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Computer Algebra Methods for Orthogonal Polynomials

Maple Worksheet

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> restart;
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Computing the Recurrence Coefficients

We consider the three highest coefficients of the orthogonal polynomial:

```
> p:=k[n]*x^n+kprime[n]*x^(n-1)+kprimeprime[n]*x^(n-2);
```

$$p := k_n\,x^n + kprime_n\,x^{(n-1)} + kprimeprime_n\,x^{(n-2)}$$

We define the polynomials σ and τ with arbitrary coefficients a,b,c,d,e:

```
> sigma:=a*x^2+b*x+c;
  tau:=d*x+e;
```

$$\sigma := a\,x^2 + b\,x + c$$

$$\tau := d\,x + e$$

The polynomial satisfies the differential equation DE=0 with:

```
> DE:=sigma*diff(p,x$2)+tau*diff(p,x)+lambda[n]*p;
```

$$DE := (a\,x^2 + b\,x + c)\left(\frac{k_n\,x^n\,n^2}{x^2} - \frac{k_n\,x^n\,n}{x^2} + \frac{kprime_n\,x^{(n-1)}\,(n-1)^2}{x^2} - \frac{kprime_n\,x^{(n-1)}\,(n-1)}{x^2} + \frac{kprimeprime_n\,x^{(n-2)}\,(n-2)^2}{x^2} - \frac{kprimeprime_n\,x^{(n-2)}\,(n-2)}{x^2}\right) + (d\,x + e)\left(\frac{k_n\,x^n\,n}{x} + \frac{kprime_n\,x^{(n-1)}\,(n-1)}{x} + \frac{kprimeprime_n\,x^{(n-2)}\,(n-2)}{x}\right) + \lambda_n\,(k_n\,x^n + kprime_n\,x^{(n-1)} + kprimeprime_n\,x^{(n-2)})$$

We collect coefficients:

```
> de:=collect(simplify(DE/x^(n-4)),x);
```

$$de := (-a\,k_n\,n + \lambda_n\,k_n + a\,k_n\,n^2 + d\,k_n\,n)\,x^4 + (\lambda_n\,kprime_n - 3\,a\,kprime_n\,n + 2\,a\,kprime_n + b\,k_n\,n^2 + a\,kprime_n\,n^2 - b\,k_n\,n + e\,k_n\,n + d\,kprime_n\,n - d\,kprime_n)\,x^3 + (-e\,kprime_n + 6\,a\,kprimeprime_n + c\,k_n\,n^2 - c\,k_n\,n + 2\,b\,kprime_n + \lambda_n\,kprimeprime_n + a\,kprimeprime_n\,n^2 - 5\,a\,kprimeprime_n\,n + e\,kprime_n\,n + d\,kprimeprime_n\,n + b\,kprime_n\,n^2 - 3\,b\,kprime_n\,n - 2\,d\,kprimeprime_n)\,x^2 + (-3\,c\,kprime_n\,n + 2\,c\,kprime_n - 2\,e\,kprimeprime_n + b\,kprimeprime_n\,n^2 + 6\,b\,kprimeprime_n + c\,kprime_n\,n^2 - 5\,b\,kprimeprime_n\,n + e\,kprimeprime_n\,n)\,x - 5\,c\,kprime_n\,n + 6\,c\,kprimeprime_n$$

$+ c\, kprimeprime_n\, n^2$

Equating the highest coefficient gives the already mentioned identity for λ:

```
> rule1:=lambda[n]=solve(coeff(de,x,4),lambda[n]);
```

$$rule1 := \lambda_n = -n\,(-a + a\,n + d)$$

This can be substituted:

```
> de:=expand(subs(rule1,de));
```

$$de := -x^2\, e\, kprime_n - 2\, x\, e\, kprimeprime_n + 2\, x^3\, a\, kprime_n + 6\, x^2\, a\, kprimeprime_n + 2\, x^2\, b\, kprime_n + 6\, x\, b\, kprimeprime_n + 2\, x\, c\, kprime_n + 6\, c\, kprimeprime_n - x^3\, d\, kprime_n - 2\, x^2\, d\, kprimeprime_n - 2\, x^3\, a\, kprime_n\, n - 4\, x^2\, a\, kprimeprime_n\, n + x^3\, b\, k_n\, n^2 - x^3\, b\, k_n\, n + x^2\, b\, kprime_n\, n^2 - 3\, x^2\, b\, kprime_n\, n + x\, b\, kprimeprime_n\, n^2 - 5\, x\, b\, kprimeprime_n\, n + x^2\, c\, k_n\, n^2 - x^2\, c\, k_n\, n + x\, c\, kprime_n\, n^2 - 3\, x\, c\, kprime_n\, n + c\, kprimeprime_n\, n^2 - 5\, c\, kprimeprime_n\, n + x^3\, e\, k_n\, n + x^2\, e\, kprime_n\, n + x\, e\, kprimeprime_n\, n$$

Equating the second highest coefficient gives k'[n] as rational multiple of k[n]:

```
> rule2:=kprime[n]=solve(coeff(de,x,3),kprime[n]);
```

$$rule2 := kprime_n = \frac{k_n\, n\,(e + b\,n - b)}{-2\,a + 2\,a\,n + d}$$

Equating the third highest coefficient gives k''[n] as rational multiple of k[n]:

```
> rule3:=kprimeprime[n]=solve(coeff(subs(rule2,de),x,2),kprimeprime[n]);
```

$$rule3 := kprimeprime_n = \frac{1}{2}\, k_n\, n\,(2\,c\,a - 4\,c\,a\,n - c\,d - e^2 - 5\,e\,b\,n + 3\,e\,b + 5\,b^2\,n - 2\,b^2 + 2\,c\,n^2\,a + c\,n\,d - 4\,b^2\,n^2 + e^2\,n + 2\,e\,n^2\,b + b^2\,n^3)\,/\,((-2\,a + 2\,a\,n + d)\,(2\,a\,n - 3\,a + d))$$

Without loss of generality we consider the monic case, hence

```
> k[n]:=1;
```

$$k_n := 1$$

and therefore

```
> rule2;
```

$$kprime_n = \frac{n\,(e + b\,n - b)}{-2\,a + 2\,a\,n + d}$$

```
> rule3;
```

$$kprimeprime_n = n\,(2\,c\,a - 4\,c\,a\,n - c\,d - e^2 - 5\,e\,b\,n + 3\,e\,b + 5\,b^2\,n - 2\,b^2 + 2\,c\,n^2\,a + c\,n\,d - 4\,b^2\,n^2 + e^2\,n + 2\,e\,n^2\,b + b^2\,n^3)\,/\,(2\,(-2\,a + 2\,a\,n + d)\,(2\,a\,n - 3\,a + d))$$

We would like to compute the coefficients a(n), b(n) and c(n) in the recurrence equation RE=0:

```
> RE:=x*P(n)-(a[n]*P(n+1)+b[n]*P(n)+c[n]*P(n-1));
```

$$RE := x\,P(n) - a_n\,P(n+1) - b_n\,P(n) - c_n\,P(n-1)$$

```
> RE:=subs({P(n)=p,P(n+1)=subs(n=n+1,p),P(n-1)=subs(n=n-1,p)},RE);
```

$$RE := x\,(x^n + kprime_n\,x^{(n-1)} + kprimeprime_n\,x^{(n-2)}) - a_n\,(x^{(n+1)} + kprime_{n+1}\,x^n + kprimeprime_{n+1}\,x^{(n-1)}) - b_n\,(x^n + kprime_n\,x^{(n-1)} + kprimeprime_n\,x^{(n-2)}) - c_n\,(x^{(n-1)} + kprime_{n-1}\,x^{(n-2)} + kprimeprime_{n-1}\,x^{(n-3)})$$

We substitute the already known formulas:

```
> RE:=subs({rule2,subs(n=n+1,rule2),subs(n=n-1,rule2),rule3,subs(n=n+1,rule3),subs(n=n-1,rule3)},RE);
```

$$RE := x\left(x^n + \frac{n\,(e+b\,n-b)\,x^{(n-1)}}{-2\,a+2\,a\,n+d} + n\,(2\,c\,a-4\,c\,a\,n-c\,d-e^2-5\,e\,b\,n+3\,e\,b + 5\,b^2\,n-2\,b^2+2\,c\,n^2\,a+c\,n\,d-4\,b^2\,n^2+e^2\,n+2\,e\,n^2\,b+b^2\,n^3)\,x^{(n-2)} / (2\,(-2\,a+2\,a\,n+d)\,(2\,a\,n-3\,a+d))\right) - a_n\left(x^{(n+1)} + \frac{(n+1)\,(e+b\,(n+1)-b)\,x^n}{-2\,a+2\,a\,(n+1)+d} + (n+1)\,(2\,c\,a-4\,c\,a\,(n+1)-c\,d-e^2-5\,e\,b\,(n+1)+3\,e\,b+5\,b^2\,(n+1)-2\,b^2 + 2\,c\,(n+1)^2\,a+c\,(n+1)\,d-4\,b^2\,(n+1)^2+e^2\,(n+1)+2\,e\,(n+1)^2\,b+b^2\,(n+1)^3)\,x^{(n-1)} / (2\,(-2\,a+2\,a\,(n+1)+d)\,(2\,a\,(n+1)-3\,a+d))\right) - b_n\left(x^n + \frac{n\,(e+b\,n-b)\,x^{(n-1)}}{-2\,a+2\,a\,n+d} + n\,(2\,c\,a-4\,c\,a\,n-c\,d-e^2-5\,e\,b\,n+3\,e\,b+5\,b^2\,n-2\,b^2 + 2\,c\,n^2\,a+c\,n\,d-4\,b^2\,n^2+e^2\,n+2\,e\,n^2\,b+b^2\,n^3)\,x^{(n-2)} / (2\,(-2\,a+2\,a\,n+d)\,(2\,a\,n-3\,a+d))\right) - c_n\left(x^{(n-1)} + \frac{(n-1)\,(e+b\,(n-1)-b)\,x^{(n-2)}}{-2\,a+2\,a\,(n-1)+d} + (n-1)\,(2\,c\,a - 4\,c\,a\,(n-1)-c\,d-e^2-5\,e\,b\,(n-1)+3\,e\,b+5\,b^2\,(n-1)-2\,b^2+2\,c\,(n-1)^2\,a + c\,(n-1)\,d-4\,b^2\,(n-1)^2+e^2\,(n-1)+2\,e\,(n-1)^2\,b+b^2\,(n-1)^3)\,x^{(n-3)} / (2\,(-2\,a+2\,a\,(n-1)+d)\,(2\,a\,(n-1)-3\,a+d))\right)$$

```
> re:=simplify(numer(normal(RE))/x^(n-3));
```

$$re := -2\,c_n\,c\,d^5 - 12\,c_n\,b^2\,d^4 + 2\,x\,c_n\,e\,d^5 - 2192\,x^2\,c_n\,a^6\,n^2 - 288\,x^3\,n^3\,b\,d^3\,a^2 + 1096\,x^3\,n^3\,b\,a^5 + 2\,x^3\,n^2\,b\,d^5 - 240\,x^3\,n^2\,b\,a^5 - 2\,x^3\,n\,b\,d^5 - 118\,x^3\,n\,b\,a^2\,d^3 - 624\,x^3\,n^3\,e\,d^2\,a^3 + 256\,c_n\,c\,n^6\,a^5 + 3\,c_n\,c\,n\,d^5 - 2\,x^2\,c_n\,d^6 + 1148\,x\,b_n\,n^4\,b^2\,a^3\,d + 1160\,c_n\,c\,d^2\,a^3\,n^2 - 2396\,x\,c_n\,b\,d^2\,a^3\,n^3 + 2654\,x\,c_n\,b\,d^2\,a^3\,n^2 + 688\,x\,c_n\,e\,d^2\,a^3\,n^3 - 1020\,x\,c_n\,e\,d^2\,a^3\,n^2 + 1200\,x^4\,a_n\,a^3\,d^3\,n^2 - 4080\,x^4\,a_n\,a^4\,d^2\,n^2 + 2400\,x^4\,a_n\,a^4\,d^2\,n^3$$

$$
\begin{aligned}
&- 480\, x^4\, a_n\, a^4\, n^4\, d^2 - 320\, x^4\, a_n\, a^3\, n^3\, d^3 + 1416\, x^3\, n^3\, e\, a^4\, d + 496\, c_n\, e\, n^5\, b\, a^3\, d \\
&- 64\, c_n\, e\, n^6\, b\, a^3\, d + 80\, x^2\, n^5\, c\, d^2\, a^3 - 60\, c_n\, e\, b\, d^3\, a + 554\, c_n\, e\, b\, a^3\, n\, d \\
&- 1622\, c_n\, e\, b\, a^3\, n^2\, d - 1488\, c_n\, e\, b\, n^4\, a^3\, d - 32\, c_n\, c\, n^7\, a^5 - c_n\, c\, n^2\, d^5 - c_n\, e^2\, n^2\, d^4 \\
&+ x^2\, a_n\, n^2\, b^2\, d^4 + 188\, x^2\, a_n\, n^2\, e^2\, a^4 - 120\, x^2\, a_n\, n\, e^2\, a^4 - x^2\, a_n\, n\, e^2\, d^4 - 384\, x^4\, a_n\, a^5\, n^5\, d \\
&- 856\, x^3\, n^2\, e\, a^4\, d - 208\, x^3\, n^2\, e\, d^3\, a^2 - 48\, c_n\, c\, a^4\, d + 20\, c_n\, c\, d^4\, a - 160\, x\, c_n\, e\, n^4\, d^2\, a^3 \\
&- 160\, x\, c_n\, b\, n^5\, d^2\, a^3 + 480\, c_n\, c\, n^4\, d^2\, a^3 - 80\, c_n\, c\, n^5\, d^2\, a^3 - 1448\, c_n\, e\, b\, a^4\, n^3 \\
&- 218\, c_n\, e\, b\, n^2\, d^3\, a + 336\, c_n\, e\, n^4\, b\, d^2\, a^2 + 100\, c_n\, e\, n^3\, b\, d^3\, a + 11\, c_n\, e\, n^2\, b\, d^4 \\
&- 120\, x^4\, a_n\, a^2\, n^2\, d^4 + 12\, c_n\, e^2\, d^3\, a + 44\, x\, c_n\, b\, d^4\, a - 22\, x\, c_n\, e\, d^4\, a - 384\, x^2\, c_n\, a^5\, n^5\, d \\
&+ 156\, x\, b_n\, n^5\, b^2\, d^2\, a^2 + 42\, x\, b_n\, n^4\, b^2\, d^3\, a - 8\, x\, b_n\, n^3\, e^2\, d^3\, a - x\, b_n\, n^2\, e^2\, d^4 - x\, b_n\, n^4\, b^2\, d^4 \\
&+ 120\, x^2\, a_n\, n\, e\, b\, a^4 + x^2\, a_n\, n\, e\, b\, d^4 + 24\, x^2\, a_n\, n^3\, e^2\, a^4 + 112\, x^2\, a_n\, n^7\, b^2\, a^4 \\
&- 268\, x^2\, a_n\, n^6\, b^2\, a^4 - 16\, x^2\, a_n\, n^8\, b^2\, a^4 + 196\, x^2\, a_n\, n^5\, b^2\, a^4 - 584\, x\, b_n\, n^5\, c\, a^5 \\
&- 32\, x\, b_n\, n^7\, c\, a^5 + 704\, x\, b_n\, n^4\, c\, a^5 - 12\, x\, b_n\, n\, c\, d^4\, a - 900\, x\, b_n\, n^3\, b^2\, a^3\, d \\
&+ 332\, x\, b_n\, n^2\, b^2\, a^3\, d + 144\, c_n\, b^2\, a^4\, n - 30\, x^2\, n\, e\, b\, d^3\, a - 400\, x^2\, n^5\, e\, b\, a^3\, d \\
&+ 74\, x^2\, n^2\, e\, b\, d^3\, a - 89\, x\, b_n\, n^2\, e^2\, d^2\, a^2 + 784\, x^2\, b_n\, n^4\, b\, d^2\, a^3 - 1332\, x^2\, b_n\, n^3\, b\, d^2\, a^3 \\
&- 160\, x^2\, b_n\, n^5\, b\, d^2\, a^3 + 464\, x\, b_n\, n^3\, b^2\, d^2\, a^2 + 980\, x\, b_n\, n^3\, e\, b\, a^3\, d + 1156\, x\, c_n\, b\, a^4\, n\, d \\
&+ 472\, c_n\, c\, a^4\, n\, d - 3492\, x\, c_n\, b\, a^4\, n^2\, d - x\, b_n\, n^2\, c\, d^5 + 416\, x^2\, b_n\, n^5\, e\, a^5 - 944\, x^2\, b_n\, n^4\, e\, a^5 \\
&- 64\, x^2\, b_n\, n^6\, e\, a^5 + 856\, x^2\, b_n\, n^3\, e\, a^5 - c_n\, b^2\, n^4\, d^4 - 22\, c_n\, e^2\, d^2\, a^2 - 232\, c_n\, b^2\, n^3\, d^3\, a \\
&- 2\, x^3\, b_n\, d^6 + 248\, x\, b_n\, n^6\, b^2\, a^3\, d - 49\, x^2\, n\, c\, a^2\, d^3 + 32\, x^2\, n^7\, b^2\, a^3\, d + 680\, x^2\, n^5\, e\, b\, a^4 \\
&+ 8\, x^2\, n^5\, b^2\, d^3\, a + 236\, x^2\, n^3\, e^2\, a^3\, d + 48\, x^2\, n^5\, e\, b\, d^2\, a^2 + x^2\, n^4\, b^2\, d^4 + 8\, x^2\, n^3\, e^2\, d^3\, a \\
&+ 2248\, x^3\, n^4\, b\, a^4\, d - 548\, x\, c_n\, e\, a^4\, n\, d - 240\, x^2\, n^4\, e\, b\, d^2\, a^2 - 32\, x^2\, a_n\, n^7\, b^2\, a^3\, d \\
&- 66\, x^2\, n^2\, b^2\, d^3\, a + 78\, x^2\, n\, c\, a^3\, d^2 - 972\, x\, b_n\, n^4\, c\, a^4\, d + 22\, x\, b_n\, n^2\, c\, d^4\, a \\
&+ 832\, x^2\, b_n\, n^4\, e\, a^4\, d - 1416\, x^2\, b_n\, n^3\, e\, a^4\, d + 856\, x^2\, b_n\, n^2\, e\, a^4\, d - 160\, x\, c_n\, e\, n^5\, a^4\, d \\
&- 936\, x^2\, n^5\, b^2\, a^4 + 168\, c_n\, e^2\, n^4\, a^3\, d + 108\, c_n\, e^2\, n^3\, a^2\, d^2 - 8\, c_n\, e^2\, n^3\, a\, d^3 \\
&+ 3600\, x^2\, c_n\, a^6\, n^3 - 2\, c_n\, e^2\, d^4 + 960\, x^2\, c_n\, a^6\, n^5 - 4\, x\, c_n\, b\, d^5 + 480\, x^2\, c_n\, a^6\, n \\
&- 128\, x^2\, c_n\, a^6\, n^6 + 30\, x^2\, c_n\, a\, d^5 + 160\, x^3\, n^5\, e\, a^4\, d + 328\, c_n\, b^2\, n^6\, a^3\, d - 120\, x^2\, a_n\, n\, c\, d\, a^4 \\
&+ 154\, x^2\, a_n\, n\, c\, d^2\, a^3 - 272\, x^2\, a_n\, n^2\, c\, d^2\, a^3 - 416\, x^3\, n^5\, e\, a^5 + 64\, x^3\, n^6\, e\, a^5 - 856\, x^3\, n^3\, e\, a^5 \\
&+ 240\, x^3\, n^2\, e\, a^5 + 2\, x^3\, n\, e\, d^5 - 480\, x^3\, n^6\, b\, a^5 + 64\, x^3\, n^7\, b\, a^5 - 1800\, x^3\, n^4\, b\, a^5 \\
&- 32\, c_n\, e^2\, n^5\, a^3\, d + 450\, x^2\, c_n\, a^3\, d^3 - 80\, x\, c_n\, b\, n^4\, d^3\, a^2 + 416\, x\, c_n\, b\, n^3\, d^3\, a^2 \\
&- 324\, c_n\, e^2\, d\, a^3\, n^3 + 282\, c_n\, e^2\, d\, a^3\, n^2 - 80\, x\, c_n\, e\, n^3\, d^3\, a^2 - 20\, x\, c_n\, e\, n^2\, d^4\, a \\
&+ 160\, x^3\, n^4\, e\, d^2\, a^3 - 264\, c_n\, b^2\, d^3\, a\, n - 3416\, x\, c_n\, b\, n^4\, a^4\, d + 1008\, x\, c_n\, b\, n^4\, a^3\, d^2 \\
&- 1540\, c_n\, c\, n^4\, a^4\, d + 2100\, c_n\, c\, n^3\, a^4\, d - 5440\, x^4\, a_n\, a^5\, n^3\, d + 4848\, x\, c_n\, b\, a^4\, n^3\, d
\end{aligned}
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\begin{aligned}
&- 1688\, x\, c_n\, e\, a^4\, n^3\, d - 32\, c_n\, b^2\, n^7\, a^3\, d + 864\, x\, c_n\, e\, n^4\, a^4\, d - 1464\, c_n\, c\, a^4\, n^2\, d \\
&+ 1472\, x\, c_n\, e\, a^4\, n^2\, d + 2234\, c_n\, b^2\, a^3\, n^2\, d - 1356\, c_n\, b^2\, n^5\, a^3\, d + 168\, x^2\, a_n\, n^6\, b^2\, a^3\, d \\
&- 252\, x^2\, a_n\, n^5\, b^2\, a^3\, d - 14\, x^2\, a_n\, n^4\, b^2\, a^3\, d + 284\, x^2\, a_n\, n^3\, b^2\, a^3\, d - 404\, x^2\, a_n\, n^4\, c\, a^4\, d \\
&+ 4\, x^2\, a_n\, n^2\, c\, d^4\, a - 236\, x^2\, a_n\, n^3\, c\, a^4\, d + 496\, x^2\, a_n\, n^2\, c\, a^4\, d - 450\, x^4\, d^3\, a^3 \\
&+ 5400\, x^4\, a_n\, a^5\, n^2\, d + 2400\, x^4\, a_n\, a^5\, n^4\, d - 10\, x^2\, a_n\, n^3\, c\, d^4\, a - 8\, x^2\, a_n\, n^3\, e^2\, d^3\, a \\
&- 71\, x^2\, a_n\, n\, c\, d^3\, a^2 + 41\, x^2\, a_n\, n^2\, c\, d^3\, a^2 + 13\, x^2\, a_n\, n^2\, e^2\, d^2\, a^2 + 71\, x^2\, a_n\, n^2\, b^2\, a^2\, d^2 \\
&- 756\, x\, b_n\, n^5\, b^2\, a^3\, d - 60\, x^2\, n\, e\, b\, d\, a^3 + 368\, x^2\, a_n\, n^5\, c\, a^4\, d + 70\, c_n\, b^2\, n^4\, d^3\, a \\
&+ 194\, c_n\, e\, b\, d^3\, a\, n - 8\, c_n\, b^2\, n^5\, d^3\, a + 992\, x^2\, b_n\, n^5\, b\, a^4\, d - 392\, x\, b_n\, n^3\, c\, a^5 \\
&- 19\, c_n\, e\, b\, n\, d^4 - 900\, x^2\, n^3\, c\, a^4\, d - 40\, x^2\, n\, c\, a^4\, d - 2896\, x\, c_n\, b\, a^5\, n^3 + 1144\, x\, c_n\, e\, a^5\, n^3 \\
&+ 2400\, x^2\, c_n\, a^4\, d^2\, n^3 - 448\, x\, b_n\, n^2\, e\, b\, a^3\, d + 60\, x\, b_n\, n^3\, e\, b\, d^3\, a - 74\, x\, b_n\, n^2\, e\, b\, d^3\, a \\
&+ 30\, x\, b_n\, n\, e\, b\, d^3\, a - 120\, x^2\, c_n\, a^2\, n^2\, d^4 + 416\, x\, c_n\, e\, n^5\, a^5 - 64\, x\, c_n\, e\, n^6\, a^5 - 2\, x\, c_n\, e\, n\, d^5 \\
&+ 544\, x\, c_n\, b\, n^6\, a^5 + 1200\, x^2\, c_n\, a^3\, d^3\, n^2 - 864\, c_n\, b^2\, a^4\, n^2 - 548\, x^2\, c_n\, a^4\, d^2 + 240\, x^2\, c_n\, a^5\, d \\
&- 170\, x^2\, c_n\, a^2\, d^4 - 1208\, c_n\, c\, a^5\, n^3 - 2\, x\, b_n\, n^3\, e\, b\, d^4 + 30\, c_n\, e^2\, n^2\, d^3\, a - 8\, x^2\, a_n\, n^5\, b^2\, d^3\, a \\
&+ 8\, x^2\, a_n\, n^3\, b^2\, d^3\, a - 40\, x^2\, a_n\, n^4\, c\, d^3\, a^2 + 72\, x^2\, a_n\, n^3\, c\, d^3\, a^2 - 8\, x\, b_n\, n^5\, b^2\, d^3\, a \\
&+ 14\, x^2\, a_n\, n\, c\, d^4\, a + 60\, x^2\, a_n\, n^3\, e^2\, d^2\, a^2 - 24\, x^2\, a_n\, n^6\, b^2\, d^2\, a^2 + 84\, x^2\, a_n\, n^5\, b^2\, d^2\, a^2 \\
&- 47\, x^2\, a_n\, n^4\, b^2\, d^2\, a^2 + 40\, c_n\, c\, n^2\, d^4\, a - 40\, c_n\, c\, n^4\, d^3\, a^2 - 10\, c_n\, c\, n^3\, d^4\, a - 23\, c_n\, b^2\, n^2\, d^4 \\
&+ 8\, c_n\, b^2\, n^3\, d^4 + 1332\, x^3\, n^3\, b\, d^2\, a^3 + 2184\, c_n\, e\, b\, d\, a^3\, n^3 + 1580\, c_n\, e\, b\, a^4\, n^4 \\
&+ 272\, c_n\, e\, n^6\, b\, a^4 - 32\, c_n\, e\, n^7\, b\, a^4 - 874\, c_n\, e\, b\, n^3\, d^2\, a^2 - 265\, x\, b_n\, n^2\, b^2\, a^2\, d^2 \\
&- 430\, x\, b_n\, n^3\, e\, b\, a^2\, d^2 + 325\, x\, b_n\, n^2\, e\, b\, a^2\, d^2 - 236\, x\, b_n\, n^3\, e^2\, a^3\, d + 10\, c_n\, e\, b\, d^4 \\
&+ 3\, c_n\, e^2\, n\, d^4 + 196\, x^2\, n^4\, e^2\, a^4 - 156\, x^2\, n^3\, e^2\, a^4 - 96\, x^2\, n^5\, e^2\, a^4 + 40\, x^2\, n^2\, e^2\, a^4 \\
&+ 584\, x^2\, n^5\, c\, a^5 - 4\, x^2\, n^3\, b^2\, d^4 - 325\, x^2\, n^2\, e\, b\, a^2\, d^2 + 6\, x^2\, a_n\, n^2\, e^2\, d^3\, a \\
&+ 14\, x^2\, a_n\, n^4\, b^2\, d^3\, a - 154\, x^2\, a_n\, n^2\, b^2\, a^3\, d + 240\, c_n\, e^2\, a^4\, n^3 - 16\, c_n\, e^2\, n^6\, a^4 \\
&+ 96\, c_n\, e^2\, n^5\, a^4 - 32\, x\, b_n\, n^5\, e^2\, a^3\, d - 32\, x\, b_n\, n^7\, b^2\, a^3\, d - 10\, x\, b_n\, n\, e^2\, d^3\, a \\
&+ 66\, x\, b_n\, n^2\, b^2\, d^3\, a - 20\, x\, b_n\, n\, b^2\, d^3\, a + 164\, x^2\, a_n\, n^4\, b^2\, a^4 - 308\, x^2\, a_n\, n^3\, b^2\, a^4 \\
&- x^2\, a_n\, n^2\, e^2\, d^4 + 2\, x^4\, d^6 - 24\, x\, b_n\, n^4\, e^2\, d^2\, a^2 - 24\, x\, b_n\, n^6\, b^2\, d^2\, a^2 + 240\, x\, b_n\, n^6\, e\, b\, a^4 \\
&- 71\, x^2\, a_n\, n\, e^2\, a^2\, d^2 - 155\, x^2\, a_n\, n^2\, e\, b\, a^2\, d^2 + 71\, x^2\, a_n\, n\, e\, b\, a^2\, d^2 - 80\, x^2\, a_n\, n^6\, c\, a^4\, d \\
&- 29\, x^2\, n\, e^2\, d^2\, a^2 - 980\, x^2\, n^3\, e\, b\, a^3\, d - 2\, x^2\, n\, b^2\, d^4 - 84\, x^2\, n^3\, e^2\, d^2\, a^2 - x^2\, a_n\, n^4\, b^2\, d^4 \\
&+ 120\, x^2\, a_n\, n^2\, b^2\, a^4 + 160\, x^3\, n^5\, b\, d^2\, a^3 - 240\, x^2\, b_n\, n^2\, e\, a^5 - 2\, x^2\, b_n\, n\, e\, d^5 \\
&+ 480\, x^2\, b_n\, n^6\, b\, a^5 - 1360\, x^2\, b_n\, n^5\, b\, a^5 - 64\, x^2\, b_n\, n^7\, b\, a^5 + 1800\, x^2\, b_n\, n^4\, b\, a^5 \\
&- 120\, x^3\, n\, b\, a^4\, d - 832\, x^3\, n^4\, e\, a^4\, d + 80\, x^3\, n^3\, e\, d^3\, a^2 + x^2\, n^2\, e^2\, d^4 - 680\, x\, b_n\, n^5\, e\, b\, a^4 \\
&- 32\, x\, b_n\, n^7\, e\, b\, a^4 + 900\, x\, b_n\, n^4\, e\, b\, a^4 - 548\, x\, b_n\, n^3\, e\, b\, a^4 + 120\, x\, b_n\, n^2\, e\, b\, a^4
\end{aligned}
$$

$$- 118\, x^2\, b_n\, n\, e\, a^2\, d^3 + 120\, x^2\, b_n\, n\, b\, a^4\, d - 214\, x^2\, b_n\, n\, b\, a^3\, d^2 + 922\, x^2\, b_n\, n^2\, b\, a^3\, d^2$$
$$- 14\, x^2\, a_n\, n^2\, b^2\, d^3\, a - 130\, x^2\, a_n\, n^2\, e^2\, a^3\, d + 1336\, x\, c_n\, b\, a^5\, n^2 - 240\, x\, c_n\, b\, a^5\, n$$
$$+ 544\, c_n\, c\, a^5\, n^2 - 96\, c_n\, c\, a^5\, n - 136\, x^2\, n^3\, c\, d^3\, a^2 - 900\, x^2\, n^4\, e\, b\, a^4 - x^2\, n\, c\, d^5$$
$$- 1148\, x^2\, n^4\, b^2\, a^3\, d + 10\, x^2\, n^3\, c\, d^4\, a - 608\, x\, c_n\, e\, a^5\, n^2 + 120\, x\, c_n\, e\, a^5\, n$$
$$+ 2700\, x^2\, c_n\, a^4\, d^2\, n - 1360\, x^2\, c_n\, a^3\, d^3\, n - 2192\, x^2\, c_n\, a^5\, d\, n - 24\, x^2\, c_n\, d^5\, a\, n$$
$$- 124\, c_n\, e^2\, a^4\, n^2 + 24\, c_n\, e^2\, a^4\, n + 32\, x^2\, n^5\, e^2\, a^3\, d + 30\, x^4\, a_n\, a\, d^5 - 2720\, x^4\, a_n\, a^6\, n^4$$
$$+ 3600\, x^4\, a_n\, a^6\, n^3 + 928\, x^2\, n^4\, e\, b\, a^3\, d - 42\, x^2\, n^4\, b^2\, d^3\, a + 756\, x^2\, n^5\, b^2\, a^3\, d - x^2\, n\, e^2\, d^4$$
$$+ 64\, x^2\, n^6\, e\, b\, a^3\, d + 900\, x^2\, n^4\, b^2\, a^4 + 16\, x^2\, n^4\, e\, b\, d^3\, a - 224\, x^2\, n^6\, c\, a^5 - 120\, x^2\, n^2\, e\, b\, a^4$$
$$+ 12\, x^2\, n\, c\, d^4\, a + 32\, x^2\, n^7\, e\, b\, a^4 + 800\, x^3\, a_n\, b\, n^5\, a^4\, d + 26\, x^3\, n\, b\, d^4\, a + 160\, x^3\, n^6\, b\, a^4\, d$$
$$- 1096\, x^2\, b_n\, n^3\, b\, a^5 - 2\, x^2\, b_n\, n^2\, b\, d^5 + 240\, x^2\, b_n\, n^2\, b\, a^5 + 2\, x^2\, b_n\, n\, b\, d^5 - 70\, c_n\, c\, d^3\, a^2$$
$$+ 244\, x\, c_n\, b\, a^3\, d^2 - 164\, x\, c_n\, b\, a^2\, d^3 + 100\, c_n\, c\, a^3\, d^2 - 2400\, x^4\, a^5\, n^4\, d - 122\, x\, c_n\, e\, a^3\, d^2$$
$$+ 1588\, c_n\, b^2\, n^3\, a^2\, d^2 + 362\, c_n\, b^2\, n^2\, a\, d^3 - 770\, x\, c_n\, b\, d^3\, a^2\, n^2 + 598\, x\, c_n\, b\, d^3\, a^2\, n$$
$$+ 256\, x\, c_n\, e\, d^3\, a^2\, n^2 + x\, b_n\, n\, c\, d^5 + 936\, x\, b_n\, n^5\, b^2\, a^4 - 900\, x\, b_n\, n^4\, b^2\, a^4$$
$$- 516\, x\, b_n\, n^6\, b^2\, a^4 + 432\, x\, b_n\, n^3\, b^2\, a^4 + 118\, x^2\, b_n\, n\, b\, a^2\, d^3 - 5\, x^2\, n^2\, e\, b\, d^4$$
$$- 1360\, x^4\, a_n\, a^3\, d^3\, n - 2192\, x^4\, a_n\, a^5\, d\, n + 300\, x^4\, a_n\, a^2\, d^4\, n + 2700\, x^4\, a_n\, a^4\, d^2\, n$$
$$+ 2\, x^2\, n^3\, e\, b\, d^4 - 156\, x^2\, n^5\, b^2\, d^2\, a^2 - 22\, x^2\, n^2\, c\, d^4\, a + 20\, x^2\, n\, e^2\, d\, a^3 + 972\, x^2\, n^4\, c\, a^4\, d$$
$$- 992\, x^3\, n^5\, b\, a^4\, d - 80\, x\, b_n\, n^2\, b^2\, a^4 + 144\, x\, b_n\, n^7\, b^2\, a^4 + 18\, x\, b_n\, n^2\, e^2\, d^3\, a$$
$$- 80\, x\, b_n\, n^3\, b^2\, d^3\, a - 80\, x^2\, b_n\, n^3\, e\, d^3\, a^2 - 220\, c_n\, e^2\, n^4\, a^4 + 1916\, c_n\, b^2\, n^5\, a^4$$
$$- 2644\, c_n\, b^2\, n^4\, a^4 - 796\, c_n\, b^2\, n^6\, a^4 + 176\, c_n\, b^2\, n^7\, a^4 + 3160\, x\, c_n\, b\, a^5\, n^4$$
$$+ 208\, x^2\, b_n\, n^2\, e\, d^3\, a^2 + 368\, x\, b_n\, n^4\, c\, d^2\, a^3 - 582\, x\, b_n\, n^3\, c\, d^2\, a^3 - 80\, x\, b_n\, n^5\, c\, d^2\, a^3$$
$$+ 624\, x^2\, b_n\, n^3\, e\, d^2\, a^3 + 326\, x^3\, n^2\, b\, d^3\, a^2 + 400\, x\, b_n\, n^5\, e\, b\, a^3\, d - 64\, x\, b_n\, n^6\, e\, b\, a^3\, d$$
$$- 928\, x\, b_n\, n^4\, e\, b\, a^3\, d - 2\, x^4\, a_n\, d^6 + 118\, x^3\, n\, e\, a^2\, d^3 - 46\, x^3\, n^2\, b\, d^4\, a + 582\, x^2\, n^3\, c\, d^2\, a^3$$
$$+ 2084\, c_n\, b^2\, a^4\, n^3 - 3460\, c_n\, b^2\, n^3\, a^3\, d - 332\, x^2\, n^2\, b^2\, a^3\, d + 40\, x^2\, n\, b^2\, d\, a^3$$
$$+ 24\, x^2\, n^4\, e^2\, d^2\, a^2 + 389\, x^2\, n^4\, b^2\, a^2\, d^2 + 16\, x^2\, n^6\, e^2\, a^4 + 480\, x^4\, a_n\, a^6\, n + 240\, x^4\, a_n\, a^5\, d$$
$$+ 32\, x^2\, n^7\, c\, a^5 - 128\, x^4\, a_n\, a^6\, n^6 + 450\, x^4\, a_n\, a^3\, d^3 - 1840\, x\, c_n\, b\, n^5\, a^5 - 64\, x\, c_n\, b\, n^7\, a^5$$
$$- 2\, x\, c_n\, b\, n^2\, d^5 + 6\, x\, c_n\, b\, n\, d^5 - 824\, c_n\, c\, n^5\, a^5 - 214\, x^3\, n\, e\, a^3\, d^2 - 50\, c_n\, c\, d^4\, a\, n$$
$$- 4080\, x^2\, c_n\, a^4\, d^2\, n^2 - 34\, c_n\, e^2\, d^3\, a\, n - 106\, x\, c_n\, b\, d^4\, a\, n + 42\, x\, c_n\, e\, d^4\, a\, n$$
$$- 320\, x^3\, b_n\, a^3\, n^3\, d^3 + 240\, x^3\, a_n\, e\, a^5 - 2\, x^3\, a_n\, e\, d^5 + 450\, x^3\, b_n\, d^3\, a^3 + 960\, x^3\, b_n\, a^6\, n^5$$
$$- 548\, x^3\, b_n\, a^4\, d^2 - 2192\, x^3\, b_n\, a^6\, n^2 + 30\, x^3\, b_n\, d^5\, a + 480\, x^3\, b_n\, a^6\, n + 240\, x^3\, b_n\, a^5\, d$$
$$- 2720\, x^3\, b_n\, a^6\, n^4 + 3600\, x^3\, b_n\, a^6\, n^3 - 170\, x^3\, b_n\, d^4\, a^2 - 128\, x^3\, b_n\, a^6\, n^6 + 1360\, x^4\, d^3\, a^3\, n$$
$$+ 2192\, x^4\, a^5\, d\, n - 300\, x^4\, d^4\, a^2\, n - 2700\, x^4\, a^4\, d^2\, n + 24\, x^4\, d^5\, a\, n - 1200\, x^4\, d^3\, a^3\, n^2$$
$$+ 4080\, x^4\, a^4\, d^2\, n^2 + 480\, x^4\, a^4\, n^4\, d^2 - 2400\, x^4\, a^4\, n^3\, d^2 + 320\, x^4\, a^3\, n^3\, d^3$$

$$- 160\, x\, c_n\, b\, n^6\, a^4\, d + 560\, c_n\, c\, n^5\, a^4\, d - 80\, c_n\, c\, n^6\, a^4\, d - 20\, x^2\, b_n\, n^3\, b\, d^4\, a$$
$$- 16\, x\, b_n\, n^8\, b^2\, a^4 + x\, b_n\, n\, e^2\, d^4 - 5\, x\, b_n\, n^2\, b^2\, d^4 + 2\, x\, b_n\, n\, b^2\, d^4 + 516\, x^2\, n^6\, b^2\, a^4$$
$$+ 450\, x^3\, a_n\, b\, n\, a^3\, d^2 - 170\, x^3\, a_n\, b\, n\, a^2\, d^3 + 1252\, x^3\, a_n\, e\, a^4\, n\, d - 240\, x^3\, a_n\, e\, a^4\, n^2\, d$$
$$- 1080\, x^3\, a_n\, e\, n^3\, a^4\, d + 800\, x^3\, a_n\, e\, n^4\, a^4\, d - 160\, x^3\, a_n\, e\, n^5\, a^4\, d - 240\, x^3\, a_n\, b\, n^3\, a^4\, d$$
$$+ 1252\, x^3\, a_n\, b\, n^2\, a^4\, d - 548\, x^3\, a_n\, b\, n\, a^4\, d + 944\, x^3\, n^4\, e\, a^5 - 16\, x^2\, a_n\, n^6\, e^2\, a^4$$
$$+ 392\, x^2\, n^3\, c\, a^5 - 80\, x^2\, n^2\, c\, a^5 + 72\, c_n\, b^2\, d^3\, a + 5400\, x^2\, c_n\, a^5\, n^2\, d - 5440\, x^2\, c_n\, a^5\, n^3\, d$$
$$+ 2400\, x^2\, c_n\, a^5\, n^4\, d - 1350\, x\, c_n\, b\, a^3\, d^2\, n - 80\, x^2\, a_n\, n^5\, c\, d^2\, a^3 - 440\, x^2\, a_n\, n^5\, e\, b\, a^4$$
$$+ 220\, x^2\, a_n\, n^4\, e\, b\, a^4 + 352\, x^2\, a_n\, n^3\, e\, b\, a^4 + 2700\, x^3\, b_n\, a^4\, d^2\, n - 1360\, x^3\, b_n\, d^3\, a^3\, n$$
$$+ 120\, x^4\, a^2\, n^2\, d^4 - 5400\, x^4\, a^5\, n^2\, d + 5440\, x^4\, a^5\, n^3\, d + 384\, x^4\, a^5\, n^5\, d + 30\, x^3\, a_n\, b\, n\, d^4\, a$$
$$+ 10\, x^3\, a_n\, e\, d^4\, a\, n + 160\, x^3\, a_n\, e\, d^3\, a^2\, n^2 + 70\, x^3\, a_n\, e\, d^3\, a^2\, n - 1080\, x^3\, a_n\, b\, n^4\, a^4\, d$$
$$+ 560\, x^3\, a_n\, b\, n^4\, a^3\, d^2 - 160\, x^3\, a_n\, b\, n^6\, a^4\, d - 784\, x^3\, n^4\, b\, d^2\, a^3 + 82\, x\, c_n\, e\, a^2\, d^3$$
$$- 132\, c_n\, b^2\, d^2\, a^2 + 12\, c_n\, e^2\, a^3\, d + 72\, c_n\, b^2\, a^3\, d + 60\, x\, c_n\, e\, a^4\, d - 120\, x\, c_n\, b\, a^4\, d$$
$$- 48\, x\, b_n\, n^5\, e\, b\, d^2\, a^2 + 240\, x\, b_n\, n^4\, e\, b\, d^2\, a^2 - 16\, x\, b_n\, n^4\, e\, b\, d^3\, a - 120\, x^2\, b_n\, n\, e\, a^4\, d$$
$$- 32\, x^2\, a_n\, n^7\, c\, a^5 - 160\, x^3\, a_n\, b\, n^5\, a^3\, d^2 - 80\, x^3\, a_n\, b\, n^4\, a^2\, d^3 - 80\, x^3\, a_n\, e\, n^3\, d^3\, a^2$$
$$- 20\, x^3\, a_n\, e\, n^2\, d^4\, a - 856\, x^3\, a_n\, e\, a^5\, n - 2\, x^3\, a_n\, b\, n\, d^5 + 704\, x^3\, a_n\, e\, a^5\, n^2$$
$$+ 440\, x^3\, a_n\, e\, a^5\, n^3 - 880\, x^3\, a_n\, e\, n^4\, a^5 + 160\, x^3\, a_n\, b\, n^3\, d^3\, a^2 - 20\, x^3\, a_n\, b\, n^3\, d^4\, a$$
$$- 26\, x^3\, n\, e\, d^4\, a - 389\, x\, b_n\, n^4\, b^2\, a^2\, d^2 - 40\, x\, b_n\, n^4\, c\, d^3\, a^2 + 136\, x\, b_n\, n^3\, c\, d^3\, a^2$$
$$- 10\, x\, b_n\, n^3\, c\, d^4\, a - 20\, x^2\, b_n\, n^2\, e\, d^4\, a - 136\, x^2\, n^2\, e^2\, a^3\, d - 18\, x^2\, n^2\, e^2\, d^3\, a$$
$$+ 152\, x\, b_n\, n^4\, e^2\, a^3\, d + 136\, x\, b_n\, n^2\, e^2\, a^3\, d + 5\, x^2\, n^2\, b^2\, d^4 + 214\, x^2\, b_n\, n\, e\, a^3\, d^2$$
$$- 920\, c_n\, e\, b\, n^5\, a^4 - 32\, x^2\, a_n\, n^5\, e^2\, a^3\, d + 136\, x^2\, a_n\, n^4\, e^2\, a^3\, d - 24\, x^2\, a_n\, n^4\, e^2\, a^2\, d^2$$
$$- 106\, c_n\, e^2\, a^3\, n\, d - x^2\, a_n\, n^2\, e\, b\, d^4 - 160\, x^2\, b_n\, n^5\, e\, a^4\, d + 548\, x^2\, n^3\, e\, b\, a^4$$
$$- 258\, x\, c_n\, e\, d^3\, a^2\, n - 464\, x^2\, n^3\, b^2\, d^2\, a^2 - 16\, x\, b_n\, n^6\, e^2\, a^4 + 16\, x^2\, n^8\, b^2\, a^4$$
$$+ 120\, x^3\, n\, e\, a^4\, d - 2720\, x^2\, c_n\, a^6\, n^4 + 10\, x^3\, a_n\, b\, n^2\, d^4\, a + 416\, x^3\, a_n\, e\, n^5\, a^5$$
$$- 64\, x^3\, a_n\, e\, n^6\, a^5 - 2\, x^3\, a_n\, e\, n\, d^5 - 880\, x^3\, a_n\, b\, n^5\, a^5 + 440\, x^3\, a_n\, b\, n^4\, a^5 + 416\, x^3\, a_n\, b\, n^6\, a^5$$
$$+ 704\, x^3\, a_n\, b\, n^3\, a^5 - 64\, x^3\, a_n\, b\, n^7\, a^5 - 856\, x^3\, a_n\, b\, n^2\, a^5 - 2\, x^3\, a_n\, b\, n^2\, d^5$$
$$+ 240\, x^3\, a_n\, b\, n\, a^5 + 4\, x\, b_n\, n^3\, b^2\, d^4 + 376\, x^2\, a_n\, n^3\, c\, a^5 - 240\, x^2\, a_n\, n^2\, c\, a^5 - x^2\, a_n\, n\, c\, d^5$$
$$+ 96\, x^2\, a_n\, n^5\, e^2\, a^4 - 172\, x^2\, a_n\, n^4\, e^2\, a^4 + 26\, x^2\, b_n\, n\, e\, d^4\, a - 160\, x^2\, b_n\, n^6\, b\, a^4\, d$$
$$- 2248\, x^2\, b_n\, n^4\, b\, a^4\, d + 2272\, x^2\, b_n\, n^3\, b\, a^4\, d + 20\, x^3\, n^2\, e\, d^4\, a - 32\, x^2\, a_n\, n^7\, e\, b\, a^4$$
$$- 84\, x^2\, a_n\, n^3\, b^2\, d^2\, a^2 + 256\, x^2\, a_n\, n^4\, c\, d^2\, a^3 - 90\, x^2\, a_n\, n^3\, c\, d^2\, a^3 - 24\, x^4\, a_n\, d^5\, a\, n$$
$$+ 192\, x^2\, a_n\, n^6\, c\, a^5 - 344\, x^2\, a_n\, n^5\, c\, a^5 + 48\, x^2\, a_n\, n^4\, c\, a^5 - 432\, x^2\, n^3\, b^2\, a^4 - 960\, x^4\, a^6\, n^5$$
$$- 48\, x^2\, a_n\, n^5\, e\, b\, d^2\, a^2 - 16\, x^2\, a_n\, n^4\, e\, b\, d^3\, a - 2\, x^2\, a_n\, n^3\, e\, b\, d^4 + 976\, x^3\, n^2\, b\, a^4\, d$$

$$+ 1184\, x\, c_n\, b\, n^5\, a^4\, d + 24\, x^2\, n^6\, b^2\, d^2\, a^2 + 3\, x^2\, n\, e\, b\, d^4 + 20\, x^2\, n\, b^2\, d^3\, a - 372\, x^2\, n^2\, c\, a^3\, d^2$$
$$- 152\, x^2\, n^4\, e^2\, a^3\, d + 46\, x^2\, b_n\, n^2\, b\, d^4\, a - x^2\, a_n\, n^2\, c\, d^5 - 3600\, x^4\, a^6\, n^3 + 2720\, x^4\, a^6\, n^4$$
$$- 240\, x^4\, a^5\, d - 480\, x^4\, a^6\, n - 30\, x^4\, d^5\, a + 2192\, x^4\, a^6\, n^2 + 548\, x^4\, a^4\, d^2 - 355\, c_n\, c\, d^3\, a^2\, n^2$$
$$+ 265\, c_n\, c\, d^3\, a^2\, n - 167\, c_n\, e^2\, d^2\, a^2\, n^2 + 105\, c_n\, e^2\, d^2\, a^2\, n + 80\, x^2\, n^6\, c\, a^4\, d$$
$$+ 40\, x^2\, n^4\, c\, d^3\, a^2 + 208\, x^2\, a_n\, n^6\, e\, b\, a^4 - 464\, x^2\, n^5\, c\, a^4\, d - 922\, x^3\, n^2\, b\, a^3\, d^2$$
$$- 428\, x^2\, a_n\, n^2\, e\, b\, a^4 + 144\, x^2\, a_n\, n^4\, e\, b\, d^2\, a^2 - 34\, x^2\, a_n\, n^3\, e\, b\, d^2\, a^2 + 20\, x^2\, a_n\, n^3\, e\, b\, d^3\, a$$
$$- 1090\, c_n\, c\, n^3\, a^3\, d^2 - 1549\, c_n\, b^2\, d^2\, a^2\, n^2 + 740\, c_n\, b^2\, d^2\, a^2\, n + 2910\, c_n\, b^2\, n^4\, a^3\, d$$
$$+ 128\, x^4\, a^6\, n^6 + 170\, x^4\, d^4\, a^2 + 214\, x^3\, n\, b\, a^3\, d^2 + 1360\, c_n\, c\, a^5\, n^4 - 1008\, x\, c_n\, e\, a^5\, n^4$$
$$- 16\, c_n\, b^2\, n^8\, a^4 - 480\, x^2\, c_n\, a^4\, n^4\, d^2 - 320\, x^2\, c_n\, a^3\, n^3\, d^3 - 708\, x^2\, b_n\, n^2\, e\, d^2\, a^3$$
$$- 160\, x^2\, b_n\, n^4\, e\, d^2\, a^3 - 80\, x^2\, b_n\, n^4\, b\, d^3\, a^2 + 288\, x^2\, b_n\, n^3\, b\, d^3\, a^2 + 84\, x\, b_n\, n^3\, e^2\, d^2\, a^2$$
$$- 120\, c_n\, e\, b\, a^4\, n + 1045\, c_n\, e\, b\, d^2\, a^2\, n^2 - 569\, c_n\, e\, b\, d^2\, a^2\, n - 24\, c_n\, b^2\, n^6\, d^2\, a^2$$
$$+ 154\, x^2\, a_n\, n\, e^2\, a^3\, d + 14\, x^2\, a_n\, n\, e^2\, d^3\, a - 368\, x^2\, a_n\, n^4\, e\, b\, a^3\, d + 304\, x^2\, a_n\, n^5\, e\, b\, a^3\, d$$
$$+ 80\, x^2\, n^2\, b^2\, a^4 - 144\, x^2\, n^7\, b^2\, a^4 - 2192\, x^3\, b_n\, a^5\, d\, n + 300\, x^3\, b_n\, d^4\, a^2\, n - 24\, x^3\, b_n\, d^5\, a\, n$$
$$+ 1200\, x^3\, b_n\, d^3\, a^3\, n^2 - 4080\, x^3\, b_n\, a^4\, d^2\, n^2 - 480\, x^3\, b_n\, a^4\, n^4\, d^2 + 2400\, x^3\, b_n\, a^4\, n^3\, d^2$$
$$- 120\, x^3\, b_n\, a^2\, n^2\, d^4 + 5400\, x^3\, b_n\, a^5\, n^2\, d - 5440\, x^3\, b_n\, a^5\, n^3\, d + 2400\, x^3\, b_n\, a^5\, n^4\, d$$
$$- 384\, x^3\, b_n\, a^5\, n^5\, d + 300\, x^2\, c_n\, a^2\, d^4\, n - 368\, x^2\, n^4\, c\, d^2\, a^3 + 89\, x^2\, n^2\, e^2\, d^2\, a^2$$
$$+ 900\, x^2\, n^3\, b^2\, a^3\, d + 1360\, x^3\, n^5\, b\, a^5 + 960\, x^4\, a_n\, a^6\, n^5 - 548\, x^4\, a_n\, a^4\, d^2 - 2192\, x^4\, a_n\, a^6\, n^2$$
$$- 170\, x^4\, a_n\, a^2\, d^4 + 80\, x^2\, n^3\, b^2\, d^3\, a - 58\, x^2\, n\, b^2\, d^2\, a^2 + x^2\, n^2\, c\, d^5 + 352\, x^2\, n^2\, c\, a^4\, d$$
$$- 570\, c_n\, c\, a^3\, d^2\, n + 614\, x\, c_n\, e\, a^3\, d^2\, n - 851\, c_n\, b^2\, n^4\, d^2\, a^2 + 228\, c_n\, b^2\, n^5\, d^2\, a^2$$
$$+ 668\, c_n\, e\, b\, a^4\, n^2 + 708\, x^3\, n^2\, e\, d^2\, a^3 - 248\, x^2\, n^6\, b^2\, a^3\, d - 20\, x\, c_n\, b\, n^3\, d^4\, a$$
$$+ 82\, x\, c_n\, b\, n^2\, d^4\, a + 200\, c_n\, c\, n^3\, d^3\, a^2 - 116\, x^2\, a_n\, n^3\, e^2\, a^3\, d + 464\, x\, b_n\, n^5\, c\, a^4\, d$$
$$- 80\, x\, b_n\, n^6\, c\, a^4\, d - 48\, c_n\, e\, n^5\, b\, d^2\, a^2 - 16\, c_n\, e\, n^4\, b\, d^3\, a - 2\, c_n\, e\, n^3\, b\, d^4$$
$$+ 110\, c_n\, e\, b\, a^2\, d^2 - 60\, c_n\, e\, b\, a^3\, d - 696\, c_n\, b^2\, a^3\, n\, d - 24\, c_n\, e^2\, n^4\, a^2\, d^2 + 10\, x^2\, n\, e^2\, d^3\, a$$
$$- 144\, x^2\, a_n\, n^3\, e\, b\, a^3\, d - 64\, x^2\, a_n\, n^6\, e\, b\, a^3\, d + 438\, x^2\, a_n\, n^2\, e\, b\, a^3\, d - 240\, x^2\, n^6\, e\, b\, a^4$$
$$+ 430\, x^2\, n^3\, e\, b\, a^2\, d^2 - 704\, x^2\, n^4\, c\, a^5 + 80\, x^3\, n^4\, b\, d^3\, a^2 + 448\, x^2\, n^2\, e\, b\, a^3\, d$$
$$+ 80\, x\, b_n\, n^2\, c\, a^5 + 265\, x^2\, n^2\, b^2\, a^2\, d^2 + 29\, x\, b_n\, n\, e^2\, d^2\, a^2 - 40\, x\, b_n\, n\, b^2\, d\, a^3$$
$$+ 58\, x\, b_n\, n\, b^2\, d^2\, a^2 + 60\, x\, b_n\, n\, e\, b\, d\, a^3 - 87\, x\, b_n\, n\, e\, b\, d^2\, a^2 + 5\, x\, b_n\, n^2\, e\, b\, d^4$$
$$- 3\, x\, b_n\, n\, e\, b\, d^4 - 145\, x\, b_n\, n^2\, c\, d^3\, a^2 - 326\, x^2\, b_n\, n^2\, b\, d^3\, a^2 - 20\, x\, b_n\, n\, e^2\, d\, a^3$$
$$+ 40\, x\, b_n\, n\, c\, a^4\, d - 78\, x\, b_n\, n\, c\, a^3\, d^2 + 372\, x\, b_n\, n^2\, c\, a^3\, d^2 + 49\, x\, b_n\, n\, c\, a^2\, d^3$$
$$+ 87\, x^2\, n\, e\, b\, d^2\, a^2 - 60\, x^2\, n^3\, e\, b\, d^3\, a + 145\, x^2\, n^2\, c\, d^3\, a^2 + 22\, x^2\, a_n\, n^2\, e\, b\, d^3\, a$$
$$- 154\, x^2\, a_n\, n\, e\, b\, a^3\, d - 14\, x^2\, a_n\, n\, e\, b\, d^3\, a - 976\, x^2\, b_n\, n^2\, b\, a^4\, d - 26\, x^2\, b_n\, n\, b\, d^4\, a$$

$$+ 900\, x\, b_n\, n^3\, c\, a^4\, d - 352\, x\, b_n\, n^2\, c\, a^4\, d + 20\, x^3\, n^3\, b\, d^4\, a - 2272\, x^3\, n^3\, b\, a^4\, d$$
$$- 196\, x\, b_n\, n^4\, e^2\, a^4 + 156\, x\, b_n\, n^3\, e^2\, a^4 + 96\, x\, b_n\, n^5\, e^2\, a^4 - 40\, x\, b_n\, n^2\, e^2\, a^4$$
$$+ 224\, x\, b_n\, n^6\, c\, a^5 + 450\, x^3\, a_n\, e\, d^2\, a^3 - 170\, x^3\, a_n\, e\, d^3\, a^2 - 548\, x^3\, a_n\, e\, a^4\, d$$
$$+ 30\, x^3\, a_n\, e\, d^4\, a - 570\, x^3\, a_n\, e\, d^2\, a^3\, n - 300\, x^3\, a_n\, e\, d^2\, a^3\, n^2 + 560\, x^3\, a_n\, e\, n^3\, d^2\, a^3$$
$$- 160\, x^3\, a_n\, e\, n^4\, d^2\, a^3 - 300\, x^3\, a_n\, b\, n^3\, d^2\, a^3 - 570\, x^3\, a_n\, b\, n^2\, a^3\, d^2 + 70\, x^3\, a_n\, b\, n^2\, a^2\, d^3$$
$$+ 28\, c_n\, b^2\, n\, d^4$$

Equating the highest coefficient gives for monic polynomials

```
> rule4:=a[n]=solve(coeff(re,x,4),a[n]);
```

$$rule4 := a_n = 1$$

and equating the second highest coefficient yields

```
> rule5:=b[n]=factor(solve(subs(rule4,coeff(re,x,3)),b[n]));
```

$$rule5 := b_n = -\frac{2\, b\, n^2\, a - 2\, b\, n\, a - 2\, e\, a + 2\, b\, n\, d + e\, d}{(-2\, a + 2\, a\, n + d)\, (2\, a\, n + d)}$$

Finally equating the third highest coefficient yields

```
> rule6:=c[n]=factor(solve(subs(rule5,subs(rule4,coeff(re,x,2))
  ),c[n]));
```

$$rule6 := c_n = -n\, (d - 2\, a + a\, n)\, (4\, a^2\, n^2\, c - 8\, a^2\, c\, n + 4\, a^2\, c - a\, b^2\, n^2 + 2\, a\, b^2\, n$$
$$+ 4\, a\, c\, n\, d - 4\, a\, c\, d + a\, e^2 - a\, b^2 - b^2\, d\, n + b^2\, d + c\, d^2 - e\, b\, d) \Big/ ((2\, a\, n - 3\, a + d)$$
$$(2\, a\, n - a + d)\, (-2\, a + 2\, a\, n + d)^2)$$

```
>
```

Zeilberger's algorithm

We load the package "hsum.mpl" from my book
"Hypergeometric Summation", Vieweg, Braunschweig/Wiesbaden, 1998

```
> read "hsum9.mpl";
```

Package "Hypergeometric Summation", Maple V - Maple 9

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We define the hypergeometric summand of the Laguerre polynomials.

```
> laguerreterm:=pochhammer(alpha+1,n)/n!*hyperterm([-n],[alpha+
  1],x,k);
```

$$laguerreterm := \frac{\text{pochhammer}(\alpha + 1, n)\ \text{pochhammer}(-n, k)\, x^k}{n!\ \text{pochhammer}(\alpha + 1, k)\, k!}$$

and use Zeilberger's algorithm to detect a recurrence equation for the sum, hence for the Laguerre polynomials.

```
> RE:=sumrecursion(laguerreterm,k,L(n));
```

$$RE := (n + 1 + \alpha)\, \mathrm{L}(n) + (-\alpha - 3 + x - 2\, n)\, \mathrm{L}(n + 1) + (2 + n)\, \mathrm{L}(2 + n) = 0$$

Next, we detect the differential equation of the Laguerre polynomials from their hypergeometric representation.

```
> DE:=sumdiffeq(laguerreterm,k,L(x));
```

$$DE := x\left(\frac{d^2}{dx^2}\mathrm{L}(x)\right) - (x-\alpha-1)\left(\frac{d}{dx}\mathrm{L}(x)\right) + \mathrm{L}(x)\,n = 0$$

To compute recurrence and differential equations for sums and products, we load the gfun package.

```
> with(gfun):
```

The following computes the recurrence equation valid for the square of the Laguerre polynomials

```
> `rec*rec`(RE,RE,L(n));
```

$$\{(10 - 2x + 29n - 4xn^2 - xn^3 + 25\alpha^2 + \alpha^4 + 9\alpha^3 + 13n^3 + 2n^4 + 27\alpha + 55n\alpha - 5x\alpha - 5xn + 30n^2 + 5n\alpha^3 + 31n\alpha^2 + 9n^2\alpha^2 + 35n^2\alpha + 7n^3\alpha - \alpha^3 x - 4\alpha^2 x - 3n\alpha^2 x - 3n^2\alpha x - 8n\alpha x)\,\mathrm{L}(n) + (-66 - 22x^2 + 70x - 149n - 23nx^2 + 62xn^2 + x^3 n - 6x^2n^2 + 11xn^3 - 47\alpha^2 + x^3\alpha - 17\alpha x^2 + 2x^3 - \alpha^4 - 11\alpha^3 - 45n^3 - 6n^4 - 91\alpha - 153n\alpha + 75x\alpha + 115xn - 124n^2 - 6n\alpha^3 - 52n\alpha^2 - 14n^2\alpha^2 - 84n^2\alpha - 15n^3\alpha + 3\alpha^3 x + 26\alpha^2 x - 3\alpha^2x^2 + 14n\alpha^2 x - 9n\alpha x^2 + 22n^2\alpha x + 82n\alpha x)\,\mathrm{L}(n+1) + (110 + 26x^2 - 102x + 219n + 25nx^2 - 70xn^2 - x^3 n + 6x^2n^2 - 11xn^3 + 22\alpha^2 + 6\alpha x^2 - 2x^3 + 2\alpha^3 + 51n^3 + 6n^4 + 82\alpha + 119n\alpha - 48x\alpha - 147xn + 160n^2 + n\alpha^3 + 21n\alpha^2 + 5n^2\alpha^2 + 57n^2\alpha + 9n^3\alpha - 6\alpha^2 x - 3n\alpha^2 x + 3n\alpha x^2 - 11n^2\alpha x - 46n\alpha x)\,\mathrm{L}(2+n) + (-8n^2\alpha - n^3\alpha - 21n\alpha - 18\alpha - 66n^2 - 19n^3 - 99n - 54 + 8xn^2 + xn^3 + 21xn + 18x - 2n^4)\,\mathrm{L}(n+3),\ \mathrm{L}(2) = \frac{1}{4}\_C_1\,x\,\_C_2 + \frac{1}{4}\_C_0\,\_C_3\,x - \_C_0\,\alpha\,\_C_3 + \frac{1}{2}\_C_0\,\alpha\,\_C_2 + \frac{1}{4}\_C_1\,\alpha^2\,\_C_3 - \frac{1}{4}\_C_1\,\alpha^2\,\_C_2 + \frac{1}{4}\_C_1\,x^2\,\_C_3 - \frac{1}{4}\_C_0\,\alpha^2\,\_C_3 - \frac{1}{2}\_C_1\,x\,\_C_3\,\alpha + \frac{1}{4}\_C_1\,x\,\_C_2\,\alpha + \frac{1}{4}\_C_0\,\alpha\,\_C_3\,x + \frac{3}{2}\_C_1\,\_C_3\,\alpha - \_C_1\,\_C_2\,\alpha - \frac{3}{2}\_C_1\,x\,\_C_3 - \frac{3}{4}\_C_1\,\_C_2 - \frac{3}{4}\_C_0\,\_C_3 + \frac{1}{4}\_C_0\,\alpha^2\,\_C_2 + \frac{1}{4}\_C_0\,\_C_2 + \frac{9}{4}\_C_1\,\_C_3,\ \mathrm{L}(0) = \_C_0\,\_C_2,\ \mathrm{L}(1) = \_C_1\,\_C_3\}$$

and this is the differential equation for the square of the Laguerre polynomials.

```
> `diffeq*diffeq`(DE,DE,L(x));
```

$$(2n - 4xn + 4n\alpha)\,\mathrm{L}(x) + (4xn + 3\alpha + 1 + 2x^2 - 4x\alpha - 4x + 2\alpha^2)\left(\frac{d}{dx}\mathrm{L}(x)\right) + (-3x^2 + 3x\alpha + 3x)\left(\frac{d^2}{dx^2}\mathrm{L}(x)\right) + \left(\frac{d^3}{dx^3}\mathrm{L}(x)\right)x^2$$

Next, we define the summand of the Hahn polynomials

```
> hahnterm:=(-1)^n*pochhammer(N-n,n)*pochhammer(beta+1,n)/n!*hy
```

```
perterm([-n,-x,alpha+beta+n+1],[beta+1,1-N],1,k);
```

$$hahnterm := (-1)^n \,\mathrm{pochhammer}(N-n, n)\, \mathrm{pochhammer}(\beta+1, n)\, \mathrm{pochhammer}(-n, k)\, \mathrm{pochhammer}(-x, k)\, \mathrm{pochhammer}(\alpha+\beta+n+1, k) / (n!\, \mathrm{pochhammer}(\beta+1, k)\, \mathrm{pochhammer}(1-N, k)\, k!)$$

and compute a recurrence equation for them

```
> sumrecursion(hahnterm,k,h(n));
```

$$(n+1-N)(n+1+\alpha)(n+1+\beta)(\alpha+4+2n+\beta)(1+N+\alpha+\beta+n)\,\mathrm{h}(n) + (2n+3+\alpha+\beta)(4+6x\beta+8x+6n+\alpha\beta+2\beta^2+5\beta n+4x\beta n+4xn^2+\beta^2 n + \beta n^2 - 6nN - 2n^2N - \beta^2 N - 2\beta nN - 2n\alpha N - \beta\alpha N - \alpha^2 - 3\beta N + 5\beta + \alpha - n\alpha + 6x\alpha + 12xn + x\beta^2 - 3\alpha N + 2n^2 - n\alpha^2 - n^2\alpha + \alpha^2 x + 2x\alpha\beta + 4n\alpha x - 4N)\,\mathrm{h}(n+1) - (2+n)(\alpha+\beta+n+2)(2n+\alpha+\beta+2)\,\mathrm{h}(2+n) = 0$$

Similarly, a difference equation w.r.t. x is obtained

```
> sumrecursion(hahnterm,k,h(x));
```

$$(x+1)(x-\alpha+1-N)\,\mathrm{h}(x) - (6x+2x^2+2\beta-\alpha-x\alpha-2xN+5-\beta N-3N+n+n\alpha+x\beta+n^2+\beta n)\,\mathrm{h}(x+1) + (x+2-N)(x+\beta+2)\,\mathrm{h}(x+2) = 0$$

```
>
```

Orthogonal Polynomial Solutions of Recurrence Equations

```
> read "retode.mpl";
```

Package "REtoDE", Maple V - Maple 8

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Example recurrence

```
> RE:=P(n+2)-(x-n-1)*P(n+1)+alpha*(n+1)^2*P(n)=0;
```

$$RE := \mathrm{P}(2+n) - (x-n-1)\,\mathrm{P}(n+1) + \alpha\,(n+1)^2\,\mathrm{P}(n) = 0$$

Classical continuous solutions

```
> REtoDE(RE,P(n),x);
```

Warning: parameters have the values, $\{a=0, b=2c, \alpha=\frac{1}{4}, e=0, c=c, d=-4c\}$

$$\left[\frac{1}{2}(2x+1)\left(\frac{\partial^2}{\partial x^2}\mathrm{P}(n,x)\right) - 2x\left(\frac{\partial}{\partial x}\mathrm{P}(n,x)\right) + 2n\,\mathrm{P}(n,x) = 0, \left[I = \left[\frac{-1}{2}, \infty\right], \rho(x) = 2\,\mathbf{e}^{(-2x)}\right], \frac{k_{n+1}}{k_n} = 1\right]$$

Classical discrete solutions

```
> REtodiscreteDE(RE,P(n),x);
```

Warning: parameters have the values, $\{f=f, \alpha=\frac{f^2-1}{4f^2}, c=-\frac{1}{4}f^2 d + \frac{1}{4}d + \frac{1}{2}g\,d\,f + \frac{1}{2}g\,d,$

$$e=-g\,d,\ g=g,\ a=0,\ b=-\frac{1}{2}f\,d-\frac{1}{2}d,\ d=d\}$$

$$\left[\frac{1}{2}\frac{(f+2\,f\,x-1)\,\Delta(\mathrm{Nabla}(\mathrm{P}(n,f\,x+g),x),x)}{f}-\frac{2\,x\,\Delta(\mathrm{P}(n,f\,x+g),x)}{f+1}\right.$$

$$+\frac{2\,n\,\mathrm{P}(n,f\,x+g)}{(f+1)\,f}=0,$$

$$\left[\sigma(x)=\frac{f}{2}+x-\frac{1}{2}-g,\ \sigma(x)+\tau(x)=\frac{(f-1)\,(f+2\,x+1-2\,g)}{2\,(f+1)}\right],\ \rho(x)=\left(\frac{f-1}{f+1}\right)^{x},$$

$$\left.\frac{k_{n+1}}{k_n}=\frac{1}{f}\right]$$

```
> strict:=true;
```

$$strict := true$$

Without translations no classical discrete solutions exist.

```
> REtodiscreteDE(RE,P(n),x);
Error, (in REtodiscreteDE) this recurrence equation has no classical
discrete orthogonal polynomial solutions
>
>
```