Tutorial ISSAC 2004, July 4, 2004

Wolfram Koepf: Power Series and Summation

```
> restart;
```

Computation of Power Series

Maple supports truncated power series

```
> series(exp(x),x);
```

$$1+x+\frac{1}{2}x^2+\frac{1}{6}x^3+\frac{1}{24}x^4+\frac{1}{120}x^5+O(x^6)$$

The following algorithm for the computation of Formal Power Series is from
Koepf, Wolfram: Power Series in Computer Algebra, Journal of Symbolic Computation 13, 1992, 581-603

```
> read "FPS.mpl";
```

Package Formal Power Series, Maple V - Maple 8

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```
> FPS(exp(x),x);
```

$$\sum_{k=0}^{\infty}\frac{x^k}{k!}$$

```
> infolevel[FPS]:=5:
> FPS(exp(x),x);
```

```
FPS/FPS:   looking for DE of degree   1
FPS/FPS:   DE of degree   1   found.
FPS/FPS:   DE =
```

$$F'(x)-F(x)=0$$

```
FPS/FPS:   RE =
```

$$a(k+1)=\frac{a(k)}{k+1}$$

```
FPS/hypergeomRE:   RE is of hypergeometric type.
FPS/hypergeomRE:   Symmetry number m :=    1
FPS/hypergeomRE:   RE:
```

$$(k+1)\,a(k+1)=a(k)$$

```
FPS/hypergeomRE:   RE valid for all k >=    0
FPS/hypergeomRE:   a(0) =   1
```

$$\sum_{k=0}^{\infty}\frac{x^k}{k!}$$

```
> FPS(exp(x^2),x);
```

```
FPS/FPS:   looking for DE of degree   1
FPS/FPS:   DE of degree   1   found.
FPS/FPS:   DE =
```

$$F'(x)-2\,x\,F(x)=0$$

```
FPS/FPS:   RE =
```

$$a(k+1) = \frac{2\,a(k-1)}{k+1}$$

```
FPS/hypergeomRE:   RE is of hypergeometric type.
FPS/hypergeomRE:   Symmetry number m :=    2
FPS/hypergeomRE:   RE:
```

$$(k+2)\,a(k+2) = 2\,a(k)$$

```
FPS/hypergeomRE:   RE valid for all k >=    -1
FPS/hypergeomRE:   a(0) =   1
```

$$\sum_{k=0}^{\infty} \frac{x^{(2k)}}{k!}$$

a Puiseux series

```
> FPS(exp(sqrt(x)),x);
FPS/FPS:    looking for DE of degree    1
FPS/FPS:    looking for DE of degree    2
FPS/FPS:    DE of degree    2    found.
FPS/FPS:    DE =
```

$$4\,x\,F''(x) + 2\,F'(x) - F(x) = 0$$

```
FPS/FPS:    RE =
```

$$a(k+1) = \frac{1}{2}\,\frac{a(k)}{(k+1)\,(2\,k+1)}$$

```
FPS/hypergeomRE:   RE is of hypergeometric type.
FPS/hypergeomRE:   Symmetry number m :=    1
FPS/hypergeomRE:   RE:
```

$$2\,(k+1)\,(2\,k+1)\,a(k+1) = a(k)$$

```
FPS/hypergeomRE:   RE modified to     k = 1/2*k
FPS/hypergeomRE:   => f :=     exp(x)
FPS/hypergeomRE:   RE is of hypergeometric type.
FPS/hypergeomRE:   Symmetry number m :=    2
FPS/hypergeomRE:   RE:
```

$$(k+2)\,(k+1)\,a(k+2) = a(k)$$

```
FPS/hypergeomRE:   RE valid for all k >=    0
FPS/hypergeomRE:   a(0) =   1
FPS/hypergeomRE:   a(1) =   1
```

$$\left(\sum_{k=0}^{\infty} \frac{x^k}{(2\,k)!}\right) + \left(\sum_{k=0}^{\infty} \frac{x^{(k+1/2)}}{(2\,k+1)!}\right)$$

```
> FPS(arcsin(x),x);
FPS/FPS:    looking for DE of degree    1
FPS/FPS:    looking for DE of degree    2
FPS/FPS:    DE of degree    2    found.
FPS/FPS:    DE =
```

$$(-1 + x^2)\,F''(x) + x\,F'(x) = 0$$

```
FPS/FPS:    RE =
```

$$a(k+2) = \frac{k^2\,a(k)}{(k+1)\,(k+2)}$$

```
FPS/hypergeomRE:   RE is of hypergeometric type.
FPS/hypergeomRE:   Symmetry number m :=    2
FPS/hypergeomRE:   RE:
```

$$-(k+1)\,(k+2)\,a(k+2) = -k^2\,a(k)$$

```
FPS/hypergeomRE:   RE valid for all k >=    0
FPS/hypergeomRE:   a(0) =   0
FPS/hypergeomRE:   a(2*j) = 0   for all j>0.
FPS/hypergeomRE:   a(1) =   1
```

$$\sum_{k=0}^{\infty} \frac{(2\,k)!\,4^{(-k)}\,x^{(2\,k+1)}}{(k!)^2\,(2\,k+1)}$$

```
> infolevel[FPS]:=0:
```

computation in steps

```
> f[0]:=arcsin(x);
```

$$f_0 := \arcsin(x)$$

$$(x+1)^k$$

$$f_0 := \arcsin(x)$$

```
> f[1]:=diff(f[0],x);
```

$$f_1 := \frac{1}{\sqrt{1-x^2}}$$

```
> normal(f[1]/f[0]);
```

$$\frac{1}{\sqrt{1-x^2}\,\arcsin(x)}$$

```
> f[2]:=diff(f[1],x);
```

$$f_2 := \frac{x}{(1-x^2)^{(3/2)}}$$

```
> ansatz:=sum(c[k]*f[k],k=0..2);
```

$$ansatz := c_0\,\arcsin(x) + \frac{c_1}{\sqrt{1-x^2}} + \frac{c_2\,x}{(1-x^2)^{(3/2)}}$$

```
> normal(subs(c[0]=0,ansatz));
```

$$-\frac{-c_1 + c_1\,x^2 - c_2\,x}{(1-x^2)^{(3/2)}}$$

```
> sol:=solve(normal(subs(c[0]=0,ansatz)),{c[1],c[2]});
```

$$sol := \{c_2 = c_2,\ c_1 = \frac{c_2\,x}{-1+x^2}\}$$

```
> DE:=c[0]*F(x)+c[1]*diff(F(x),x)+c[2]*diff(F(x),x$2);
```

$$DE := c_0\,\mathrm{F}(x) + c_1\left(\frac{d}{dx}\,\mathrm{F}(x)\right) + c_2\left(\frac{d^2}{dx^2}\,\mathrm{F}(x)\right)$$

```
> collect(numer(normal(subs(sol,c[0]=0,DE/c[2]))),diff)=0;
```

$$x\left(\frac{d}{dx}\,\mathrm{F}(x)\right) + (-1+x^2)\left(\frac{d^2}{dx^2}\,\mathrm{F}(x)\right) = 0$$

procedure combining these steps

```
> DE:=HolonomicDE(arcsin(x),F(x));
```

$$DE := x\left(\frac{d}{dx}\mathrm{F}(x)\right)+(x-1)(x+1)\left(\frac{d^2}{dx^2}\mathrm{F}(x)\right)=0$$

```
> dsolve(DE,F(x));
```

$$\mathrm{F}(x)=\_C1+\ln(x+\sqrt{-1+x^2})\,\_C2$$

```
> RE:=SimpleRE(arcsin(x),x,a(k));
```

$$RE := -(k+1)(k+2)\,\mathrm{a}(k+2)+k^2\,\mathrm{a}(k)=0$$

```
> rsolve({RE,a(0)=0,a(1)=1},a(k));
```

$$\begin{cases} 0 & k::even \\ \dfrac{\Gamma\left(\frac{k}{2}\right)}{k\sqrt{\pi}\,\Gamma\left(\frac{k}{2}+\frac{1}{2}\right)} & k::odd \end{cases}$$

some final examples: a Laurent series

```
> FPS(arcsin(x)^2/x^5,x);
```

$$\sum_{k=0}^{\infty}\frac{(k!)^2\,4^k\,x^{(2k-3)}}{(1+2k)!\,(k+1)}$$

a complicated example that cannot be found in Gradshteyn/Ryshik

```
> FPS(exp(arcsin(x)),x);
```

$$\left(\sum_{k=0}^{\infty}\frac{\left(\prod_{j=0}^{k}(4j^2+1)\right)x^{(2k)}}{(4k^2+1)\,(2k)!}\right)+\left(\sum_{k=0}^{\infty}\frac{\left(\prod_{j=0}^{k}(1+2j+2j^2)\right)2^k\,x^{(2k+1)}}{(2k^2+2k+1)\,(2k+1)!}\right)$$

and an asymptotic series

```
> FPS((erf(x)-1)*exp(x^2),x=infinity);
```

$$-\frac{\displaystyle\sum_{k=0}^{\infty}\frac{(-1)^k\,(2k)!\,4^{(-k)}\left(\frac{1}{x}\right)^{(2k+1)}}{k!}}{\sqrt{\pi}}$$

Also covered are holonomic special functions

```
> FPS(LegendreP(n,x),x);
```

$$\frac{2\sqrt{\pi}\left(\displaystyle\sum_{k=0}^{\infty}\frac{\mathrm{pochhammer}\left(-\frac{n}{2},k\right)\mathrm{pochhammer}\left(\frac{n}{2}+\frac{1}{2},k\right)4^k\,x^{(2k)}}{(2k)!}\right)}{\Gamma\left(\frac{1}{2}-\frac{n}{2}\right)\Gamma\left(\frac{n}{2}\right)n}$$

$$-\frac{2\sqrt{\pi}\left(\sum_{k=0}^{\infty}\frac{\mathrm{pochhammer}\left(\frac{1}{2}-\frac{n}{2},k\right)\mathrm{pochhammer}\left(1+\frac{n}{2},k\right)4^k x^{(2k+1)}}{(2k+1)!}\right)}{\Gamma\left(\frac{n}{2}+\frac{1}{2}\right)\Gamma\left(-\frac{n}{2}\right)}$$

```
> FPS(LegendreP(n,x),x=1);
```

$$\sum_{k=0}^{\infty}\frac{(-1)^k 2^{(-k)}\,\mathrm{pochhammer}(n+1,k)\,\mathrm{pochhammer}(-n,k)\,(x-1)^k}{(k!)^2}$$

```
>
```

Computation of Holonomic Differential Equations

Find a holonomic differential equation for f(x)=sin(x)*exp(x)

```
> f[0]:=sin(x)*exp(x);
```

$$f_0 := \sin(x)\,\mathbf{e}^x$$

```
> f[1]:=diff(f[0],x);
```

$$f_1 := \cos(x)\,\mathbf{e}^x + \sin(x)\,\mathbf{e}^x$$

```
> normal(f[1]/f[0]);
```

$$\frac{\cos(x)+\sin(x)}{\sin(x)}$$

```
> f[2]:=diff(f[1],x);
```

$$f_2 := 2\cos(x)\,\mathbf{e}^x$$

```
> ansatz:=expand(sum(c[k]*f[k],k=0..2));
```

$$ansatz := c_0\sin(x)\,\mathbf{e}^x + c_1\cos(x)\,\mathbf{e}^x + c_1\sin(x)\,\mathbf{e}^x + 2\,c_2\cos(x)\,\mathbf{e}^x$$

```
> ansatz:=collect(ansatz,{cos(x),sin(x)});
```

$$ansatz := (c_0\,\mathbf{e}^x + c_1\,\mathbf{e}^x)\sin(x) + (c_1\,\mathbf{e}^x + 2\,c_2\,\mathbf{e}^x)\cos(x)$$

```
> coeffs(ansatz,{cos(x),sin(x)});
```

$$c_0\,\mathbf{e}^x + c_1\,\mathbf{e}^x,\ c_1\,\mathbf{e}^x + 2\,c_2\,\mathbf{e}^x$$

```
> sol:=solve({coeffs(ansatz,{cos(x),sin(x)})},{c[0],c[1],c[2]})
;
```

$$sol := \{c_0 = 2\,c_2,\ c_1 = -2\,c_2,\ c_2 = c_2\}$$

```
> DE:=c[0]*F(x)+c[1]*diff(F(x),x)+c[2]*diff(F(x),x$2);
```

$$DE := c_0\,\mathrm{F}(x) + c_1\left(\frac{d}{dx}\mathrm{F}(x)\right) + c_2\left(\frac{d^2}{dx^2}\mathrm{F}(x)\right)$$

```
> DE:=collect(numer(normal(subs(sol,DE/c[0]))),diff)=0;
```

$$DE := 2\,\mathrm{F}(x) - 2\left(\frac{d}{dx}\mathrm{F}(x)\right) + \left(\frac{d^2}{dx^2}\mathrm{F}(x)\right) = 0$$

```
> f:='f':
>
```

Algebra of Holonomic Functions

```
> read "FPS.mpl";
```

Package Formal Power Series, Maple V - Maple 8

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```
> with(gfun);
```

[*Laplace, algebraicsubs, algeqtodiffeq, algeqtoseries, algfuntoalgeq, borel, cauchyproduct, diffeq*diffeq, diffeq+diffeq, diffeqtable, diffeqtohomdiffeq, diffeqtorec, guesseqn, guessgf, hadamardproduct, holexprtodiffeq, invborel, listtoalgeq, listtodiffeq, listtohypergeom, listtolist, listtoratpoly, listtorec, listtoseries, maxdegcoeff, maxdegeqn, maxordereqn, mindegcoeff, mindegeqn, minordereqn, optionsgf, poltodiffeq, poltorec, ratpolytocoeff, rec*rec, rec+rec, rectodiffeq, rectohomrec, rectoproc, seriestoalgeq, seriestodiffeq, seriestohypergeom, seriestolist, seriestoratpoly, seriestorec, seriestoseries*]

The function sin(x)*exp(x), again:

The differential equation of sin(x):

```
> DE1:=diff(F(x),x$2)+F(x)=0;
```

$$DE1 := \left(\frac{d^2}{dx^2} F(x)\right) + F(x) = 0$$

The differential equation of exp(x):

```
> DE2:=diff(F(x),x)-F(x)=0;
```

$$DE2 := \left(\frac{d}{dx} F(x)\right) - F(x) = 0$$

```
> `diffeq*diffeq`(DE1,DE2,F(x));
```

$$2\,F(x) - 2\left(\frac{d}{dx} F(x)\right) + \left(\frac{d^2}{dx^2} F(x)\right)$$

and the sum sin(x)+exp(x) satisfies

```
> `diffeq+diffeq`(DE1,DE2,F(x));
```

$$\left(\frac{d^3}{dx^3} F(x)\right) + \left(\frac{d}{dx} F(x)\right) - \left(\frac{d^2}{dx^2} F(x)\right) - F(x)$$

Now a more complicated example: exp(x)*Ai(x)

```
> DE1:=diff(F(x),x)-F(x)=0;
```

$$DE1 := \left(\frac{d}{dx} F(x)\right) - F(x) = 0$$

```
> DE2:=HolonomicDE(AiryAi(x),F(x));
```

$$DE2 := \left(\frac{d^2}{dx^2} F(x)\right) - x\,F(x) = 0$$

```
> `diffeq*diffeq`(DE1,DE2,F(x));
```

$$(1 - x)\,F(x) + \left(\frac{d^2}{dx^2} F(x)\right) - 2\left(\frac{d}{dx} F(x)\right)$$

and the sum exp(x)+Ai(x) satisfies

```
> 'diffeq+diffeq'(DE1,DE2,F(x));
```

$$\{(D^{(2)})(F)(0) = \_C_0,$$

$$(-x+1+x^2)\,F(x) + (x-x^2)\left(\frac{d}{dx}F(x)\right) - x\left(\frac{d^2}{dx^2}F(x)\right) + (x-1)\left(\frac{d^3}{dx^3}F(x)\right)\}$$

Similarly, HolonomicDE yields

```
> HolonomicDE(exp(x)+AiryAi(x),F(x));
```

$$(-x+1+x^2)\,F(x) - x\,(x-1)\left(\frac{d}{dx}F(x)\right) - x\left(\frac{d^2}{dx^2}F(x)\right) + (x-1)\left(\frac{d^3}{dx^3}F(x)\right) = 0$$

Similar algorithms exist for sequences and recurrence equations. Assume we want to find a recurrence equation w.r.t. k for

```
> binomial(n,k)+binomial(k,n);
```

$$\mathrm{binomial}(n, k) + \mathrm{binomial}(k, n)$$

The binomial coefficient S(k)=binomial(n,k) (first summand) satisfies the equation

```
> S(k+1)/S(k)=expand(binomial(n,k+1)/binomial(n,k));
```

$$\frac{S(k+1)}{S(k)} = \frac{n-k}{k+1}$$

w.r.t. k. This gives the holonomic recurrence equation

```
> RE1:=collect(numer(normal(S(k+1)-expand(binomial(n,k+1)/binomial(n,k))*S(k))),S,factor);
```

$$RE1 := (k+1)\,S(k+1) + (k-n)\,S(k)$$

The binomial coefficient S(k)=binomial(k,n) (second summand) satisfies the equation

```
> S(k+1)/S(k)=expand(binomial(k+1,n)/binomial(k,n));
```

$$\frac{S(k+1)}{S(k)} = \frac{k+1}{k+1-n}$$

w.r.t. k. This gives the holonomic recurrence equation

```
> RE2:=collect(numer(normal(S(k+1)-expand(binomial(k+1,n)/binomial(k,n))*S(k))),S,factor);
```

$$RE2 := (k+1-n)\,S(k+1) + (-1-k)\,S(k)$$

Therefore we get for the sum

```
> 'rec+rec'(RE1,RE2,S(k));
```

$$\{S(1) = n\,\_C_0 + \_C_1,\ S(0) = \_C_0 - \_C_1\,n + \_C_1,\ (-2\,k^4 - 9\,k^3 - 13\,k^2 - 3\,n^2 + n^3$$
$$+ 14\,k^2\,n - 3\,k^2\,n^2 - 6\,k\,n^2 + k\,n^3 + 16\,k\,n - 6\,k + 6\,n + 4\,k^3\,n)\,S(k) + (5\,n^2 - 2\,k^3 + n^4$$
$$- 8\,k^2 - 2\,n - 4\,n^3 - 12\,k^2\,n + 12\,k\,n^2 - 4\,k\,n^3 - 4\,k^3\,n + 6\,k^2\,n^2 - 4 - 10\,k - 10\,k\,n)$$
$$S(k+1) + (2\,k^4 + 11\,k^3 + 21\,k^2 - 4\,k^3\,n - 16\,k^2\,n + 16\,k - 19\,k\,n + 3\,k^2\,n^2 + 9\,k\,n^2 - 6\,n$$
$$+ 6\,n^2 - k\,n^3 - 2\,n^3 + 4)\,S(k+2)\}$$

Just for fun we compute the recurrence equation for the product which - of course - is much simpler

```
> 'rec*rec'(RE1,RE2,S(k));
```

$$(k-n)\,S(k) + (k+1-n)\,S(k+1)$$

```
>
```

Hypergeometric Functions

```
> hypergeom([a,b],[c],x);
```

$$\mathrm{hypergeom}([a, b], [c], x)$$

```
> sumtools[hyperterm]([a,b],[c],x,k);
```

$$\frac{\mathrm{pochhammer}(a, k)\, \mathrm{pochhammer}(b, k)\, x^k}{\mathrm{pochhammer}(c, k)\, k!}$$

```
> sum(sumtools[hyperterm]([a,b],[c],x,k),k=0..infinity);
```

$$\mathrm{hypergeom}([a, b], [c], x)$$

```
> hypergeom([a,b],[c],1);
```

$$\mathrm{hypergeom}([a, b], [c], 1)$$

```
> simplify(hypergeom([a,b],[c],1));
```

$$\frac{\Gamma(c)\, \Gamma(c-a-b)}{\Gamma(c-a)\, \Gamma(c-b)}$$

```
>
```

Identification of Hypergeometric Functions

We are interested in

```
> s:=Sum((-1)^k/(2*k+1)!*x^(2*k+1),k=0..infinity);
```

$$s := \sum_{k=0}^{\infty} \frac{(-1)^k\, x^{(2\,k+1)}}{(2\,k+1)!}$$

```
> F:=k->(-1)^k/(2*k+1)!*x^(2*k+1);
```

$$F := k \rightarrow \frac{(-1)^k\, x^{(2\,k+1)}}{(2\,k+1)!}$$

```
> r:=F(k+1)/F(k);
```

$$r := \frac{(-1)^{(k+1)}\, x^{(2\,k+3)}\, (2\,k+1)!}{(2\,k+3)!\, (-1)^k\, x^{(2\,k+1)}}$$

```
> expand(r);
```

$$-\frac{x^2}{(2\,k+2)\,(2\,k+3)}$$

Hence

```
> s=F(0)*hypergeom([],[3/2],-x^2/4);
```

$$\sum_{k=0}^{\infty} \frac{(-1)^k\, x^{(2\,k+1)}}{(2\,k+1)!} = x\, \mathrm{hypergeom}\left([\;], \left[\frac{3}{2}\right], -\frac{x^2}{4}\right)$$

Check

```
> convert(s,hypergeom);
```

$$\sin(x)$$

Maple simplifies completely, hence we don’t see the hypergeometric form. The same applies to

```
> simplify(x*hypergeom([],[3/2],-x^2/4));
```

$$\sin(x)$$

The following procedure uses the given algorithm and gives therefore the hypergeometric form:

```
> sumtools[Sumtohyper](F(k),k);
```

$$x \,\mathrm{Hypergeom}\left([\ ], \left[\frac{3}{2}\right], -\frac{x^2}{4}\right)$$

Another example

```
> F:=binomial(n,k)*binomial(-n-1,k)*((1-x)/2)^k;
```

$$F := \mathrm{binomial}(n, k)\,\mathrm{binomial}(-n-1, k)\left(\frac{1}{2}-\frac{x}{2}\right)^k$$

```
> Sum(F,k=0..n)=sumtools[Sumtohyper](F,k);
```

$$\sum_{k=0}^{n} \mathrm{binomial}(n, k)\,\mathrm{binomial}(-n-1, k)\left(\frac{1}{2}-\frac{x}{2}\right)^k = \mathrm{Hypergeom}\left([-n, n+1], [1], \frac{1}{2}-\frac{x}{2}\right)$$

Details of this algorithm and an implementation can be found in the book
Wolfram Koepf: *Hypergeometric Summation*, Vieweg, Braunschweig/Wiesbaden, 1998

We can combine the FPS and the identification algorithm:

```
> s:=FPS(exp(x),x,k);
```

$$s := \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

```
> op(1,s);
```

$$\frac{x^k}{k!}$$

```
> sumtools[Sumtohyper](op(1,s),k);
```

$$\mathrm{Hypergeom}([\ ], [\ ], x)$$

```
> s:='s':
```

Write cos(x) in hypergeometric notation.

```
> fps:=FPS(cos(x),x,k);
```

$$fps := \sum_{k=0}^{\infty} \frac{(-1)^k x^{(2k)}}{(2k)!}$$

```
> sumtools[Sumtohyper](op(1,fps),k);
```

$$\mathrm{Hypergeom}\left([\ ], \left[\frac{1}{2}\right], -\frac{x^2}{4}\right)$$

```
>
```

Computation of Recurrence Equations for Hypergeometric Functions

How does one generate the result

```
> Sum(binomial(n,k),k=0..n)=
```

```
sum(binomial(n,k),k=0..n);
```

$$\sum_{k=0}^{n} \mathrm{binomial}(n, k) = 2^n$$

We do the following more complicated example with Maple:

```
> Sum(k*binomial(n,k),k=0..n)=
  sum(k*binomial(n,k),k=0..n);
```

$$\sum_{k=0}^{n} k\,\mathrm{binomial}(n, k) = \frac{2^n\, n}{2}$$

```
> F:=(n,k)->k*binomial(n,k);
```

$$F := (n, k) \to k\,\mathrm{binomial}(n, k)$$

```
> ansatz:=sum(sum(a(j,i)*F(n+j,k+i),i=0..1),j=0..1);
```

$$ansatz := \mathrm{a}(0, 0)\, k\,\mathrm{binomial}(n, k) + \mathrm{a}(0, 1)\,(k+1)\,\mathrm{binomial}(n, k+1) + \mathrm{a}(1, 0)\, k\,\mathrm{binomial}(n+1, k) + \mathrm{a}(1, 1)\,(k+1)\,\mathrm{binomial}(n+1, k+1)$$

```
> ansatz:=ansatz/F(n,k);
```

$$ansatz := (\mathrm{a}(0, 0)\, k\,\mathrm{binomial}(n, k) + \mathrm{a}(0, 1)\,(k+1)\,\mathrm{binomial}(n, k+1) + \mathrm{a}(1, 0)\, k\,\mathrm{binomial}(n+1, k) + \mathrm{a}(1, 1)\,(k+1)\,\mathrm{binomial}(n+1, k+1)) / (k\,\mathrm{binomial}(n, k))$$

```
> ansatz:=expand(ansatz);
```

$$ansatz := \mathrm{a}(0, 0) + \frac{\mathrm{a}(0, 1)\, n}{k+1} - \frac{k\,\mathrm{a}(0, 1)}{k+1} + \frac{\mathrm{a}(0, 1)\, n}{k\,(k+1)} - \frac{\mathrm{a}(0, 1)}{k+1} + \frac{\mathrm{a}(1, 0)\, n}{n-k+1} + \frac{\mathrm{a}(1, 0)}{n-k+1} + \frac{\mathrm{a}(1, 1)\, n}{k+1} + \frac{\mathrm{a}(1, 1)}{k+1} + \frac{\mathrm{a}(1, 1)\, n}{k\,(k+1)} + \frac{\mathrm{a}(1, 1)}{k\,(k+1)}$$

```
> ansatz:=normal(ansatz);
```

$$ansatz := -(-k^2\,\mathrm{a}(0, 0) + k^2\,\mathrm{a}(0, 1) + \mathrm{a}(0, 0)\, k\, n - \mathrm{a}(1, 1)\, n\, k - 2\,\mathrm{a}(0, 1)\, n\, k - \mathrm{a}(1, 1)\, k + \mathrm{a}(1, 0)\, n\, k + \mathrm{a}(1, 0)\, k + \mathrm{a}(0, 0)\, k - k\,\mathrm{a}(0, 1) + \mathrm{a}(1, 1) + \mathrm{a}(0, 1)\, n^2 + 2\,\mathrm{a}(1, 1)\, n + \mathrm{a}(1, 1)\, n^2 + \mathrm{a}(0, 1)\, n) / ((-n+k-1)\, k)$$

```
> ansatz:=numer(ansatz);
```

$$ansatz := k^2\,\mathrm{a}(0, 0) - k^2\,\mathrm{a}(0, 1) - \mathrm{a}(0, 0)\, k\, n + \mathrm{a}(1, 1)\, n\, k + 2\,\mathrm{a}(0, 1)\, n\, k + \mathrm{a}(1, 1)\, k - \mathrm{a}(1, 0)\, n\, k - \mathrm{a}(1, 0)\, k - \mathrm{a}(0, 0)\, k + k\,\mathrm{a}(0, 1) - \mathrm{a}(1, 1) - \mathrm{a}(0, 1)\, n^2 - 2\,\mathrm{a}(1, 1)\, n - \mathrm{a}(1, 1)\, n^2 - \mathrm{a}(0, 1)\, n$$

```
> eqs:={coeffs(ansatz,k)};
```

$$eqs := \{\mathrm{a}(0, 0) - \mathrm{a}(0, 1), -\mathrm{a}(1, 1)\, n^2 - \mathrm{a}(0, 1)\, n - \mathrm{a}(1, 1) - \mathrm{a}(0, 1)\, n^2 - 2\,\mathrm{a}(1, 1)\, n, \mathrm{a}(1, 1) - \mathrm{a}(0, 0)\, n + \mathrm{a}(1, 1)\, n + 2\,\mathrm{a}(0, 1)\, n + \mathrm{a}(0, 1) - \mathrm{a}(1, 0)\, n - \mathrm{a}(1, 0) - \mathrm{a}(0, 0)\}$$

```
> sol:=solve(eqs,{seq(seq(a(j,i),j=0..1),i=0..1)});
```

$$sol := \left\{\mathrm{a}(1, 0) = 0, \mathrm{a}(0, 0) = -\frac{(n+1)\,\mathrm{a}(1, 1)}{n}, \mathrm{a}(0, 1) = -\frac{(n+1)\,\mathrm{a}(1, 1)}{n}, \mathrm{a}(1, 1) = \mathrm{a}(1, 1)\right\}$$

```
> re:=sum(sum(a(j,i)*f(n+j,k+i),i=0..1),j=0..1);
```

$$re := a(0,0)\,f(n,k) + a(0,1)\,f(n,k+1) + a(1,0)\,f(n+1,k) + a(1,1)\,f(n+1,k+1)$$

```
> re:=subs(sol,re);
```

$$re := -\frac{(n+1)\,a(1,1)\,f(n,k)}{n} - \frac{(n+1)\,a(1,1)\,f(n,k+1)}{n} + a(1,1)\,f(n+1,k+1)$$

```
> re:=numer(normal(re/a(1,1)));
```

$$re := -f(n,k)\,n - f(n,k) - f(n,k+1)\,n - f(n,k+1) + f(n+1,k+1)\,n$$

```
> RE:=subs({seq(seq(f(n+j,k+i)=s(n+j),i=0..1),j=0..1)},re);
```

$$RE := -2\,s(n)\,n - 2\,s(n) + s(n+1)\,n$$

```
> RE:=map(factor,collect(RE,s))=0;
```

$$RE := -2\,(n+1)\,s(n) + s(n+1)\,n = 0$$

Now we use the implementation from the book
Wolfram Koepf: *Hypergeometric Summation*, Vieweg, Braunschweig/Wiesbaden, 1998

```
> restart; read "hsum9.mpl";
```

Package "Hypergeometric Summation", Maple V - Maple 9

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```
> libname:=libname,"C:/Dokumente und Einstellungen/koepf/Eigene Dateien/Koepf/Maple/Software/hsum";
```

libname := "C:\Programme\Maple 9/lib",
"C:/Dokumente und Einstellungen/koepf/Eigene Dateien/Koepf/Maple/Software/hsum"

```
> ?hsum
> fasenmyer(k*binomial(n,k),k,s(n),1,1);
```

$$n\,s(n+1) - 2\,s(n)\,(n+1) = 0$$

```
> fasenmyer(binomial(n,k)^2,k,s(n),1,1);
Error, (in kfreerec) No kfree recurrence equation of order (1,1) exists
```

```
> fasenmyer(binomial(n,k)^2,k,s(n),2,1);
```

$$(n+2)\,s(n+2) - 2\,s(n+1)\,(2n+3) = 0$$

```
> fasenmyer(binomial(n-k,k),k,s(n),2,1);
```

$$s(n+2) - s(n) - s(n+1) = 0$$

```
> [seq(sum(binomial(n-k,k),k=0..n),n=0..10)]; n:='n':
```

$$[1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89]$$

```
> fasenmyer((-1)^k*binomial(n,k)^2,k,s(n),2,2);
```

$$(n+2)\,s(n+2) + 4\,s(n)\,(n+1) = 0$$

```
> fasenmyer(binomial(n,k)^3,k,s(n),2,1);
Error, (in kfreerec) No kfree recurrence equation of order (2,2) exists
```

```
> fasenmyer(binomial(n,k)^3,k,s(n),3,1);
```

$$(3n+4)\,(n+3)^2\,s(n+3) - 2\,(9n^3 + 57n^2 + 116n + 74)\,s(n+2)$$
$$- (3n+5)\,(15n^2 + 55n + 48)\,s(n+1) - 8\,(3n+7)\,(n+1)^2\,s(n) = 0$$

Legendre polynomials

```
> Sum(binomial(n,k)*binomial(-n-1,k)*((1-x)/2)^k,k=0..n);
```

$$\sum_{k=0}^{n} \mathrm{binomial}(n,k)\,\mathrm{binomial}(-n-1,k)\left(\frac{1}{2}-\frac{x}{2}\right)^k$$

This corresponds to the hypergeometric representation

```
> Sumtohyper(binomial(n,k)*binomial(-n-1,k)*((1-x)/2)^k,k);
```

$$\mathrm{Hypergeom}\left([-n, n+1], [1], \frac{1}{2}-\frac{x}{2}\right)$$

```
> fasenmyer(binomial(n,k)*binomial(-n-1,k)*((1-x)/2)^k,k,s(n),2
  ,1);
```

$$(n+2)\,s(n+2) - x\,(2\,n+3)\,s(n+1) + (n+1)\,s(n) = 0$$

Compute a three-term recurrence equation for the Laguerre polynomials.

The Laguerre polynomials have the representation

```
> LaguerreL(n,x)=
  sum((-1)^k/k!*binomial(n,k)*x^k,k=0..infinity);
```

$$\mathrm{LaguerreL}(n,x) = \mathrm{hypergeom}([-n], [1], x)$$

Therefore we get

```
> fasenmyer((-1)^k/k!*binomial(n,k)*x^k,k,s(n),2,1);
```

$$(n+2)\,s(n+2) - (2\,n - x + 3)\,s(n+1) + (n+1)\,s(n) = 0$$

The generalized Laguerre polynomials have the hypergeometric representation

```
> LaguerreL(n,alpha,x)=sum((-1)^k/k!*binomial(n+alpha,n-k)*x^k,
  k=0..infinity);
```

$$\mathrm{LaguerreL}(n,\alpha,x) = \mathrm{binomial}(n+\alpha, n)\,\mathrm{hypergeom}([-n], [\alpha+1], x)$$

Therefore we get

```
> fasenmyer((-1)^k/k!*binomial(n+alpha,n-k)*x^k,k,s(n),2,1);
```

$$(n+2)\,s(n+2) - (3+\alpha+2\,n - x)\,s(n+1) + (n+\alpha+1)\,s(n) = 0$$

```
>
```

Indefinite Summation

Indefinite sum of k*k!

```
> s:=sum(k*k!,k);
```

$$s := k!$$

Check:

```
> difference:=subs(k=k+1,s)-s;
```

$$\mathit{difference} := (k+1)! - k!$$

```
> simplify(difference);
```

$$k\,\Gamma(k+1)$$

```
> simplify(difference-k*k!);
```

$$0$$

Maple's simplify treats binomials etc. badly:

```
> simplify(binomial(n,k)/k!);
```

$$\frac{\Gamma(n+1)}{\Gamma(k+1)^2\,\Gamma(n+1-k)}$$

We can check the algorithms internally used:

```
> infolevel[sum]:=5:
> sum((-1)^k*binomial(n,k),k);
sum/indefnew:   indefinite summation
sum/extgosper:   applying Gosper algorithm to a(    k    ):=   (-1)^k*binomial
(n,k)
sum/gospernew/internal:   a(   k    )/a(    k   -1):=    (-n-1+k)/k
sum/gospernew/internal:   Gosper's algorithm applicable
sum/gospernew/internal:   p:=   1
sum/gospernew/internal:   q:=   -n-1+k
sum/gospernew/internal:   r:=   k
sum/gospernew/internal:   degreebound:=   0
sum/gospernew/internal:   solving equations to find f
sum/gospernew/internal:   Gosper's algorithm successful
sum/gospernew/internal:   f:=   -1/n
sum/indefnew:   indefinite summation finished
```

$$-\frac{k\,(-1)^k\,\mathrm{binomial}(n,k)}{n}$$

```
> with(sumtools);
Warning, these previously assigned names now have a global binding:
Sumtohyper, extended_gosper, gosper, hyperterm, simpcomb, sumrecursion
```

[*Hypersum*, *Sumtohyper*, *extended_gosper*, *gosper*, *hyperrecursion*, *hypersum*, *hyperterm*, *simpcomb*, *sumrecursion*, *sumtohyper*]

```
> gosper((-1)^k*binomial(n,k),k);
sum/gospernew/internal:   a(   k    )/a(    k   -1):=    (-n-1+k)/k
sum/gospernew/internal:   Gosper's algorithm applicable
sum/gospernew/internal:   p:=   1
sum/gospernew/internal:   q:=   -n-1+k
sum/gospernew/internal:   r:=   k
sum/gospernew/internal:   degreebound:=   0
sum/gospernew/internal:   solving equations to find f
sum/gospernew/internal:   Gosper's algorithm successful
sum/gospernew/internal:   f:=   -1/n
```

$$-\frac{k\,(-1)^k\,\mathrm{binomial}(n,k)}{n}$$

Example from SIAM Reviews 36, 1994, Problem 94-2

```
> Sum((-1)^(k+1)*(4*k+1)*(2*k)!/(k!*4^k*(2*k-1)*(k+1)!),k=1..in
  finity);
```

$$\sum_{k=1}^{\infty} \frac{(-1)^{(k+1)}\,(4\,k+1)\,(2\,k)!}{k!\,4^k\,(2\,k-1)\,(k+1)!}$$

```
> sum((-1)^(k+1)*(4*k+1)*(2*k)!/(k!*4^k*(2*k-1)*(k+1)!),k);
sum/indefnew:   indefinite summation
sum/extgosper:   applying Gosper algorithm to a(    k    ):=   (-1)^(k+1)*(4*k
+1)*(2*k)!/k!/(4^k)/(2*k-1)/(k+1)!
sum/gospernew/internal:   a(   k    )/a(    k   -1):=    -1/2*(4*k+1)/(4*k-3)/
(k+1)*(2*k-3)
sum/gospernew/internal:   Gosper's algorithm applicable
sum/gospernew/internal:   p:=   4*k+1
sum/gospernew/internal:   q:=   -2*k+3
sum/gospernew/internal:   r:=   2*k+2
sum/gospernew/internal:   degreebound:=   0
sum/gospernew/internal:   solving equations to find f
sum/gospernew/internal:   Gosper's algorithm successful
```

```
sum/gospernew/internal:   f:=   -1
sum/indefnew:   indefinite summation finished
```

$$-\frac{2\,(k+1)\,(-1)^{(k+1)}\,(2\,k)!}{k!\,4^k\,(2\,k-1)\,(k+1)!}$$

```
> sum((-1)^(k+1)*(4*k+1)*(2*k)!/(k!*4^k*(2*k-1)*(k+1)!),k=1..infinity);
sum/infinite:   infinite summation
```

$$1$$

```
> infolevel[sum]:=0:
```

We do a more complicated example

```
> s:=k!*binomial(n,k)/(n-k);
```

$$s := \frac{k!\,\mathrm{binomial}(n,k)}{n-k}$$

```
> a:=subs(k=k+3,s)-s;
```

$$a := \frac{(k+3)!\,\mathrm{binomial}(n,k+3)}{n-k-3}-\frac{k!\,\mathrm{binomial}(n,k)}{n-k}$$

```
> b:=gosper(a,k);
```

$$b := -(-n+k+3)\,(2-n^3+3\,n^2\,k-3\,n\,k^2-4\,n^3\,k+6\,n^2\,k^2-4\,n\,k^3+3\,k-3\,n+k^3+n^4+k^4)\left(\frac{(k+3)!\,\mathrm{binomial}(n,k+3)}{n-k-3}-\frac{k!\,\mathrm{binomial}(n,k)}{n-k}\right)\Big/\left((n^4-4\,n^3\,k-3\,n^3+6\,n^2\,k^2+9\,n^2\,k+2\,n^2-4\,n\,k^3-9\,n\,k^2-4\,n\,k+k^4+3\,k^3+2\,k^2-n+k+3)\,(-n+k+2)\,(-n+k+1)\right)$$

```
> gosper(b,k);
```

$$\mathit{FAIL}$$

```
> restart; read "hsum9.mpl";
```

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```
> s:=k!*binomial(n,k)/(n-k);
```

$$s := \frac{k!\,\mathrm{binomial}(n,k)}{n-k}$$

```
> a:=subs(k=k+3,s)-s;
```

$$a := \frac{(k+3)!\,\mathrm{binomial}(n,k+3)}{n-k-3}-\frac{k!\,\mathrm{binomial}(n,k)}{n-k}$$

```
> b:=gosper(a,k);
```

$$b := -(-n+k+3)\,(2+n^4-n^3+k^4+k^3-4\,n^3\,k+3\,k-3\,n+6\,n^2\,k^2+3\,n^2\,k-4\,n\,k^3-3\,n\,k^2)\left(\frac{(k+3)!\,\mathrm{binomial}(n,k+3)}{n-k-3}-\frac{k!\,\mathrm{binomial}(n,k)}{n-k}\right)\Big/\left((n^4-4\,n^3\,k+6\,n^2\,k^2-4\,n\,k^3-3\,n^3+9\,n^2\,k-9\,n\,k^2+k^4+3\,k^3+2\,n^2-4\,n\,k+2\,k^2-n+k+3)\,(-n+k+2)\right.$$

$(-n+k+1))$

```
> gosper(b,k);
Error, (in gosper) No hypergeometric term antidifference exists
> a:='a': b:='b':
>
```

Gosper's Algorithm in Detail

```
> read "hsum9.mpl";
```

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first example

```
> a:=k*k!;
```

$$a := k\,k!$$

```
> rat:=subs(k=k+1,a)/a;
```

$$rat := \frac{(k+1)\,(k+1)!}{k\,k!}$$

```
> rat:=normal(expand(rat));
```

$$rat := \frac{k^2+2\,k+1}{k}$$

```
> q:=numer(rat);
```

$$q := k^2+2\,k+1$$

```
> r:=denom(rat);
```

$$r := k$$

```
> p:=1;
```

$$p := 1$$

q(k) and r(k+j) have a nontrivial gcd for j=1:

```
> gcd(q,subs(k=k+1,r));
```

$$k+1$$

```
> pqr:=update(p,subs(k=k-1,q),subs(k=k-1,r),k);
```

$$pqr := [k, k, 1]$$

```
> p:=op(1,pqr); q:=op(2,pqr); r:=op(3,pqr);
```

$$p := k$$

$$q := k$$

$$r := 1$$

```
> f:='f':
> RE:=subs(k=k+1,q)*f(k)-subs(k=k+1,r)*f(k-1)=p;
```

$$RE := (k+1)\,\mathrm{f}(k) - \mathrm{f}(k-1) = k$$

```
> rsolve(RE,f(k));
```

$$1 + \frac{-1+\mathrm{f}(0)}{\Gamma(k+2)}$$

```
> f:=findf(p,q,r,k);
```

$$f := 1$$

```
> s:=r/p*subs(k=k-1,f)*a;
```

$$s := k!$$

second example

```
> a:=(-1)^k*binomial(n,k);
```

$$a := (-1)^k \operatorname{binomial}(n, k)$$

```
> rat:=subs(k=k+1,a)/a;
```

$$rat := \frac{(-1)^{(k+1)} \operatorname{binomial}(n, k+1)}{(-1)^k \operatorname{binomial}(n, k)}$$

```
> rat:=normal(expand(rat));
```

$$rat := \frac{-n+k}{k+1}$$

```
> q:=numer(rat);
```

$$q := -n+k$$

```
> r:=denom(rat);
```

$$r := k+1$$

```
> p:=1;
```

$$p := 1$$

q(k) and r(k+j) have no nontrivial gcd for n a symbol, but for negative integer n. We will come back to this case later.

```
> pqr:=update(p,subs(k=k-1,q),subs(k=k-1,r),k);
```

$$pqr := [1, -n+k-1, k]$$

```
> p:=op(1,pqr); q:=op(2,pqr); r:=op(3,pqr);
```

$$p := 1$$

$$q := -n+k-1$$

$$r := k$$

```
> f:='f':
> RE:=subs(k=k+1,q)*f(k)-subs(k=k+1,r)*f(k-1)=p;
```

$$RE := (-n+k)\, \mathrm{f}(k) - (k+1)\, \mathrm{f}(k-1) = 1$$

```
> sol:=rsolve(RE,f(k));
```

$$sol := \frac{(\mathrm{f}(0)\, n + 1 + \mathrm{f}(0))\, \Gamma(-n+1)\, \Gamma(k+2)}{(1+n)\, \Gamma(-n+k+1)} - \frac{1}{1+n}$$

```
> f:=findf(p,q,r,k);
```

$$f := -\frac{1}{n}$$

```
> s:=r/p*subs(k=k-1,f)*a;
```

$$s := -\frac{k\, (-1)^k \operatorname{binomial}(n, k)}{n}$$

Now we consider the particular case n=-10.

```
> a:=(-1)^k*binomial(-10,k);
```

$$a := (-1)^k \operatorname{binomial}(-10, k)$$

```
> rat:=subs(k=k+1,a)/a;
```

$$rat := \frac{(-1)^{(k+1)} \operatorname{binomial}(-10, k+1)}{(-1)^k \operatorname{binomial}(-10, k)}$$

```
> rat:=normal(expand(rat));
```

$$rat := \frac{10+k}{k+1}$$

```
> q:=numer(rat);
```

$$q := 10+k$$

```
> r:=denom(rat);
```

$$r := k+1$$

```
> p:=1;
```

$$p := 1$$

q(k) and r(k+j) have a nontrivial gcd for j=9:

```
> gcd(q,subs(k=k+9,r));
```

$$10+k$$

```
> pqr:=update(p,subs(k=k-1,q),subs(k=k-1,r),k);
```

$$pqr := [(k+9)(k+8)(k+7)(k+6)(k+5)(k+4)(k+3)(k+2)(k+1), 1, 1]$$

```
> p:=op(1,pqr); q:=op(2,pqr); r:=op(3,pqr);
```

$$p := (k+9)(k+8)(k+7)(k+6)(k+5)(k+4)(k+3)(k+2)(k+1)$$

$$q := 1$$

$$r := 1$$

```
> f:='f':
> RE:=subs(k=k+1,q)*f(k)-subs(k=k+1,r)*f(k-1)=p;
```

$$RE := \mathrm{f}(k) - \mathrm{f}(k-1) = (k+9)(k+8)(k+7)(k+6)(k+5)(k+4)(k+3)(k+2)(k+1)$$

```
> sol:=rsolve(RE,f(k));
```

$$sol := \mathrm{f}(0) - 362880 + 362880\,(k+1)\left(\frac{k}{2}+1\right)\left(\frac{k}{3}+1\right)\left(\frac{k}{4}+1\right)\left(\frac{k}{5}+1\right)\left(\frac{k}{6}+1\right)\left(\frac{k}{7}+1\right)\left(\frac{k}{8}+1\right)\left(\frac{k}{9}+1\right)\left(1+\frac{k}{10}\right)$$

```
> f:=findf(p,q,r,k);
```

$$f := 1062864\,k + \frac{6376788}{5}k^2 + 840950\,k^3 + 341693\,k^4 + \frac{180411}{2}k^5 + \frac{157773}{10}k^6 + 1815\,k^7 + 132\,k^8 + \frac{11}{2}k^9 + \frac{1}{10}k^{10}$$

```
> specials:=r/p*subs(k=k-1,f)*a;
```

$$specials := \left(1062864\,k - 1062864 + \frac{6376788\,(k-1)^2}{5} + 840950\,(k-1)^3\right.$$

$$+ 341693\,(k-1)^4 + \frac{180411\,(k-1)^5}{2} + \frac{157773\,(k-1)^6}{10} + 1815\,(k-1)^7 + 132\,(k-1)^8$$

$$+ \frac{11\,(k-1)^9}{2} + \frac{(k-1)^{10}}{10}\Bigg) (-1)^k \,\mathrm{binomial}(-10, k) \,/\, ((k+9)\,(k+8)\,(k+7)\,(k+6)$$

$$(k+5)\,(k+4)\,(k+3)\,(k+2)\,(k+1))$$

```
> difference:=simplify(specials-subs(n=-10,s));
```

$$difference := \frac{362880\,(-1)^{(k+1)}\,\mathrm{binomial}(-10, k)}{(k+9)\,(k+8)\,(k+7)\,(k+6)\,(k+5)\,(k+4)\,(k+3)\,(k+2)\,(k+1)}$$

```
> simplify(difference);
```

$$\frac{362880\,(-1)^{(k+1)}\,\mathrm{binomial}(-10, k)}{(k+9)\,(k+8)\,(k+7)\,(k+6)\,(k+5)\,(k+4)\,(k+3)\,(k+2)\,(k+1)}$$

```
> [seq(difference,k=1..10)]; k:='k':
```

$$[-1, -1, -1, -1, -1, -1, -1, -1, -1, -1]$$

third example

```
> a:=binomial(n,k);
```

$$a := \mathrm{binomial}(n, k)$$

```
> rat:=subs(k=k+1,a)/a;
```

$$rat := \frac{\mathrm{binomial}(n, k+1)}{\mathrm{binomial}(n, k)}$$

```
> rat:=normal(expand(rat));
```

$$rat := -\frac{-n+k}{k+1}$$

```
> q:=numer(rat);
```

$$q := n - k$$

```
> r:=denom(rat);
```

$$r := k + 1$$

```
> p:=1;
```

$$p := 1$$

```
> pqr:=update(p,subs(k=k-1,q),subs(k=k-1,r),k);
```

$$pqr := [1, n-k+1, k]$$

```
> p:=op(1,pqr); q:=op(2,pqr); r:=op(3,pqr);
```

$$p := 1$$

$$q := n - k + 1$$

$$r := k$$

```
> f:='f':
> RE:=subs(k=k+1,q)*f(k)-subs(k=k+1,r)*f(k-1)=p;
```

$$RE := (n-k)\,\mathrm{f}(k) - (k+1)\,\mathrm{f}(k-1) = 1$$

```
> rsolve(RE,f(k));
```

$$\frac{(-1)^k\,(-n-2+2^{(1+n)}+f(0)\,n+f(0)\,n^2)\,\Gamma(-n-1)\,\Gamma(k+2)}{\Gamma(-n+k+1)}$$

$$-\frac{\text{hypergeom}([1,-n+k+1],[k+3],-1)}{k+2}$$

```
> f:=findf(p,q,r,k);
Error, (in findf) No polynomial f exists
> gosper(a,k);
Error, (in gosper) No hypergeometric term antidifference exists
> a:='a': s:='s': p:='p': q:='q': r:='r': f:='f':
>
```

Zeilberger's Algorithm

```
> read "hsum9.mpl";
```

Package "Hypergeometric Summation", Maple V - Maple 9

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```
> sumrecursion(k*binomial(n,k),k,s(n));
```

$$2\,(1+n)\,s(n)-n\,s(1+n)=0$$

```
> sumrecursion((-1)^k*binomial(n,k)^2,k,s(n));
```

$$4\,(1+n)\,s(n)+(2+n)\,s(2+n)=0$$

```
> sumrecursion(binomial(n,k)^3,k,s(n));
```

$$8\,(1+n)^2\,s(n)+(7\,n^2+21\,n+16)\,s(1+n)-(2+n)^2\,s(2+n)=0$$

With Zeilberger's algorithm, we can do more complicated examples.

The Apéry numbers

```
> Sum(binomial(n,k)^2*binomial(n+k,k)^2,k=0..n);
```

$$\sum_{k=0}^{n}\text{binomial}(n,k)^2\,\text{binomial}(k+n,k)^2$$

satisfy the recurrence equation

```
> sumrecursion(binomial(n,k)^2*binomial(n+k,k)^2,k,A(n));
```

$$(1+n)^3\,A(n)-(3+2\,n)\,(17\,n^2+51\,n+39)\,A(1+n)+(2+n)^3\,A(2+n)=0$$

The power sums of the binomial coefficients were worth a paper in the 1980s:

```
> sumrecursion(binomial(n,k)^4,k,s(n));
```

$$4\,(4\,n+5)\,(3+4\,n)\,(1+n)\,s(n)+2\,(3+2\,n)\,(3\,n^2+9\,n+7)\,s(1+n)$$
$$-(2+n)^3\,s(2+n)=0$$

```
> sumrecursion(binomial(n,k)^5,k,s(n));
```

$$32\,(55\,n^2+253\,n+292)\,(1+n)^4\,s(n)-$$
$$(2682770\,n^2+900543\,n^4+1827064\,n+19415\,n^6+205799\,n^5+2082073\,n^3+514048)$$
$$s(1+n)-$$
$$(1155\,n^6+14553\,n^5+205949\,n^3+310827\,n^2+75498\,n^4+245586\,n+79320)\,s(2+n)$$
$$+(55\,n^2+143\,n+94)\,(3+n)^4\,s(3+n)=0$$

```
> sumrecursion(binomial(n,k)^6,k,s(n));
```

$$24\,(6\,n+5)\,(3+2\,n)\,(6\,n+7)\,(91\,n^3+637\,n^2+1491\,n+1167)\,(1+n)^3\,s(n)-(22934340+187916733\,n^5+378741807\,n^3+280311768\,n^2+327503034\,n^4+153881\,n^9+2462096\,n^8+120507876\,n+17419983\,n^7+71536002\,n^6)\,s(1+n)-(2+n)\,(3458\,n^8+57057\,n^7+408555\,n^6+1656761\,n^5+4158211\,n^4+6610054\,n^3+6496560\,n^2+3609252\,n+868140)\,s(2+n)+(2+n)\,(91\,n^3+364\,n^2+490\,n+222)\,(3+n)^5\,s(3+n)=0$$

```
> sumrecursion(binomial(n,k)^7,k,s(n));
```

$$128\,(427721\,n^8+9776480\,n^7+97373115\,n^6+551893883\,n^5+1946706314\,n^4+4375566933\,n^3+6119692458\,n^2+4869142152\,n+1687389120)\,(2+n)^2\,(1+n)^6\,s(n)-(114791322401632464\,n^3+54690808998655008\,n^2+21938070699816960+176624649389228512\,n^5+166377205614902736\,n^4+2283968506414\,n^{14}+25606027648545\,n^{13}+198784165636833\,n^{12}+1132823172700850\,n^{11}+142107402452328480\,n^6+16415798739266369\,n^9+43010799826545440\,n^8+4900968186516568\,n^{10}+16071328274727552\,n+126062821360\,n^{15}+3244263785\,n^{16}+88420368230599884\,n^7)\,s(2+n)-(129212210111012\,n^6+203258395972016\,n^5+198216442561728\,n^3+112552666603632\,n^2+236869167238448\,n^4+30368191\,n^{14}+1088916563\,n^{13}+17971912105\,n^{12}+180879396742\,n^{11}+6117625957887\,n^9+22406262825083\,n^8+1239681510073\,n^{10}+38805627231072\,n+6127621340928+61845130443640\,n^7)\,(3+n)^2\,s(3+n)+(45209280+209877096\,n+421557546\,n^2+478442631\,n^3+335597294\,n^4+149008897\,n^5+40913943\,n^6+6354712\,n^7+427721\,n^8)\,(3+n)^2\,(4+n)^6\,s(4+n)+(26380423880989287\,n^6+37123771902845896\,n^5+29207641278240480\,n^3+14968213677069888\,n^2+38801484010527532\,n^4+15821827511\,n^{14}+504038219279\,n^{13}+671258737065984+7388757320392\,n^{12}+66049812430419\,n^{11}+1764446202422005\,n^9+5750836202090468\,n^8+402186441422282\,n^{10}+4674653721868800\,n+14144725417173505\,n^7)\,(2+n)^2\,s(1+n)=0$$

```
> sumrecursion(binomial(n,k)^8,k,s(n));
```

$$16\,(8\,n+13)\,(8\,n+7)\,(8\,n+9)\,(8\,n+11)\,(2+n)\,(102375360\,n^{11}+3186433080\,n^{10}+44960611518\,n^9+379608257007\,n^8+2130886001250\,n^7+8350001129322\,n^6+23306855546382\,n^5+46339428278457\,n^4+64315605847158\,n^3+59346884858090\,n^2+32767840545852\,n+8201727801720)\,(1+n)^5\,s(n)-12\,(858467294792387280\,0+4325426980028204773202\,n^6+3591739108587502596080\,n^5+3882809750612836159 68\,n^2+531359186174014492 89\,n^{13}+1313108333525255 6274\,n^{14}+2325420945730194232698\,n^4+280508486400\,n^{21}$$

$$+ 334201973882040\, n^{19} + 14060487880800\, n^{20} + 3251191961324982788923\, n^{8}$$
$$+ 2084807859419201931337\, n^{9} + 4850739466331070894 11\, n^{11}$$
$$+ 1765911402240850944 02\, n^{12} + 5009309431465140\, n^{18} + 2637180986865085374\, n^{15}$$
$$+ 423581680113805917\, n^{16} + 8392551049648310774 4\, n + 53113860263695806\, n^{17}$$
$$+ 1104564579231841006148\, n^{10} + 4161525693270481443599\, n^{7}$$
$$+ 1130722271587064275368\, n^{3})\, s(2 + n) - 2\, (4170731594507388838\, n^{6}$$
$$+ 4325192582660019738\, n^{5} + 2022592897697984984\, n^{3} + 828656447429098560\, n^{2}$$
$$+ 3438871758751753182\, n^{4} + 770715264100878\, n^{14} + 5194295369065098\, n^{13}$$
$$+ 25145115503187680 + 6961524480\, n^{18} + 26874700372746516\, n^{12}$$
$$+ 1091213736330860 19\, n^{11} + 9119302147464052 78\, n^{9} + 1894748039012557464\, n^{8}$$
$$+ 352423811632001922\, n^{10} + 310658029920\, n^{17} + 211038635712599424\, n$$
$$+ 6500512066104\, n^{16} + 84729238051860\, n^{15} + 3153073563533903151\, n^{7})\, (3 + n)^{3}$$
$$s(3 + n) + (54585830156 + 350689467812\, n + 1017700462466\, n^{2} + 1760584594380\, n^{3}$$
$$+ 2017065459849\, n^{4} + 1606745735736\, n^{5} + 907992479736\, n^{6} + 364013859042\, n^{7}$$
$$+ 101460307545\, n^{8} + 18726925518\, n^{9} + 2060304120\, n^{10} + 102375360\, n^{11})\, (3 + n)^{3}$$
$$(4 + n)^{7}\, s(4 + n) + 8\, (2 + n)\, (7072908871680\, n^{20} + 315628558398720\, n^{19}$$
$$+ 6650661243415104\, n^{18} + 87979206823913808\, n^{17} + 819439991165553516\, n^{16}$$
$$+ 5711991395289139404\, n^{15} + 30917972174651220597\, n^{14}$$
$$+ 133070276638133809227\, n^{13} + 462516691604036543940\, n^{12}$$
$$+ 1311025295092282143740\, n^{11} + 3047209515781789762641\, n^{10}$$
$$+ 5817899143713103665172\, n^{9} + 9108545400676905550771\, n^{8}$$
$$+ 11630327275776577718556\, n^{7} + 11993481346952514494264\, n^{6}$$
$$+ 9835369404711553127321\, n^{5} + 6263916978444480644973\, n^{4}$$
$$+ 2986089280124489341048\, n^{3} + 1002446238942897024570\, n^{2}$$
$$+ 211318335235609832268\, n + 21039060801453294600)\, s(1 + n) = 0$$

Four different representations of the Legendre polynomials:

(a) We consider the summand:

```
> legendre1:=binomial(n,k)*binomial(-n-1,k)*((1-x)/2)^k;
```

$$legendre1 := \mathrm{binomial}(n, k)\, \mathrm{binomial}(-n - 1, k) \left(\frac{1}{2} - \frac{x}{2}\right)^{k}$$

The sum

```
> Sum(legendre1,k=0..n);
```

$$\sum_{k=0}^{n} \mathrm{binomial}(n, k)\, \mathrm{binomial}(-n - 1, k) \left(\frac{1}{2} - \frac{x}{2}\right)^{k}$$

has the hypergeometric representation

```
> Sumtohyper(legendre1,k);
```

$$\mathrm{Hypergeom}\left([-n, 1+n], [1], \frac{1}{2} - \frac{x}{2}\right)$$

and satisfies the recurrence equation

```
> sumrecursion(legendre1,k,P(n));
```

$$(1+n)\,P(n) - x\,(2\,n+3)\,P(1+n) + (2+n)\,P(2+n) = 0$$

(b) We consider the summand:

```
> legendre2:=1/2^n*binomial(n,k)^2*(x-1)^(n-k)*(x+1)^k;
```

$$legendre2 := \frac{\mathrm{binomial}(n, k)^2\,(-1+x)^{(n-k)}\,(x+1)^k}{2^n}$$

The sum

```
> Sum(legendre2,k=0..n);
```

$$\sum_{k=0}^{n} \frac{\mathrm{binomial}(n, k)^2\,(-1+x)^{(n-k)}\,(x+1)^k}{2^n}$$

has the hypergeometric representation

```
> Sumtohyper(legendre2,k);
```

$$\frac{(-1+x)^n\,\mathrm{Hypergeom}\left([-n, -n], [1], \frac{x+1}{-1+x}\right)}{2^n}$$

and satisfies the recurrence equation

```
> sumrecursion(legendre2,k,P(n));
```

$$(1+n)\,P(n) - x\,(2\,n+3)\,P(1+n) + (2+n)\,P(2+n) = 0$$

(c) We consider the summand:

```
> legendre3:=1/2^n*(-1)^k*binomial(n,k)*binomial(2*n-2*k,n)*x^(
  n-2*k);
```

$$legendre3 := \frac{(-1)^k\,\mathrm{binomial}(n, k)\,\mathrm{binomial}(2\,n-2\,k, n)\,x^{(n-2\,k)}}{2^n}$$

The sum

```
> Sum(legendre3,k=0..floor(n/2));
```

$$\sum_{k=0}^{\mathrm{floor}\left(\frac{n}{2}\right)} \frac{(-1)^k\,\mathrm{binomial}(n, k)\,\mathrm{binomial}(2\,n-2\,k, n)\,x^{(n-2\,k)}}{2^n}$$

has the hypergeometric representation

```
> Sumtohyper(legendre3,k);
```

$$\frac{\Gamma(2\,n+1)\,x^n\,\mathrm{Hypergeom}\left(\left[-\frac{n}{2}, \frac{1}{2}-\frac{n}{2}\right], \left[-n+\frac{1}{2}\right], \frac{1}{x^2}\right)}{2^n\,\Gamma(1+n)^2}$$

and satisfies the recurrence equation

```
> sumrecursion(legendre3,k,P(n));
```

$$(1+n)\,P(n) - x\,(2\,n+3)\,P(1+n) + (2+n)\,P(2+n) = 0$$

(d) We consider the summand:

```
> legendre4:=x^n*hyperterm([-n/2,(1-n)/2],[1],1-1/x^2,k);
```

$$legendre4 := \frac{x^n \operatorname{pochhammer}\left(-\frac{n}{2}, k\right) \operatorname{pochhammer}\left(\frac{1}{2}-\frac{n}{2}, k\right)\left(1-\frac{1}{x^2}\right)^k}{(k!)^2}$$

The sum

```
> Sum(legendre4,k=0..floor(n/2));
```

$$\sum_{k=0}^{\operatorname{floor}\left(\frac{n}{2}\right)} \frac{x^n \operatorname{pochhammer}\left(-\frac{n}{2}, k\right) \operatorname{pochhammer}\left(\frac{1}{2}-\frac{n}{2}, k\right)\left(1-\frac{1}{x^2}\right)^k}{(k!)^2}$$

has the hypergeometric representation

```
> Sumtohyper(legendre4,k);
```

$$x^n \operatorname{Hypergeom}\left(\left[-\frac{n}{2}, \frac{1}{2}-\frac{n}{2}\right], [1], \frac{(-1+x)(x+1)}{x^2}\right)$$

and satisfies the recurrence equation

```
> sumrecursion(legendre4,k,P(n));
```

$$(1+n) \operatorname{P}(n) - x(2n+3) \operatorname{P}(1+n) + (2+n) \operatorname{P}(2+n) = 0$$

```
>
```

Proof of Clausen’s formula by Cauchy product:

```
> summand:=j->hyperterm([a,b],[a+b+1/2],1,j);
```

$$summand := j \rightarrow \operatorname{hyperterm}\left([a, b], \left[a+b+\frac{1}{2}\right], 1, j\right)$$

```
> Closedform(summand(j)*summand(k-j),j,k);
```

$$\operatorname{Hyperterm}\left([2b, 2a, a+b], \left[2b+2a, a+b+\frac{1}{2}\right], 1, k\right)$$

Proof of Clausen’s formula by differential equations:

The left hand factor satisfies the differential equation

```
> DE:=sumdiffeq(summand(j)*x^j,j,C(x));
```

$$DE := 2(-1+x)x\left(\frac{d^2}{dx^2}\operatorname{C}(x)\right) + (2xa-1-2b-2a+2xb+2x)\left(\frac{d}{dx}\operatorname{C}(x)\right) + 2\operatorname{C}(x)ab = 0$$

Therefore the left hand side satisfies the differential equation

```
> with(gfun):
> LHS:='diffeq*diffeq'(DE,DE,C(x));
```

$$LHS := (8ab^2+8a^2b)\operatorname{C}(x) + (16abx+4xb^2+2x+4xa^2+6xa+6xb-4b^2-8ab-4a^2-2a-2b)\left(\frac{d}{dx}\operatorname{C}(x)\right) + (6x^2a+6x^2b+6x^2-3x-6xa-6xb)\left(\frac{d^2}{dx^2}\operatorname{C}(x)\right)$$

$$+\left(-2x^2+2x^3\right)\left(\frac{d^3}{dx^3}\mathrm{C}(x)\right)$$

On the other hand the right hand side satisfies the differential equation

```
> RHS:=sumdiffeq(hyperterm([2*a,2*b,a+b],[2*a+2*b,a+b+1/2],x,k)
,k,C(x));
```

$$RHS := 8\,\mathrm{C}(x)\,a\,b\,(a+b) + 2\left(2xb^2+2xa^2+8abx+x-2b^2-2a^2-b-a+3xb-4ab+3xa\right)\left(\frac{d}{dx}\mathrm{C}(x)\right) + 3x\left(2xa-1-2b-2a+2xb+2x\right)\left(\frac{d^2}{dx^2}\mathrm{C}(x)\right)+2(-1+x)x^2\left(\frac{d^3}{dx^3}\mathrm{C}(x)\right)=0$$

These are equal:

```
> expand(LHS-op(1,RHS));
```

$$0$$

```
>
```

Differential Equations for Hypergeometric Sums

The differential equation of the sine function:

```
> sumdiffeq((-1)^k/(2*k+1)!*x^(2*k+1),k,s(x));
```

$$\mathrm{s}(x)+\left(\frac{d^2}{dx^2}\mathrm{s}(x)\right)=0$$

The four different hypergeometric representations of the Legendre polynomials all lead to the same differential equation:

```
> legendre1:=binomial(n,k)*binomial(-n-1,k)*((1-x)/2)^k;
```

$$legendre1 := \mathrm{binomial}(n,k)\,\mathrm{binomial}(-n-1,k)\left(\frac{1}{2}-\frac{x}{2}\right)^k$$

```
> sumdiffeq(legendre1,k,P(x));
```

$$-(x+1)(-1+x)\left(\frac{d^2}{dx^2}\mathrm{P}(x)\right)-2x\left(\frac{d}{dx}\mathrm{P}(x)\right)+\mathrm{P}(x)\,n\,(1+n)=0$$

```
> legendre2:=1/2^n*binomial(n,k)^2*(x-1)^(n-k)*(x+1)^k;
```

$$legendre2 := \frac{\mathrm{binomial}(n,k)^2\,(-1+x)^{(n-k)}\,(x+1)^k}{2^n}$$

```
> sumdiffeq(legendre2,k,P(x));
```

$$-(x+1)(-1+x)\left(\frac{d^2}{dx^2}\mathrm{P}(x)\right)-2x\left(\frac{d}{dx}\mathrm{P}(x)\right)+\mathrm{P}(x)\,n\,(1+n)=0$$

```
> legendre3:=1/2^n*(-1)^k*binomial(n,k)*binomial(2*n-2*k,n)*x^(
n-2*k);
```

$$legendre3 := \frac{(-1)^k\,\mathrm{binomial}(n,k)\,\mathrm{binomial}(2n-2k,n)\,x^{(n-2k)}}{2^n}$$

```
> sumdiffeq(legendre3,k,P(x));
```

$$-(x+1)(-1+x)\left(\frac{d^2}{dx^2}\mathrm{P}(x)\right)-2x\left(\frac{d}{dx}\mathrm{P}(x)\right)+\mathrm{P}(x)\,n\,(1+n)=0$$

```
> legendre4:=x^n*hyperterm([-n/2,(1-n)/2],[1],1-1/x^2,k);
```

$$legendre4 := \frac{x^n \operatorname{pochhammer}\left(-\frac{n}{2}, k\right) \operatorname{pochhammer}\left(\frac{1}{2}-\frac{n}{2}, k\right)\left(1-\frac{1}{x^2}\right)^k}{(k!)^2}$$

```
> sumdiffeq(legendre4,k,P(x));
```

$$-(x+1)(-1+x)\left(\frac{d^2}{dx^2}\mathrm{P}(x)\right)-2x\left(\frac{d}{dx}\mathrm{P}(x)\right)+\mathrm{P}(x)\,n\,(1+n)=0$$

```
>
```

A Generating Function Problem

```
> read "hsum9.mpl";
```

Package "Hypergeometric Summation", Maple V - Maple 9

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```
> RE:=sumrecursion(binomial(alpha+n-1,n)*legendre4*z^n,n,s(k));
```

$RE :=$

$$z^2(-1+x)(x+1)(2k+\alpha+1)(2k+\alpha)\,\mathrm{s}(k)-4(k+1)^2(xz-1)^2\,\mathrm{s}(k+1)=0$$

```
> sol:=rsolve(RE,s(k));
```

$$sol := \frac{(z^2)^k(-1+x)^k(x+1)^k\left(\frac{1}{(xz-1)^2}\right)^k 4^{(-k)}\,\Gamma(2k+\alpha)\,\mathrm{s}(0)}{\Gamma(\alpha)\,\Gamma(k+1)^2}$$

We compute the initial value:

```
> s(0)=Sum(binomial(alpha+n-1,n)*subs(k=0,legendre4)*z^n,n=0..i
  nfinity);
```

$$\mathrm{s}(0)=\sum_{n=0}^{\infty}\frac{\operatorname{binomial}(\alpha+n-1, n)\, x^n \operatorname{pochhammer}\left(-\frac{n}{2}, 0\right)\operatorname{pochhammer}\left(\frac{1}{2}-\frac{n}{2}, 0\right) z^n}{(0!)^2}$$

```
> aw:=s(0)=sum(binomial(alpha+n-1,n)*subs(k=0,legendre4)*z^n,n=
  0..infinity);
```

$$aw := \mathrm{s}(0)=\frac{1}{(1-xz)^\alpha}$$

Therefore we get the solution:

```
> sol:=subs(aw,sol);
```

$$sol := \frac{(z^2)^k(-1+x)^k(x+1)^k\left(\frac{1}{(xz-1)^2}\right)^k 4^{(-k)}\,\Gamma(2k+\alpha)}{\Gamma(\alpha)\,\Gamma(k+1)^2(1-xz)^\alpha}$$

which we put into hypergeometric form:

```
> Sumtohyper(sol,k);
```

$$\frac{\mathrm{Hypergeom}\left(\left[\frac{\alpha}{2}, \frac{\alpha}{2}+\frac{1}{2}\right], [1], \frac{z^2(-1+x)(x+1)}{(xz-1)^2}\right)}{(1-xz)^{\alpha}}$$

```
>
```

Combining the algorithms

```
> read "hsum9.mpl";
```

Package "Hypergeometric Summation", Maple V - Maple 9

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```
> read "FPS.mpl";
```

Package Formal Power Series, Maple V - Maple 8

Copyright 1995, Dominik Gruntz, University of Basel

Copyright 2002, Detlef Müller & Wolfram Koepf, University of Kassel

For

```
> Sum(x^(3*k)/(3*k)!,k=0..infinity)=sum(x^(3*k)/(3*k)!,k=0..infinity);
```

$$\sum_{k=0}^{\infty} \frac{x^{(3k)}}{(3k)!} = \mathrm{hypergeom}\left([\ ], \left[\frac{1}{3}, \frac{2}{3}\right], \frac{x^3}{27}\right)$$

Zeilberger's algorithm detects the differential equation

```
> DE:=sumdiffeq(x^(3*k)/(3*k)!,k,F(x));
```

$$DE := \mathrm{F}(x) - \left(\frac{d^3}{dx^3}\mathrm{F}(x)\right) = 0$$

Maple's internal differential equation solver can solve this equation

```
> f:=rhs(dsolve({DE,F(0)=1,D(F)(0)=0,(D@@2)(F)(0)=0},F(x)));
```

$$f := \frac{1}{3}\mathbf{e}^x + \frac{2}{3}\mathbf{e}^{\left(-\frac{x}{2}\right)}\cos\left(\frac{\sqrt{3}\,x}{2}\right)$$

Reversely, the FPS algorithm redetects the differential equation from this representation

```
> HolonomicDE(f,F(x));
```

$$\left(\frac{d^3}{dx^3}\mathrm{F}(x)\right) - \mathrm{F}(x) = 0$$

and recomputes the power series representation of f

```
> FPS(f,x);
```

$$\sum_{k=0}^{\infty} \frac{x^{(3k)}}{(3k)!}$$

```
>
```

Infinite Sums

```
> read "hsum9.mpl";
```

Package "Hypergeometric Summation", Maple V - Maple 9

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```
> read "infhsum.mpl";
```

This is a Maple package for computing recurrence relations,

closed form expressions and uniformly bounded convergence of

non-terminating hypergeometric series; written by R. Vidunas

" Version 4.25, 27-05-2002."

Supported by NWO, project number 613-06-565

The help function is envoked by " infhsumhelp( ) "

Gauss identity

```
> infclosedform(hyperterm([a,b],[c],1,k),k,c);
Warning, The condition(s) for uniformly bounded convergence are: 0 <
Re(-a-b+c)
```

$$\frac{\Gamma(c)\,\Gamma(-a-b+c)}{\Gamma(c-a)\,\Gamma(c-b)}$$

Kummer's identity

```
> infclosedform(hyperterm([a,b],[1+a-b],-1,k),k,a);
Warning, The condition(s) for uniformly bounded convergence are: Re(b) < 0
```

$$\frac{2^{(-a)}\sqrt{\pi}\,\Gamma(1+a-b)}{\Gamma\left(\frac{a}{2}+\frac{1}{2}\right)\Gamma\left(1+\frac{a}{2}-b\right)}$$

Pfaff-Saalschütz identity

```
> infclosedform(hyperterm([a,b,c],[d,1+a+b+c-d],1,k),k,d);
Warning, The condition(s) for uniformly bounded convergence are: Re(a+b+c) <
1
```

$$\frac{\Gamma(-b-c+d)\,\Gamma(d)\,\Gamma(-c+d-a)\,\Gamma(d-a-b)}{\Gamma(-b+d)\,\Gamma(-a+d)\,\Gamma(d-a-b-c)\,\Gamma(-c+d)}+(b+a+c-2\,d)\,\mathrm{Hypergeom}\left(\left[-\frac{b}{2}-\frac{a}{2}-\frac{c}{2}+d+1,\,-a+d,\,-c+d,\,-b+d,\,1\right],\right.$$
$$\left.\left[-a-c+d+1,\,-\frac{b}{2}-\frac{a}{2}-\frac{c}{2}+d,\,d+1-a-b,\,-b-c+d+1\right],\,\text{-}1\right)\Gamma(1+a+b+c-d)$$
$$\Gamma(d)\,/\,((a+c-d)\,(-d+a+b)\,(b+c-d)\,\Gamma(c)\,\Gamma(b)\,\Gamma(a))$$

Note that this is an non-obvious generalization of the Pfaff-Saalschütz identity.

```
>
```

Petkovsek's Algorithm

```
> read "hsum9.mpl";
```

Package "Hypergeometric Summation", Maple V - Maple 9

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For the following sum Zeilberger's algorithm finds a recurrence equation of order c-1 instead of 1:

```
> Sum((-1)^k*binomial(n,k)*binomial(c*k,n),k=0..n)=(-c)^k;
```

$$\sum_{k=0}^{n} (-1)^k \,\mathrm{binomial}(n, k)\, \mathrm{binomial}(c\, k, n) = (-c)^k$$

We compute:

```
> rec:=sumrecursion((-1)^k*binomial(n,k)*binomial(4*k,n),k,s(n)
  );
```

$$rec := 64\,(3\,n+7)\,(2+n)\,(1+n)\,\mathrm{s}(n) + 4\,(3\,n+4)\,(37\,n^2+180\,n+218)\,\mathrm{s}(2+n) + 3\,(3\,n+7)\,(3\,n+4)\,(3\,n+8)\,\mathrm{s}(n+3) + 16\,(2+n)\,(33\,n^2+125\,n+107)\,\mathrm{s}(1+n) = 0$$

We use my package's implementation of Petkovsek's algorithm, and deduce the hypergeometric term solution:

```
> TIME:=time():
  rechyper(rec,s(n));
  time()-TIME;
```

$$\{-4\}$$

$$1.532$$

Alternatively, we load a package which includes an implementation of a much faster algorithm than Petkovsek's by Mark van Hoeij:

```
> TIME:=time():
  `LREtools/hsols`(rec,s(n));
  time()-TIME;
```

$$[(-4)^n]$$

$$0.291$$

For c=5, we get

```
> rec:=sumrecursion((-1)^k*binomial(n,k)*binomial(5*k,n),k,s(n)
  );
```

$$rec := 625\,(2\,n+7)\,(4\,n+13)\,(4\,n+9)\,(n+3)\,(2+n)\,(1+n)\,\mathrm{s}(n) + 25\,(4\,n+5)\,(n+3)\,(1048\,n^4+12242\,n^3+52919\,n^2+100279\,n+70302)\,\mathrm{s}(2+n) + 8\,(2\,n+7)\,(2\,n+5)\,(4\,n+13)\,(4\,n+9)\,(4\,n+5)\,(4\,n+15)\,\mathrm{s}(n+4) + 5\,(9\,n+31)\,(4\,n+9)\,(4\,n+5)\,(2\,n+5)\,(41\,n^2+283\,n+486)\,\mathrm{s}(n+3) + 125\,(4\,n+13)\,(n+3)\,(2+n)\,(152\,n^3+1098\,n^2+2437\,n+1623)\,\mathrm{s}(1+n) = 0$$

```
> # TIME:=time():
  # rechyper(rec,s(n));
  # time()-TIME;
> TIME:=time():
  `LREtools/hsols`(rec,s(n));
  time()-TIME;
```

$$[(-5)^n]$$

$$0.441$$

Wolfram Koepf: Hypergeometric Summation, Exercise 9.3 (a):

```
> rec:=
  sumrecursion(hyperterm([-n,a,a+1/2,b],[2*a,(b-n+1)/2,(b-n)/2+
  1],1,k),k,s(n));
```

$$rec := (1+n)(b+n)(2a+1-b+n)(-b+n+2a)\,s(n) + 2(b+n+1)(b-n)(a+1+n)(2a+1-b+n)\,s(1+n) + (b+n+2)(b-n)(b-n-1)(2a+1+n)\,s(2+n) = 0$$

```
> TIME:=time():
  res2:=`LREtools/hsols`(rec,s(n));
  time()-TIME;
```

$$res2 := \left[\frac{\Gamma(-b+n+2a)\,\Gamma(1+n)}{\Gamma(n-b)\,\Gamma(n+2a)\,(b+n)}, \frac{\Gamma(-b+n+2a)}{\Gamma(n-b)\,(b+n)}\right]$$

$$0.580$$

```
>
```

Hyperexponential Integration

```
> read "hsum9.mpl";
```

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Continuous version of Gosper's algorithm.

Does the function

```
> f:=exp(x^2);
```

$$f := \mathbf{e}^{(x^2)}$$

have a hyperexponential antiderivative? The answer is

```
> contgosper(exp(x^2),x);
Error, (in contgosper) No hyperexponential antiderivative exists
```

The situation is different for

```
> contgosper(x*exp(x^2),x);
```

$$\frac{1}{2}\mathbf{e}^{(x^2)}$$

Let's do a more complicated example:

```
> term:=diff((1+x^2)/(1-x^10),x);
```

$$term := \frac{2x}{1-x^{10}} + \frac{10(1+x^2)x^9}{(1-x^{10})^2}$$

```
> res:=contgosper(term,x);
```

$$res := -\frac{1+x^2}{(x^6-x^5+x-1)(x^4+x^3+x^2+x+1)}$$

```
> res:=normal(res);
```

$$res := -\frac{1+x^2}{(x^6-x^5+x-1)(x^4+x^3+x^2+x+1)}$$

```
> res:=normal(res,expanded);
```

$$res := \frac{-1-x^2}{-1+x^{10}}$$

Let’s check Maple’s internal integrator:

```
> res:=int(term,x);
```

$$res := -\frac{2}{5}\frac{\arctan\left(\frac{4\,x+1+\sqrt{5}}{\sqrt{10-2\sqrt{5}}}\right)}{\sqrt{10-2\sqrt{5}}}$$

$$+\frac{(-8\sqrt{5}-(\sqrt{5}-5)(\sqrt{5}-1))\,x-2\sqrt{5}\,(\sqrt{5}-1)-4\sqrt{5}+20}{5\,(10+2\sqrt{5})\,(2\,x^2-x+\sqrt{5}\,x+2)}$$

$$+\frac{4}{5}\frac{\arctan\left(\frac{4\,x+1-\sqrt{5}}{\sqrt{10+2\sqrt{5}}}\right)\sqrt{5}}{(10+2\sqrt{5})^{(3/2)}}+\frac{2}{5}\frac{\arctan\left(\frac{4\,x-1-\sqrt{5}}{\sqrt{10-2\sqrt{5}}}\right)}{\sqrt{10-2\sqrt{5}}}-\frac{4\arctan\left(\frac{4\,x-1-\sqrt{5}}{\sqrt{10-2\sqrt{5}}}\right)}{(10-2\sqrt{5})^{(3/2)}}$$

$$+\frac{4\arctan\left(\frac{4\,x+1+\sqrt{5}}{\sqrt{10-2\sqrt{5}}}\right)}{(10-2\sqrt{5})^{(3/2)}}+\frac{2}{5}\frac{\arctan\left(\frac{4\,x-1+\sqrt{5}}{\sqrt{10+2\sqrt{5}}}\right)}{\sqrt{10+2\sqrt{5}}}-\frac{4}{5}\frac{\arctan\left(\frac{4\,x+1+\sqrt{5}}{\sqrt{10-2\sqrt{5}}}\right)\sqrt{5}}{(10-2\sqrt{5})^{(3/2)}}$$

$$+\frac{(8\sqrt{5}-(\sqrt{5}-5)(-\sqrt{5}+1))\,x+2\sqrt{5}\,(-\sqrt{5}+1)-4\sqrt{5}+20}{5\,(10+2\sqrt{5})\,(2\,x^2+x-\sqrt{5}\,x+2)}$$

$$-\frac{2}{5}\frac{\arctan\left(\frac{4\,x+1-\sqrt{5}}{\sqrt{10+2\sqrt{5}}}\right)}{\sqrt{10+2\sqrt{5}}}+\frac{4}{5}\frac{\arctan\left(\frac{4\,x-1-\sqrt{5}}{\sqrt{10-2\sqrt{5}}}\right)\sqrt{5}}{(10-2\sqrt{5})^{(3/2)}}$$

$$-\frac{4}{5}\frac{\arctan\left(\frac{4\,x-1+\sqrt{5}}{\sqrt{10+2\sqrt{5}}}\right)\sqrt{5}}{(10+2\sqrt{5})^{(3/2)}}+\frac{1}{5\,(x+1)}+\frac{4\arctan\left(\frac{4\,x+1-\sqrt{5}}{\sqrt{10+2\sqrt{5}}}\right)}{(10+2\sqrt{5})^{(3/2)}}-\frac{1}{5\,(-1+x)}$$

$$+\frac{(-8\sqrt{5}-(-\sqrt{5}-5)(\sqrt{5}+1))\,x-2\sqrt{5}\,(\sqrt{5}+1)+4\sqrt{5}+20}{5\,(10-2\sqrt{5})\,(2\,x^2+x+\sqrt{5}\,x+2)}$$

$$-\frac{4\arctan\left(\frac{4\,x-1+\sqrt{5}}{\sqrt{10+2\sqrt{5}}}\right)}{(10+2\sqrt{5})^{(3/2)}}$$

$$+\frac{(8\sqrt{5}-(-\sqrt{5}-5)(-\sqrt{5}-1))\,x+2\sqrt{5}\,(-\sqrt{5}-1)+4\sqrt{5}+20}{5\,(10-2\sqrt{5})\,(2\,x^2-x-\sqrt{5}\,x+2)}$$

```
> res:=normal(res);
```

$$res := 320\,(1+x^2) \Big/ \left((5+\sqrt{5})\,(2x^2-x+\sqrt{5}\,x+2)\,(2x^2+x-\sqrt{5}\,x+2)\,(x+1)\,(-1+x)\,(\sqrt{5}-5)\,(2x^2+x+\sqrt{5}\,x+2)\,(2x^2-x-\sqrt{5}\,x+2)\right)$$

```
> res:=normal(res,expanded);
```

$$res := \frac{-1-x^2}{-1+x^{10}}$$

Let's check Risch's algorithm:

```
> `int/risch`(term,x);
```

$$\frac{2\left(-\frac{1}{2}-\frac{x^2}{2}\right)}{-1+x^{10}}$$

```
>
```

Differential and Recurrence Equations for Integrals

```
> read "hsum9.mpl";
```

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We would like to compute:

```
> Int(x^2/(x^4+t^2)/(1+t^2),t=0..infinity);
```

$$\int_0^{\infty} \frac{x^2}{(x^4+t^2)\,(1+t^2)}\,dt$$

```
> integrand:=x^2/(x^4+t^2)/(1+t^2);
```

$$integrand := \frac{x^2}{(x^4+t^2)\,(1+t^2)}$$

The integrand is a hyperexponential term:

```
> contratio(integrand,t);
```

$$-\frac{2\,t\,(1+2\,t^2+x^4)}{(1+t^2)\,(x^4+t^2)}$$

What type of result should we expect?

```
> [ratio(integrand,x),contratio(integrand,x)];
```

$$\left[\frac{(x+1)^2\,(x^4+t^2)}{(x^4+4\,x^3+6\,x^2+4\,x+1+t^2)\,x^2}, -\frac{2\,(-t+x^2)\,(x^2+t)}{x\,(x^4+t^2)}\right]$$

Application of the continuous version of Zeilberger's algorithm:

```
> RE:=intrecursion(integrand,t,S(x));
Error, (in intrecursion) Algorithm finds no recurrence equation of order <=
5
```

```
> DE:=intdiffeq(integrand,t,S(x));
```

$$DE := (-1+x)\,(x+1)\,(1+x^2)\left(\frac{d^2}{dx^2}\,S(x)\right)x + (1+7\,x^4)\left(\frac{d}{dx}\,S(x)\right) + 8\,S(x)\,x^3 = 0$$

```
> dsolve(DE,S(x));
```

$$S(x) = \frac{\_C1}{x^4 - 1} + \frac{\_C2\, x^2}{x^4 - 1}$$

```
> res:=int(integrand,t=0..infinity);
```

$$res := \frac{1}{2} \frac{\pi\, (-\mathrm{csgn}(x^{-2}) + x^2)}{x^4 - 1}$$

```
> assume(x>0):
> res:=normal(res);
```

$$res := \frac{\pi}{2\, (x\sim^2 + 1)}$$

Which recurrence equation is valid for the result S(x)?

```
> ratio(res,x);
```

$$\frac{x\sim^2 + 1}{x\sim^2 + 2\, x\sim + 2}$$

```
> rat:=factor(ratio(res,x),I);
```

$$rat := \frac{(x\sim - I)\, (x\sim + I)}{(x\sim + 1 - I)\, (x\sim + 1 + I)}$$

Hence the recurrence equation for S(x) is

```
> denom(rat)*S(x+1)-numer(rat)*S(x)=0;
```

$$(-x\sim - 1 + I)\, (x\sim + 1 + I)\, S(x\sim + 1) - (-x\sim + I)\, (x\sim + I)\, S(x\sim) = 0$$

```
> x:='x':
>
```

Rodrigues Formulas

```
> read "hsum9.mpl";
```

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Rodrigues formula of the Legendre polynomials

```
> P(n,x)=(-1)^n/2^n/n!*diff((1-x^2)^n,x$n);
```

$$P(n, x) = \frac{(-1)^n \left(\frac{\partial^n}{\partial x^n} (1 - x^2)^n \right)}{2^n\, n!}$$

The following function computes the recurrence equation of the family by Cauchy's integral formula

```
> RE:=rodriguesrecursion((-1)^n/2^n/n!,(1-x^2)^n,x,P(n));
```

$$RE := (2 + n)\, P(2 + n) - x\, (3 + 2\, n)\, P(1 + n) + (1 + n)\, P(n) = 0$$

Similarly, we get the differential equation

```
> DE:=rodriguesdiffeq((-1)^n/2^n/n!,(1-x^2)^n,n,P(x));
```

$$DE := -(-1 + x)\, (x + 1) \left(\frac{d^2}{dx^2} P(x) \right) - 2\, x \left(\frac{d}{dx} P(x) \right) + P(x)\, n\, (1 + n) = 0$$

The holonomic recurrence equation defines the Legendre polynomials uniquely up to the initial values

```
> P(0,x)=eval(subs(n=0,(-1)^n/2^n/n!*(1-x^2)^n));
```

$$P(0, x) = 1$$

and

```
> P(1,x)=eval(subs(n=1,(-1)^n/2^n/n!*diff((1-x^2)^n,x$n)));
```

$$P(1, x) = x$$

Rodrigues formula of the generalized Laguerre polynomials

```
> L(n,alpha,x)=exp(x)/n!/x^alpha*diff(exp(-x)*x^(alpha+n),x$n);
```

$$L(n, \alpha, x) = \frac{e^x \left(\frac{\partial^n}{\partial x^n} (e^{(-x)} x^{(\alpha+n)}) \right)}{n!\, x^\alpha}$$

The following function computes the recurrence equation of the family by Cauchy's integral formula

```
> RE:=rodriguesrecursion(exp(x)/n!/x^alpha,exp(-x)*x^(alpha+n),
  x,L(n));
```

$$RE := (2+n)\, L(2+n) + (-\alpha - 3 - 2\, n + x)\, L(1+n) + (\alpha + n + 1)\, L(n) = 0$$

Similarly, we get the differential equation

```
> DE:=rodriguesdiffeq(exp(x)/n!/x^alpha,exp(-x)*x^(alpha+n),n,L
  (x));
```

$$DE := x \left(\frac{d^2}{dx^2} L(x) \right) - (x - \alpha - 1) \left(\frac{d}{dx} L(x) \right) + L(x)\, n = 0$$

The holonomic recurrence equation defines the Legendre polynomials uniquely up to the initial values

```
> L(0,alpha,x)=simplify(subs(n=0,exp(x)/n!/x^alpha*exp(-x)*x^(a
  lpha+n)));
```

$$L(0, \alpha, x) = 1$$

and

```
> L(1,alpha,x)=simplify(subs(n=1,exp(x)/n!/x^alpha*diff((exp(-x
  )*x^(alpha+n),x$n))));
```

$$L(1, \alpha, x) = -x + \alpha + 1$$

```
>
```

Generating Functions

```
> read "hsum9.mpl";
```

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The generating function of the generalized Laguerre polynomials satisfies the recurrence equation

```
> GFrecursion((1-z)^(-alpha-1)*exp((x*z)/(z-1)),1,z,L(n));
```

$$(2+n)\, L(2+n) + (-\alpha - 3 - 2\, n + x)\, L(1+n) + (\alpha + n + 1)\, L(n) = 0$$

compare:

```
> RE;
```

$$(2+n)\,L(2+n)+(-\alpha-3-2\,n+x)\,L(1+n)+(\alpha+n+1)\,L(n)=0$$

and the differential equation

```
> GFdiffeq((1-z)^(-alpha-1)*exp((x*z)/(z-1)),1,z,n,L(x));
```

$$x\left(\frac{d^2}{dx^2}L(x)\right)-(x-\alpha-1)\left(\frac{d}{dx}L(x)\right)+L(x)\,n=0$$

compare:

```
> DE;
```

$$x\left(\frac{d^2}{dx^2}L(x)\right)-(x-\alpha-1)\left(\frac{d}{dx}L(x)\right)+L(x)\,n=0$$

The initial values:

```
> series((1-z)^(-alpha-1)*exp((x*z)/(z-1)),z=0,3);
```

$$1+(1+\alpha-x)\,z+\left(-x+\frac{x^2}{2}-\frac{(\alpha+1)(-\alpha-2)}{2}-(\alpha+1)\,x\right)z^2+O(z^3)$$

```
>
```