Computer Algebra Methods for Orthogonal Polynomials

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CASC 2004, St. Petersburg
July 12-19, 2004

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Online Demonstrations with Computer Algebra

- I will use the computer algebra system *Maple* to demonstrate and program the algorithms presented.
- Of course, we could also easily use any other system like *Mathematica*, MuPAD or Reduce.
- The following algorithms are most prominently used: linear algebra techniques, multivariate polynomial factorization and the solution of nonlinear equations, e. g. by Gröbner basis techniques.

Scalar Products

- Given: a scalar product
$$\langle f,g\rangle := \int_a^b f(x)g(x)\,d\mu(x)$$
with non-negative measure $\mu(x)$ supported in an interval $[a,b]$.

- Particular cases:
  - absolutely continuous measure $d\mu(x) = \rho(x)\,dx$ with weight function $\rho(x)$,
  - discrete measure $\mu(x) = \rho(x)$ with support in $\mathbb{Z}$,
  - discrete measure $\mu(x) = \rho(x)$ with support in $q^{\mathbb{Z}}$.

Orthogonal Polynomials

- A family $P_n(x)$ of polynomials

$$P_n(x) = k_n x^n + k_n' x^{n-1} + k_n'' x^{n-2} + \cdots, \quad k_n \neq 0$$

is called orthogonal w. r. t. the positive definite measure $\mu(x)$, if

$$\langle P_m, P_n \rangle = \begin{cases} 0 & \text{if } m \neq n \\ d_n^2 \neq 0 & \text{if } m = n \end{cases}$$

Classical Families

- The classical orthogonal polynomials can be defined as the polynomial solutions of the differential equation:

$$\sigma(x)P_n''(x) + \tau(x)P_n'(x) + \lambda_n P_n(x) = 0 \, .$$

- Conclusions:
  - $n = 1$ implies $\tau(x) = dx + e, d \neq 0$
  - $n = 2$ implies $\sigma(x) = ax^2 + bx + c$
  - coefficient of x^n implies $\lambda_n = -n(a(n-1) + d)$

Classification

- The classical systems can be classified according to the scheme (Bochner 1929):
- $\sigma(x) = 0$ powers x^n
- $\sigma(x) = 1$ Hermite polynomials
- $\sigma(x) = x$ Laguerre polynomials
- $\sigma(x) = x^2$ powers, Bessel polynomials
- $\sigma(x) = x^2 - 1$ Jacobi polynomials




Hermite, Laguerre, Jacobi and Bessel

Weight function

- The weight function $\rho(x)$ corresponding to the differential equation satisfies Pearson's differential equation

$$\frac{d}{dx}\Big(\sigma(x)\rho(x)\Big) = \tau(x)\rho(x)\ .$$

- Hence it is given as

$$\rho(x) = \frac{C}{\sigma(x)} e^{\int \frac{\tau(x)}{\sigma(x)} dx}\ .$$

Classical Discrete Families

- The classical discrete orthogonal polynomials can be defined as the polynomial solutions of the difference equation: $(\Delta f(x) = f(x+1) - f(x), \nabla f(x) = f(x) - f(x-1))$

$$\sigma(x)\Delta\nabla P_n(x) + \tau(x)\Delta P_n(x) + \lambda_n P_n(x) = 0\ .$$

- Conclusions:
  - $n = 1$ implies $\tau(x) = dx + e, d \neq 0$
  - $n = 2$ implies $\sigma(x) = ax^2 + bx + c$
  - coefficient of x^n implies $\lambda_n = -n(a(n-1) + d)$

Classification

- The classical discrete systems can be classified according to the scheme (Nikiforov, Suslov, Uvarov 1991):
- $\sigma(x) = 0$ — falling factorials
  $x^{\underline{n}} = x(x-1)\cdots(x-n+1)$
- $\sigma(x) = 1$ — translated Charlier polynomials
- $\sigma(x) = x$ — falling factorials, Charlier, Meixner, Krawtchouk polynomials
- $\deg(\sigma(x), x) = 2$ — Hahn polynomials

Weight function

- The weight function $\rho(x)$ corresponding to the difference equation satisfies Pearson's difference equation

$$\Delta\Big(\sigma(x)\rho(x)\Big) = \tau(x)\rho(x)\ .$$

- Hence it is given as

$$\frac{\rho(x+1)}{\rho(x)} = \frac{\sigma(x)+\tau(x)}{\sigma(x+1)}\ .$$

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Hypergeometric Functions

- The power series

$$ {}_pF_q\left(\begin{matrix} a_1, \ldots, a_p \\ b_1, \ldots, b_q \end{matrix} \middle| \, z\right) = \sum_{k=0}^{\infty} A_k \, z^k \ , $$

whose coefficients $\alpha_k = A_k z^k$ have rational term ratio

$$ \frac{\alpha_{k+1}}{\alpha_k} = \frac{A_{k+1} \, z^{k+1}}{A_k \, z^k} = \frac{(k+a_1)\cdots(k+a_p)}{(k+b_1)\cdots(k+b_q)} \frac{z}{(k+1)} $$

is called the generalized hypergeometric function.

Hypergeometric Terms

- The summand $\alpha_k = A_k z^k$ of a hypergeometric series is called a hypergeometric term w. r. t. k.

- The relation

$$\frac{\rho(x+1)}{\rho(x)} = \frac{\sigma(x) + \tau(x)}{\sigma(x+1)}$$

therefore states that the weight functions $\rho(x)$ of classical discrete orthogonal polynomials are hypergeometric terms w. r. t. the variable x.

Coefficients of Hypergeometric Functions

- For the coefficients of the hypergeometric function we get the formula

$$ {}_pF_q\left(\left.\begin{matrix} a_1, \ldots, a_p \\ b_1, \ldots, b_q \end{matrix}\right| z\right) = \sum_{k=0}^{\infty} \frac{(a_1)_k \cdots (a_p)_k}{(b_1)_k \cdots (b_q)_k} \frac{z^k}{k!}\,, $$

  where $(a)_k = a(a+1)\cdots(a+k-1)$ is called the Pochhammer symbol (or shifted factorial).

Examples of Hypergeometric Functions

$$e^z = {}_0F_0(z)$$

$$\sin z = z \cdot {}_0F_1\left(\left.\begin{matrix} - \\ 3/2 \end{matrix}\right| -\frac{z^2}{4}\right)$$

- Further examples: $\cos(z), \arcsin(z), \arctan(z)$, $\ln(1+z), \operatorname{erf}(z), L_n^{(\alpha)}(z)$, . . . , but for example not $\tan(z)$.

Classical Discrete Orthogonal Polynomials of Hahn Class as Hypergeometric Functions

- From the differential or difference equation, one can determine a hypergeometric representation.
- To get this representation, one determines by linear algebra the coefficients of the following identities

$$
\begin{aligned}
\textbf{(RE)} \qquad x\,P_n(x) &= a_n\,P_{n+1}(x) + b_n\,P_n(x) + c_n\,P_{n-1}(x) \\
\textbf{(DR)} \qquad \sigma(x)\,P_n'(x) &= \alpha_n\,P_{n+1}(x) + \beta_n\,P_n(x) + \gamma_n\,P_{n-1}(x) \\
\textbf{(SR)} \qquad P_n(x) &= \widehat{a}_n\,P_{n+1}'(x) + \widehat{b}_n\,P_n'(x) + \widehat{c}_n\,P_{n-1}'(x)
\end{aligned}
$$

in terms of the given numbers a, b, c, d and e.

Classical Discrete Orthogonal Polynomials of Hahn Class as Hypergeometric Functions

- Combining these equations one obtains for the coefficients $C_k(n)$ of the power series for the monic polynomials
$$\widetilde{P}_n(x) = \sum_{k=0}^{n} C_k(n)\, x^n$$
(again by linear algebra) the recurrence equation
$$\begin{aligned}&(k-n)(an+d-a+ak)C_k(n)\\&+(k+1)(bk+e)C_{k+1}(n)\\&+c(k+1)(k+2)C_{k+2}(n) = 0\,.\end{aligned}$$

Classical Discrete Orthogonal Polynomials of Hahn Class as Hypergeometric Functions

- From these general results, we get, for example, for the Laguerre polynomials

$$L_n^{\alpha}(x) = \frac{(\alpha+1)_n}{n!}\, {}_1F_1\left(\left.\begin{matrix} -n \\ \alpha+1 \end{matrix}\right| x\right) ,$$

and the Hahn polynomials are given by $h_n^{(\alpha,\beta)}(x,N) =$

$$\frac{(-1)^n (N-n)_n (\beta+1)_n}{n!}\, {}_3F_2\left(\left.\begin{matrix} -n, -x, \alpha+\beta+n+1 \\ \beta+1, 1-N \end{matrix}\right| 1\right).$$

Zeilberger's Algorithm

- In 1990 Zeilberger developed an algorithm to detect a holonomic recurrence equation for hypergeometric sums

$$s_n = \sum_{k=-\infty}^{\infty} F(n,k) \; .$$

- A recurrence equation is called holonomic, if it is homogeneous, linear and has polynomial coefficients.
- The holonomic recurrence equation constitutes a normal form for holonomic sequences.

Zeilberger's Algorithm

- A similar algorithm detects a holonomic differential equation for sums of the form
$$s(x) = \sum_{k=-\infty}^{\infty} F(x,k) \; .$$
- The holonomic differential equation constitutes a normal form for holonomic functions.
- Holonomic functions form an algebra, i.e. sum and product of holonomic functions are holonomic, and there are linear algebra algorithms to compute the corresponding differential / recurrence equations.

Application to Orthogonal Polynomials

- As examples, we apply Zeilberger's algorithm to the Laguerre polynomials

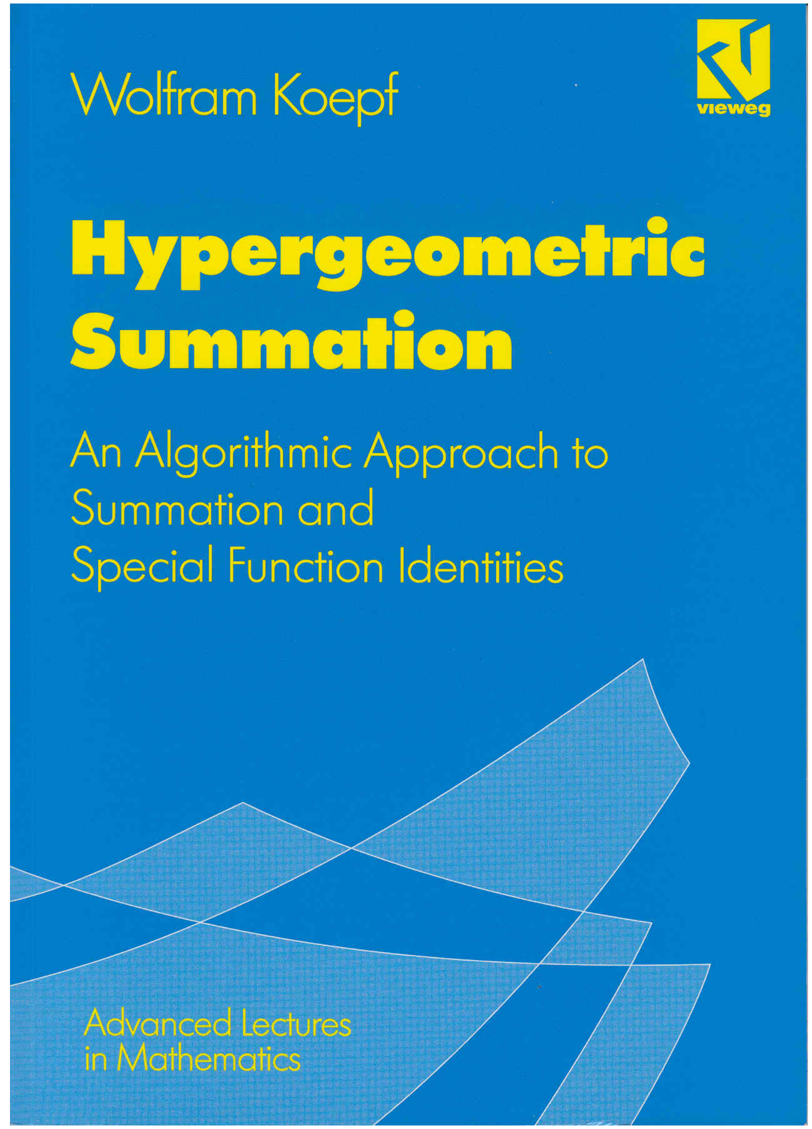
$$L_n^\alpha(x) = \frac{(\alpha+1)_n}{n!}\, {}_1F_1\left(\left.\begin{matrix} -n \\ \alpha+1 \end{matrix}\right| x\right)$$

and to the Hahn polynomials $h_n^{(\alpha,\beta)}(x,N) =$

$$\frac{(-1)^n (N-n)_n (\beta+1)_n}{n!}\, {}_3F_2\left(\left.\begin{matrix} -n, -x, \alpha+\beta+n+1 \\ \beta+1, 1-N \end{matrix}\right| 1\right).$$

The software used was developed for my book

Hypergeometric Summation, Vieweg, 1998, Braunschweig/Wiesbaden



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Computation of the Differential Equation from the Recurrence Equation

- We have shown how the recurrence equation can be explicitly expressed in terms of the coefficients of the differential / difference equation.

- If one uses this information in the opposite direction, then the corresponding differential / difference equation can be obtained from a given three-term recurrence.

Example

- Let the recurrence

$$P_{n+2}(x) - (x - n - 1)\, P_{n+1}(x) + \alpha(n+1)^2 P_n(x) = 0$$

  be given.

- We can compute that for $\alpha = 1/4$ this corresponds to translated Laguerre polynomials, and for $\alpha < 1/4$ Meixner and Krawtchouk polynomial solutions occur.

The End

Thank you very much for your attention!