{2} u^{3} z^{4} x^{5} - w v^{2} u^{3} x^{4} z^{5} - w v^{2} z^{3} x^{4} y^{5} + w v^{2} z^{3} x^{4} u^{5}$$
$$+ w v^{2} z^{3} y^{5} u^{4} + w v^{2} z^{3} y^{4} x^{5} - w v^{2} z^{3} y^{4} u^{5} - w v^{2} z^{3} u^{4} x^{5} + w v^{2} u^{3} x^{4} y^{5} - w v^{2} u^{3} y^{4} x^{5}$$
$$+ w x^{2} v^{3} y^{4} z^{5} - w x^{2} v^{3} y^{4} u^{5} - w x^{2} v^{3} u^{4} z^{5} - w x^{2} v^{3} y^{5} z^{4} + w x^{2} v^{3} y^{5} u^{4} + w x^{2} v^{3} u^{5} z^{4}$$
$$- w x^{2} y^{3} u^{4} v^{5} - w x^{2} y^{3} v^{4} z^{5} - w x^{2} u^{3} z^{4} v^{5} + w x^{2} u^{3} v^{4} z^{5} + w x^{2} z^{3} v^{4} y^{5} - w x^{2} z^{3} v^{4} u^{5}$$
$$- w x^{2} z^{3} y^{4} v^{5} + w x^{2} z^{3} u^{4} v^{5} - w x^{2} u^{3} v^{4} y^{5} + w x^{2} u^{3} y^{4} v^{5} - w u^{2} v^{3} y^{4} z^{5} + w u^{2} v^{3} y^{5} z^{4}$$
$$+ w u^{2} z^{3} y^{4} v^{5} - w y^{2} z^{3} v^{4} x^{5} + w y^{2} v^{3} u^{4} z^{5} - w y^{2} v^{3} u^{5} z^{4} + w y^{2} u^{3} z^{4} v^{5} - w y^{2} u^{3} v^{4} z^{5}$$
$$+ w y^{2} z^{3} v^{4} u^{5} - w y^{2} z^{3} u^{4} v^{5} - w y^{2} v^{3} x^{4} z^{5} + w y^{2} v^{3} x^{4} u^{5} + w y^{2} v^{3} z^{4} x^{5} - w y^{2} x^{3} z^{4} v^{5}$$
$$+ w y^{2} x^{3} u^{4} v^{5} + w y^{2} x^{3} v^{4} z^{5} - w y^{2} x^{3} v^{4} u^{5} + w y^{2} z^{3} x^{4} v^{5} + w y^{2} u^{3} v^{4} x^{5} - w y^{2} u^{3} x^{4} v^{5}$$
$$+ w u^{2} z^{3} v^{4} x^{5} + w u^{2} v^{3} x^{4} z^{5} - w u^{2} v^{3} z^{4} x^{5} - w u^{2} x^{3} v^{4} z^{5} - w u^{2} z^{3} x^{4} v^{5} + w z^{2} v^{3} y^{4} u^{5}$$
$$- w z^{2} v^{3} y^{5} u^{4} + w z^{2} y^{3} u^{4} v^{5} - w z^{2} y^{3} v^{4} u^{5} + w z^{2} u^{3} v^{4} y^{5} - w z^{2} u^{3} y^{4} v^{5} - w z^{2} v^{3} x^{4} u^{5}$$
$$+ w z^{2} v^{3} u^{4} x^{5} - w z^{2} x^{3} u^{4} v^{5} + w z^{2} x^{3} v^{4} u^{5} - w z^{2} u^{3} v^{4} x^{5} + w z^{2} v^{3} x^{4} y^{5} - w z^{2} v^{3} y^{4} x^{5}$$
$$+ w z^{2} x^{3} y^{4} v^{5} + w z^{2} y^{3} v^{4} x^{5} - w z^{2} y^{3} x^{4} v^{5} + w u^{2} x^{3} v^{4} y^{5} - w u^{2} y^{3} v^{4} x^{5} + w u^{2} y^{3} x^{4} v^{5}$$
$$- u v^{2} x^{3} y^{5} z^{4} + u v^{2} y^{3} z^{4} x^{5} - u v^{2} y^{3} x^{4} z^{5} - u v^{2} z^{3} y^{4} x^{5} + u x^{2} v^{3} y^{5} z^{4} - u x^{2} y^{3} z^{4} v^{5}$$
$$+ u x^{2} y^{3} v^{4} z^{5} - u x^{2} z^{3} v^{4} y^{5} + u x^{2} z^{3} y^{4} v^{5} + u y^{2} z^{3} v^{4} x^{5} - u y^{2} v^{3} z^{4} x^{5} + u y^{2} x^{3} z^{4} v^{5}$$
$$- u y^{2} z^{3} x^{4} v^{5} + u z^{2} v^{3} y^{4} x^{5} - u z^{2} x^{3} y^{4} v^{5} + u z^{2} x^{3} v^{4} y^{5} - u z^{2} y^{3} v^{4} x^{5} + u z^{2} y^{3} x^{4} v^{5}$$
$$+ x w^{2} u^{3} y^{4} z^{5} - x w^{2} u^{3} y^{5} z^{4} + x w^{2} y^{3} u^{5} z^{4} - x w^{2} y^{3} u^{4} z^{5} + x w^{2} z^{3} y^{5} u^{4} - x w^{2} z^{3} y^{4} u^{5}$$
$$+ x z^{2} w^{3} y^{4} u^{5} + x z^{2} y^{3} u^{4} w^{5} - x z^{2} u^{3} w^{4} v^{5} - x z^{2} v^{3} w^{4} y^{5} - x z^{2} w^{3} y^{5} u^{4} - x z^{2} w^{3} y^{4} v^{5}$$
$$+ x u^{2} w^{3} y^{5} z^{4} + x u^{2} y^{3} z^{4} v^{5} + x u^{2} z^{3} v^{4} y^{5} - x z^{2} y^{3} v^{4} w^{5} - x u^{2} y^{3} z^{4} w^{5} - x z^{2} y^{3} w^{4} u^{5}$$
$$+ x z^{2} u^{3} v^{4} w^{5} + x z^{2} w^{3} v^{4} y^{5} + x z^{2} v^{3} y^{4} w^{5} - x u^{2} y^{3} v^{4} z^{5} + x u^{2} y^{3} w^{4} z^{5} - x u^{2} z^{3} w^{4} y^{5}$$
$$+ x u^{2} z^{3} y^{4} w^{5} + x y^{2} w^{3} u^{4} z^{5} - x y^{2} w^{3} u^{5} z^{4} + x y^{2} u^{3} z^{4} w^{5} - x y^{2} u^{3} w^{4} z^{5} + x y^{2} z^{3} w^{4} u^{5}$$

$$- x y^2 z^3 u^4 w^5 + x z^2 u^3 w^4 y^5 - x z^2 u^3 y^4 w^5 - x v^2 u^3 y^4 z^5 + x v^2 u^3 y^5 z^4 - x v^2 y^3 u^5 z^4$$
$$- x v^2 z^3 y^5 u^4 + x v^2 z^3 y^4 u^5 + x u^2 v^3 y^4 z^5 - x u^2 v^3 y^5 z^4 - x u^2 z^3 y^4 v^5 - x y^2 v^3 u^4 z^5$$
$$- x y^2 u^3 z^4 v^5 + x y^2 u^3 v^4 z^5 - x y^2 z^3 v^4 u^5 + x y^2 z^3 u^4 v^5 - x z^2 v^3 y^4 u^5 + x z^2 v^3 y^5 u^4$$
$$- x z^2 y^3 u^4 v^5 + x z^2 y^3 v^4 u^5 - x z^2 u^3 v^4 y^5 + x z^2 u^3 y^4 v^5 + x v^2 w^3 y^4 z^5 - x v^2 w^3 y^4 u^5$$
$$- x v^2 w^3 u^4 z^5 - x v^2 w^3 y^5 z^4 + x v^2 w^3 y^5 u^4 + x v^2 w^3 u^5 z^4 + x v^2 y^3 z^4 w^5 - x v^2 y^3 u^4 w^5$$
$$- x v^2 y^3 w^4 z^5 + x v^2 y^3 w^4 u^5 - x v^2 u^3 z^4 w^5 + x v^2 u^3 w^4 z^5 + x v^2 z^3 w^4 y^5 - x v^2 z^3 w^4 u^5$$
$$- x v^2 z^3 y^4 w^5 + x v^2 z^3 u^4 w^5 - x v^2 u^3 w^4 y^5 + x v^2 u^3 y^4 w^5 + x w^2 v^3 y^4 u^5 + x w^2 v^3 u^4 z^5$$
$$+ x w^2 v^3 y^5 z^4 - x w^2 v^3 y^5 u^4 - x w^2 v^3 u^5 z^4 - x w^2 y^3 z^4 v^5 + x w^2 y^3 u^4 v^5 + x w^2 y^3 v^4 z^5$$
$$- x w^2 y^3 v^4 u^5 + x w^2 u^3 z^4 v^5 - x w^2 u^3 v^4 z^5 - x w^2 z^3 v^4 y^5 + x w^2 z^3 v^4 u^5 + x w^2 z^3 y^4 v^5$$
$$- x w^2 z^3 u^4 v^5 + x w^2 u^3 v^4 y^5 - x w^2 u^3 y^4 v^5 + x y^2 v^3 w^4 z^5 - x y^2 v^3 w^4 u^5 - x y^2 v^3 z^4 w^5$$
$$+ x y^2 w^3 z^4 v^5 - x y^2 w^3 u^4 v^5 - x y^2 w^3 v^4 z^5 + x y^2 w^3 v^4 u^5 + x y^2 z^3 v^4 w^5 - x y^2 z^3 w^4 v^5$$
$$- x y^2 u^3 v^4 w^5 + x y^2 u^3 w^4 v^5 - x u^2 v^3 w^4 z^5 + x u^2 v^3 z^4 w^5 - x u^2 w^3 z^4 v^5 + x u^2 w^3 v^4 z^5$$
$$- x u^2 z^3 v^4 w^5 + x u^2 z^3 w^4 v^5 + x z^2 v^3 w^4 u^5 - x z^2 v^3 u^4 w^5 - x z^2 w^3 v^4 u^5 + x z^2 y^3 w^4 v^5$$
$$- x u^2 v^3 y^4 w^5 + x u^2 w^3 y^4 v^5 - x u^2 w^3 v^4 y^5 + x u^2 y^3 v^4 w^5 - x u^2 y^3 w^4 v^5 + u z^2 v^3 w^4 y^5$$
$$+ u z^2 w^3 y^4 v^5 + u z^2 y^3 v^4 w^5 - u z^2 w^3 v^4 y^5 - u v^2 w^3 y^4 z^5 - u v^2 y^3 z^4 w^5 + u v^2 y^3 w^4 z^5$$
$$+ u v^2 z^3 y^4 w^5 + u w^2 v^3 y^4 z^5 - u w^2 v^3 y^5 z^4 - u w^2 y^3 v^4 z^5 - u w^2 z^3 y^4 v^5 - u y^2 v^3 w^4 z^5$$
$$+ u y^2 v^3 z^4 w^5 - u y^2 w^3 z^4 v^5 + u y^2 w^3 v^4 z^5 - u y^2 z^3 v^4 w^5 + u y^2 z^3 w^4 v^5 - u z^2 y^3 w^4 v^5$$
$$+ y w^2 x^3 u^4 z^5 - y w^2 x^3 u^5 z^4 + y w^2 u^3 z^4 x^5 - y w^2 u^3 x^4 z^5 + y w^2 z^3 x^4 u^5 - y w^2 z^3 u^4 x^5$$
$$- y x^2 w^3 u^4 z^5 + y x^2 w^3 u^5 z^4 - y x^2 u^3 z^4 w^5 + y x^2 u^3 w^4 z^5 - y u^2 x^3 w^4 z^5 + y z^2 u^3 w^4 v^5$$
$$+ y x^2 z^3 u^4 w^5 - y x^2 z^3 w^4 u^5 - y z^2 u^3 v^4 w^5 + y x^2 v^3 w^4 u^5 + y u^2 w^3 x^4 z^5 - y u^2 w^3 z^4 x^5$$
$$+ y u^2 z^3 w^4 x^5 - y u^2 z^3 x^4 w^5 - y z^2 w^3 x^4 u^5 + y z^2 w^3 u^4 x^5 - y z^2 x^3 u^4 w^5 + y z^2 x^3 w^4 u^5$$
$$- y z^2 u^3 w^4 x^5 + y z^2 u^3 x^4 w^5 - y v^2 x^3 u^4 z^5 - y v^2 u^3 z^4 x^5 + y v^2 u^3 x^4 z^5 - y v^2 z^3 x^4 u^5$$
$$+ y v^2 z^3 u^4 x^5 + y x^2 v^3 u^4 z^5 - y x^2 v^3 u^5 z^4 + y x^2 u^3 z^4 v^5 - y x^2 u^3 v^4 z^5 - y x^2 z^3 u^4 v^5$$
$$- y u^2 z^3 v^4 x^5 - y u^2 v^3 x^4 z^5 - y u^2 x^3 z^4 v^5 + y u^2 x^3 v^4 z^5 + y u^2 z^3 x^4 v^5 + y z^2 v^3 x^4 u^5$$
$$- y z^2 v^3 u^4 x^5 + y z^2 x^3 u^4 v^5 - y z^2 x^3 v^4 u^5 + y z^2 u^3 v^4 x^5 - y z^2 u^3 x^4 v^5 + y v^2 w^3 u^4 z^5$$
$$- y v^2 w^3 u^5 z^4 + y v^2 u^3 z^4 w^5 - y v^2 u^3 w^4 z^5 - y v^2 z^3 u^4 w^5 - y w^2 v^3 u^4 z^5 + y w^2 v^3 u^5 z^4$$
$$- y w^2 u^3 z^4 v^5 + y w^2 u^3 v^4 z^5 - y w^2 z^3 v^4 u^5 + y w^2 z^3 u^4 v^5 + y u^2 v^3 w^4 z^5 + y u^2 w^3 z^4 v^5$$
$$- y u^2 w^3 v^4 z^5 + y u^2 z^3 v^4 w^5 - y u^2 z^3 w^4 v^5 - y z^2 v^3 w^4 u^5 + y z^2 v^3 u^4 w^5 - y z^2 w^3 u^4 v^5$$
$$+ y z^2 w^3 v^4 u^5 - y v^2 w^3 x^4 z^5 + y v^2 w^3 x^4 u^5 - y v^2 w^3 u^4 x^5 - y v^2 x^3 z^4 w^5 + y v^2 x^3 w^4 z^5$$
$$- y v^2 x^3 w^4 u^5 - y v^2 z^3 w^4 x^5 + y v^2 z^3 x^4 w^5 + y v^2 u^3 w^4 x^5 - y v^2 u^3 x^4 w^5 + y w^2 z^3 v^4 x^5$$
$$+ y w^2 v^3 x^4 z^5 - y w^2 v^3 x^4 u^5 - y w^2 v^3 z^4 x^5 + y w^2 v^3 u^4 x^5 + y w^2 x^3 z^4 v^5 - y w^2 x^3 u^4 v^5$$
$$- y w^2 x^3 v^4 z^5 + y w^2 x^3 v^4 u^5 - y w^2 z^3 x^4 v^5 - y w^2 u^3 v^4 x^5 + y w^2 u^3 x^4 v^5 - y x^2 v^3 w^4 z^5$$
$$+ y x^2 v^3 z^4 w^5 - y x^2 v^3 u^4 w^5 - y x^2 w^3 z^4 v^5 + y x^2 w^3 u^4 v^5 + y x^2 w^3 v^4 z^5 - y x^2 w^3 v^4 u^5$$
$$- y x^2 z^3 v^4 w^5 + y x^2 z^3 w^4 v^5 + y x^2 u^3 v^4 w^5 - y x^2 u^3 w^4 v^5 + y z^2 w^3 x^4 v^5 + y z^2 v^3 w^4 x^5$$
$$- y z^2 v^3 x^4 w^5 + y z^2 x^3 v^4 w^5 - y z^2 x^3 w^4 v^5 - y u^2 w^3 x^4 v^5 - y u^2 v^3 w^4 x^5 + y u^2 v^3 x^4 w^5$$

$$+ y u^2 w^3 v^4 x^5 - y u^2 x^3 v^4 w^5 + y u^2 x^3 w^4 v^5 + u v^2 w^3 x^4 z^5 - u v^2 w^3 z^4 x^5 + u v^2 x^3 z^4 w^5$$
$$- u v^2 x^3 w^4 z^5 + u v^2 z^3 w^4 x^5 - u v^2 z^3 x^4 w^5 - u w^2 z^3 v^4 x^5 - u w^2 v^3 x^4 z^5 + u w^2 v^3 z^4 x^5$$
$$- u w^2 x^3 z^4 v^5 + u w^2 x^3 v^4 z^5 + u w^2 z^3 x^4 v^5 - u x^2 v^3 z^4 w^5 - u x^2 w^3 v^4 z^5 + u x^2 z^3 v^4 w^5$$
$$- u x^2 z^3 w^4 v^5 - u z^2 v^3 w^4 x^5 + u z^2 v^3 x^4 w^5 - u z^2 x^3 v^4 w^5 + u z^2 x^3 w^4 v^5 - z w^2 x^3 y^5 u^4$$
$$- z w^2 y^3 x^4 u^5 + z w^2 u^3 x^4 y^5 - z x^2 w^3 y^4 u^5 + z x^2 w^3 y^5 u^4 - z x^2 y^3 u^4 w^5 + z x^2 y^3 w^4 u^5$$
$$+ z u^2 v^3 x^4 y^5 - z x^2 u^3 w^4 y^5 - z u^2 v^3 y^4 x^5 + z x^2 u^3 y^4 w^5 + z w^2 x^3 y^4 u^5 + z u^2 x^3 y^4 v^5$$
$$- z x^2 v^3 w^4 u^5 + z y^2 w^3 x^4 u^5 - z y^2 w^3 u^4 x^5 + z y^2 x^3 u^4 w^5 - z y^2 x^3 w^4 u^5 + z y^2 u^3 w^4 x^5$$
$$- z y^2 u^3 x^4 w^5 - z u^2 w^3 x^4 y^5 + z u^2 w^3 y^4 x^5 - z u^2 x^3 y^4 w^5 + z u^2 x^3 w^4 y^5 - z u^2 y^3 w^4 x^5$$
$$- z v^2 x^3 y^4 u^5 + z v^2 x^3 y^5 u^4 - z v^2 y^3 u^4 x^5 + z v^2 y^3 x^4 u^5 - z v^2 u^3 x^4 y^5 + z v^2 u^3 y^4 x^5$$
$$+ z x^2 v^3 y^4 u^5 - z x^2 v^3 y^5 u^4 + z x^2 y^3 u^4 v^5 - z x^2 y^3 v^4 u^5 + z x^2 u^3 v^4 y^5 - z x^2 u^3 y^4 v^5$$
$$- z y^2 v^3 x^4 u^5 + z y^2 v^3 u^4 x^5 - z y^2 x^3 u^4 v^5 + z y^2 x^3 v^4 u^5 - z y^2 u^3 v^4 x^5 - z u^2 x^3 v^4 y^5$$
$$+ z u^2 y^3 v^4 x^5 + z v^2 y^3 u^4 w^5 - z v^2 y^3 w^4 u^5 + z v^2 u^3 w^4 y^5 - z v^2 u^3 y^4 w^5 + z w^2 v^3 y^5 u^4$$
$$- z w^2 y^3 u^4 v^5 + z w^2 y^3 v^4 u^5 - z w^2 u^3 v^4 y^5 + z w^2 u^3 y^4 v^5 + z y^2 v^3 w^4 u^5 - z y^2 v^3 u^4 w^5$$
$$+ z y^2 w^3 u^4 v^5 - z y^2 w^3 v^4 u^5 + z y^2 u^3 v^4 w^5 - z y^2 u^3 w^4 v^5 - z u^2 v^3 w^4 y^5 + z u^2 v^3 y^4 w^5$$
$$- z u^2 w^3 y^4 v^5 - z u^2 y^3 v^4 w^5 + z u^2 y^3 w^4 v^5 - z v^2 w^3 x^4 u^5 + z v^2 w^3 u^4 x^5 - z v^2 x^3 u^4 w^5$$
$$+ z v^2 x^3 w^4 u^5 - z v^2 u^3 w^4 x^5 + z v^2 u^3 x^4 w^5 + z w^2 v^3 x^4 u^5 + z w^2 x^3 u^4 v^5 - z w^2 x^3 v^4 u^5$$
$$- z w^2 u^3 x^4 v^5 + z x^2 v^3 u^4 w^5 - z x^2 w^3 u^4 v^5 + z x^2 w^3 v^4 u^5 - z x^2 u^3 v^4 w^5 + z u^2 w^3 x^4 v^5$$
$$+ z u^2 v^3 w^4 x^5 - z u^2 w^3 v^4 x^5 + z u^2 x^3 v^4 w^5 - z u^2 x^3 w^4 v^5 + z v^2 w^3 x^4 y^5 - z v^2 w^3 y^4 x^5$$
$$+ z v^2 x^3 y^4 w^5 - z v^2 x^3 w^4 y^5 + z v^2 y^3 w^4 x^5 - z v^2 y^3 x^4 w^5 - z w^2 v^3 x^4 y^5 + z w^2 v^3 y^4 x^5$$
$$- z w^2 x^3 y^4 v^5 + z w^2 x^3 v^4 y^5 - z w^2 y^3 v^4 x^5 + z w^2 y^3 x^4 v^5 + z x^2 v^3 w^4 y^5 - z x^2 v^3 y^4 w^5$$
$$+ z x^2 w^3 y^4 v^5 - z x^2 w^3 v^4 y^5 - z x^2 y^3 w^4 v^5 - z y^2 w^3 x^4 v^5 - z y^2 v^3 w^4 x^5 + z y^2 v^3 x^4 w^5$$
$$+ z y^2 w^3 v^4 x^5 - z y^2 x^3 v^4 w^5 + z y^2 x^3 w^4 v^5 - u v^2 w^3 x^4 y^5 + u v^2 w^3 y^4 x^5 - u v^2 x^3 y^4 w^5$$
$$+ u v^2 x^3 w^4 y^5 - u v^2 y^3 w^4 x^5 + u v^2 y^3 x^4 w^5 + u w^2 v^3 x^4 y^5 - u w^2 v^3 y^4 x^5 - u w^2 x^3 v^4 y^5$$
$$+ u w^2 y^3 v^4 x^5 - u x^2 v^3 w^4 y^5 + u x^2 v^3 y^4 w^5 - u x^2 w^3 y^4 v^5 + u x^2 w^3 v^4 y^5 - u x^2 y^3 v^4 w^5$$
$$+ u x^2 y^3 w^4 v^5 + u y^2 w^3 x^4 v^5 + u y^2 v^3 w^4 x^5 - u y^2 v^3 x^4 w^5 + u y^2 x^3 v^4 w^5 - u y^2 x^3 w^4 v^5$$

```
> factor(%);
```

$$-(z-w)(y-w)(y-z)(x-w)(x-z)(x-y)(-w+u)(u-z)(u-y)(u-x)(-w+v)(v-z)(v-y)(v-x)(v-u)$$

Equations

```
> s:=[solve(x^2-3*x-1=0,x)];
```

$$s := \left[\frac{3}{2}+\frac{\sqrt{13}}{2}, \frac{3}{2}-\frac{\sqrt{13}}{2}\right]$$

```
> s:=[solve(x^3-3*x-1=0,x)];
```

$$s := \left[\frac{(4+4I\sqrt{3})^{(1/3)}}{2}+\frac{2}{(4+4I\sqrt{3})^{(1/3)}}, -\frac{(4+4I\sqrt{3})^{(1/3)}}{4}-\frac{1}{(4+4I\sqrt{3})^{(1/3)}}\right.$$

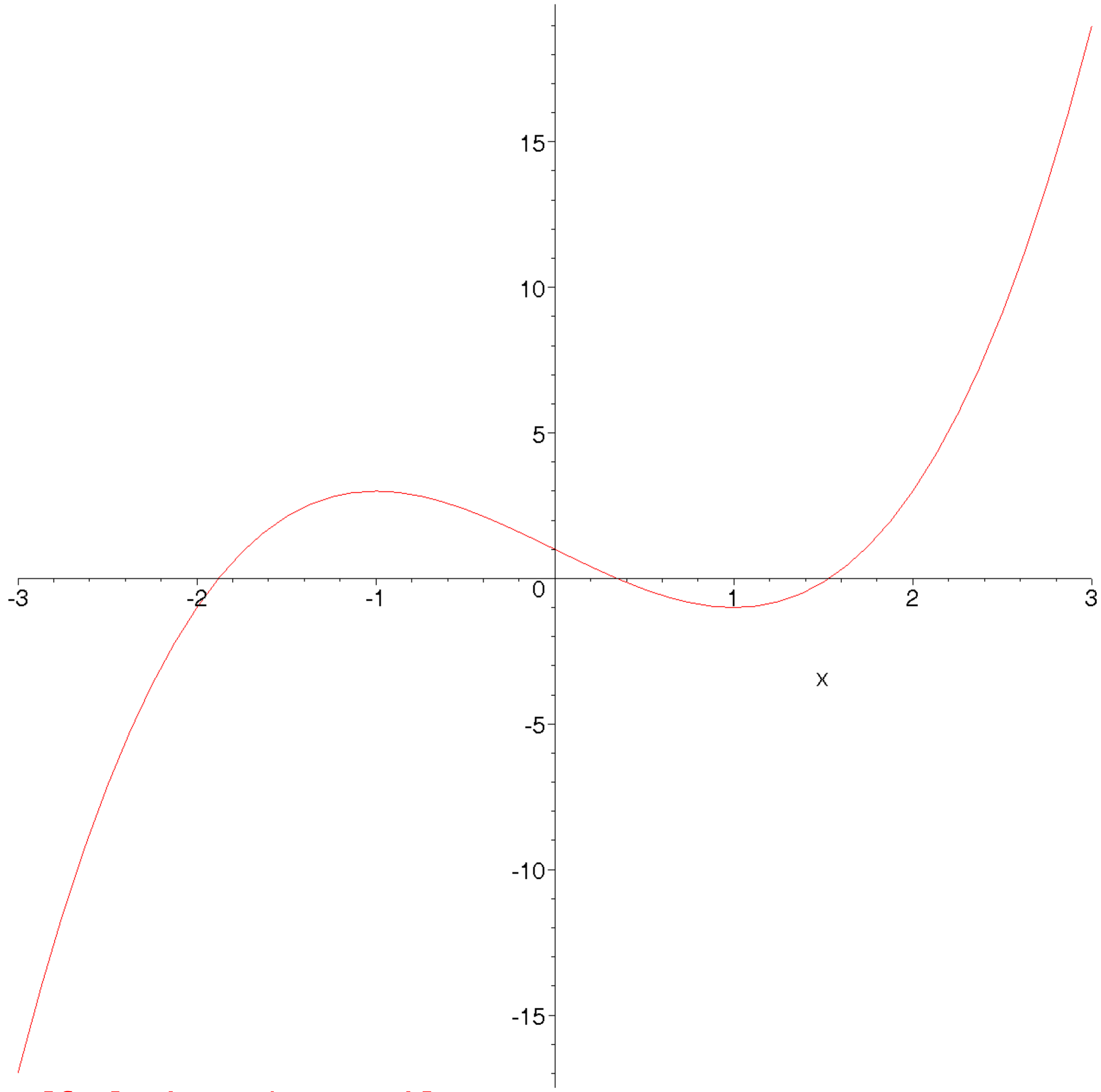
$$+\frac{1}{2} I\sqrt{3}\left(\frac{(4+4I\sqrt{3})^{(1/3)}}{2}-\frac{2}{(4+4I\sqrt{3})^{(1/3)}}\right), -\frac{(4+4I\sqrt{3})^{(1/3)}}{4}$$

$$-\frac{1}{(4+4I\sqrt{3})^{(1/3)}}-\frac{1}{2} I\sqrt{3}\left(\frac{(4+4I\sqrt{3})^{(1/3)}}{2}-\frac{2}{(4+4I\sqrt{3})^{(1/3)}}\right)\Bigg]$$

```
> evalf(s);
```

$$[1.879385242-0.1\ 10^{-9}\, I, -1.532088886+0.2732050808\ 10^{-9}\, I, -0.3472963554-0.732050808\ 10^{-10}\, I]$$

```
> plot(x^3-3*x+1,x=-3..3);
```




```
> [fsolve(x^3-3*x-1=0,x)];
```

$$[-1.532088886, -0.3472963553, 1.879385242]$$

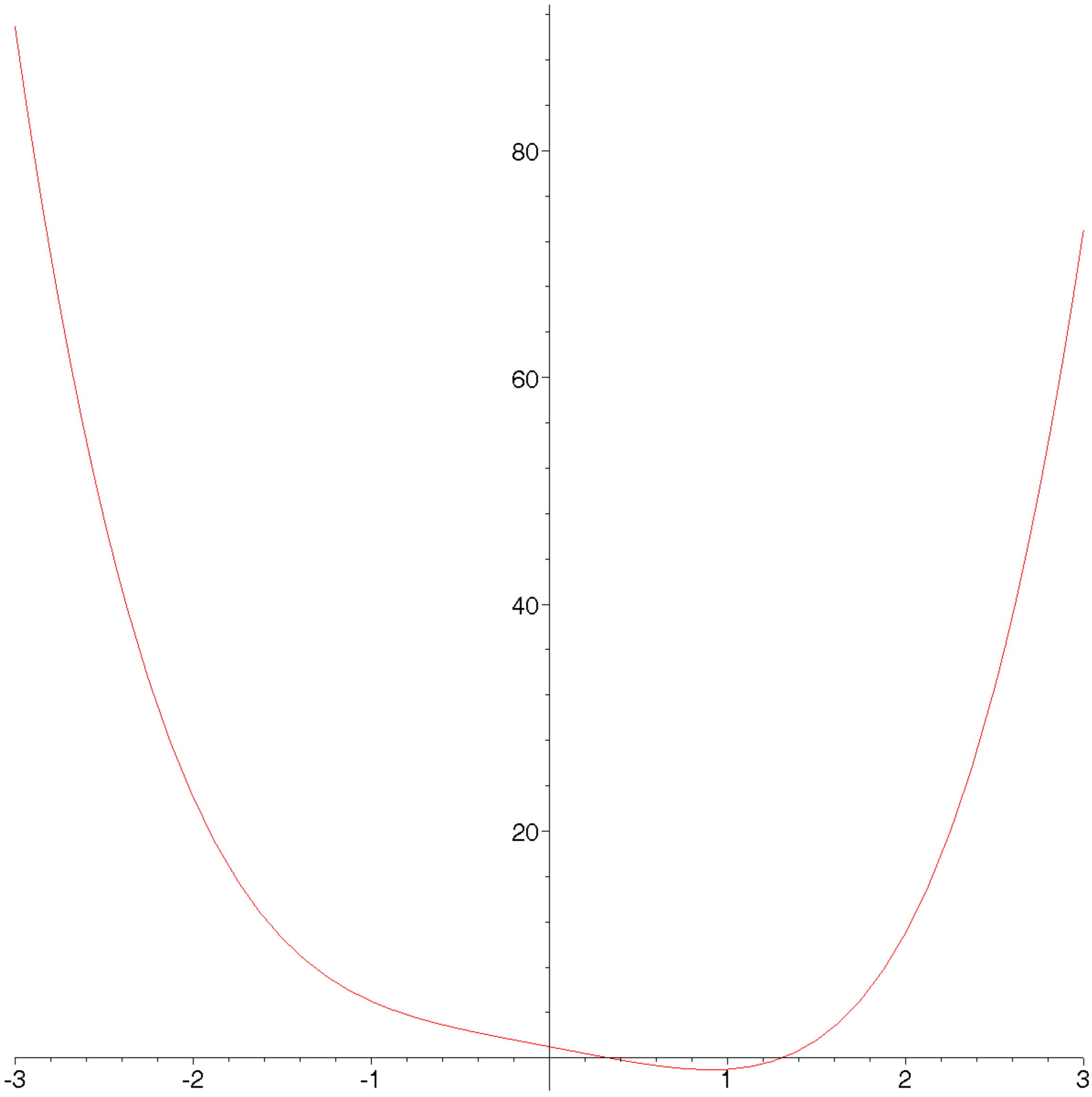
```
> s:=[solve(x^4-3*x-1=0,x)];
```

$$s := [\mathrm{RootOf}(\_Z^4-3\,\_Z-1, index=1), \mathrm{RootOf}(\_Z^4-3\,\_Z-1, index=2), \mathrm{RootOf}(\_Z^4-3\,\_Z-1, index=3), \mathrm{RootOf}(\_Z^4-3\,\_Z-1, index=4)]$$

```
> evalf(s);
```

[1.539613346, -0.6051024089 + 1.267126096 I, -0.3294085282,
-0.6051024089 – 1.267126096 I]

```
> plot(x^4-3*x+1,x=-3..3);
```




```
> [fsolve(x^4-3*x-1=0,x)];
```

[-0.3294085282, 1.539613346]

```
> _EnvExplicit:=true;
```

$\_EnvExplicit := true$

```
> s:=[solve(x^4-3*x-1=0,x)];
```

$$s := \left[\frac{\sqrt{6}\sqrt{\frac{(972+12\sqrt{7329})^{(2/3)} - 48}{(972+12\sqrt{7329})^{(1/3)}}}}{12} + \left(\left(\right.\right.\right.$$

$$-6\sqrt{\frac{(972+12\sqrt{7329})^{(2/3)}-48}{(972+12\sqrt{7329})^{(1/3)}}}\,(972+12\sqrt{7329})^{(2/3)}$$

$$+288\sqrt{\frac{(972+12\sqrt{7329})^{(2/3)}-48}{(972+12\sqrt{7329})^{(1/3)}}}+216\sqrt{6}\,(972+12\sqrt{7329})^{(1/3)}\Bigg)\Bigg/\Bigg($$

$$(972+12\sqrt{7329})^{(1/3)}\sqrt{\frac{(972+12\sqrt{7329})^{(2/3)}-48}{(972+12\sqrt{7329})^{(1/3)}}}\Bigg)\Bigg)^{(1/2)}\Big/12,$$

$$\frac{\sqrt{6}\sqrt{\frac{(972+12\sqrt{7329})^{(2/3)}-48}{(972+12\sqrt{7329})^{(1/3)}}}}{12}-\Bigg(\Bigg($$

$$-6\sqrt{\frac{(972+12\sqrt{7329})^{(2/3)}-48}{(972+12\sqrt{7329})^{(1/3)}}}\,(972+12\sqrt{7329})^{(2/3)}$$

$$+288\sqrt{\frac{(972+12\sqrt{7329})^{(2/3)}-48}{(972+12\sqrt{7329})^{(1/3)}}}+216\sqrt{6}\,(972+12\sqrt{7329})^{(1/3)}\Bigg)\Bigg/\Bigg($$

$$(972+12\sqrt{7329})^{(1/3)}\sqrt{\frac{(972+12\sqrt{7329})^{(2/3)}-48}{(972+12\sqrt{7329})^{(1/3)}}}\Bigg)\Bigg)^{(1/2)}\Big/12,$$

$$-\frac{\sqrt{6}\sqrt{\frac{(972+12\sqrt{7329})^{(2/3)}-48}{(972+12\sqrt{7329})^{(1/3)}}}}{12}+\frac{1}{12}I\Bigg(\Bigg($$

$$6\sqrt{\frac{(972+12\sqrt{7329})^{(2/3)}-48}{(972+12\sqrt{7329})^{(1/3)}}}\,(972+12\sqrt{7329})^{(2/3)}$$

$$-288\sqrt{\frac{(972+12\sqrt{7329})^{(2/3)}-48}{(972+12\sqrt{7329})^{(1/3)}}}+216\sqrt{6}\,(972+12\sqrt{7329})^{(1/3)}\Bigg)\Bigg/\Bigg($$

$$(972+12\sqrt{7329})^{(1/3)}\sqrt{\frac{(972+12\sqrt{7329})^{(2/3)}-48}{(972+12\sqrt{7329})^{(1/3)}}}\Bigg)\Bigg)^{(1/2)},$$

$$-\frac{\sqrt{6}\sqrt{\frac{(972+12\sqrt{7329})^{(2/3)}-48}{(972+12\sqrt{7329})^{(1/3)}}}}{12}-\frac{1}{12}I\Bigg(\Bigg($$

$$6\sqrt{\frac{(972+12\sqrt{7329})^{(2/3)}-48}{(972+12\sqrt{7329})^{(1/3)}}}\,(972+12\sqrt{7329})^{(2/3)}$$

$$-288\sqrt{\frac{(972+12\sqrt{7329})^{(2/3)}-48}{(972+12\sqrt{7329})^{(1/3)}}}+216\sqrt{6}\,(972+12\sqrt{7329})^{(1/3)}\Bigg)\Bigg/\Bigg($$

$$(972+12\sqrt{7329})^{(1/3)}\sqrt{\frac{(972+12\sqrt{7329})^{(2/3)}-48}{(972+12\sqrt{7329})^{(1/3)}}}\Bigg)\Bigg)^{(1/2)}\Bigg]$$

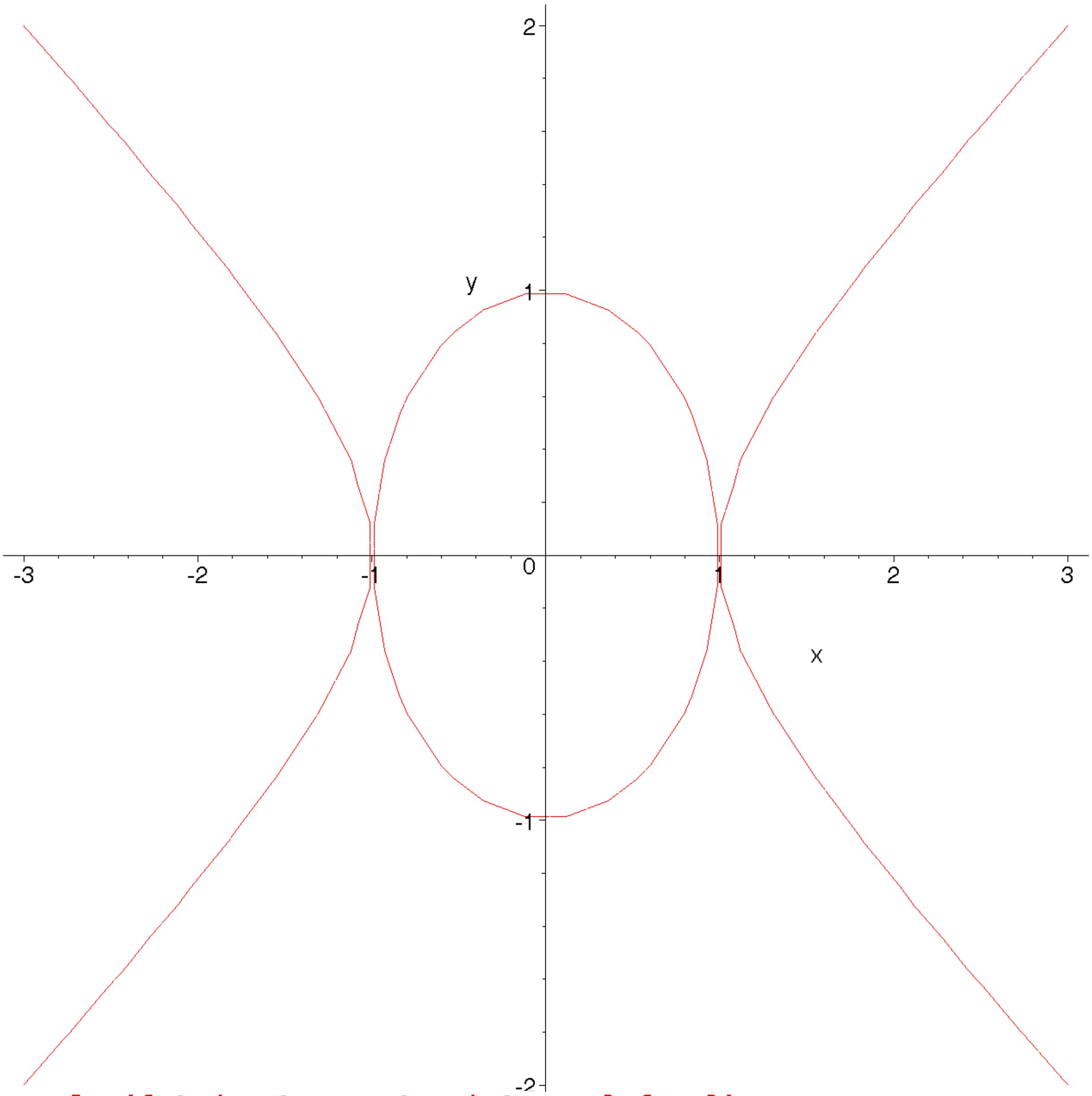
```
> s:=[solve(x^5-3*x-1=0,x)];
```

$s := [\mathrm{RootOf}(\_Z^5-3\_Z-1, index=1), \mathrm{RootOf}(\_Z^5-3\_Z-1, index=2),$
$\mathrm{RootOf}(\_Z^5-3\_Z-1, index=3), \mathrm{RootOf}(\_Z^5-3\_Z-1, index=4),$
$\mathrm{RootOf}(\_Z^5-3\_Z-1, index=5)]$

```
> solve({x^2+y^2=1,-x^2+2*y^2+1=0},{x,y});
```

$$\{y=0, x=1\}, \{x=-1, y=0\}$$

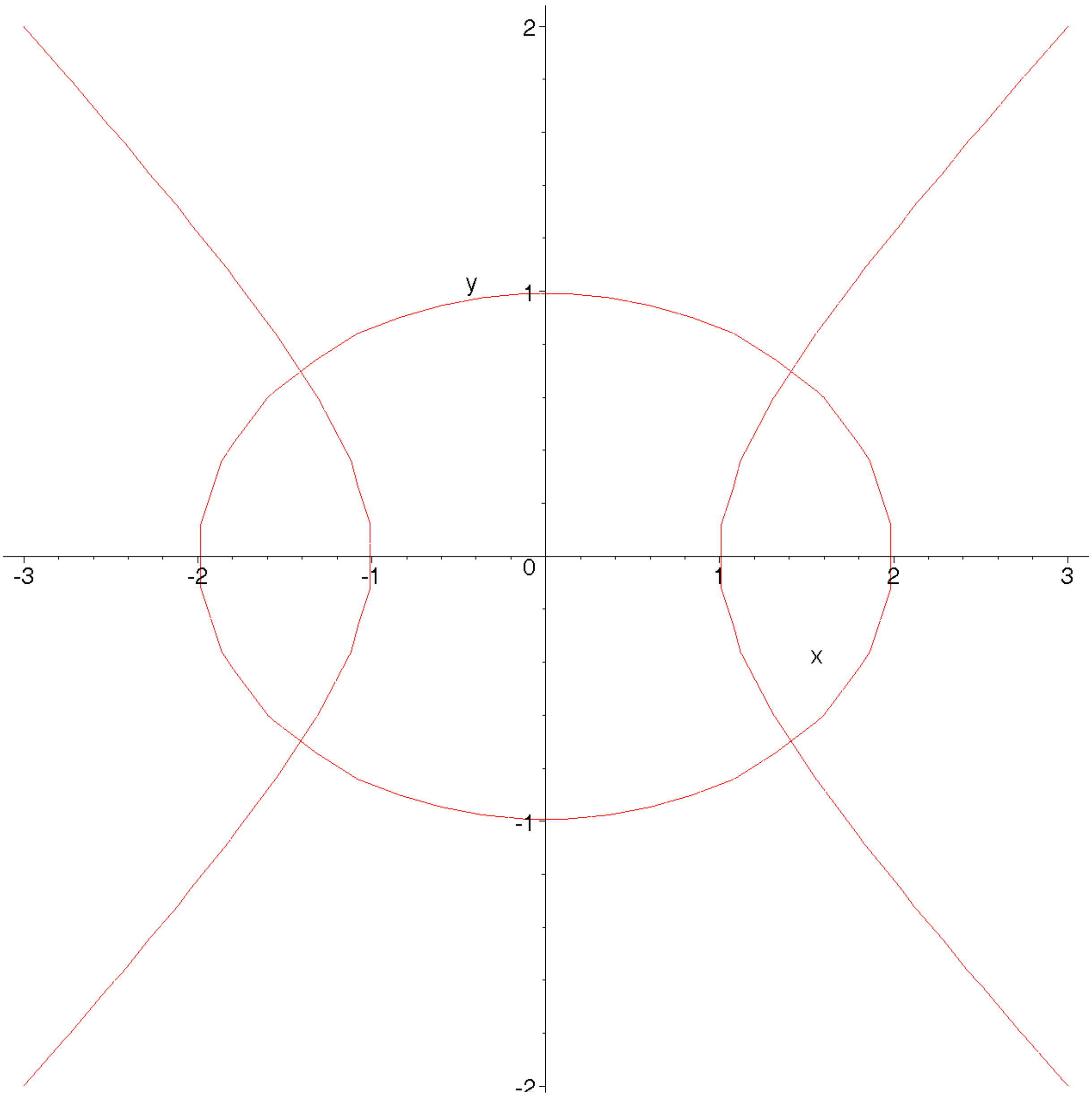
```
> with(plots):
  implicitplot({x^2+y^2=1,-x^2+2*y^2+1=0},x=-3..3,y=-3..3);
Warning, the name changecoords has been redefined
```




```
> solve({x^2/4+y^2=1,-x^2+2*y^2+1=0},{x,y});
```

$$\{y=\frac{\sqrt{2}}{2}, x=\sqrt{2}\}, \{y=\frac{\sqrt{2}}{2}, x=-\sqrt{2}\}, \{y=-\frac{\sqrt{2}}{2}, x=\sqrt{2}\}, \{y=-\frac{\sqrt{2}}{2}, x=-\sqrt{2}\}$$

```
> implicitplot({x^2/4+y^2=1,-x^2+2*y^2+1=0},x=-3..3,y=-3..3);
```



>

Calculus

Limits and Power Series Expansions

```
> limit((exp(x)-1)/x,x=0);
```

$$1$$

```
> series((exp(x)-1)/x,x=0);
```

$$1+\frac{1}{2}x+\frac{1}{6}x^2+\frac{1}{24}x^3+\frac{1}{120}x^4+O(x^5)$$

```
> series((exp(x)-1)/x,x=0,20);
```

$$1+\frac{1}{2}x+\frac{1}{6}x^2+\frac{1}{24}x^3+\frac{1}{120}x^4+\frac{1}{720}x^5+\frac{1}{5040}x^6+\frac{1}{40320}x^7+\frac{1}{362880}x^8+\frac{1}{3628800}x^9+$$

$$\frac{1}{39916800}x^{10}+\frac{1}{479001600}x^{11}+\frac{1}{6227020800}x^{12}+\frac{1}{87178291200}x^{13}+\frac{1}{1307674368000}$$

$$x^{14}+\frac{1}{20922789888000}x^{15}+\frac{1}{355687428096000}x^{16}+\frac{1}{6402373705728000}x^{17}+\frac{1}{121645100408832000}x^{18}+O(x^{19})$$

```
> read("FPS.mpl");
```

Package Formal Power Series, Maple V-8

Copyright 1995, Dominik Gruntz, University of Basel

Copyright 2002, Detlef Müller & Wolfram Koepf, University of Kassel

```
> FPS((exp(x)-1)/x,x=0);
```

$$\sum_{k=0}^{\infty}\frac{x^k}{(k+1)!}$$

Derivatives and Integrals

```
> diff((exp(x)-1)/x,x);
```

$$\frac{e^x}{x}-\frac{e^x-1}{x^2}$$

```
> derivative:=diff((exp(x)-1)/x,x$10);
```

$$derivative := \frac{e^x}{x}-\frac{10\,e^x}{x^2}+\frac{90\,e^x}{x^3}-\frac{720\,e^x}{x^4}+\frac{5040\,e^x}{x^5}-\frac{30240\,e^x}{x^6}+\frac{151200\,e^x}{x^7}-\frac{604800\,e^x}{x^8}+\frac{1814400\,e^x}{x^9}-\frac{3628800\,e^x}{x^{10}}+\frac{3628800\,(e^x-1)}{x^{11}}$$

```
> int(%,x);
```

$$\frac{362880}{x^{10}}+\frac{72\,e^x}{x^3}-\frac{9\,e^x}{x^2}+\frac{e^x}{x}-\frac{362880\,e^x}{x^{10}}+\frac{362880\,e^x}{x^9}-\frac{181440\,e^x}{x^8}+\frac{60480\,e^x}{x^7}-\frac{15120\,e^x}{x^6}+\frac{3024\,e^x}{x^5}-\frac{504\,e^x}{x^4}$$

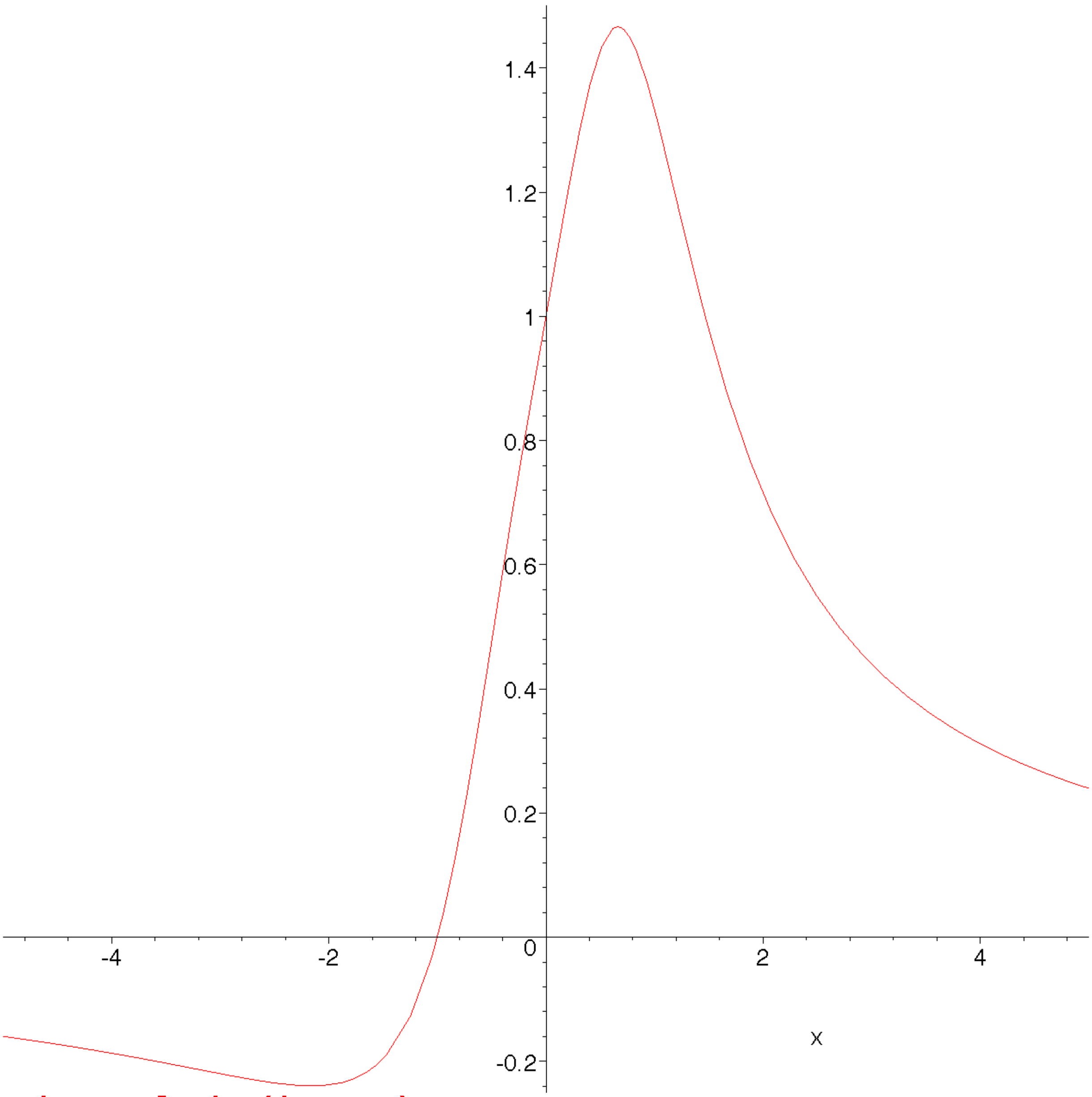
```
> ((y->int(y,x))@@10)(derivative);
```

$$-\frac{1}{x}+\frac{e^x}{x}$$

```
> input:=(x^3+x^2+x+1)/(x^4+x^2+1);
```

$$input := \frac{x^3+x^2+x+1}{x^4+x^2+1}$$

```
> plot(input,x=-5..5);
```




```
> integral:=int(input,x);
```

$$integral := \frac{1}{4}\ln(1+x^2+x)+\frac{1}{6}\sqrt{3}\arctan\left(\frac{(2x+1)\sqrt{3}}{3}\right)+\frac{1}{4}\ln(x^2-x+1)+\frac{1}{2}\sqrt{3}\arctan\left(\frac{(2x-1)\sqrt{3}}{3}\right)$$

```
> result:=diff(integral,x);
```

$$result := \frac{2x+1}{4(1+x^2+x)}+\frac{1}{3\left(1+\frac{(2x+1)^2}{3}\right)}+\frac{2x-1}{4(x^2-x+1)}+\frac{1}{1+\frac{(2x-1)^2}{3}}$$

```
>
```

Simplification

```
> normal(result);
```

$$\frac{x^3 + x^2 + x + 1}{(1 + x^2 + x)\,(x^2 - x + 1)}$$

```
> simplify(result);
```

$$\frac{x^3 + x^2 + x + 1}{(1 + x^2 + x)\,(x^2 - x + 1)}$$

```
> normal(result-input);
```

$$0$$

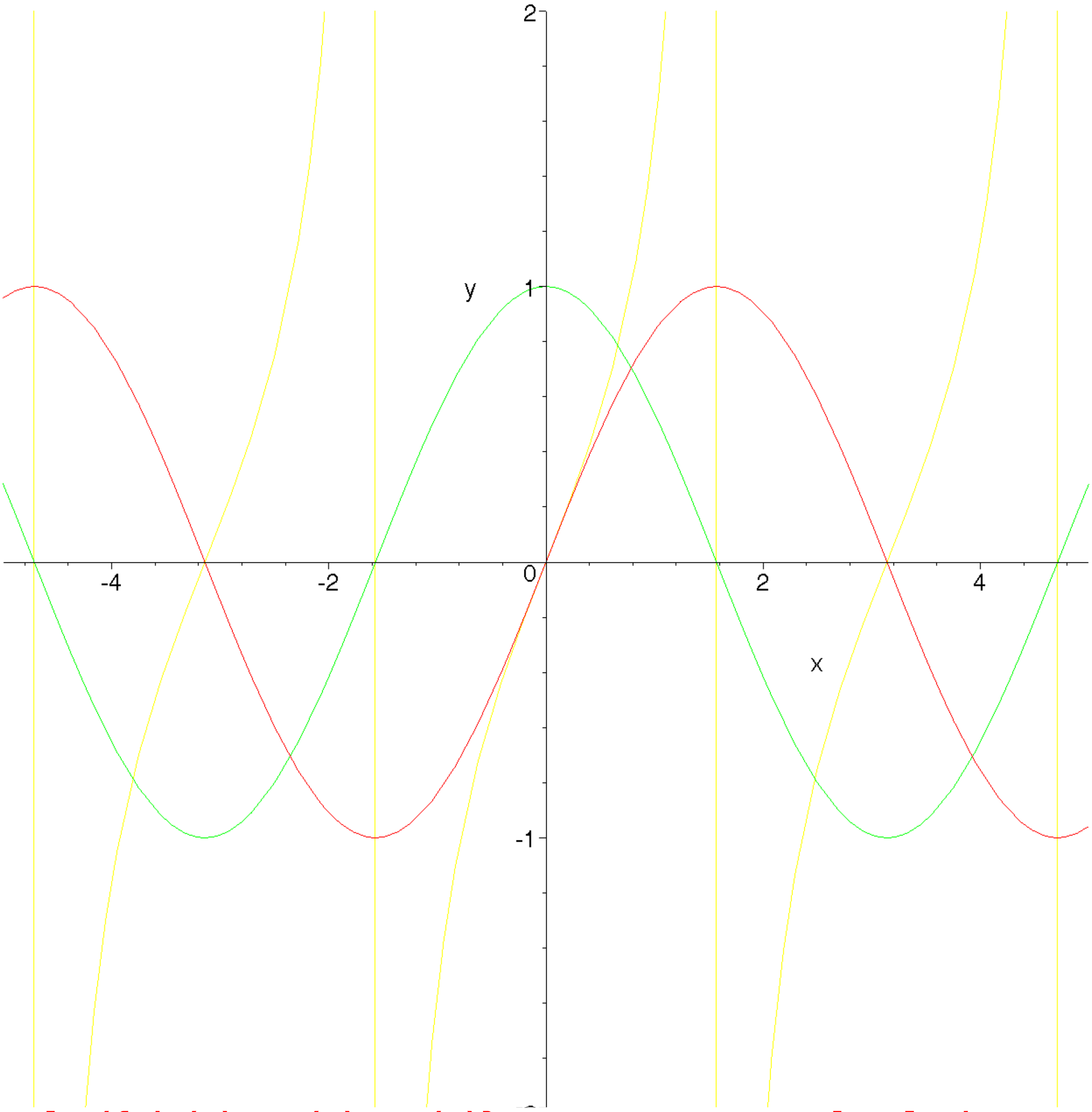
```
>
```

Computer Algebra in Education

Graphical Representations and Animations

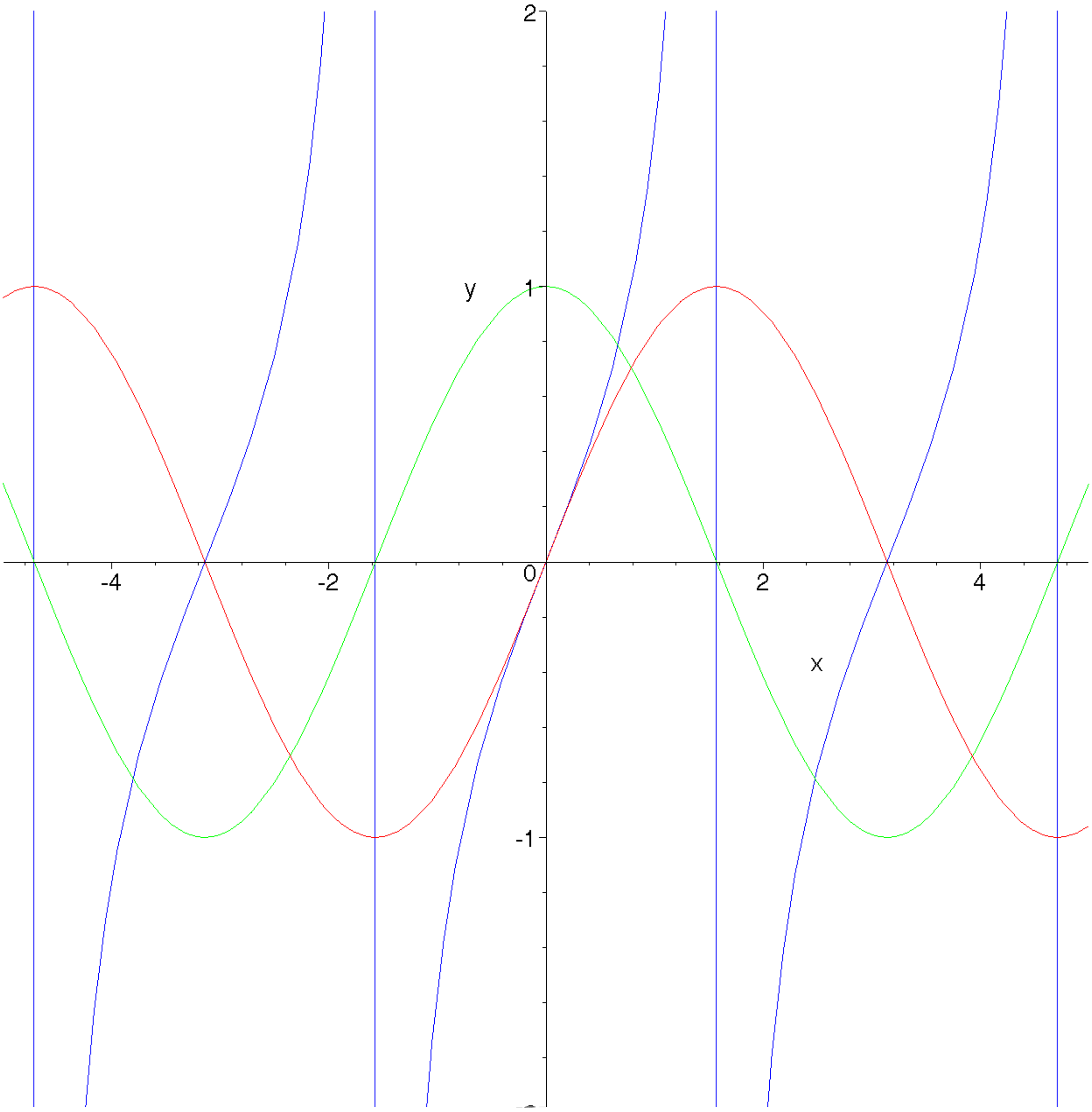
One can easily study plots of mathematical functions

```
> plot({sin(x),cos(x),tan(x)},x=-5..5,y=-2..2);
```




```
> plot({sin(x),cos(x),tan(x)},x=-5..5,y=-2..2,color=[red,green,
```

```
blue]);
```

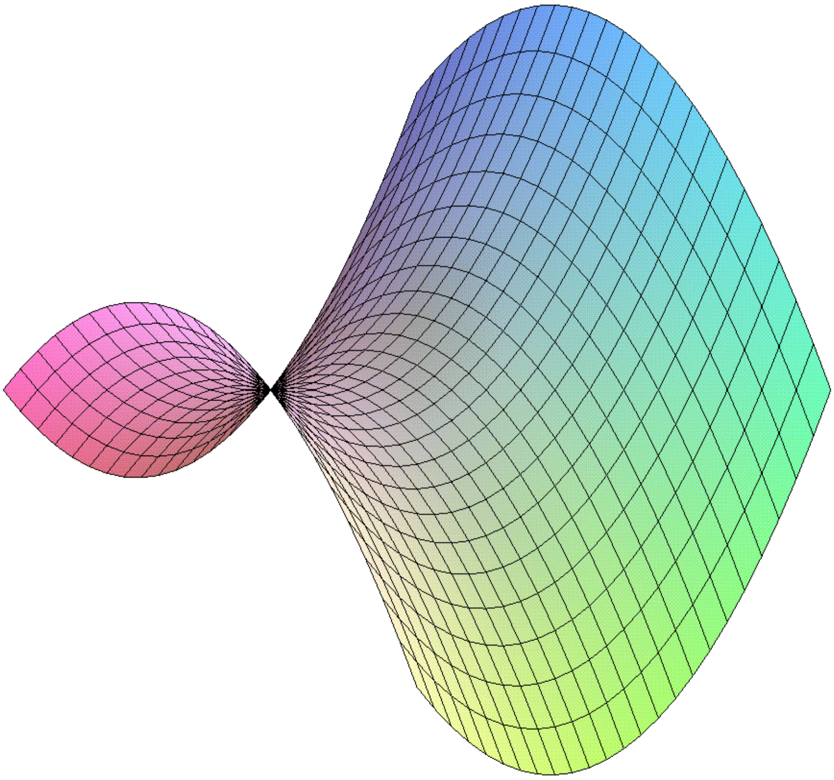


For plots there are many options available

```
> ?plot[options];
```

The following is a picture of a saddle point

```
> plot3d(x^2-y^2,x=-1..1,y=-1..1);
```

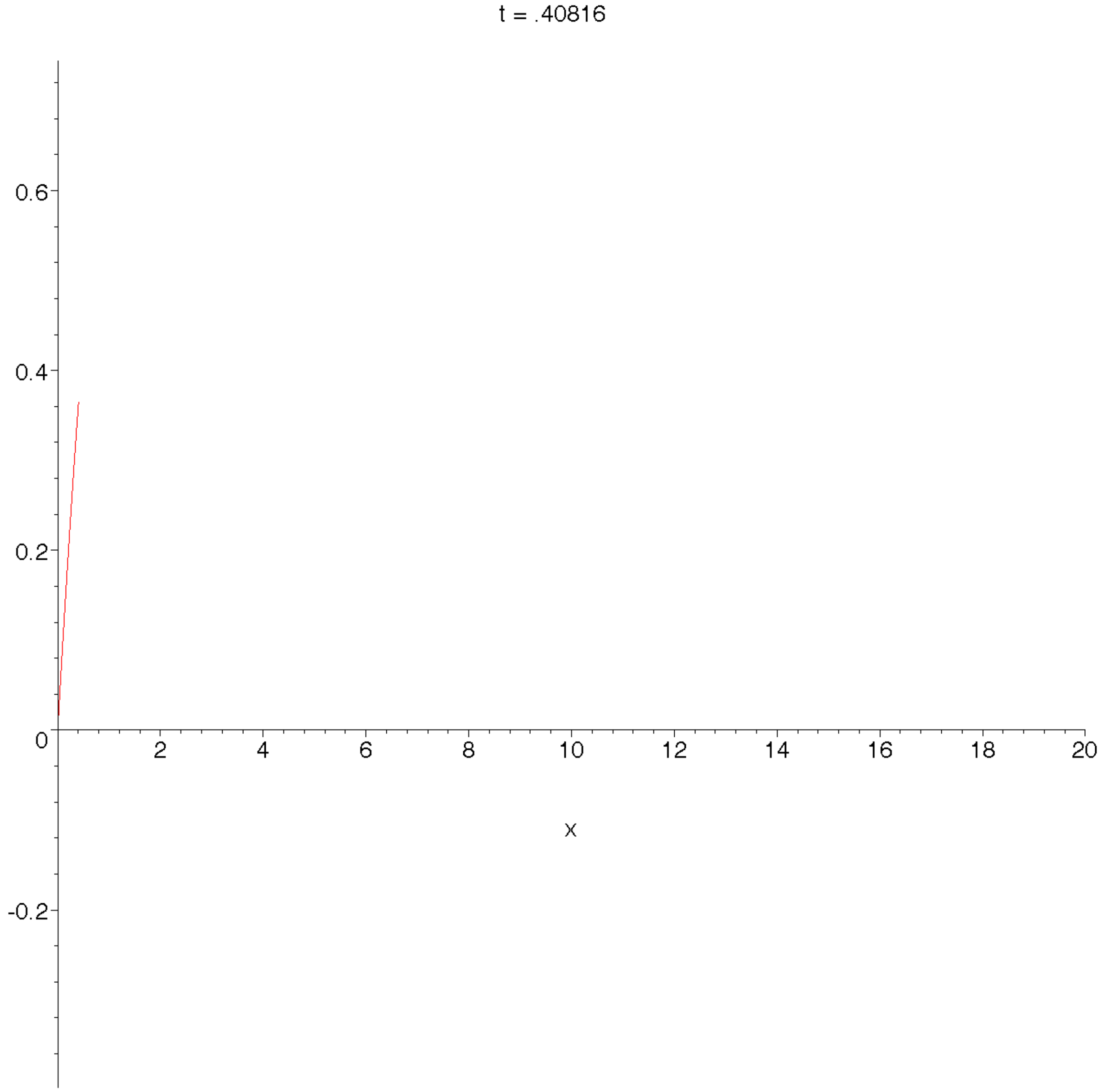


```
> with(plots);
```

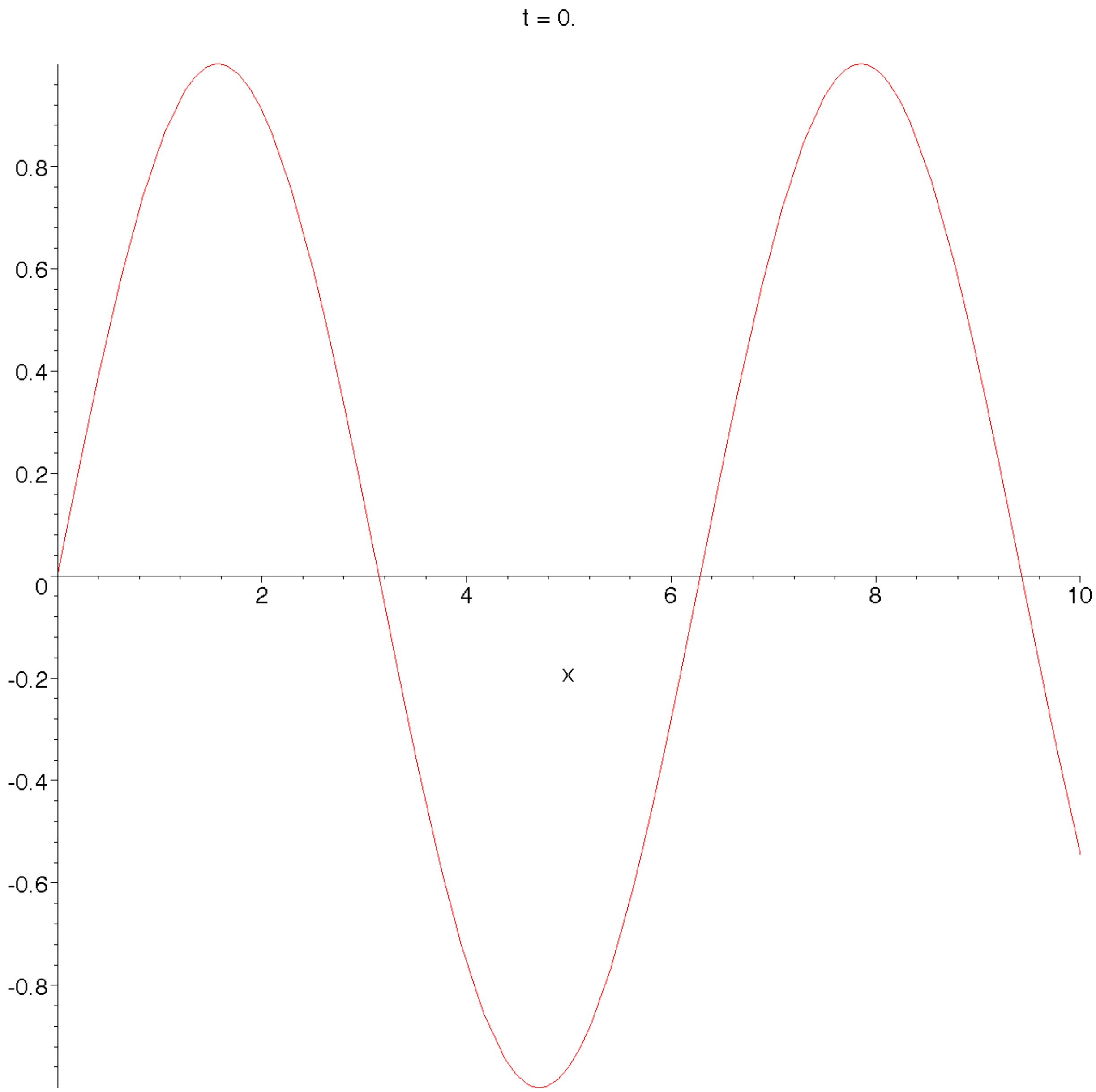
[*animate*, *animate3d*, *animatecurve*, *arrow*, *changecoords*, *complexplot*, *complexplot3d*, *conformal*, *conformal3d*, *contourplot*, *contourplot3d*, *coordplot*, *coordplot3d*, *cylinderplot*, *densityplot*, *display*, *display3d*, *fieldplot*, *fieldplot3d*, *gradplot*, *gradplot3d*, *graphplot3d*, *implicitplot*, *implicitplot3d*, *inequal*, *interactive*, *interactiveparams*, *listcontplot*, *listcontplot3d*, *listdensityplot*, *listplot*, *listplot3d*, *loglogplot*, *logplot*, *matrixplot*, *multiple*, *odeplot*, *pareto*, *plotcompare*, *pointplot*, *pointplot3d*, *polarplot*, *polygonplot*, *polygonplot3d*, *polyhedra_supported*, *polyhedraplot*, *replot*, *rootlocus*, *semilogplot*, *setoptions*, *setoptions3d*, *spacecurve*, *sparsematrixplot*, *sphereplot*, *surfdata*, *textplot*, *textplot3d*, *tubeplot*]

One can also animate graphs. Here a damped wave is animated.

```
> animate(plot,[sin(x)*exp(-x/5),x=0..t],t=0..20,frames=50 );
```




```
> animate(plot,[sin(x)*exp(-x*t),x=0..10],t=0..1, frames=50);
```



One can study the differential equation of the damped wave.

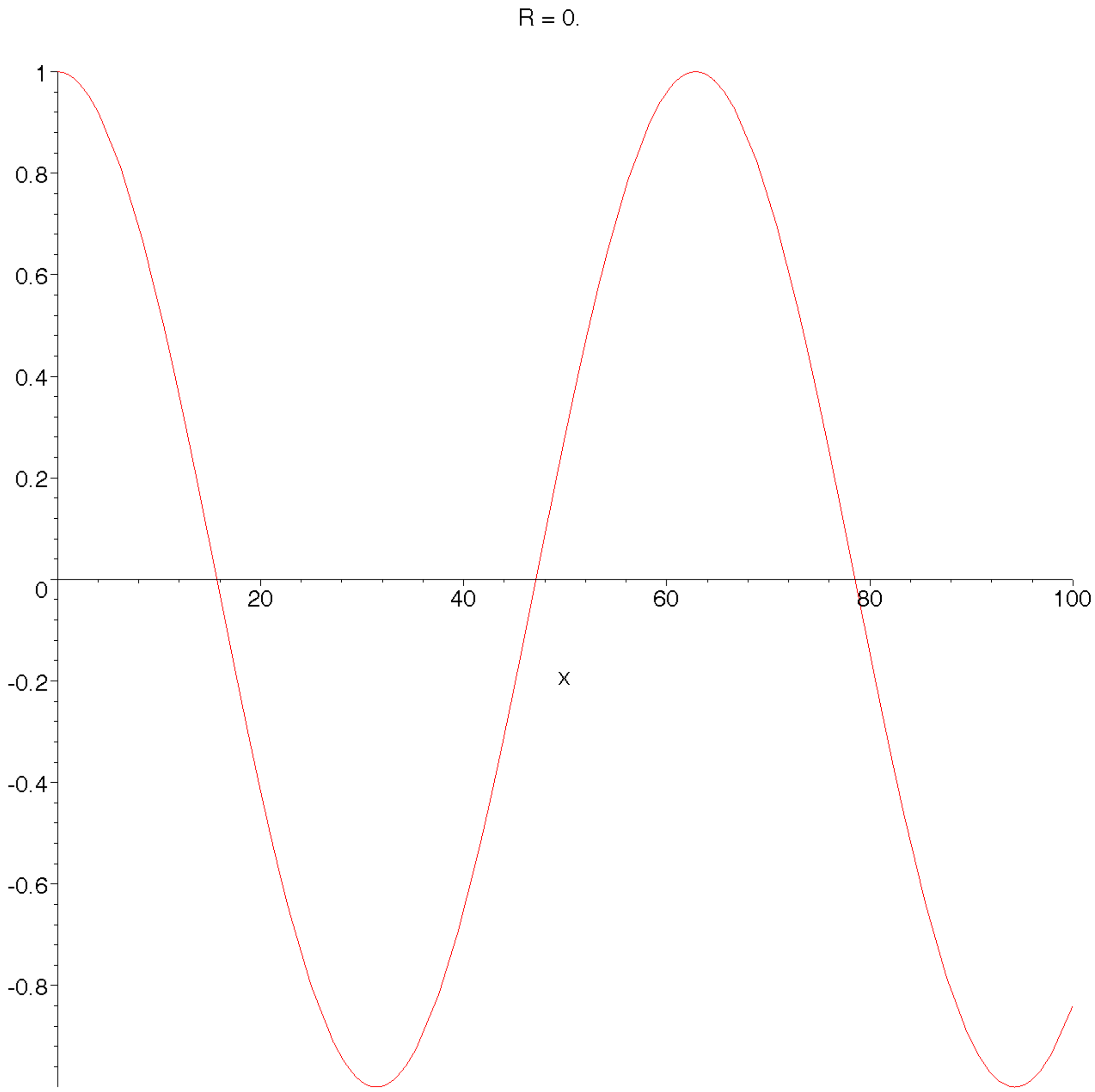
```
> sol:=dsolve(m*diff(y(x),x$2)+R*diff(y(x),x)+D*y(x)=0,y(x));
```

$$sol := y(x) = \_C1\, \mathbf{e}^{\left(-\frac{(R-\sqrt{R^2-4\,D\,m})\,x}{2\,m}\right)} + \_C2\, \mathbf{e}^{\left(-\frac{(R+\sqrt{R^2-4\,D\,m})\,x}{2\,m}\right)}$$

```
> f:=subs(_C1=1,_C2=0,subs(sol,y(x)));
```

$$f := \mathbf{e}^{\left(-\frac{(R-\sqrt{R^2-4\,D\,m})\,x}{2\,m}\right)}$$

```
> animate(plot,[subs(m=10,D=0.1,Re(f)),x=0..100],R=0..4,frames=
  50);
```

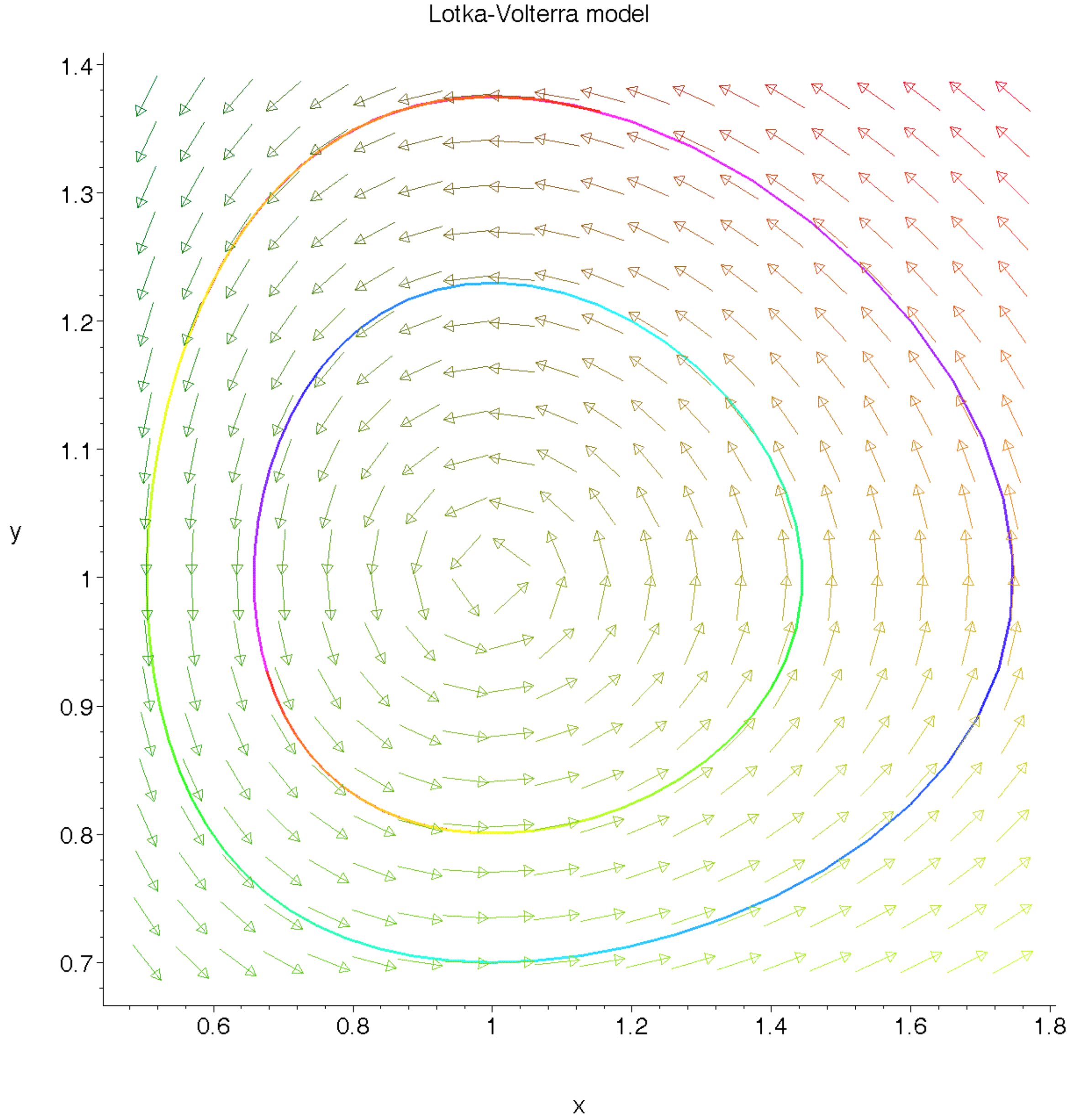



```
> with(ODETools);
```

[*DEnormal, DEplot, DEplot3d, DEplot_polygon, DFactor, DFactorLCLM, DFactorsols, Dchangevar, FunctionDecomposition, GCRD, LCLM, MeijerGsols, PDEchangecoords, RiemannPsols, Xchange, Xcommutator, Xgauge, Zeilberger, abelsol, adjoint, autonomous, bernoullisol, buildsol, buildsym, canoni, caseplot, casesplit, checkrank, chinisol, clairautsol, constcoeffsols, convertAlg, convertsys, dalembertsol, dcoeffs, de2diffop, dfieldplot, diff_table, diffop2de, dperiodic_sols, dpolyform, dsubs, eigenring, endomorphism_charpoly, equinv, eta_k, eulersols, exactsol, expsols, exterior_power, firint, firtest, formal_sol, gen_exp, generate_ic, genhomosol, gensys, hamilton_eqs, hypergeomsols, hyperode, indicialeq, infgen, initialdata, integrate_sols, intfactor, invariants, kovacicsols, leftdivision, liesol, line_int, linearsol, matrixDE, matrix_riccati, maxdimsystems, moser_reduce, muchange, mult, mutest, newton_polygon, normalG2, ode_int_y, ode_y1, odeadvisor, odepde, parametricsol, phaseportrait, poincare, polysols, power_equivalent, ratsols, redode, reduceOrder, reduce_order, regular_parts, regularsp,* *remove_RootOf*, *riccati_system*, *riccatisol*, *rifread*, *rifsimp*, *rightdivision*, *rtaylor*, *separablesol*, *singularities*, *solve_group*, *super_reduce*, *symgen*, *symmetric_power*, *symmetric_product*, *symtest*, *transinv*, *translate*, *untranslate*, *varparam*, *zoom*]

The Lotka-Volterra system cannot be solved by elementary functions. But we can plot numerically computed solutions. The following is the phase portrait.

```
> DEplot([diff(x(t),t)=x(t)*(1-y(t)),
  diff(y(t),t)=.3*y(t)*(x(t)-1)],[x(t),y(t)],
  t=-7..7,[[x(0)=1.2,y(0)=1.2],[x(0)=1,y(0)=.7]],
  stepsize=.2,title=`Lotka-Volterra model`,
  color=[.3*y(t)*(x(t)-1),x(t)*(1-y(t)),.1],
  linecolor=t/2,arrows=MEDIUM,method=rkf45);
```

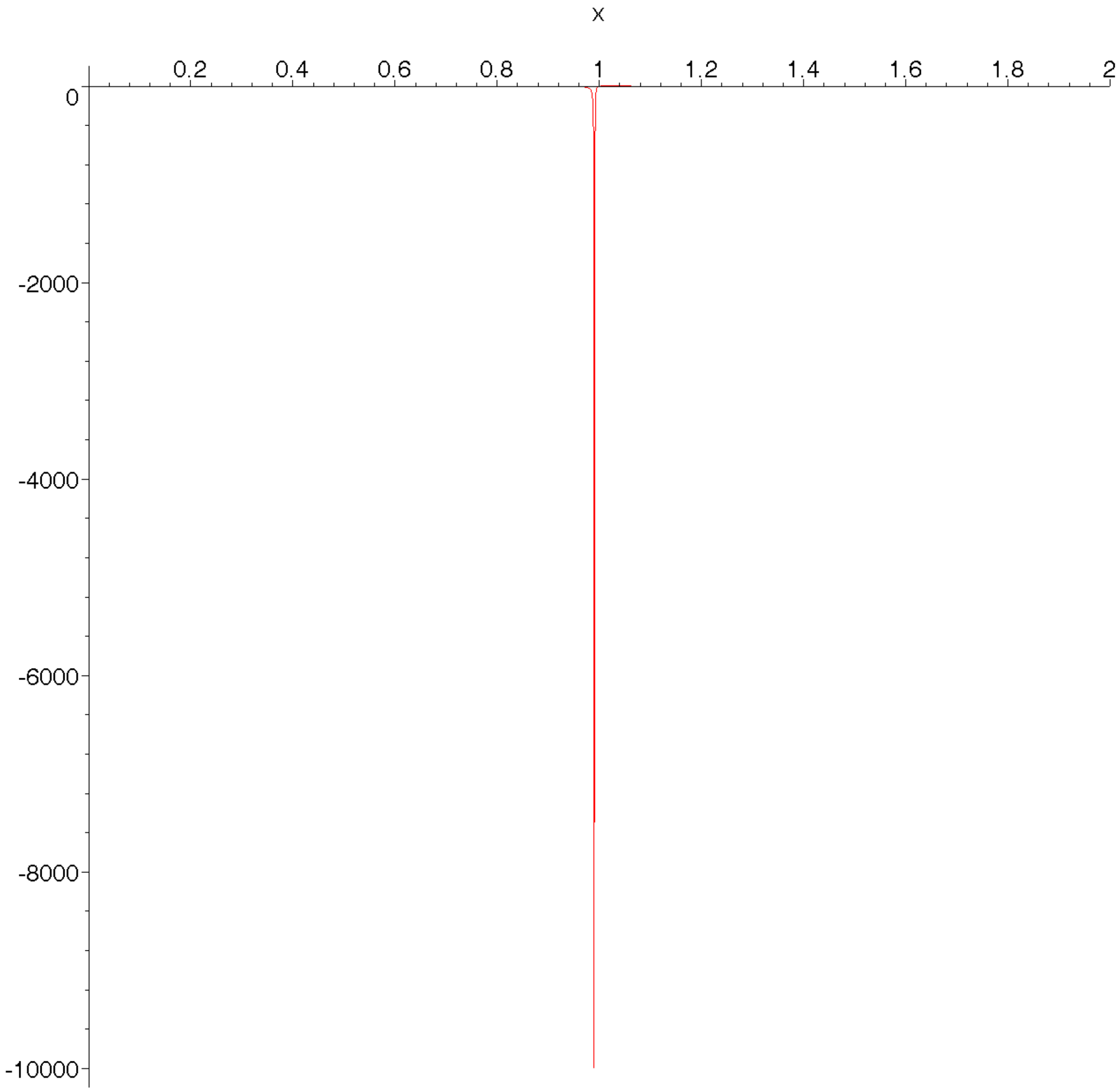


Where is the second pole?

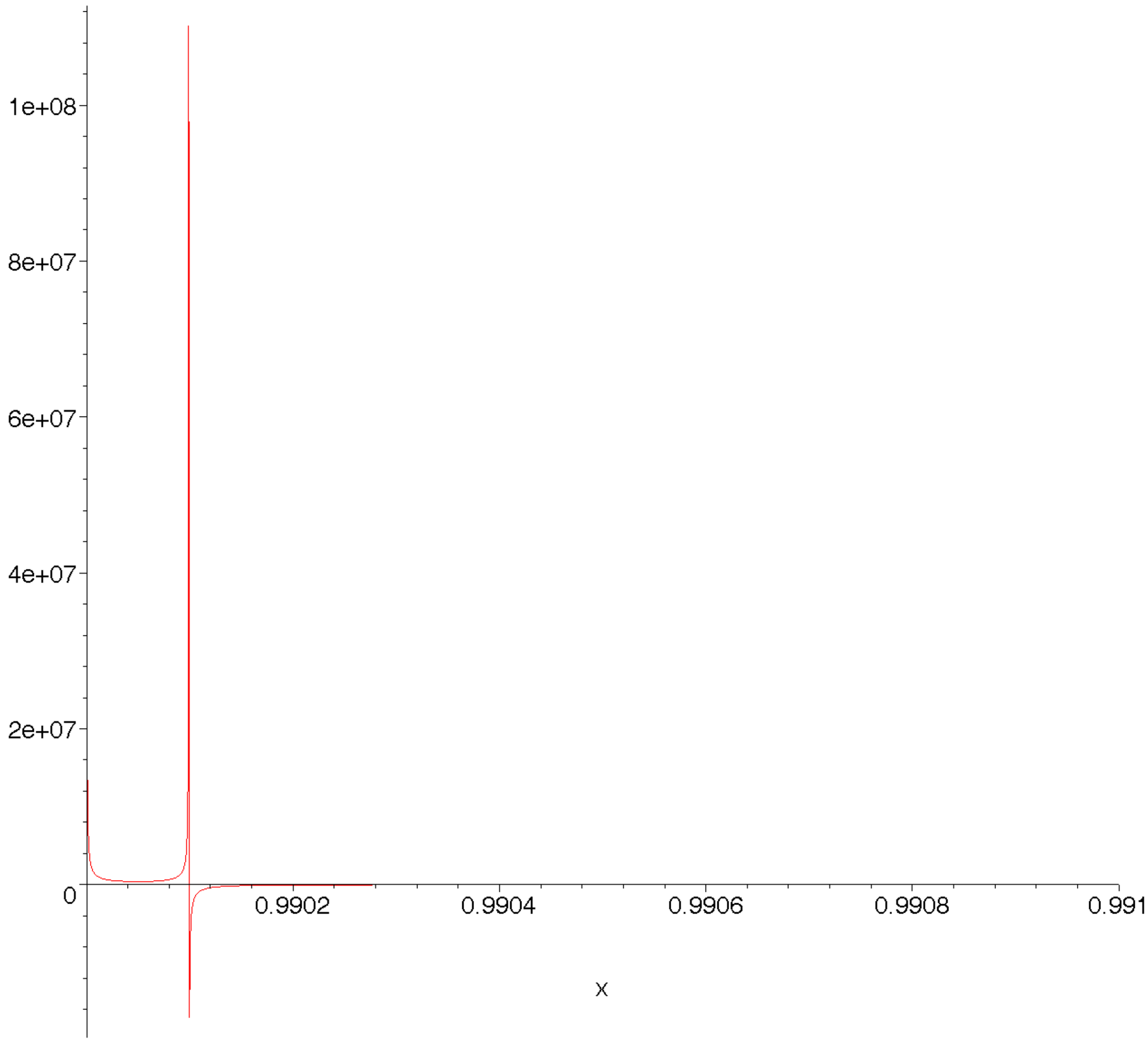
```
> f:=1000*(x-1)/((101*x-100)*(100*x-99));
```

$$f := \frac{1000\,(x-1)}{(101\,x-100)\,(100\,x-99)}$$

```
> plot(f,x=0..2);
```




```
> plot(f,x=0.99..0.991);
```




```
> sol:=[solve(diff(f,x),x)];
```

$$sol := \left[1 - \frac{\sqrt{101}}{1010},\ 1 + \frac{\sqrt{101}}{1010}\right]$$

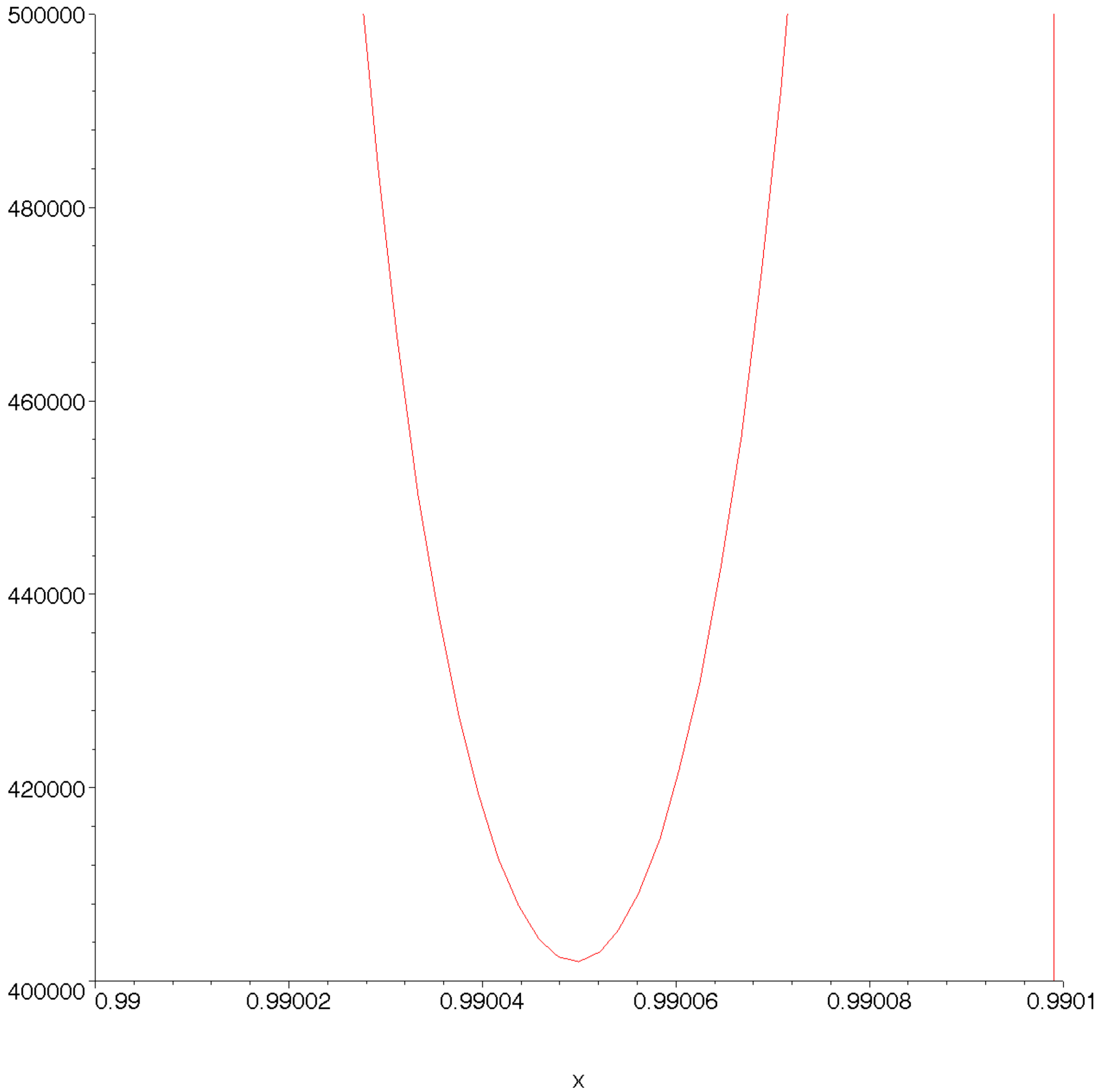
```
> value:=subs(x=op(1,sol),f);
```

$$value := -\frac{100\sqrt{101}}{101\left(1 - \frac{\sqrt{101}}{10}\right)\left(1 - \frac{10\sqrt{101}}{101}\right)}$$

```
> evalf(value);
```

$$401997.5125$$

```
> plot(f,x=0.99..0.9901,view=[0.99..0.9901, 400000..500000]);
```




```
>
```

Symbolic Computation

Study of linear equations

```
> eq1:=780*x+563*y=217;
```

$$eq1 := 780\,x + 563\,y = 217$$

```
> eq2:=913*x+659*y=254;
```

$$eq2 := 913\,x + 659\,y = 254$$

```
> solve({eq1,eq2},{x,y});
```

$$\{x = 1, y = -1\}$$

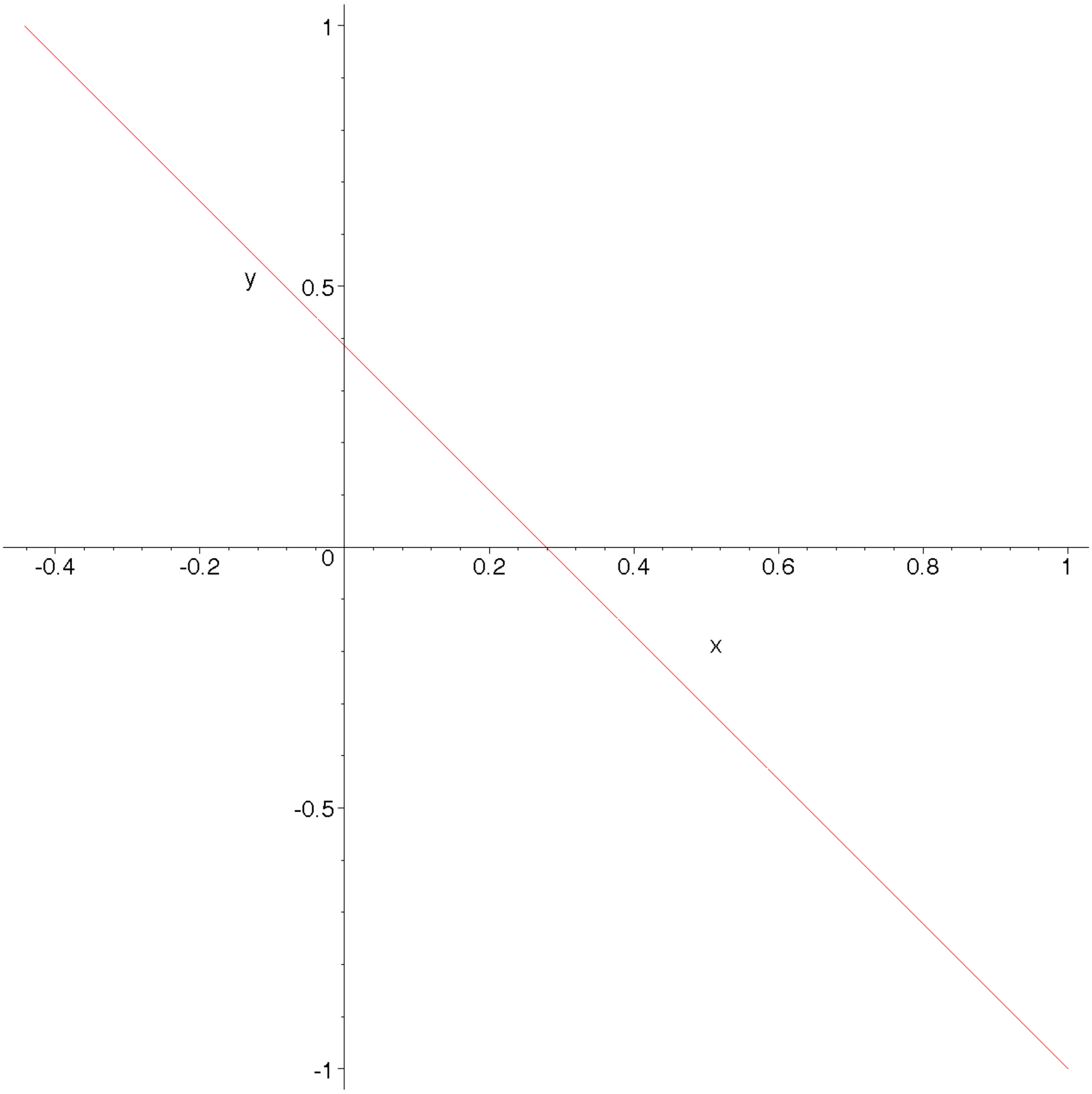
```
> eq1a:=781*x+563*y=217;
```

$$eq1a := 781\,x + 563\,y = 217$$

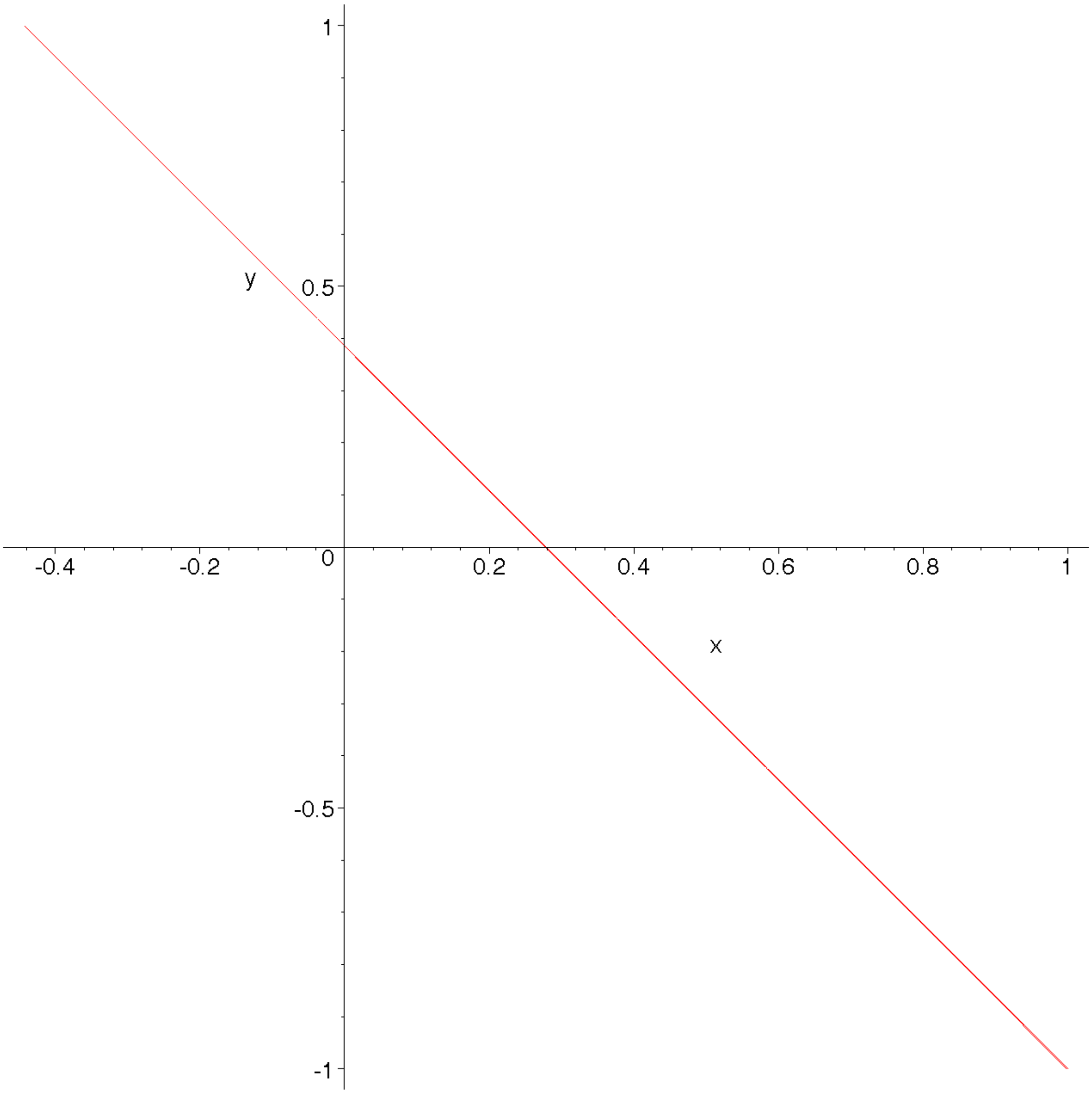
```
> solve({eq1a,eq2},{x,y});
```

$$\{x = \frac{1}{660}, y = \frac{23}{60}\}$$

```
> plots[implicitplot]([eq1,eq2],x=-1..1,y=-1..1);
```




```
> plots[implicitplot]([eq1a,eq2],x=-1..1,y=-1..1);
```



How can a teacher prepare good examples? We want to develop a 3x3 linear system with nice results. Therefore we need an integer 3x3-matrix with determinant +-1.

```
> with(LinearAlgebra);
```

[*&x, Add, Adjoint, BackwardSubstitute, BandMatrix, Basis, BezoutMatrix, BidiagonalForm, BilinearForm, CharacteristicMatrix, CharacteristicPolynomial, Column, ColumnDimension, ColumnOperation, ColumnSpace, CompanionMatrix, ConditionNumber, ConstantMatrix, ConstantVector, Copy, CreatePermutation, CrossProduct, DeleteColumn, DeleteRow, Determinant, Diagonal, DiagonalMatrix, Dimension, Dimensions, DotProduct, EigenConditionNumbers, Eigenvalues, Eigenvectors, Equal, ForwardSubstitute, FrobeniusForm, GaussianElimination, GenerateEquations, GenerateMatrix, GetResultDataType, GetResultShape, GivensRotationMatrix, GramSchmidt, HankelMatrix, HermiteForm, HermitianTranspose, HessenbergForm, HilbertMatrix, HouseholderMatrix, IdentityMatrix, IntersectionBasis, IsDefinite, IsOrthogonal, IsSimilar, IsUnitary, JordanBlockMatrix, JordanForm, LA_Main,* *LUDecomposition*, *LeastSquares*, *LinearSolve*, *Map*, *Map2*, *MatrixAdd*, *MatrixExponential*, *MatrixFunction*, *MatrixInverse*, *MatrixMatrixMultiply*, *MatrixNorm*, *MatrixPower*, *MatrixScalarMultiply*, *MatrixVectorMultiply*, *MinimalPolynomial*, *Minor*, *Modular*, *Multiply*, *NoUserValue*, *Norm*, *Normalize*, *NullSpace*, *OuterProductMatrix*, *Permanent*, *Pivot*, *PopovForm*, *QRDecomposition*, *RandomMatrix*, *RandomVector*, *Rank*, *RationalCanonicalForm*, *ReducedRowEchelonForm*, *Row*, *RowDimension*, *RowOperation*, *RowSpace*, *ScalarMatrix*, *ScalarMultiply*, *ScalarVector*, *SchurForm*, *SingularValues*, *SmithForm*, *SubMatrix*, *SubVector*, *SumBasis*, *SylvesterMatrix*, *ToeplitzMatrix*, *Trace*, *Transpose*, *TridiagonalForm*, *UnitVector*, *VandermondeMatrix*, *VectorAdd*, *VectorAngle*, *VectorMatrixMultiply*, *VectorNorm*, *VectorScalarMultiply*, *ZeroMatrix*, *ZeroVector*, *Zip*]

```
> M:=RandomMatrix(3,3,generator=-5..5);
```

$$M := \begin{bmatrix} 0 & -1 & 5 \\ 1 & 0 & 2 \\ 2 & -4 & 1 \end{bmatrix}$$

```
> while not(Determinant(M)=1 or Determinant(M)=-1) do
  M:=RandomMatrix(3,3,generator=-5..5): od:
> M;
```

$$\begin{bmatrix} -5 & 0 & 1 \\ -3 & 1 & -3 \\ 3 & 1 & -4 \end{bmatrix}$$

```
> Determinant(M);
```

$$-1$$

We solve the linear system given as M.x=<1,2,3> which now has an integer result.

```
> LinearSolve(M,<1,2,3>);
```

$$\begin{bmatrix} 2 \\ 41 \\ 11 \end{bmatrix}$$

If you need a matrix whose eigenvalues are integers, you can do as follows:

```
> Eigenvalues(M);
```

$$\left[\frac{(-1036+36\,I\sqrt{643})^{(1/3)}}{6}+\frac{62}{3\,(-1036+36\,I\sqrt{643})^{(1/3)}}-\frac{8}{3}\right]$$

$$\left[-\frac{(-1036+36\,I\sqrt{643})^{(1/3)}}{12}-\frac{31}{3\,(-1036+36\,I\sqrt{643})^{(1/3)}}-\frac{8}{3}\right.$$

$$\left.+\frac{1}{2}\,I\sqrt{3}\left(\frac{(-1036+36\,I\sqrt{643})^{(1/3)}}{6}-\frac{62}{3\,(-1036+36\,I\sqrt{643})^{(1/3)}}\right)\right]$$

$$\left[-\frac{(-1036+36\,I\sqrt{643})^{(1/3)}}{12}-\frac{31}{3\,(-1036+36\,I\sqrt{643})^{(1/3)}}-\frac{8}{3}\right.$$

$$-\frac{1}{2} I \sqrt{3} \left(\frac{(-1036 + 36 I \sqrt{643})^{(1/3)}}{6} - \frac{62}{3 (-1036 + 36 I \sqrt{643})^{(1/3)}} \right) \Bigg]$$

```
> f:=factor(CharacteristicPolynomial(M,t)):
  while not(type(f,`*`) and nops(f)=3) do
  M:=RandomMatrix(3,3,generator=-5..5):
  f:=factor(CharacteristicPolynomial(M,t)): od:
> M;
```

$$\begin{bmatrix} 1 & 1 & 1 \\ -1 & 5 & 1 \\ 1 & 1 & 5 \end{bmatrix}$$

```
> factor(CharacteristicPolynomial(M,t));
```

$$(t-1)(t-4)(t-6)$$

```
> Eigenvalues(M);
```

$$\begin{bmatrix} 1 \\ 4 \\ 6 \end{bmatrix}$$

```
> Eigenvectors(M);
```

$$\begin{bmatrix} 1 \\ 4 \\ 6 \end{bmatrix}, \begin{bmatrix} -3 & 0 & \frac{1}{3} \\ -1 & -1 & \frac{2}{3} \\ 1 & 1 & 1 \end{bmatrix}$$

With computer algebra, one generates Fortran code etc. for better performance.

```
> p:=randpoly(x,terms=8,degree=10);
```

$$p := -62 + 15 x^{10} - 49 x^7 - 37 x^6 - 62 x^5 - 53 x^4 - 27 x^3 - 90 x$$

```
> codegen[horner](p,x);
```

$$-62 + (-90 + (-27 + (-53 + (-62 + (-37 + (-49 + 15 x^3) x) x) x) x) x^2) x$$

Robertson's Conjecture (1989): Does the function sqrt((exp(x)-1)/x) have nonnegative Taylor coefficients?

```
> f:=sqrt((exp(x)-1)/x);
```

$$f := \sqrt{\frac{\mathbf{e}^x - 1}{x}}$$

```
> series(f,x=0);
```

$$1 + \frac{1}{4} x + \frac{5}{96} x^2 + \frac{1}{128} x^3 + \frac{79}{92160} x^4 + \frac{3}{40960} x^5 + \mathrm{O}(x^6)$$

```
> series(f,x=0,15);
```

$$1 + \frac{1}{4} x + \frac{5}{96} x^2 + \frac{1}{128} x^3 + \frac{79}{92160} x^4 + \frac{3}{40960} x^5 + \frac{71}{12386304} x^6 + \frac{113}{247726080} x^7 +$$
$$\frac{3053}{118908518400} x^8 + \frac{1}{22649241600} x^9 + \frac{17}{930128855040} x^{10} + \frac{19}{744103084032} x^{11} +$$
$$\frac{935917}{1218840851644416000} x^{12} - \frac{20287103}{4387827065919897600 0} x^{13} - \frac{2452337}{2106156991641550848 00} x^{14}$$

$+ O(x^{15})$

How is relativistic energy related to classical kinetic energy?

```
> E:=m*c^2*(1/sqrt(1-v^2/c^2)-1);
```

$$E := m\,c^2 \left(\frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} - 1 \right)$$

```
> series(E,v=0);
```

$$\frac{m}{2} v^2 + \frac{3\,m}{8\,c^2} v^4 + O(v^6)$$

```
> read("FPS.mpl");
```

Package Formal Power Series, Maple V-8

Copyright 1995, Dominik Gruntz, University of Basel

Copyright 2002, Detlef Müller & Wolfram Koepf, University of Kassel

```
> FPS(E,v);
```

$$\frac{1}{2} m \left(\sum_{k=0}^{\infty} \frac{4^{(-k)} \left(\frac{1}{c^2}\right)^k (1 + 2\,k)!\, v^{(2\,k+2)}}{(k!)^2 (k+1)} \right)$$

```
>
```

Programming

Pascal-like Programming Language

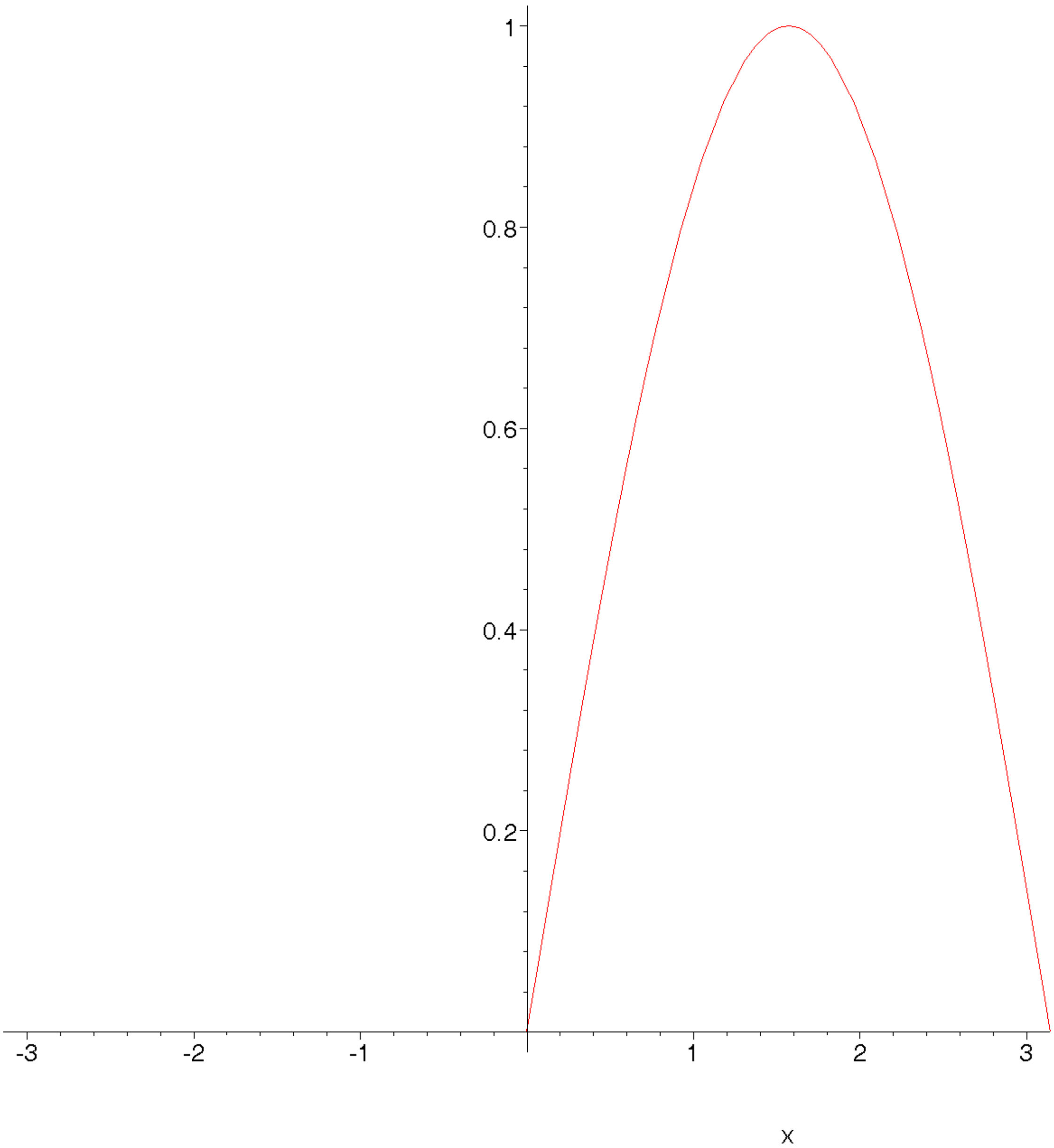
if then else construct

```
> SIN:=proc(x) if x>0 then sin(x) else 0 fi end:
```

The following is a typical example of the importance of the order of evaluation in symbolic computation

```
> plot(SIN(x),x=-Pi..Pi);
Error, (in SIN) cannot determine if this expression is true or false: 0 < x
```

```
> plot('SIN(x)',x=-Pi..Pi);
```



for and while loops: Computation of factorial

```
> x:=1: for i from 1 to 100 do x:=x*i od: x;
```

93326215443944152681699238856266700490715968264381621468592963895217599\
93229915608941463976156518286253697920827223758251185210916864000000000\
00000000000000

```
> i:=1: x:=1: while i<=100 do x:=x*i; i:=i+1; od: x;
```

93326215443944152681699238856266700490715968264381621468592963895217599\
93229915608941463976156518286253697920827223758251185210916864000000000\
00000000000000

However, in many instances there are high level constructs available:

```
> 100!;
```

93326215443944152681699238856266700490715968264381621468592963895217599\
93229915608941463976156518286253697920827223758251185210916864000000000\
00000000000000

```
> product(i,i=1..100);
```

```
Error, (in product) product variable previously assigned, second argument
evaluates to 101 = 1 .. 100
```

```
> i;
```

101

```
> product('i','i'=1..100);
```

933262154439441526816992388562667004907159682643816214685929638952175999\
932299156089414639761565182862536979208272237582511852109168640000000000\
00000000000000

```
> i:='i':
> product(i,i=1..100);
```

933262154439441526816992388562667004907159682643816214685929638952175999\
932299156089414639761565182862536979208272237582511852109168640000000000\
00000000000000

```
> x:='x':
>
```

Recursive Program

Euclidean Algorithm

```
> GCD:=(a,b)->
  if a<0 or b<0 then GCD(abs(a),abs(b))
  elif a<b then GCD(b,a)
  elif b=0 then a
  else GCD(b,modp(a,b))
  fi:
> GCD(-1234567,1604137);
```

127

```
> random := rand(1..10^200):
> a:=12373*random();
```

a := 35269285936377113445342725515168872687976312480989398462616357471086\
961688157373498241375310115402737938876769250918428519447102181406952039\
5401669230831034965416121503816308636972976281347612381619139542

```
> b:=12373*random();
```

b := 51499453913773444787532198436184161342318326847823103595550086177541\
958805365775332917024841213712242090258421739642912095464285750686042032\
1055737588392312677488362064468634303823699947472415725455593624

```
> GCD(a,b);
```

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Modular Powers

```
> powermod:=proc(a,n,p)
  option remember;
  if n=0 then 1
  elif type(n,even) then modp(powermod(a,n/2,p)^2,p)
  else modp(powermod(a,n-1,p)*a,p)
  fi: end:
```

```
> powermod(a,b,b);
```

5059402303946780531336442023515064794186381613623459970915234889302170024781438071451931831699402647943852999170823716881427902922882677470111283569234524995319292599769705222066630642705131165728449010664

```
> powermod(b,a,a);
```

409827047515173443647903039112479341137117669082706157085830026976182555351299319967301449347557644339313782347688650177910408196172032146599892051491929204131363168974579673686539171393015238829945366566

Fermat Test

```
> a;
```

352692859363771134453427255151688726879763124809893984626163574710869616881573734982413753101154027379388767692509184285194471021814069520395401669230831034965416121503816308636972976281347612381619139542

```
> powermod(2,a,a);
```

278522674153249474533268514803495577296451087048523875124656086795665999813584480960190543551873997589223739380066763635467244829578312478192793506206153453820375470726244612531915740195107505727613874656

```
> isprime(a);
```

false

```
> powermod(2,b,b);
```

285636536411887671816010506336597473672134020592679589317131667931699864157019580313821096958198962706949851882612716109268753106259704601079803748754629245144313385786636554077117061197521394101181699248

```
> isprime(b);
```

false

```
> ifactor(a);
Warning, computation interrupted
```

RSA Cryptosystem

auxiliary functions

```
> rest:=proc(liste)
  local k;
  [seq(op(k,liste),k=2..nops(liste))] end:
  comb:=proc(liste)
  if nops(liste)=1 then op(1,liste) else
  1000*comb(ListTools[Reverse]
  (rest(ListTools[Reverse](liste)))) + op(nops(liste),liste) fi
  end:
  with(StringTools):
  Convert:=proc(str)
  comb(map(Ord,Explode(str)))
  end:
  makelist:=proc(zahl)
```

```
if zahl<1000 then [zahl]
else
[op(makelist((zahl-modp(zahl,1000))/1000)),modp(zahl,1000)]
fi end:
Convertback:=proc(zahl)
local ch;
ch:=map(Char,makelist(zahl));
cat(seq(op(k,ch),k=1..nops(ch))) end:
Warning, the name Map has been rebound

Warning, the assigned names Fibonacci and Group now have a global binding
```

```
> EncryptRSA:=proc(nachricht) powermod(Convert(nachricht),e,n)
end:
DecryptRSA:=proc(zahl) Convertback(powermod(zahl,d,n)) end:
> InitializeRSA:=proc()
global p,q,n,e,d; local
phi;
p:=nextprime(random());
q:=nextprime(random());
n:=p*q;
phi:=(p-1)*(q-1);
e:=phi;
while not(gcd(e,phi)=1) do e:=nextprime(random()) od;
d:=e^(-1) mod phi;
NULL end:
```

Computation of keys

```
> InitializeRSA();
```

Public key

```
> [e,n];
[16126391227292452904905099376279845111164477391290912508411583520888076\
    91907921730700859081233261369697357149339103627259443480781180255667638 2\
    451516263456290070045725511636840978525925125707537058373, 549134602933466
    60207680036301255043046024227817967229991417776114623478011065169783993 0\
    91275093859057918620318412219470706118859850433882537290523487472771552 4\
    38871300710017619109340067670582784784813434716330821899733615550546072 1\
    13303916308566999661647078833089149229833927029780563970359714117430655 5\
    85199739080539197848840242553563809107444234556142208104854278011469192 2\
    1045240967450804820310277]
```

Private key

```
> d;
25015254598381365733305922623250442096387851795326360877711857545034743 4\
    33862337434526514121674408009735823797455857019679134490267953533679711 6\
    28183544019146061725122244192447372886901625026550756256347800928447395 2\
    26083949436666151897154154364777293388762011554622901768168473393965532 0\
```

944042736859841897921395275055103504507801103945428586990495514575445570\
640456695238907332282460866286893198167 7

encryption by RSA

```
> message:="This is my message";
```

message := "This is my message"

```
> result:=EncryptRSA(message);
```

result := 347894773334326272975921220876802375344249205275425869702400230 74\
716206193179870778281924801695850736945017077650381407980991087380531044\
720384606002472708354865083898134404197099986728980232526301986413477190\
798264584489376647321167673413250238006491692678438176584363373316593347\
117266593238701330638087128015501275845731635071193703669275191028013214\
212875706127730975550295385607000425408279788 12

```
> DecryptRSA(result);
```

"This is my message"

```
> DecryptRSA(result+1);
Error, (in StringTools:-Char) integer argument is too large
```

a few other messages

```
> result:=EncryptRSA("What about this message?");
```

result := 486197740811195219831478649809627467648116221043055644351348234 67\
346336483222931569295158592655156404921296020941027557270255463217293217\
404109428791615340973056862679100539159382947061007343387728270314761510\
294545910333806331530259686616481078501476124692114348250209294577780950\
682051119556854920483673443964682667407681495059835869124941408742845785\
462487066498385855850733812660219764800461514 88

```
> DecryptRSA(result);
```

"What about this message?"

```
> result:=EncryptRSA("");
Error, (in ListTools:-Reverse) too many levels of recursion
```

```
> DecryptRSA(result);
```

"What about this message?"

Finally a really secret message

```
> read("message.mpl");
> result:=EncryptRSA(message);
```

result := 219217978349561922429598335077266026155217179631965543122759478 77\
578237298889500460333340959975197554059240752049400425559516193638112950\
045703466167185620881161668602016074360843372332539181869432028933937509\
117364433178103559482459789696737031606014981162380425240128712150353960\
245131676993078084992990384019264945354748051380608402453988591948438103\
945278321257333470130175074349823265423027649 61

```
> DecryptRSA(result);
```

"This is my last message. Thank you very much for your attention!"

>