ON THE SOLUTIONS OF HOLONOMIC SECOND-ORDER DIFFERENTIAL EQUATIONS IN TERMS OF BESSEL FUNCTIONS

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Dedications

To my beloved parents *MOUAFO JOSEPH* and *KENGNE VICTORINE* , and my brothers and sisters.

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Contents

List of Algorithms

List of Notations

$\boldsymbol{k}$	extension field of $\mathbb{Q}$	23
$\mathbf{C_K}$	constant field of K	3
$\mathbf{C_K}(\boldsymbol{x})$	field of rational functions over $\mathrm{C_K}$	3
$\mathbf{C_K}((\boldsymbol{x}))$	field of formal Laurent series over $\mathrm{C_K}$	3
$\overline{\mathbf{C_K}}$	algebraic closure of $\mathrm{C_K}$	5
$\overline{\mathbf{C_K}((\boldsymbol{x}))}$	union of $\mathrm{C_K}((x^{1/n}))$, $n \in \mathbb{N}^*$	27
$\mathbb{C}[\boldsymbol{x}]$	ring of polynomials over $\mathbb{C}$	45
$\mathbb{C}[[\boldsymbol{x}]]$	ring of formal power series over $\mathbb{C}$	27
$\mathbb{C}(\boldsymbol{x})$	field of rational functions over $\mathbb{C}$	27
$\mathbb{C}((\boldsymbol{x}))$	field of formal Laurent series over $\mathbb{C}$	25
$\boldsymbol{\partial}$	derivation $\frac{d}{dx}$	3
$\boldsymbol{\delta}$	derivation $x\frac{d}{dx}$	26
$\boldsymbol{t_p}$	local parameter at $x = p$	5
$\mathbf{L}$	differential operator	3
$\mathbf{L}_{\boldsymbol{B}}$	modified Bessel operator	32
$\mathbf{K}[\boldsymbol{\partial}]$	ring of differential operators over K	4
$\mathbf{V(L)}$	solution space of an operator L	4
$\boldsymbol{deg}(.)$	degree of a polynomial or an operator	3
$\boldsymbol{d_A}$	the upper bound of degree of numerator of change of variable f.......	47

$\boldsymbol{\nu}$	parameter of the Bessel function	7
$\mathbf{B}_{\boldsymbol{\nu}}$	one of the Bessel functions	20
$\boldsymbol{\Gamma}$	Gamma function	10
${}_{\boldsymbol{p}}\boldsymbol{F}_{\boldsymbol{q}}$	hypergeometric function	17
$\boldsymbol{U}$	the universal extension ring of a differential field	25
$\boldsymbol{r_x}$	ramification index of x	27
$\boldsymbol{gexp}(\mathbf{L}, \boldsymbol{p})$	the set of generalized exponents of L at $x = p$	27
$\longrightarrow_{\boldsymbol{C}}$	change of variables	29
$\longrightarrow_{\boldsymbol{E}}$	exp-product	29
$\longrightarrow_{\boldsymbol{G}}$	gauge transformation	29
$\longrightarrow_{\boldsymbol{EG}}$	composition of gauge and exp-product	30
$\longrightarrow$	composition of $\longrightarrow_C$, $\longrightarrow_E$ and $\longrightarrow_G$	29
$\boldsymbol{m_p}$	the multiplicity or pole order at $x = p$	33
$\boldsymbol{S_{reg}}$	set of exp-regular points	36
$\boldsymbol{S_{irr}}$	set of exp-irregular points	36
$\boldsymbol{\Delta}(\mathbf{L}, \boldsymbol{p})$	exponent-difference of L at $x = p$	30
$\boldsymbol{\mathcal{N}_p}$	possibilities for ν corresponding to $p \in S_{reg}$	38
$\boldsymbol{\mathcal{N}(d)}$	possibilities for ν corresponding to $d \in \mathbb{N}^*$	38
$\boldsymbol{\mathcal{N}}$	possibilities for the Bessel parameter ν	38
$\boldsymbol{A_1}$	the known part of $f^2 := \frac{CA_1A_2^d}{B}$	48
$\boldsymbol{A_2^d}$	the disappearing part of $f^2 := \frac{CA_1A_2^d}{B}$	48
$\boldsymbol{C}$	the constant part of $f^2 := \frac{CA_1A_2^d}{B}$	48
$\boldsymbol{monic}(\mathbf{L})$	the operator obtained by divided L by it leading coefficient	23

Abstract

In this master thesis, we first present Bessel functions as solutions of a certain second-order linear homogeneous differential equation, called " Bessel's equation " .
Secondly, we show that some families of second-order differential equations can be solved by means of Bessel functions, this throught specific transformations such as

- the change of variables,
- the exp-product and
- the Gauge transformation.

We complete the work by providing explicit examples for each transformation.

Résumé

Dans cette thèse de master, nous présentons prémièrement les fonctions de Bessel comme solutions d'une certaine équation différentielle linéaire homogène du second ordre, appelée équation de Bessel.
Deuxièmement, nous montrons que certaines familles d'équations différentielles du second ordre peuvent être résolues à l'aide des fonctions de Bessel, ceci à travers des transformations spécifiques telles que

- le changement de variables,
- le produit exponentiel et
- la transformation de Gauge.

Nous complétons le travail en donnant des exemples explicites pour chaque transformation.

Introduction

Ordinary differential equations have always been of interest since they occur in many applications. Although there is no general algorithm to solve every equation, there are many methods, such as integrating factors, symmetry method.

A special class of ordinary differential equations is the class of linear homogeneous differential equations $\mathrm{L}y = 0$, where L is a linear differential operator

$$\mathrm{L} = \sum_{i=0}^{n} a_i \partial^i,$$

with coefficients a_i in some differential field K, e.g. $\mathrm{K} = \mathbb{Q}(x)$ or $\mathrm{K} = \mathbb{C}(x)$ and $\partial = \frac{d}{dx}$. Information on the solutions of the differential equation $\mathrm{L}(y) = 0$, can be obtained by studying algebraic properties of the operator L, e.g. in [7].

For reducible operators, Beke's algorithm and the algorithm in [7] can factor L into irreducible factors. After factoring, if we have a right factor of first order, then we have a solution which is a Liouvillian solution.

But not all operators have liouvillian solutions. For example, the Bessel operator

$$\mathrm{L}_{B_1} = x^2\partial^2 + x\partial + (x^2 - \nu^2)$$

is irreducible in $\mathbb{C}(x)[\partial]$ and has no Liouvillian solution when $\nu \notin \frac{1}{2} + \mathbb{Z}$. Although some irreducible operators have no Liouvillian solutions, their solutions may correspond to special functions. For example, the solutions of the Bessel operator are Bessel functions. Because of the availability of various studies relative to special functions, it is useful to find solutions of second-order differential equations in terms of special functions, along side with algebraic operations and exponential integrals.

The approach we develop in this thesis will be restricted to Bessel functions. The Bessel functions were first used by Fredrich Bessel in 1824 to describe three body motion, with the Bessel functions appearing in the series expansion on planetary perturbation. They are solutions of an equation which appears frequently in applications and solutions to physical situations. A linear differential equation with rational function coefficients has a Bessel type solution when it is solvable in terms of the Bessel functions. The idea for algorithm to solve linear differential equations in terms of Bessel functions is by Mark van Hoeij, and was developed in collaboration with Ruben Debeerst in 2006.

We only consider irreducible operators with order two, because if the second order-ope rator is reducible, then it has Liouvillian solutions. So we can solve it by Kovacic's algorithm [4]. If the order is higher than two, one looks for Eulerian solution, that is, a solution which can be expressed as products of second-order operators (using sums, products, field operations, algebraic extensions, integrals, differentiations, exponential, logarithm and change of variables).

Singer [6] showed that solving such operator L can be reduced to solving second-order operators through factoring operators, or reducing operators to tensor products of lower order operators. An algorithm and implementation for such reduction (order three to order two) is given in [9]. Such reduction to order two is valuable, if we can actually solve such second-order equations. That is why we focus on second-order operators.

The main problem here is to decide whether an irreducible operator of order two can be obtained from the Bessel operator by certain transformations. To have solution, first we will give some preliminaries about differential operators, they singularities, and an overview over Bessel functions. Chapter two will deal with formal solutions and generalized exponents, then will describe the transformations that we use and show how they are obtained in the case $\mathrm{K} = \mathbb{C}(x)$. Those transformations associated with parameters are: (i) change of variables $x \to f(x)$, (ii) an exp-product $y \to exp(\int r\,dx)y$, and (iii) a gauge transformation $y \to r_0 y + r_1 y'$, where y' is $\frac{dy}{dx}$. We will also handle the constant parameter ν of the Bessel function. In the last chapter we will apply the resolution algorithm developed case by case with explicit examples.

Chapter One

Preliminaries

We first introduce some facts about differential operators, their solution spaces and corresponding differential equations. After that, we give an overview over the operator singularities and the solutions around those singularities. Finally we give some basic properties of Bessel functions and their corresponding differential operator. For the proofs of some of the statements given here, we will refer to specific literatures.

1.1 Differential Operators and Singularities

1.1.1 Differential Operators

Definition 1.1.1. *Let* K *be a field. A derivation on* K *is a linear map* $\mathrm{D}:\mathrm{K}\to\mathrm{K}$ *satisfying the product rule*

$$\mathrm{D}(ab)=a\mathrm{D}(b)+b\mathrm{D}(a),\qquad \forall a,b\in\mathrm{K}.$$

A field K with a derivation D is called *differential field*.

Theorem 1.1.1. *Let* K *be a differential field with derivation* D*, then* $\mathrm{C_K}:=\{a\in\mathrm{K}\mid\mathrm{D}(a)=0\}$ *is also a field. It is called the constant field of* K.

Proof: The proof is trivial and can be found in [7]. $\square$

Example 1.1.1. *Let us assume that* $\mathrm{C_K}$ *is an extension field of* $\mathbb{Q}$*, and* $\mathrm{D}=\partial:=\frac{d}{dx}$*, then*

- $\mathrm{C_K}(x)$ *is a differential field called the field of rational functions over* $\mathrm{C_K}$*;*

- $\mathrm{C_K}((x))$ *is a differential field called the field of formal Laurent series over* $\mathrm{C_K}$.

In our context we will consider functions in terms of variable x with the " usual " derivation $\partial{:=}\frac{d}{dx}$.

Definition 1.1.2. *Let* K *be a differential field with derivation* ∂*, then*

$$\mathrm{L}=\sum_{i=0}^{n}a_i\partial^i,\quad a_i\in\mathrm{K}$$

is called differential operator. When the coefficient $a_n \neq 0$*, then* n *is the degree of* L *denoted by* $deg(\mathrm{L})$*. In case* $\mathrm{L} = 0$ *we define the degree to be* $-\infty$*. The leading coefficient of* L *refers to the coefficient* a_n*.*

The ring of differential operators with coefficients in K, denoted by $\mathrm{K}[\partial]$, is an Euclidean ring since for $\mathrm{L}_1, \mathrm{L}_2 \in \mathrm{K}[\partial]$ with $\mathrm{L}_1 \neq 0$, there are unique differential operators $\mathrm{Q}, \mathrm{R} \in \mathrm{K}[\partial]$ such that $\mathrm{L}_2 = \mathrm{Q}\mathrm{L}_1 + \mathrm{R}$ and $deg\mathrm{R} < deg\mathrm{L}_1$. The addition is canonical, i.e. $a\partial^i + b\partial^i = (a+b)\partial^i$, and the multiplication is completely determined by the prescribed rule $\partial a = a\partial + a'$ where $\partial(a) = a'$. In general, since there exists an element $a \in \mathrm{K}$ with $a' \neq 0$ the ring $\mathrm{K}[\partial]$ is not commutative. For example $\partial x = x\partial + 1$

Every differential operator L corresponds to a homogeneous differential equation $\mathrm{L}y = 0$ and vice versa. Hence, when talking about differential equations, the term order is commonly used for the degree of the corresponding operator. We will always assume that $\mathrm{L} \neq 0$.

Definition 1.1.3. *By the solutions of a differential operator* L *we mean the solutions of the homogeneous linear differential equation* $\mathrm{L}y = 0$*. The vector space of solutions, which is denoted as* $\mathrm{V}(\mathrm{L})$*, is called the solution space of* L*.*

Remark 1.1.1. *The set* $\mathrm{V}(\mathrm{L})$ *is a vector space of dimension at most* $deg(\mathrm{L})$ *and a set of* $deg(\mathrm{L})$ *linearly independent solutions of* L *is called fundamental system of* L*.*

Note that a linear differential equation is commonly solved by transforming it into a matrix equation of order one.

Let k be a field and $\tilde{k}$ an extension field of k. Let us consider, for $a \in \tilde{k}$, the homomorphism

$$\begin{array}{rccc} \varphi_a : & k[X] & \longrightarrow & \tilde{k} \\ & P & \longmapsto & P(a) \end{array}$$

and $Ker\varphi_a = \{P \in k[X] | \, \varphi_a(P) = 0\}$ where $k[X]$ is the ring of polynomials with unknown variable X and coefficients in k.

Definition 1.1.4. *We say that* a *is algebraic over* k *if* $Ker\varphi_a \neq \{0\}$*.*

The set of algebraic elements of $\tilde{k}$ over k is a sub-field of $\tilde{k}$ containing k. We call it the algebraic closure of k, denoted by $\overline{k}$.

Theorem 1.1.2. *Let* k *be a sub-field of* $\mathbb{C}$*, then*

$$\overline{k((x))} = \bigcup_{n \in \mathbb{N}^*} \overline{k}((x^{1/n}))$$

Proof: The proof can be found in [7]. $\square$

1.1.2 Singular Points

Let $y(x)$ be a function with values in $\mathbb{C}$.

Definition 1.1.5. *A function $y(x)$ is called*

(i) *regular at $p \in \mathbb{C}$ if there exists a neighborhood O of p such that $y(x)$ is continuous on O,*

(ii) *regular at ∞ if $y\left(\frac{1}{x}\right)$ is regular at 0,*

(iii) *holomorphic at $p \in \mathbb{C}$ if $y(x)$ is differentiable in a open set around p,*

(iv) *analytic at $p \in \mathbb{C}$ if $y(x)$ can be represented as a power series*

$$y(x) = \sum_{i=0}^{+\infty} a_i (x-p)^i, \qquad a_i \in \mathbb{C}.$$

Definition 1.1.6. *Let K be a differential field, $\mathrm{C_K}$ it constant field and $\overline{\mathrm{C_K}}$ the algebraic closure of $\mathrm{C_K}$. We call a point $p \in \overline{\mathrm{C_K}}$ a singularity of the differential operator $\mathrm{L} \in \mathrm{K}[\partial]$, if p is a zero of the leading coefficient of L or p is a pole of one of the other coefficients. All other points are called regular.*

Remark 1.1.2. - *∞ is also a singular point of L; to understand it, one can always use the change of variables $x \to \frac{1}{x}$ and deal with 0.*

- *At all regular points of L we can find a fundamental system of power series solutions.*

If p is a singularity of a solution of L, then p must be a singularity of L. But the converse is not true (see apparent singularity in the definition after this following definition).

Definition 1.1.7. *If $p \in \overline{\mathrm{C_K}} \cup \{\infty\}$, we define the local parameter t_p as*

$$t_p = \begin{cases} x - p & \text{if } p \neq \infty \\ \frac{1}{x} & \text{if } p = \infty \end{cases}.$$

Definition 1.1.8. *Let $\mathrm{L} = \sum_{i=0}^{n} a_i \partial^i \in \mathrm{K}[\partial]$ with $a_n = 1$. A singularity p of L is called*

(i) *apparent singularity if all solutions of L are regular at p,*

(ii) *regular singular ($p \neq \infty$) if ${t_p}^i a_{n-i}$ is regular at p for $1 \leq i \leq n$,*

(iii) *regular singular ($p = \infty$) if $\frac{a_{n-i}}{t_\infty^i}$ is regular at ∞ for $1 \leq i \leq n$, and*

(iv) *irregular singular otherwise.*

Theorem 1.1.3. *Let* $\mathrm{L} = \partial^2 + P(x)\partial + Q(x) \in \mathrm{K}[\partial]$ *and let* $p \in \mathbb{C}$ *be a point with local parameter* $t_p = x - p$.

(i) If L *is regular or apparent singular at* p*, then all solutions are analytic at* $x = p$*. Hence, they can be written as convergent power series. Therefore there exists a unique solution* $y(x) = \sum_{j=0}^{+\infty} a_j {t_p}^j$ *of* L *satisfying the initial conditions* $y(p) = c_0$ *and* $y'(p) = c_1$*, where* c_0 *and* c_1 *are given arbitrary constants.*

(ii) If L *is regular singular at* p*, then there exists the two linearly independent solutions*

$$\begin{aligned} y_1(x) =& {t_p}^{e_1} \sum_{j=0}^{+\infty} a_j {t_p}^j, \quad a_0 \neq 0 \\ \textit{and} \quad y_2(x) =& {t_p}^{e_2} \sum_{j=0}^{+\infty} b_j {t_p}^j + c y_1(x) \ln(t_p), \quad \textit{when } b_0 \textit{ and } c \textit{ are not both zero}, \end{aligned}$$

with $e_1, e_2, a_j, b_j, c \in \overline{\mathrm{C_K}}$ *are constants and* $c = 0$ *if* $e_1 - e_2 \notin \mathbb{Z}$.

(iii) If L *is irregular singular at* p*, two linearly independent solutions are*

$$\begin{aligned} y_1(x) =& \exp\left(\int \frac{e_1}{t_p}\, dt_p\right) \sum_{j=0}^{+\infty} a_j {t_p}^{\frac{j}{m}}, \quad a_0 \neq 0 \\ \textit{and} \quad y_2(x) =& \exp\left(\int \frac{e_2}{t_p}\, dt_p\right) \sum_{j=0}^{+\infty} b_j {t_p}^{\frac{j}{m}} + c y_1(x) \ln(t_p), \quad b_0 \textit{ and } c \textit{ are not both zero}, \end{aligned}$$

with $a_i, b_i, c \in \overline{\mathrm{C_K}}$, $e_1, e_2 \in \overline{\mathrm{C_K}}[t_p^{\frac{-1}{m}}]$, $c = 0$ *if* $e_1 - e_2 \notin \mathbb{Z}$ *and* m *is* 1 *or* 2 *(because the order of* L *is two).*

__Proof:__ The proof can be found in [10]. □

Definition 1.1.9. *In the previous theorem, if* $c = 0$*, then the solutions of* L *do not contain logarithmic terms. If* $c \neq 0$*, then we say that* L *has logarithmic solutions at* $x = p$.

Remark 1.1.3.

- *Note that* L *can only have logarithm solutions at* $x = p$ *if* $e_1 - e_2 \in \mathbb{Z}$.
- *In the regular singular case, the constants* e_1 *and* e_2 *can be found by solving the indicial equation*

$$\lambda(\lambda - 1) + p_0\lambda + q_0 = 0$$

where p_0 *resp.* q_0 *is the constant coefficient of the power series expansion of* $(z - z_0)P(x)$ *resp.* $(z - z_0)^2 Q(x)$ *at* $z = z_0$:

$$p_0 = \lim_{x \to p} t_p P(x), \quad q_0 = \lim_{x \to p} t_p^2 Q(x).$$

- *In both cases, regular singularity and irregular singularity,* e_1 *and* e_2 *are generalized exponents, which will be explained in the next chapter.*

1.2 Bessel Functions

1.2.1 Bessel's Differential Equation

In the Sturm-Liouville Boundary value problem, there is an important special case called Bessel's Differential Equation which arises in numerous problem, especially in polar and cylindral coordinates.

One of the most important of all variable-coefficient differential equations is

$$t^2\frac{d^2y}{dt^2}+t\frac{dy}{dt}+\left(\lambda^2t^2-\nu^2\right)y=0, \tag{1.1}$$

which is known as Bessel's ordinary equation of order ν with parameter λ, where $\nu,\ \lambda\in\mathbb{C}$.

By doing a change of variables from t to x using the substitution $x=\lambda t$ we get:

$$x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}+\left(x^2-\nu^2\right)y=0, \tag{1.2}$$

which is known as Bessel's differential equation of order ν. ν is called the Bessel's parameter. The solutions of this equation are called Bessel function of order ν. This equation has singularities at 0 and ∞.

Bessel Functions of the first and second kind

Let us be in the vicinity of $x=0$ with $\nu\in\mathbb{C}$. Since the equation (1.2) has a regular singular point at $x=0$, we can assume a solution of the form

$$y=x^c\sum_{i=0}^{+\infty}a_ix^i=\sum_{i=0}^{+\infty}a_ix^{i+c} \tag{1.3}$$

with $a_0\neq 0$ by theorem 1.1.3, and apply Frobenius method using this series. The substitution of (1.3) in (1.2) gives

$$\sum_{i=0}^{+\infty}[(i+c)^2-\nu^2]a_ix^{i+c}+\sum_{i=0}^{+\infty}a_ix^{i+c+2}=0.$$

A schift in index, replacing i with $i-2$ in the second term, gives

$$\sum_{i=0}^{+\infty}\left[(i+c)^2-\nu^2\right]a_ix^{i+c}+\sum_{i=2}^{+\infty}a_{i-2}x^{i+c}=0.$$

The equations for determining the parameter c and the coefficients a_i are:

$$\begin{cases} i=0 & : \ (c^2-\nu^2)\,a_0=0,\\ i=1 & : \ \left[(1+c)^2-\nu^2\right]a_1=0,\\ i\geq 2 & : \ \left[(i+c)^2-\nu^2\right]a_i+a_{i-2}=0.\end{cases} \tag{1.4}$$

The indicial equation is then given by

$$c^2 - \nu^2 = 0,$$

and it's roots are $c_1 = \nu$ and $c_2 = -\nu$.

Since $a_0 \neq 0$, then $c = \pm\nu$

case 1: $c = \nu$

Substituting $c = \nu$ into equations (1.4) we get

$$\begin{aligned} 0.a_0 &= 0, \\ (1+2\nu).a_1 &= 0, \\ i(i+2\nu)a_i + a_{i-2} &= 0, \qquad i \geq 2 \end{aligned}$$

The last equation gives us the recurrence relation

$$a_i = -\frac{1}{i(i+2\nu)}a_{i-2}, \qquad i \geq 2 \; and \; \nu \notin -\frac{1}{2}\left(\mathbb{N}\backslash\{0,1\}\right). \tag{1.5}$$

1- If $a_1 \neq 0$, then $\nu = -\frac{1}{2}$ and the recurrence relation (1.5) give us:
for $i = 2k$, where $k = 1, 2, ...$

$$\begin{aligned} a_{2k} &= -\frac{1}{2k(2k-1)}a_{2k-2} \\ &= \frac{(-1)^k}{[2k(2k-1)]\,[2(k-1)(2k-3)]\,...\,[2.1.1]}a_0 \\ &= \frac{(-1)^k}{2^k k! \prod_{j=1}^{k}(2j-1)}a_0 \\ &= \frac{(-1)^k}{\prod_{j=1}^{k}(2j)\prod_{j=1}^{k}(2j-1)}a_0 \qquad \left(using\ the\ identity\ \ 2^k k! = \prod_{j=1}^{k}(2j)\right) \\ &= \frac{(-1)^k}{(2k)!}a_0. \end{aligned}$$

Similarly, for $i = 2k+1$, where $k = 1, 2, ...$

$$\begin{aligned} a_{2k+1} &= -\frac{1}{2k(2k+1)}a_{2k-1} \\ &= \frac{(-1)^k}{[2k(2k+1)]\,[2(k-1)(2k-1)]\,...\,[2.1.3]}a_1 \\ &= \frac{(-1)^k}{2^k k! \prod_{j=0}^{k}(2j+1)}a_1 \\ &= \frac{(-1)^k}{\prod_{j=1}^{k}(2j)\prod_{j=0}^{k}(2j+1)}a_1 \qquad \left(using\ the\ identity\ \ 2^k k! = \prod_{j=1}^{k}(2j)\right) \\ &= \frac{(-1)^k}{(2k+1)!}a_1. \end{aligned}$$

Plugging back into (1.3) give us

$$\begin{aligned} y &= x^{-\frac{1}{2}} \sum_{i=0}^{+\infty} a_i x^i \\ &= x^{-\frac{1}{2}} \left[\sum_{k=0}^{+\infty} a_{2k} x^{2k} + \sum_{k=0}^{+\infty} a_{2k+1} x^{2k+1} \right] \\ &= x^{-\frac{1}{2}} \left[a_0 \sum_{k=0}^{+\infty} \frac{(-1)^k}{(2k)!} x^{2k} + a_1 \sum_{k=0}^{+\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1} \right] \\ &= x^{-\frac{1}{2}} \left(a_0 \cos x + a_1 \sin x \right). \end{aligned}$$

2- If $a_1 = 0$, then all terms with odd subscript will be zero.

* For $\nu = -\frac{1}{2}$ we have $(k = 1, 2, ...)$

$$a_{2k} = \frac{(-1)^k}{(2k)!} a_0 \quad and \quad a_{2k+1} = 0.$$

Therefore

$$y = x^{-\frac{1}{2}} a_0 \cos x.$$

* For $\nu = \frac{1}{2}$ we have $(k = 1, 2, ...)$

$$\begin{aligned} a_{2k} &= -\frac{1}{2k(2k+1)} a_{2k-2} \\ &= \frac{(-1)^k}{[2k(2k+1)]\,[2(k-1)(2k-1)] ... [2.1.3]} a_0 \\ &= \frac{(-1)^k}{2^k k! \prod_{j=0}^{k} (2j+1)} a_0 \\ &= \frac{(-1)^k}{\prod_{j=1}^{k} (2j) \prod_{j=0}^{k} (2j+1)} a_0 \qquad \left(using\ the\ identity\ 2^k k! = \prod_{j=1}^{k} (2j) \right) \\ &= \frac{(-1)^k}{(2k+1)!} a_0. \end{aligned}$$

Plugging back into (1.3) send us

$$\begin{aligned} y = x^{\frac{1}{2}} \sum_{i=0}^{+\infty} a_i x^i = x^{\frac{1}{2}} \left[\sum_{k=0}^{+\infty} a_{2k} x^{2k} \right] = x^{\frac{1}{2}} a_0 \sum_{k=0}^{+\infty} \frac{(-1)^k}{(2k+1)!} x^{2k} \\ = x^{-\frac{1}{2}} \sum_{k=0}^{+\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1} = x^{-\frac{1}{2}} a_0 \sin x. \end{aligned}$$

$*$ For $\mid \nu \mid \neq \frac{1}{2}$ we have:

$$\begin{aligned} a_2 &= -\frac{1}{2(2+2\nu)} a_0 \\ a_4 &= -\frac{1}{4(4+2\nu)} a_2 \\ a_6 &= -\frac{1}{6(6+2\nu)} a_4 \\ &-- - -- \\ a_{2k} &= -\frac{1}{2k(2k+2\nu)} a_{2k-2}. \end{aligned}$$

Multiplying these equations together and simplifying we get

$$a_{2k} = \frac{(-1)^k}{2.4.6....(2k)\left[(2+2\nu)(4+2\nu)(6+2\nu)...(2k+2\nu)\right]} a_0$$

or

$$a_{2k} = \frac{(-1)^k}{2^{2k} k! (\nu+1)(\nu+2)(\nu+3)....(\nu+k)} a_0.$$

This represents the expression for the coefficients.
Let us make some modifications for purposes of simplification. Let $\Gamma(x)$ denotes the Gamma function

$$\Gamma(x) := \int_0^{+\infty} t^{x-1} \exp(-t) dt, \quad Re(x) > 0$$

and multiply both numerator and denominator of a_{2k} by $\Gamma(\nu+1)$. This gives

$$a_{2k} = \frac{(-1)^k \Gamma(\nu+1)}{2^{2k} k! \Gamma(\nu+1)(\nu+1)\left[(\nu+2)(\nu+3)....(\nu+k)\right]} a_0.$$

Since $\Gamma(\nu+1)\left[(\nu+2)(\nu+3)....(\nu+k)\right] = \Gamma(\nu+k+1)$ we have

$$a_{2k} = \frac{(-1)^k \Gamma(\nu+1)}{2^{2k} k! \Gamma(\nu+k+1)} a_0.$$

Let us now multiply the numerator and denominator by 2^ν, we obtain

$$a_{2k} = \frac{(-1)^k 2^\nu \Gamma(\nu+1)}{2^{2k+\nu} k! \Gamma(\nu+k+1)} a_0.$$

Hence the solution can then be written as

$$y = 2^\nu a_0 \Gamma(\nu+1) \sum_{k=0}^{+\infty} \frac{(-1)^k}{k! \Gamma(\nu+k+1)} \left(\frac{x}{2}\right)^{\nu+2k}.$$

case 2: $c = -\nu$

Substituting $c = -\nu$ into equations (1.4) we get

$$\begin{aligned} 0.a_0 &= 0 \\ (1-2\nu).a_1 &= 0 \\ i(i-2\nu)a_i + a_{i-2} &= 0 \qquad i \geq 2 \end{aligned}$$

The last equation gives us the recurrence relation

$$a_i = -\frac{1}{i(i-2\nu)}a_{i-2} \quad i \geq 2 \ and \ \nu \notin \frac{1}{2}\left(\mathbb{N}\backslash\{0,1\}\right). \tag{1.6}$$

1- If $a_1 \neq 0$, then $\nu = \frac{1}{2}$ and the recurrence relation (1.6) give us $(k = 1, 2, ...)$

$$a_{2k} = \frac{(-1)^k}{(2k)!}a_0 \quad and \quad a_{2k+1} = \frac{(-1)^k}{(2k+1)!}a_1.$$

Therefore

$$y = x^{-\frac{1}{2}}\left(a_0 \cos x + a_1 \sin x\right).$$

2- If $a_1 = 0$, then all terms with odd subscript will be zero.

$*$ For $\nu = \frac{1}{2}$ we have $(k = 1, 2, ...)$

$$a_{2k} = \frac{(-1)^k}{(2k)!}a_0 \quad and \quad a_{2k+1} = 0.$$

Therefore

$$y = x^{-\frac{1}{2}}a_0 \cos x$$

$*$ For $\nu = -\frac{1}{2}$ we have $(k = 1, 2, ...)$

$$\begin{aligned}
a_{2k} &= -\frac{1}{2k(2k+1)}a_{2k-2} \\
&= \frac{(-1)^k}{[2k(2k+1)]\,[2(k-1)(2k-1)]\,...\,[2.1.3]}a_0 \\
&= \frac{(-1)^k}{2^k k! \prod_{j=0}^{k}(2j+1)}a_0 \\
&= \frac{(-1)^k}{\prod_{j=1}^{k}(2j)\prod_{j=0}^{k}(2j+1)}a_0 \qquad using\ the\ identity\ 2^k k! = \prod_{j=1}^{k}(2j) \\
&= \frac{(-1)^k}{(2k+1)!}a_0.
\end{aligned}$$

Plugging back into (1.3) send us to

$$\begin{aligned}
y = x^{\frac{1}{2}}\sum_{i=0}^{+\infty}a_i x^i = x^{\frac{1}{2}}\left[\sum_{k=0}^{+\infty}a_{2k}x^{2k}\right] &= x^{\frac{1}{2}}a_0\sum_{k=0}^{+\infty}\frac{(-1)^k}{(2k+1)!}x^{2k} \\
&= x^{-\frac{1}{2}}\sum_{k=0}^{+\infty}\frac{(-1)^k}{(2k+1)!}x^{2k+1} = x^{-\frac{1}{2}}a_0 \sin x.
\end{aligned}$$

∗ For $\mid \nu \mid \neq \frac{1}{2}$ we have:

$$a_2 = -\frac{1}{2(2-2\nu)}a_0$$
$$a_4 = -\frac{1}{4(4-2\nu)}a_2$$
$$a_6 = -\frac{1}{6(6-2\nu)}a_4$$
$$-- - --$$
$$a_{2k} = -\frac{1}{2k(2k-2\nu)}a_{2k-2}.$$

Multiplying these equations together and simplifying we get

$$a_{2k} = \frac{(-1)^k}{2.4.6....(2k)\left[(2-2\nu)(4-2\nu)(6-2\nu)...(2k-2\nu)\right]}a_0$$

or

$$a_{2k} = \frac{(-1)^k}{2^{2k}k!(1-\nu)(2-\nu)(3-\nu)....(k-\nu)}a_0.$$

Let us multiply both numerator and denominator of a_{2k} by $\Gamma(1-\nu)$: this gives

$$a_{2k} = \frac{(-1)^k\Gamma(1-\nu)}{2^{2k}k!\Gamma(1-\nu)(1-\nu)\left[(2-\nu)(3-\nu)....(k-\nu)\right]}a_0.$$

Since $\Gamma(1-\nu)(1-\nu)\left[(2-\nu)(c+3)....(k-\nu)\right] = \Gamma(k-\nu+1)$ we have

$$a_{2k} = \frac{(-1)^k\Gamma(1-\nu)}{2^{2k}k!\Gamma(k-\nu+1)}a_0.$$

Let us now multiply the numerator and denominator by $2^{-\nu}$, we obtain

$$a_{2k} = \frac{(-1)^k 2^{-\nu}\Gamma(1-\nu)}{2^{2k-\nu}k!\Gamma(k-\nu+1)}a_0.$$

Hence the solution can then be written as

$$y = 2^{-\nu}a_0\Gamma(1-\nu)\sum_{k=0}^{+\infty}\frac{(-1)^k}{k!\Gamma(k-\nu+1)}\left(\frac{x}{2}\right)^{-\nu+2k}.$$

The constant a_0 is arbitrary and since we are only looking for a particular solution let us assign a_0 the value

$$a_0 = \frac{1}{2^{\nu}\Gamma(\nu+1)} \quad when \quad c = \nu \ ,$$
$$and \quad a_0 = \frac{1}{2^{-\nu}\Gamma(-\nu+1)} \quad when \quad c = -\nu \ .$$

Hence, we obtain as solution, in the case $c = \nu$, the function

$$J_\nu(x) = \begin{cases} \sum_{k=0}^{+\infty}\frac{(-1)^k}{k!\Gamma(\nu+k+1)}\left(\frac{x}{2}\right)^{\nu+2k} & for \quad \mid \nu \mid \neq \frac{1}{2} \\ \sqrt{\frac{2}{\pi x}}\cos(x) & for \quad \nu = -\frac{1}{2} \\ \sqrt{\frac{2}{\pi x}}\sin(x) & for \quad \nu = \frac{1}{2}. \end{cases} \tag{1.7}$$

which is called the Bessel function of the first kind of order ν. Since the Bessel's equation has no finite singular points except the origin, the series

$$\sum_{k=0}^{+\infty} \frac{(-1)^k}{k!\Gamma(\nu+k+1)} \left(\frac{x}{2}\right)^{\nu+2k},$$

converge for all $x \neq 0$. By the D'Alembert criteria we have the convergence for all x if $\nu \geq 0$. We can relate $J_m(x)$ and $J_{-m}(x)$ (when m is an integer). Let $k = k' + m$ by writing

$$\begin{aligned} J_{-m}(x) &= \sum_{k=0}^{+\infty} \frac{(-1)^k}{k!\Gamma(-m+k+1)} \left(\frac{x}{2}\right)^{-m+2k} \\ &= \sum_{k'+m=0}^{+\infty} \frac{(-1)^{k'+m}}{k'!\Gamma(m+k'+1)} \left(\frac{x}{2}\right)^{m+2k'} \qquad \textit{with } k = k'+m \\ &= \sum_{k'=-m}^{-1} \frac{(-1)^{k'+m}}{k'!\Gamma(m+k'+1)} \left(\frac{x}{2}\right)^{m+2k'} + \sum_{k'=0}^{+\infty} \frac{(-1)^{k'+m}}{k'!\Gamma(m+k'+1)} \left(\frac{x}{2}\right)^{m+2k'}. \end{aligned}$$

But $k'! := \infty$ for $k' = -m, ..., -1$, so the denominator is infinite and the terms on the left are zero. We therefore have

$$\begin{aligned} J_{-m}(x) &= \sum_{k'=0}^{+\infty} \frac{(-1)^{k'+m}}{k'!\Gamma(m+k'+1)} \left(\frac{x}{2}\right)^{m+2k'} \\ &= (-1)^m J_m(x). \end{aligned} \tag{1.8}$$

Note that the Bessel differential equation is second-order, so there must be two linearly independent solutions.
The wronkian of J_ν and $J_{-\nu}$ is

$$w = J'_{-\nu} J_\nu - J'_\nu J_{-\nu} = \frac{2 \sin \nu\pi}{\pi},$$

hence

- for $\nu \in \mathbb{Z}$ we have $\sin \nu\pi = 0$ and then J_ν and $J_{-\nu}$ are linearly dependent (as we have seen before);
- for $\nu \notin \mathbb{Z}$ we have $\sin \nu\pi \neq 0$ and then J_ν and $J_{-\nu}$ are linearly independent.

Our problem now is to find a second solution which is linearly independent with J_ν in both cases (when ν is an integer and when it is not).
Let us define the function

$$Y_\nu(x) = \frac{J_\nu(x)\cos(\pi\nu) - J_{-\nu}(x)}{\sin(\pi\nu)};$$

it is clear that if ν is not an integer $Y_\nu(x)$ must be a solution of (1.2) but if not it will be like an indefined form since $\cos(\pi\nu) = (-1)^\nu$ and $\sin(\pi\nu) = 0$. Let $m \in \mathbb{Z}$, by the application of Hospital rule:

$$\begin{aligned} Y_m(x) &= \lim_{\nu\to m} \frac{-\pi \sin \pi\nu J_\nu(x) + \cos(\pi\nu) J_\nu(x)' - J_{-\nu}(x)'}{\pi\cos(\pi\nu)} \\ &= \frac{1}{\pi}\left[\lim_{\nu\to m} J_\nu(x)'\right] - (-1)^m \frac{1}{\pi}\left[\lim_{\nu\to m} J_{-\nu}(x)'\right]. \end{aligned} \tag{1.9}$$

$$\begin{aligned} \frac{dJ_\nu(x)}{d\nu} &= \frac{d}{d\nu}\left[\left(\frac{x}{2}\right)^\nu \sum_{k=0}^{+\infty} \frac{(-1)^k}{k!\Gamma(\nu+k+1)}\left(\frac{x}{2}\right)^{2k}\right] \\ &= \frac{d}{d\nu}\left[\exp\left(\nu \ln\left(\frac{x}{2}\right)\right) \sum_{k=0}^{+\infty} \frac{(-1)^k}{k!\Gamma(\nu+k+1)}\left(\frac{x}{2}\right)^{2k}\right] \\ &= \ln\left(\frac{x}{2}\right)\exp\left(\nu \ln\left(\frac{x}{2}\right)\right) \sum_{k=0}^{+\infty} \frac{(-1)^k}{k!\Gamma(\nu+k+1)}\left(\frac{x}{2}\right)^{2k} \\ &\quad - \left(\frac{x}{2}\right)^\nu \sum_{k=0}^{+\infty} \frac{(-1)^k\Gamma'(\nu+k+1)}{k!\Gamma^2(\nu+k+1)}\left(\frac{x}{2}\right)^{2k} \\ &= \ln\left(\frac{x}{2}\right) J_\nu(x) - \sum_{k=0}^{+\infty} \frac{(-1)^k}{k!\Gamma(\nu+k+1)} \frac{\Gamma'(\nu+k+1)}{\Gamma(\nu+k+1)}\left(\frac{x}{2}\right)^{2k+\nu}. \end{aligned}$$

When the value of ν tends towards m, the logarithmic derivative of Γ is the digamma function ψ:

$$\frac{\Gamma'(\nu+k+1)}{\Gamma(\nu+k+1)} = \psi(\nu+k+1) \ ;$$

hence

$$\lim_{\nu\to m} \frac{dJ_\nu(x)}{d\nu} = \ln\left(\frac{x}{2}\right) J_m(x) - \sum_{k=0}^{+\infty} \frac{(-1)^k}{k!(m+k)!}\psi(m+k+1)\left(\frac{x}{2}\right)^{2k+m}. \tag{1.10}$$

For $J_{-\nu}(x)$ we use the complements formula:

$$\Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin \pi x}$$

which give us

$$\Gamma(1-\nu+k) = \frac{\pi}{\Gamma(\nu-k)\sin\pi(\nu-k)} = \frac{\pi}{(-1)^k\Gamma(\nu-k)\sin\pi\nu} \tag{1.11}$$

hence if m is the integer greater that ν and nearest to it

$$\begin{aligned}
\frac{dJ_{-\nu}(x)}{d\nu} &= \frac{d}{d\nu}\left[\left(\frac{x}{2}\right)^{-\nu}\sum_{k=0}^{m-1}\frac{(-1)^k}{k!\Gamma(-\nu+k+1)}\left(\frac{x}{2}\right)^{2k}\right.\\
&\quad\left.+\left(\frac{x}{2}\right)^{-\nu}\sum_{k=m}^{+\infty}\frac{(-1)^k}{k!\Gamma(-\nu+k+1)}\left(\frac{x}{2}\right)^{2k}\right]\\
&= \frac{d}{d\nu}\left[\left(\frac{x}{2}\right)^{-\nu}\sum_{k=0}^{m-1}\frac{\Gamma(\nu-k)}{k!}\frac{\sin\pi\nu}{\pi}\left(\frac{x}{2}\right)^{2k}\right.\\
&\quad\left.+\left(\frac{x}{2}\right)^{-\nu}\sum_{k=m}^{+\infty}\frac{(-1)^k}{k!\Gamma(-\nu+k+1)}\left(\frac{x}{2}\right)^{2k}\right] \quad using\ (1.11)\\
&= -J_{-\nu}(x)\log\left(\frac{x}{2}\right)+\sum_{k=0}^{m-1}\frac{1}{k!}\left(\Gamma'(\nu-k)\frac{\sin\pi\nu}{\pi}+\Gamma(\nu-k)\cos\pi\nu\right)\left(\frac{x}{2}\right)^{-\nu+2k}\\
&\quad-\sum_{k=m}^{+\infty}\frac{(-1)^k\Gamma'(-\nu+k+1)}{k!\Gamma^2(-\nu+k+1)}\left(\frac{x}{2}\right)^{-\nu+2k},
\end{aligned}$$

$$\begin{aligned}
so\quad \frac{dJ_{-\nu}(x)}{d\nu} &= -J_{-\nu}(x)\log\left(\frac{x}{2}\right)+\sum_{k=0}^{m-1}\frac{1}{k!}\left(\Gamma'(\nu-k)\frac{\sin\pi\nu}{\pi}+\Gamma(\nu-k)\cos\pi\nu\right)\left(\frac{x}{2}\right)^{-\nu+2k}\\
&\quad-\sum_{k=m}^{+\infty}\frac{(-1)^k}{k!\Gamma(-\nu+k+1)}\psi(-\nu+k+1)\left(\frac{x}{2}\right)^{-\nu+2k}.
\end{aligned}$$

When we make the value of ν tend towards m,

$$\begin{aligned}
\lim_{\nu\to m}\frac{dJ_{-\nu}(x)}{d\nu} &= -J_{-m}(x)\log\left(\frac{x}{2}\right)+(-1)^m\sum_{k=0}^{m-1}\frac{\Gamma(m-k)}{k!}\left(\frac{x}{2}\right)^{-m+2k}\\
&\quad-\sum_{k=m}^{+\infty}\frac{(-1)^k}{k!\Gamma(-m+k+1)}\psi(-m+k+1)\left(\frac{x}{2}\right)^{-m+2k}.
\end{aligned}$$

If we use (1.8) and the relation between Γ and factorial function we get by replacing $k-m$ with k in the last term

$$\begin{aligned}
\lim_{\nu\to m}\frac{dJ_{-\nu}(x)}{d\nu} &= -(-1)^m J_m(x)\log\left(\frac{x}{2}\right)+(-1)^m\sum_{k=0}^{m-1}\frac{\Gamma(m-k)}{k!}\left(\frac{x}{2}\right)^{-m+2k}\\
&\quad-(-1)^m\sum_{k=0}^{+\infty}\frac{(-1)^k}{(k+m)!k!}\psi(k+1)\left(\frac{x}{2}\right)^{m+2k}. \qquad (1.12)
\end{aligned}$$

Now, when we replace (1.10) and (1.12) in (1.9) we get:

$$\begin{aligned} Y_m(x) &= \frac{2}{\pi} J_m(x) \log\left(\frac{x}{2}\right) - \frac{1}{\pi} \sum_{k=0}^{m-1} \frac{(m-k-1)!}{k!} \left(\frac{x}{2}\right)^{2k-m} \\ &- \frac{1}{\pi} \sum_{k=0}^{+\infty} \frac{(-1)^k}{(k+m)!k!} \left[\psi(m+k+1) - \psi(k+1)\right] \left(\frac{x}{2}\right)^{2k+m} . \qquad (1.13) \end{aligned}$$

So our functions $J_\nu(x)$ and $Y_\nu(x)$ for all $\nu \in \mathbb{R}$ are two linear independent solutions of the equation (1.2). The function Y_ν is what we call the Bessel function of the second kind (Neumann function or Weber function). The general (real) solution is then of the form:

$$z_\nu \equiv c_1 J_\nu(x) + c_2 Y_\nu(x)$$

where c_1 and c_2 are constants.

The Bessel function $J_\nu(x)$ can be expressing in term of hypergeometric function.

Definition 1.2.1. *A generalized hypergeometric series ${}_pF_q$ is defined by*

$$ {}_pF_q\left(\begin{array}{c} \alpha_1, \alpha_2, ..., \alpha_p \\ \beta_1, \beta_2, ..., \beta_q \end{array} \middle| \, x \right) = \sum_{k=0}^{+\infty} \frac{(\alpha_1)_k.(\alpha_2)_k...(\alpha_P)_k}{(\beta_1)_k.(\beta_2)_k...(\beta_q)_k.k!} x^k \ ,$$

where $(\lambda)_k$ denotes the Pochhammer symbol

$$(\lambda)_k := \begin{cases} 1 & if \ k = 0 \\ \lambda.(\lambda+1)...(\lambda+k-1) & if \ k > 0 \ . \end{cases}$$

They satisfy the differential equation:

Theorem 1.2.1. *The generalized hypergeometric series ${}_pF_q$ in the previous definition satisfies the differential equation*

$$\delta(\delta+\beta_1-1)...(\delta+\beta_q-1)y(x) = x(\delta+\alpha_1)...(\delta+\alpha_p)y(x)$$

where $\delta = x\frac{d}{dx}$.

Remarks 1.2.1. *1. For $p \leq q$ the series ${}_pF_q$ is convergent for all x. For $p > q+1$ the radius of convergence is zero, and for $p = q+1$ the series converges for $\mid x \mid < 1$.*

2. For $p \leq q+1$ the series and its analytic continuation is called hypergeometric function.

$$\begin{aligned} J_\nu(x) &= \left(\frac{x}{2}\right)^\nu \sum_{k=0}^{+\infty} \frac{(-1)^k}{k!\Gamma(\nu+k+1)} \left(\frac{x}{2}\right)^{2k} \\ &= \left(\frac{x}{2}\right)^\nu \frac{1}{\Gamma(1+\nu)} \sum_{k=0}^{+\infty} \frac{\Gamma(1+\nu)}{k!\Gamma(\nu+k+1)} \left(\frac{-x^2}{4}\right)^k \\ &= \left(\frac{x}{2}\right)^\nu \frac{1}{\Gamma(1+\nu)} \sum_{k=0}^{+\infty} \frac{1}{k!(1+\nu)_k} \left(\frac{-x^2}{4}\right)^k . \end{aligned}$$

which is getting by using the Pochhammer symbol

$$\Gamma(\nu+1+k) = (\nu+1)_k\Gamma(\nu+1).$$

So in terms of a confluent hypergeometric function of the first kind, the Bessel function is written

$$J_\nu(x) = \left(\frac{x}{2}\right)^\nu \frac{1}{\Gamma(\nu+1)} {}_0F_1\left(\begin{matrix} - \\ \nu+1 \end{matrix} \middle| -\frac{1}{4}x^2\right)$$

.

Hankel's Bessel functions of the third kind

We can define two new linearly independent functions

$$H_\nu^{(1)}(x) = J_\nu(x) + iY_\nu(x) \tag{1.14}$$

$$H_\nu^{(2)}(x) = J_\nu(x) - iY_\nu(x) \tag{1.15}$$

which are obviously solutions of the Bessl's equation and therefore the general solution can be written as

$$y = aH_\nu^{(1)}(x) + bH_\nu^{(2)}(x)$$

where a and b are arbitrary constants. The functions $H_\nu^{(1)}(x)$ and $H_\nu^{(2)}(x)$ are called Hankel's Bessel functions of the third kind.

We can remark that if $J_\nu(x)$ and $Y_\nu(x)$ are taking as cosine and sine functions respectively, $H_\nu^{(1)}(x)$ and $H_\nu^{(2)}(x)$ can be seen as exponential functions.
Using the expression of $Y_\nu(x)$ in terms of $J_\nu(x)$ and $J_{-\nu}(x)$ we have

$$\begin{aligned} H_\nu^{(1)}(x) =& J_\nu(x) + i\left(\frac{\cos(\pi\nu)J_\nu(x) - J_{-\nu}(x)}{\sin(\pi\nu)}\right) \\ =& i\frac{J_\nu(x)\exp(-i\pi\nu) - J_{-\nu}(x)}{\sin(\pi\nu)} \end{aligned}$$

$$\begin{aligned} H_\nu^{(2)}(x) =& J_\nu(x) - i\left(\frac{\cos(\pi\nu)J_\nu(x) - J_{-\nu}(x)}{\sin(\pi\nu)}\right) \\ =& -i\frac{J_\nu(x)\exp(i\pi\nu) - J_{-\nu}(x)}{\sin(\pi\nu)}. \end{aligned}$$

The expressions of $J_\nu(x)$ and $Y_\nu(x)$ in terms of Hankel's Bessel functions of the third kind are:

$$\begin{aligned} J_\nu(x) =& \frac{1}{2}\left[H_\nu^{(1)}(x) + H_\nu^{(2)}(x)\right] \\ Y_\nu(x) =& \frac{1}{2i}\left[H_\nu^{(1)}(x) - H_\nu^{(2)}(x)\right]. \end{aligned}$$

Modified Bessel Functions of the first and second kind

Definition 1.2.2. *The modified Bessel equation of order $\nu \in \mathbb{C}$ is the differential equation:*

$$x^2y'' + xy' - (x^2 + \nu^2)\, y = 0$$

obtained by replacing x by ix (where $i^2 = -1$) in the Bessel equation of order ν.

By the same computations as in the Bessel's differential equation, we get two linearly independent solutions

$$\begin{aligned} I_\nu(x) &= \sum_{k=0}^{+\infty} \frac{1}{k!\Gamma(\nu+k+1)} \left(\frac{x}{2}\right)^{\nu+2k} \\ \textit{and} \quad K_\nu(x) &= \frac{\pi\left(I_{-\nu}(x) - I_\nu(x)\right)}{2\sin(\nu\pi)} \end{aligned}$$

which are called modified Bessel functions of first and second kind respectively. Hence the general solution to this equation is of the form:

$$y \equiv aI_\nu(x) + bK_\nu(x)$$

where a and b are arbitrary constants. These solutions are connected to the solutions of the Bessel functions by the following relations:

Lemma 1.1. *Let $\nu \in \mathbb{C}$ and $i^2 = -1$, we have:*

(i) $I_\nu(x) = i^{-\nu} J_\nu(ix)$,

(ii) $K_\nu(x) = \frac{\pi}{2} i^{\nu+1} \left[J_\nu(ix) + iY_\nu(ix)\right]$

Proof:

(i) By the definition

$$\begin{aligned} J_\nu(ix) &= \sum_{k=0}^{+\infty} \frac{(-1)^k}{k!\Gamma(\nu+k+1)} \left(\frac{ix}{2}\right)^{\nu+2k} \\ &= \sum_{k=0}^{+\infty} \frac{(-1)^k i^{\nu+2k}}{k!\Gamma(\nu+k+1)} \left(\frac{x}{2}\right)^{\nu+2k} \\ &= i^\nu I_\nu(x) \\ \textit{so} \quad I_\nu(x) &= i^{-\nu} J_\nu(ix). \end{aligned} \tag{1.16}$$

(*ii*) By the definition

$$\begin{aligned}
K_\nu(x) &= \frac{\pi\left(I_{-\nu}(x) - I_\nu(x)\right)}{2\sin(\nu\pi)} \\
&= \frac{\pi\left(i^\nu J_{-\nu}(ix) - i^{-\nu} J_\nu(ix)\right)}{2\sin(\nu\pi)} \quad \textit{using the proof of (i)} \\
&= \frac{\pi}{2} i^{\nu+1}\left[\frac{-iJ_{-\nu}(ix) + i^{-2\nu+1} J_\nu(ix)}{\sin(\nu\pi)}\right] \\
&= \frac{\pi}{2} i^{\nu+1}\left[\frac{-iJ_{-\nu}(ix) + \exp(\frac{-2\nu+1}{2}\pi i) J_\nu(ix)}{\sin(\nu\pi)}\right];
\end{aligned}$$

$$\begin{aligned}
\textit{i.e.} \quad K_\nu(x) &= \frac{\pi}{2} i^{\nu+1}\left[\frac{-iJ_{-\nu}(ix) + \exp\left(\left(-\nu\pi + \frac{\pi}{2}\right) i\right) J_\nu(ix)}{\sin(\nu\pi)}\right] \\
&= \frac{\pi}{2} i^{\nu+1}\left[\frac{-iJ_{-\nu}(ix) + \left(\cos\left(-\nu\pi + \frac{\pi}{2}\right) + i\sin\left(-\nu\pi + \frac{\pi}{2}\right)\right) J_\nu(ix)}{\sin(\nu\pi)}\right] \\
&= \frac{\pi}{2} i^{\nu+1}\left[\frac{-iJ_{-\nu}(ix) + \left(\sin(\nu\pi) + i\cos(\nu\pi)\right) J_\nu(ix)}{\sin(\nu\pi)}\right] \\
&= \frac{\pi}{2} i^{\nu+1}\left[\frac{J_\nu(ix)\sin(\nu\pi) + i\left(J_\nu(ix)\cos(\nu\pi) - J_{-\nu}(ix)\right)}{\sin(\nu\pi)}\right] \\
&= \frac{\pi}{2} i^{\nu+1}\left[J_\nu(ix) + i\left(\frac{J_\nu(ix)\cos(\nu\pi) - J_{-\nu}(ix)}{\sin(\nu\pi)}\right)\right]
\end{aligned}$$

so

$$K_\nu(x) = \frac{\pi}{2} i^{\nu+1}\left[J_\nu(ix) + iY_\nu(ix)\right]. \tag{1.17}$$

□

By the fact that $I_\nu(x) = i^{-\nu} J_\nu(ix)$, if $\nu \notin \mathbb{Z}$ then $I_\nu(x)$ and $I_{-\nu}(x)$ are also two linearly independent solutions of the modified Bessel's equation.

Recurrence Relations

There are various recurrence equations and relationships for the Bessel function of the first kind and its derivative.

Lemma 1.2. *Let $\nu, u \in \mathbb{C}$. The Bessel functions of the first kind satisfy*

(i) $\frac{d}{du}\left[u^\nu J_\nu(u)\right] = u^\nu J_{\nu-1}(u)$,

(ii) $\frac{d}{du}\left[u^{-\nu} J_\nu(u)\right] = -u^{-\nu} J_{\nu+1}(u)$,

(iii) $uJ'_\nu(u) = uJ_{\nu-1}(u) - \nu J_\nu(u)$,

(iv) $uJ'_\nu(u) = \nu J_\nu(u) - uJ_{\nu+1}(u)$,

(v) $2J'_\nu(u) = J_{\nu-1}(u) - J_{\nu+1}(u)$,

(vi) $2\nu J_\nu(u) = u(J_{\nu+1}(u) + J_{\nu-1}(u))$.

Proof: The proof can be found in [1] (we use the D'Alembert criteria to prove that $J_\nu(u)$ is analytic over $\mathbb{C}$, and the rest is just the computations). □

By the definition of $Y_\nu(u)$ in terms of Bessel functions of first kind, similar equations are satisfying by $Y_\nu(u)$. Since Hankel functions are linear combinations of $J_\nu(u)$ and $Y_\nu(u)$, they satisfy the same recurrence relationships.

Lemma 1.3. *Let $\nu, u \in \mathbb{C}$. The modified Bessel functions of the first kind satisfy*

(i) $\dfrac{d}{du}\left[u^\nu I_\nu(u)\right] = u^\nu I_{\nu-1}(u)$,

(ii) $\dfrac{d}{du}\left[u^{-\nu} I_\nu(u)\right] = u^{-\nu} I_{\nu+1}(u)$,

(iii) $uI'_\nu(u) = uI_{\nu-1}(u) - \nu I_\nu(u)$,

(iv) $uI'_\nu(u) = \nu I_\nu(u) + uI_{\nu+1}(u)$,

(v) $2I'_\nu(u) = I_{\nu-1}(u) + I_{\nu+1}(u)$,

(vi) $2\nu I_\nu(u) = -u(I_{\nu+1}(u) - I_{\nu-1}(u))$.

Proof: Use the fact that $I_\nu(u) = i^{-\nu} J_\nu(iu)$ and lemma 1.2. □

By the definition of $K_\nu(u)$ in terms of modified Bessel functions of first kind, $(-1)^\nu K_\nu(u)$ satisfies the same equations as $I_\nu(x)$.

Zeroes of Bessel Functions

The zeroes of Bessel functions are of great importance in many applications. Bessel functions of the first and second kind have an infinite number of zeroes as the values of x goes to ∞.

The modified Bessel functions of the first kind ($I_\nu(x)$) have only one zero at the point $x = 0$, and the modified Bessel equations of the second kind ($K_\nu(x)$) functions do not have zeroes.

Notation: B_ν refers to any element of $\{J_\nu, Y_\nu, I_\nu, K_\nu\}$. For example, the following lemma hold for all four elements:

Lemma 1.4. *Consider $S := \mathbb{C}(x)B_\nu + \mathbb{C}(x)B'_\nu$, where $B'_\nu = \frac{d}{dx}B_\nu$. The space S is invariant under the substitutions $\nu \to \nu + 1$ and $\nu \to -\nu$.*

Proof: It follows from the two last lemmas (See [2] Corollary 1.23 for more details). □

1.2.2 Bessel Operator

Definition 1.2.3. *Let $\nu \in \mathbb{C}$ and $\partial := \dfrac{d}{dx}$.*

(i) The Bessel's differential equation of order ν corresponds to the operator

$$\mathrm{L} = x^2\partial^2 + x\partial + (x^2 - \nu^2)$$

which is called the Bessel operator and denoted by L_{B_1}.

(ii) The modified Bessel's differential equation of order ν corresponds to the operator

$$\mathrm{L} = x^2\partial^2 + x\partial - (x^2 + \nu^2)$$

which is called the modified Bessel operator and denoted by L_{B_2}.

Lemma 1.5. *The Bessel functions with parameter $\nu \in \frac{1}{2} + \mathbb{Z}$ are hyperexponential functions and in that case L_{B_1} and L_{B_2} are reducible.*

Proof: Let $i^2 = -1$. For $x \neq 2k\pi, \quad k \in \mathbb{Z}$ we have

$$\frac{J_{\frac{1}{2}}(x)'}{J_{\frac{1}{2}}(x)} = -\frac{\frac{1}{2x}\sqrt{\frac{2}{\pi x}}\sin(x) - \sqrt{\frac{2}{\pi x}}\cos(x)}{\sqrt{\frac{2}{\pi x}}\sin(x)} = -\left(\frac{1}{2x} - \coth(x)\right) \in \mathbb{C}(x).$$

Similarly for $x \neq \frac{\pi}{2} + 2k\pi, \quad k \in \mathbb{Z}$ we have

$$\frac{J_{-\frac{1}{2}}(x)'}{J_{-\frac{1}{2}}(x)} = -\left(\frac{1}{2x} + \tan(x)\right) \in \mathbb{C}(x).$$

Hence, $J_{\frac{1}{2}}(x)$ *and* $J_{-\frac{1}{2}}(x)$ are hyperexponential functions.
If $\nu \in \left\{\frac{1}{2}, -\frac{1}{2}\right\}$ then the Bessel operator and the modified Bessel operator can be factored:

$$\mathrm{L}_{B_1} = x^2\left(\partial - 1 + \frac{1}{2x}\right)\left(\partial + 1 + \frac{1}{2x}\right)$$
$$\mathrm{L}_{B_2} = x^2\left(\partial + i + \frac{1}{2x}\right)\left(\partial - i + \frac{1}{2x}\right);$$

so L_{B_1} and L_{B_2} are reducible.
Using the invariance of $\mathbb{C}(x)J_\nu(x) + \mathbb{C}(x)J_\nu(x)'$ under $\nu \to 1+\nu$ and $\nu \to -\nu$, we can say that for all integers m, $J_{\frac{1}{2}+m}(x)$ and $J_{-\frac{1}{2}+m}(x)$ are hyperexponential functions and for $\nu \in \left\{\frac{1}{2} + m, -\frac{1}{2} + m\right\}$ L_{B_1} and L_{B_2} are reducible.

By the expressions of $Y_\nu(x)$, $I_\nu(x)$ and $K_\nu(x)$ in terms of $J_\nu(x)$ we can also say that $Y_{\frac{1}{2}+m}(x)$, $Y_{-\frac{1}{2}+m}(x)$, $I_{\frac{1}{2}+m}(x)$, $I_{-\frac{1}{2}+m}(x)$, $K_{\frac{1}{2}+m}(x)$ and $K_{-\frac{1}{2}+m}(x)$ are hyperexponential functions. $\square$

Since we only consider irreducible operators, we will exclude the case $\nu \in \frac{1}{2} + \mathbb{Z}$ from this thesis.

Lemma 1.6. *Let $\nu \notin \frac{1}{2}+\mathbb{Z}$. The change of variables $x \to ix$ where $i^2=-1$ sends $\mathrm{V}(\mathrm{L}_{B_2})$ to $\mathrm{V}(\mathrm{L}_{B_1})$ and vice versa.*

Proof: Let $y(x)$ be a solution of L_{B_2} and consider $g(x)=y(ix)$, then

$$\begin{aligned} g'(x) &= i\,\frac{d}{dt}y(t)\Big|_{t=ix} \\ \text{and}\quad g''(x) &= -\frac{d^2}{dt^2}y(t)\Big|_{t=ix} = \frac{1}{(ix)^2}\left(ix\,\frac{d}{dt}y(t)\Big|_{t=ix} - \left((ix)^2+\nu^2\right)y(ix)\right). \end{aligned}$$

By using the expression of $y''(x)$ in the equation satisfied by y:

$$x^2y''(x)+xy'(x)-\left(x^2+\nu^2\right)y(x)=0.$$

A general differential operator for $g(x)$ is $\mathrm{L}=\partial^2+a_1\partial+a_0$. Using the equation of g' and g'' we can transform $\mathrm{L}g=0$ into

$$\left(\frac{1}{ix}+ia_1\right)\frac{d}{dt}y(t)\Big|_{t=ix}+\left(\frac{-1}{x^2}\left(x^2-\nu^2\right)+a_0\right)y(ix)=0.$$

Since L_{B_2} is an irreducible operator, if we equate coefficients we obtain $a_1=1/x$ and $a_0=\left(x^2-\nu^2\right)/x^2$. Then $\mathrm{L}=1/x^2\mathrm{L}_{B_1}$ and $g(x)\in\mathrm{V}(\mathrm{L})=\mathrm{V}(\mathrm{L}_{B_1})$. So $g(x)$ is a solution of L_{B_1}. The reverse work in a similar way. □

Remarks 1.2.2.

1. *L_{B_1} and L_{B_2} have only two singularities, 0 and ∞.*
2. *The generalized exponents of L_{B_1} are $\pm\nu$ at 0 and $\pm\frac{i}{t_\infty}+\frac{1}{2}$ at ∞ where $t_\infty=\frac{1}{x}$.*
3. *The generalized exponents of L_{B_2} are $\pm\nu$ at 0 and $\pm\frac{1}{t_\infty}+\frac{1}{2}$ at ∞ where $t_\infty=\frac{1}{x}$.*
4. *For L_{B_1} and L_{B_2} at $p=\infty$, the two generalized exponents belong to different submodules and there are no logarithmic solutions.*
5. *For L_{B_1} and L_{B_2} at $p=0$, there can be logarithmic solutions only if $\nu=-\nu$ modulo $\mathbb{Z}$.*
6. *For Bessel functions and modified Bessel functions, the generalized exponents are unramified (i.e. the ramification index is always 1).*

Note that the modified Bessel operator is easier to handle since the generalized exponents do not create new algebraic extensions.

Lemma 1.7. *The change of variables $y(x)\to y(\sqrt{x})$ reduces L_{B_2} to $\mathrm{L}_{\breve{B}}=x^2\partial^2+x\partial-\dfrac{1}{4}(x+\nu^2)$ which is still in $\mathbb{Q}(x)[\partial]$, when $\nu\notin\frac{1}{2}+\mathbb{Z}$.*

Proof: Let $y(x)$ be a solution of L_{B_2} and consider $g(x) = y(\sqrt{x})$, then

$$\begin{aligned} g'(x) &= \frac{1}{2\sqrt{x}} \frac{d}{dt} y(t)\Big|_{t=\sqrt{x}} \\ \text{and} \;\; g''(x) &= -\frac{1}{4x\sqrt{x}} \frac{d}{dt} y(t)\Big|_{t=\sqrt{x}} + \frac{1}{4x} \frac{d^2}{dt^2} y(t)\Big|_{t=\sqrt{x}} \\ &= -\frac{1}{2x\sqrt{x}} \frac{d}{dt} y(t)\Big|_{t=\sqrt{x}} + \frac{1}{4x^2}(x+\nu^2) y\left(\sqrt{x}\right) \end{aligned}$$

by using the expression of $y''(x)$ in the equation satisfies by y:

$$x^2 y''(x) + xy'(x) - (x^2+\nu^2) y(x) = 0.$$

A general differential operator for $g(x)$ is $\mathrm{L} = \partial^2 + a_1\partial + a_0$. Using the equation of g' and g'' we can transform $\mathrm{L}g = 0$ into

$$\frac{1}{2\sqrt{x}} \left(a_1 - \frac{1}{x}\right) \frac{d}{dt} y(t)\Big|_{t=\sqrt{x}} + \left(a_0 + \frac{1}{4x^2}\left(x+\nu^2\right)\right) y\left(\sqrt{x}\right) = 0.$$

Since L_{B_2} is an irreducible operator, if we equate coefficients we obtain $a_1 = 1/x$ and $a_0 = -\dfrac{1}{4x^2}\left(x+\nu^2\right)$. Then $\mathrm{L} = 1/x^2 \mathrm{L}_{\check{B}}$ which is equivalent to $\mathrm{L}_{\check{B}}$. $\square$

Let $CV(\mathrm{L}, f)$ denote the operator obtained from L by change of variables $x \to f$.

Lemma 1.8. *$CV(\mathrm{L}_{B_2}, f)$ can be written as $CV(\mathrm{L}_{\check{B}}, f^2)$.*

Proof: Obvious. $\square$

Lemma 1.9. *Let k be a field extension of $\mathbb{Q}$ and $\mathrm{K} = k(x)$. Let f, ν be elements of a differential field extension of* K, *and ν be constant. Then*

$$CV(\mathrm{L}_{B_2}, f) \in \mathrm{K}[\partial] \Longleftrightarrow f^2 \in \mathrm{K} \;\; \textit{and} \;\; \nu^2 \in k.$$

Proof:

$\Rightarrow$) Let ν be a constant and $\mathrm{M} := monic(CV(\mathrm{L}_{B_2}, f)) = \partial^2 + a_1\partial + a_0$. We have to prove

$$a_0, a_1 \in \mathrm{K} \Longrightarrow f^2, \nu^2 \in \mathrm{K}$$

and so we assume $a_0, a_1 \in \mathrm{K}$. The previous lemma give us

$$\begin{aligned} \mathrm{M} &= monic(CV(\mathrm{L}_{\check{B}}, f^2)) = \partial^2 + a_1\partial + a_0 \\ \textit{with} \;\; a_0 &= -\left[(f'(x))^2 + \left(\frac{f'(x)\nu}{f(x)}\right)^2\right], \quad a_1 = \frac{f'(x)}{f(x)} - \frac{f''(x)}{f'(x)}. \end{aligned}$$

$$let \quad a_2 = \frac{(a_0' + 2a_0a_1)'}{a_0' + 2a_0a_1} + 3a_1, \quad a_3 = -4\frac{a_0}{a_2^2} \quad and \quad a_4 = a_3\left(2a_1 + \frac{a_0'}{a_0}\right)$$

where $'$ means $\frac{d}{dx}$. We have $a_2, a_3, a_4 \in \mathrm{K}$ since $a_0, a_1 \in \mathrm{K}$. Direct substitution shows that:

$$a_2 = 2\frac{f'}{f}, \quad a_3 = f^2 + \nu^2 \; and \; \; a_4 = 2ff'.$$

Hence, $f^2 = a_4/a_2 \in \mathrm{K}$ and $\nu^2 = a_3 - f^2 \in \mathrm{K}$.

($\Leftarrow$ If $f^2 \in \mathrm{K}$ and $\nu^2 \in k$ then $CV(\mathrm{L}_{\check{B}}, f^2) \in \mathrm{K}[\partial]$. By the previous lemma we have $CV(\mathrm{L}_{B_2}, f) \in \mathrm{K}[\partial]$.

□

Chapter two

Solving Differential Equations in Terms of Bessel Functions

In this chapter, we apply theory developed in chapter one to solve some second order differential equations in terms of Bessel functions.

The question of solving an equating in term of Bessel functions is equivalent to the question whether two differential operators can be transformed into each other by certain transformations.

2.1 Formal Solutions and Generalized Exponents

In this section, we introduce the idea of generalized exponents. Generally, the generalized exponents give us the asymptotic local information about solutions. In this section we consider operators in $\mathbb{C}((x))[\partial]$. Since we work with solutions of differential operators we have to be sure that we can have all of them; that is why we have to construct the universal Picard-Vessiot ring of $\mathbb{C}((x))$ that contains a fundamental system of solution of differential operators. Before that, let us give the general definition of the universal extension U (universal Picard-Vessiot ring) of a differential field:
Let K be a differential field, with $\mathrm{C_K}$ as it field of constants

Definition 2.1.1. *A universal extension U of K, is a minimal (simple) differential ring in which every operator $\mathrm{L} \in \mathrm{K}[\partial]$ has precisely $deg(\mathrm{L})$ $\overline{\mathrm{C_K}}$-linear independent solutions ($\overline{\mathrm{C_K}}$ the algebraic closure of $\mathrm{C_K}$). It exists if K has an algebraically closed field $\mathrm{C_K}$ of constants of characteristic zero.*

Hence the universal extension U of $\mathbb{C}((x))$ exists and for every nonzero operator $\mathrm{L} \in \mathbb{C}((x))[\partial]$, we define the solution space of L, which has dimension $deg(\mathrm{L})$, as follow

$$\mathrm{V}(\mathrm{L}) = \{y \in U \,|\, \mathrm{L}(y) = 0\}.$$

From now, we will take $\mathrm{K} = \mathbb{C}((x))$, i.e. $\mathrm{C_K} = \mathbb{C}$.

At the point $x = 0$ we have the following construction of a universal extension U of $\mathbb{C}((x))$:

First we denote $\Omega = \bigcup_{m\in\mathbb{N}^*} x^{-1/m}\mathbb{C}[[x^{-1/m}]]$, $M \subset \mathbb{C}$ such that $M \oplus \mathbb{Q} = \mathbb{C}$, and $\overline{\mathbb{C}((x))}$ the algebraic closure of $\mathbb{C}((x))$ given by $\overline{\mathbb{C}((x))} = \bigcup_{n\in\mathbb{N}^*} \mathbb{C}((x^{1/n}))$.

Theorem 2.1.1. *1- Define the ring* $R = \overline{\mathbb{C}((x))}\,[\{X^a\}_{a\in M}, \{E(q)\}_{q\in\Omega}, l]$ *as the polynomial ring over* $\mathbb{C}((x))$ *in the infinite collection of variables* $\{X^a\}_{a\in M} \bigcup \{E(q)\}_{q\in\Omega} \bigcup \{l\}$.

2- Define the differentiation δ *on* R *by:* δ *is* $x\frac{d}{dx}$ *on* $\overline{\mathbb{C}((x))}$, $\delta X^a = aX^a$, $\delta E(q) = qE(q)$, *and* $\delta l = 1$. *This turns* R *into a differential ring.*

3- Let $I \subset R$ *denote the ideal generated by the elements*

$$X^0 - 1,\ X^{a+b} - X^a X^b,\ E(0) - 1,\ E(q_1 + q_2) - E(q_1)E(q_2).$$

Hence, I *is a differential ideal and* $I \neq R$.

4- Put $U := R/I$;
then U *is a universal extension of* $\mathbb{C}((x))$ *which means:*

$*$ *the constant field of* U *is* $\mathbb{C}$;

$*$ *if* L *has order* n, *then* $\mathrm{V(L)} := ker(\mathrm{L} :\ U \longrightarrow U)$ *is a* $\mathbb{C}$*-vector space of dimension* n.

Proof: The proof and other details of universal extension can be found in [7]. □

We can think of $E(q)$, X^a and l as

$$E(q) = \exp\left(\int \frac{q}{x}\,dx\right),\ \ X^a = \exp\left(a\ln(x)\right)\ \textit{and}\ l = \ln(x)$$

because $x\frac{d}{dx}$ acts:

- on $E(q)$ as multiplication by q,
- on X^a as multiplication by a, and
- on l as the solution of the equation $x\frac{dy}{dx} = 1$.

Hence, at $x = 0$ we have:

Theorem 2.1.2. *The universal extension* U *of* K *is unique and has the form*

$$U = \overline{\mathrm{K}}\left[\{x^a\}_{a\in M}, \{e(q)\}_{q\in\Omega}, l\right],$$

where $M \subset \mathbb{C}$ *is such that* $M \oplus \mathbb{Q} = \mathbb{C}$, $\Omega = \bigcup_{m\geq 1} x^{-1/m}\mathbb{C}[[x^{-1/m}]]$, l *the solution of the equation* $x\frac{dy}{dx} = 1$, *and the following rules hold:*

(i) The only relations between the symbols are $x^0 = 1, x^{a+b} = x^a x^b, e(0) = 1$ *and* $e(q_1 + q_2) = e(q_1)e(q_2)$.

(ii) The differentiation in U is given by $\delta x^a = ax^a, \delta e(q) = qe(q)$ and $\delta l = 1$ where $\delta = x\frac{d}{dx}$.

Proof: To give a complete proof we would have to introduce to many details about differential ring. This is why we refer to [7] where we the proof. □

A solution whose formal representation in the universal extension U involves $l = \ln(x)$ is called logarithmic solution.

A more detailed structure of the universal extension is given by the following lemma.

Lemma 2.1. *The universal extension U of $\mathbb{C}((x))$ is a $\mathbb{C}((x))[\partial]$-module which can be written as a direct sum of $\mathbb{C}((x))[\partial]$-module:*

$$\begin{aligned} U &= \bigoplus_{q\in\Omega} e(q)\overline{\mathbb{C}((x))}[\{x^a\}_{a\in\mathbb{C}/\mathbb{Q}}, l] \\ &= \bigoplus_{q\in\Omega} \bigoplus_{a\in\mathbb{C}/(\frac{1}{r_q}\mathbb{Z})} e(q)x^a\mathbb{C}((x^{1/r_q}))[l], \end{aligned}$$

where $\Omega = \bigcup_{m\in\mathbb{N}^*} x^{-1/m}\mathbb{C}[[x^{-1/m}]]$ and, in the latter equation, r_q is the ramification index of q, i.e. the smallest number such that $q \in \mathbb{C}[[x^{-1/r_q}]]$.

Proof: The first equation is proven in [[7], chapter 3.2] and for the second we refer to [[8], chapter 2.8]. □

Let $R_q := \overline{\mathbb{C}((x))}[\{x^a\}_{a\in\mathbb{C}/\mathbb{Q}}, l]e(q)$, then $U = \bigoplus_{q\in\Omega} R_q$. Put $\mathrm{V(L)}_q = \mathrm{V(L)} \cap R_q$; since the action of L on U leaves each R_q invariant, one has $\mathrm{V(L)} = \bigoplus_{q\in\Omega} \mathrm{V(L)}_q$.

$$\begin{aligned} y \in \mathrm{V(L)}_q &\Longrightarrow y = e(q)x^bS, \quad S \in \mathbb{C}((x^{1/r_q}))[\ln(x)] \text{ and } b \in \mathbb{C} \\ &\Longrightarrow y = \exp\left(\int \frac{q+b}{x}\, dx\right) S; \end{aligned}$$

$e := q + b \in \mathbb{C}[[x^{-1/r_q}]]$ is what we will call generalized exponent of L in the next definition (at $x = 0$).

Note that this construction of U at the point $x = 0$ can also be performed at other points $x = p$ by replacing x with the local parameter t_p which is $t_p = x - p$ for a point $p \in \mathbb{C}$ and $t_p = \frac{1}{x}$ for $p = \infty$.

Definition 2.1.2. *Let $\mathrm{L} \in \mathbb{C}(x)[\partial]$ and let p be a point with local parameter t_p. An element $e \in \mathbb{C}[[{t_p}^{-1/r_e}]]$, $r_e \in \mathbb{N}^*$ is called a generalized exponent of L at the point p if there exists a formal solution of L of the form*

$$y(x) = \exp\left(\int \frac{e}{t_p}\, dt_p\right) S, \quad S \in \mathbb{C}(({t_p}^{1/r_e}))[\ln(t_p)], \tag{2.1}$$

where the constant term of the Puiseux series S is non-zero. For a given solution this representation is unique and $r_e \in \mathbb{N}$ is called the ramification index of e.

The set of generalized exponent at a point p is denoted by $gexp(\mathrm{L}, p)$.

Similarly, we call e a generalized exponent of the solution y at the point p if $y = y(x)$ has the representation (2.1) for some $S \in \mathbb{C}((t_p{}^{1/r}))[\ln(t_p)]$.

For a given generalized exponent there is a unique solution of the form (2.1) if we require the constant term of the series to be one.

If $e \in \mathbb{C}$ we just get a solution $x^e S$, in this case e is called an exponent. If $r = 1$, then e is unramified, otherwise it is ramified.

Remark 2.1.1. *Since our main work in this thesis is based on second order differential operators,* r_e *in the definition can be only* 1 *or* 2 *because* $\mathrm{V(L)} = \bigoplus_{q\in\Omega} \mathrm{V(L)}_q$ *and the dimension of* $\mathrm{V(L)}_q$ *is* r_q *(the ramification index of* q*)*

Theorem 2.1.3. *Let* $\mathrm{L} \in \mathrm{K}[\partial]$, $n = deg(\mathrm{L})$, $r \in \mathbb{N}^*$ *and let* p *be a point with local parameter* t_p.*. Suppose that the ramification indices of the generalized exponents at* p *divide* r*. Then there exists a basis* $y_1, ..., y_n$ *of* $\mathrm{V(L)}$ *which satisfies the condition:* $\forall i \in \{1, ..., n\}$

$$y_i = \exp\left(\int \frac{e_i}{t_p}\,dx\right) S_i \quad for\ some \quad S_i \in \mathbb{C}((t_p^{1/r}))[\ln(x)]$$

where $e_1, ..., e_n \in \mathbb{C}[[t_p^{-1/r}]]$ *are generalized exponents and the constant term of* S_i *is non-zero.*

<u>Proof</u>: We just use the definition of the universal extension and the fact that, for $i = 1, ..., n$

$$r_{e_i} \mid r \Longrightarrow \begin{cases} e_i \in \mathbb{C}[[t_p{}^{-1/r}]] \\ S_i \in \mathbb{C}((t_p{}^{1/r}))[\ln(t_p)] \ . \end{cases}$$

The details can be found in [[8], Chapter 4.3.3, theorem 5]. $\square$ For every regular point p of L the generalized exponents are $0, 1, ..., n-1$, where n is the degree of L.

Remarks 2.1.1. *By using the definition* 1.1.8*, theorem* 1.1.3 *and the definition of generalized exponent we can finally summarize what we learn from this section for operators of degree two:*

- * *At every point* p *there are two generalized exponents* e_1 *and* e_2 *such that the solution space is generated by two solutions of the form (2.1).*
- * *If* e_1 *and* e_2 *are both non-negative integers, then the local solutions are power series and* p *is either a regular point or an apparent singularity.*
- * *If* p *is a non-apparent singularity and* e_1,$e_2 \in \mathbb{C}$ *are both constants,* p *is regular singular. If* e_1, $e_2 \notin \mathbb{C}$, p *is irregular singular.*
- * *Each generalized exponent* $e \in (e_1, e_2)$ *is unique modulo* $\frac{1}{r_e}\mathbb{Z}$*, where* r_e *is the ramification index of the generalized exponent* e*, and by the last theorem the set of generalized exponent at a point* p *is unique modulo* $\frac{1}{r}\mathbb{Z}$.

* *If $e_1 \neq e_2$ modulo $\frac{1}{r_{e_1}}\mathbb{Z}$, the generalized exponents belong to different submodules of the universal extension and there are no logarithmic local solutions at the point p.*

* *If $e_1 = e_2$ modulo $\frac{1}{r_{e_1}}\mathbb{Z}$, there can be logarithmic solutions.*

This remark is restricted to second order differential equations.

2.2 Transformations

Here, we will discuss the transformations that preserve second-order differential operators. Then we can clarify the method of solving differential equations in terms of solutions of another equation (in this master thesis, the Bessel equation). We will also discuss the invariance under the transformations, which we use to connect the Bessel operator to the operator we want to solve. In this section, we also assume the order of the differential operators is two and the operators are irreducible.

2.2.1 Types of transformations

Definition 2.2.1. *A transformation between two differential operators $\mathrm{L}_1, \mathrm{L}_2 \in \mathbb{C}(x)[\partial]$ is a map from the solution space $\mathrm{V}(\mathrm{L}_1)$ onto the solution space $\mathrm{V}(\mathrm{L}_2)$.*

The transformation is invertible if there also exists a map from $\mathrm{V}(\mathrm{L}_2)$ onto $\mathrm{V}(\mathrm{L}_1)$. There are three known types of transformations that preserve the differential field and preserve order two. They are:

Definition 2.2.2. *Let $\mathrm{L}_1 \in \mathbb{C}(x)[\partial]$ be a differential operator of degree two. For $y = y(x) \in \mathrm{V}(\mathrm{L}_1)$ we have:*

(i) change of variables: $y(x) \to y(f(x)),\;\; f^2 \in \mathbb{C}(x)\backslash\mathbb{C}$,

(ii) exp-product: $y \to \exp\left(\int r\,dx\right) y,\;\; r \in \mathbb{C}(x)$, and

(iii) gauge transformation: $y \to r_0 y + r_1 y',\quad r_0,\, r_1 \in \mathbb{C}(x)$.

They are denoted by $\longrightarrow_C$, $\longrightarrow_E$, $\longrightarrow_G$ respectively and for the resulting operator $\mathrm{L}_2 \in \mathbb{C}(x)[\partial]$ we write $\mathrm{L}_1 \xrightarrow{f}_C \mathrm{L}_2$, $\mathrm{L}_1 \xrightarrow{r}_E \mathrm{L}_2$, $\mathrm{L}_1 \xrightarrow{r_0,r_1}_G \mathrm{L}_2$, respectively. Furthermore, we write $\mathrm{L}_1 \longrightarrow \mathrm{L}_2$ if there exists a sequence of those transformations that sends L_1 to L_2.
The rational functions f, r, r_0 and r_1 will be called parameters of the transformation, and in case (ii) the function $\exp\left(\int r\,dx\right)$ is a hyperexponential function.

Remark 2.2.1. *We can consider $\longrightarrow_C$, $\longrightarrow_E$ and $\longrightarrow_G$ as binary relations on $\mathbb{C}(x)[\partial]$. Hence, $\longrightarrow_E$ and $\longrightarrow_G$ are equivalence relations, but $\longrightarrow_C$ is not: the symmetry of $\longrightarrow_C$ would require algebraic functions as parameter. For example, to cancel the operation $x \mapsto x^2$, we would need $x \mapsto \sqrt{x}$.*

An important question when searching for transformations between two operators L_1 and L_2 is whether we can restrict our search to a specific order of transformations $\longrightarrow_C, \longrightarrow_E$ and $\longrightarrow_G$.

Lemma 2.2. *Let* $L_1, L_2, L_3 \in \mathbb{C}(x)[\partial]$ *be three differential operators of degree two such that* $L_1 \longrightarrow_G L_2 \longrightarrow_E L_3$. *Then there exists a differential operator* $M \in \mathbb{C}(x)[\partial]$ *such that* $L_1 \longrightarrow_E M \longrightarrow_G L_3$.

Similarly, if $L_1 \longrightarrow_E L_2 \longrightarrow_G L_3$ *we find* $M \in \mathbb{C}(x)[\partial]$ *such that* $L_1 \longrightarrow_G M \longrightarrow_E L_3$.

Proof: See [2] Lemma 2.7 . □

We write $\longrightarrow_{EG}$ for any sequence of those transformations. Since they are equivalence relations, $\longrightarrow_{EG}$ is also.

Definition 2.2.3. *We say* $L_1 \in \mathbb{C}(x)[\partial]$ *is gauge equivalent to* L_2 *if and only if* $L_1 \longrightarrow_G L_2$. *And* $L_1 \in \mathbb{C}(x)[\partial]$ *is projectively equivalent to* L_2 *if and only if* $L_1 \longrightarrow_{EG} L_2$.

Lemma 2.3. *Let* $L_1, L_2, L_3 \in \mathbb{C}(x)[\partial]$ *be three differential operators of degree two . The following holds:*

(i) $L_1 \longrightarrow_E L_2 \longrightarrow_C L_3 \Longrightarrow \exists M \in \mathbb{C}(x)[\partial]$*:* $L_1 \longrightarrow_C M \longrightarrow_E L_3$

(ii) $L_1 \longrightarrow_G L_2 \longrightarrow_C L_3 \Longrightarrow \exists M \in \mathbb{C}(x)[\partial]$*:* $L_1 \longrightarrow_C M \longrightarrow_G L_3$

Proof: See [2] Theorem 2.10 . □

Note that the converse of (i) and (ii) is not generally true since $\longrightarrow_C$ is not symmetric.

By those two lemmas above, we can then have the following statement:

Lemma 2.4. *Let* $L_1, L_2 \in \mathbb{C}(x)[\partial]$ *be two differential operators of degree two such that* $L_1 \longrightarrow L_2$. *Then there exists an operator* $M \in \mathbb{C}(x)[\partial]$ *such that* $L_1 \longrightarrow_C M \longrightarrow_{EG} L_2$.

Proof: We use the two lemmas above and the rest follow immediately. □

2.2.2 The Exponent Difference

Definition 2.2.4. *Let* $L \in \mathbb{C}(x)[\partial]$ *be a differential operator, let* p *be any point, and let* e_1 *and* e_2 *be two generalized exponents of* L *at* p. *Then the difference* $e_1 - e_2$ *is called an exponent difference of* L *at* p.

If $deg(L) = 2$ there exists just two generalized exponents at each point and we define

$$\Delta(L, p) := \pm(e_1 - e_2).$$

We define Δ modulo a factor -1 to make it well-defined because we have no ordering in the generalized exponents we compute.

By the two following lemmas, we will see how exponent behave in exp-product and gauge transformations.

Lemma 2.5. *Let* $\mathrm{L}, \mathrm{M} \in \mathbb{C}(x)[\partial]$ *be two differential operators such that* $\mathrm{M} \xrightarrow{r}_E \mathrm{L}$ *and let* e *be an exponent of* M *at the point* p *with the ramification index* $n \in \mathbb{N}^*$. *Furthermore, let* r *have the series representation*

$$r = \sum_{i=m}^{+\infty} r_i t_p^{\frac{i}{n}}, \quad m \in \mathbb{Z} \text{ and } m \leq -1, \quad r_i \in \mathbb{C}.$$

Then $e + \sum_{i=m}^{-1} r_i t_p^{\frac{i}{n}+1}$ *is an exponent of* L *at* p.

Proof: See [2] Lemma 2.12 . □

Lemma 2.6. *Let* $\mathrm{L}, \mathrm{M} \in \mathbb{C}(x)[\partial]$ *be two differential operators such that* $\mathrm{M} \longrightarrow_G \mathrm{L}$ *and let* e *be an generalized exponent of* M *at the point* p. *The operator* L *has at* p *a generalized exponent* $\overline{e}$ *such that* $\overline{e} = e \bmod \frac{1}{n}\mathbb{Z}$, *where* $n \in \mathbb{N}^*$ *is the ramification index of* e.

Proof: See [2] Lemma 2.14 . □

Hence, the exponent difference Δ has the following property:

Corollary 2.2.1. *Let* $\mathrm{L} \in \mathbb{C}(x)[\partial]$ *of order two and* p *be a point. The exponent difference* $\Delta(\mathrm{L}, p) \bmod \frac{1}{m}\mathbb{Z}$ *is invariant under projective equivalence* ($\longrightarrow_{EG}$), *where* $m \in \mathbb{Z}^*$ *is the ramification index. Here* $m = 1$ *if the generalized exponents are unramified, and* $m = 2$ *if they are ramified.*

Proof: Use theorem 2.1.3 and the two previous lemmas. □

2.2.3 The meaning of our problem

With what we have learnt in this section, we can state what we mean by solving equations in terms of Bessel functions.

Definition 2.2.5. *Assume* y *is a solution of a differential operator* L_0, *we say we can solve differential operator* L *in terms of* y *when we can find the transformations*

$$\mathrm{L}_0 \xrightarrow{f}_C \mathrm{M} \longrightarrow_{EG} \mathrm{L}$$

where M *is an operator.*

To solve differential equations $\mathrm{L}(y) = 0$ in terms of the Bessel functions means to find a transformation from the Bessel operator L_{B_1} to the operator L. Since we only focus on second-order differential operators, we only need to find combinations of $\longrightarrow_C$, $\longrightarrow_E$, and $\longrightarrow_G$ which send L_{B_1} to L:

$$\mathrm{L}_{B_1} \xrightarrow{f}_C \mathrm{M} \longrightarrow_{EG} \mathrm{L}.$$

M should be projectively equivalent to L. So for computing f, the only information retrieved from L that we can use is information on invariance under projective equivalence. The invariant we use is the difference of the generalized exponents of L.

2.3 Resolution

The main problem we consider here is the following: for a given operator $\mathrm{L} \in \mathbb{C}(x)[\partial]$ of degree two, how to find transformations that send the Bessel or modified Bessel operator to L if they exist. Note that we also need to find the parameter of the Bessel functions involved. Since L_{B_1} and L_{B_2} are closely connected we just need to consider one of the two. We take L_{B_2} because it is easier to handle. From now on, L_B will refer to the modified Bessel operator L_{B_2}. Using the results of the previous section we just have to consider the transformations

$$\mathrm{L}_B \longrightarrow_C \mathrm{M} \longrightarrow_{EG} \mathrm{L}. \tag{2.2}$$

So this section will be concentred to the determination of the parameters of this sequence of transformations.

There are two steps to find Bessel type solutions of L. The first step is to find the middle operator M (i.e. the change of variables f). If M (or equivalently f) is known, then the next step is to find the map from M to L, which is projective equivalence: this step has been called the equivalence of differential operators. We will only discuss the first step because Barkatou, Pflugel, Van Der Put and Singer have studied and given algorithms to find projective equivalence, this is not the case for $\longrightarrow_C$.

In the following, we will take a closed look at the part $\mathrm{L}_B \longrightarrow_C \mathrm{M}$. Once we found the Bessel parameter ν and the parameter f we can obtain M from L_B. For fixed $\mathrm{M} \in \mathbb{C}(x)[\partial]$ we can already solve the question of equivalence between M and L. We can then finally solve (2.2).

Since we work with the modified Bessel operator, all generalized exponents should be unramified.

2.3.1 The Parameter of the change of variables is in $\mathbb{C}(x)$

Let $\mathrm{M} \in \mathbb{C}(x)[\partial]$ be given. We want to know whether there exists $f = f(x) \in \mathbb{C}(x)$ and $\nu \in \mathbb{C}$ such that

$$\mathrm{L}_B \xrightarrow{f}_C \mathrm{M}$$

holds.

Let us assume that $\mathrm{B}_\nu(x)$ is a solution of L_B. Then $\mathrm{B}_\nu(f(x))$ is a solution of M. Since $\mathrm{B}_\nu(x)$ has singularities at 0 and ∞ it is obvious that the singularities of $\mathrm{B}_\nu(f(x))$ are at those points p where $f(p) = 0$ or $f(p) = \infty$, i.e. at the zeros and poles of $f(x)$. Hence, the zeros and poles of $f(x)$ must be singularities of M.

We don't consider apparent singularities of M because there might exists an operator $\tilde{\mathrm{M}}$ which has degree two with $\mathrm{V(M)} \subset \mathrm{V}(\tilde{\mathrm{M}})$ and which has no singularities at those points. So the others singularities of M are exactly the zeros and poles of f.

If exp-product and gauge transformations are involved we cannot find all zeros of f by only looking at the singularities of L which are not apparent singularities. But we will soon see that we can use a similar approach by developing a property which is invariant under exp-products and gauge transformations, which can then be used to find f. This property is the exponent difference modulo $\frac{1}{m}\mathbb{Z}$, where m is as in corollary 2.2.1.

The Exponent Difference

We will analyze $\Delta(\mathrm{M},p)$ because using the invariance of the exponent difference modulo $\mathbb{Z}$ under $\longrightarrow_{EG}$ we can then apply results not only to $\mathrm{L}_B \xrightarrow{f}_C \mathrm{M}$ but also to $\mathrm{L}_B \xrightarrow{f}_C \mathrm{M} \longrightarrow_{EG} \mathrm{L}$.

Theorem 2.3.1. *Let* $\mathrm{M} \in \mathbb{C}(x)[\partial]$ *such that* $\mathrm{L}_B \xrightarrow{f}_C \mathrm{M}$

(i) If p is a zero of f with multiplicity $m_p \in \mathbb{N}$, then p is a regular singularity of M *and* $\Delta(\mathrm{M},p) = \pm 2m_p\nu$.

(ii) If p is a pole of f with multiplicity $m_p \in \mathbb{N}$ such that

$$f = \sum_{i=-m_p}^{+\infty} f_i t_p^i,$$

then p is an irregular singularity of M *and*

$$\Delta(\mathrm{M},p) = \pm 2 \sum_{i=-m_p}^{-1} i f_i t_p^i.$$

<u>Proof</u>: Let t be the local parameter t_p.

(i) Let p be a zero of f with multiplicity $m_p > 0$, then f has the representation $f = t^{m_p} \sum_{i=0}^{+\infty} f_i t^i$ with $f_i \in \mathbb{C}$ and $f_0 \neq 0$. Furthermore, let $y \in \mathrm{V}(\mathrm{L}_B)$ be a local solution at $x = 0$ of the form

$$y = x^\nu \sum_{i=0}^{+\infty} a_i x^i, \quad a_i \in \mathbb{C}, \ a_0 \neq 0.$$

If we now replace x by f, we get

$$z = f^\nu \sum_{i=0}^{+\infty} a_i f^i$$

which is a local solution of M at p. To compute the generalized exponent at p we rewrite z such that

$$z = \exp\left(\int \frac{e}{t}\, dt\right) \sum_{i=0}^{+\infty} b_i t^i, \quad b_i \in \mathbb{C},\ b_0 \neq 0,\ e \in \mathbb{C}[[t^{-1}]].$$

The fact that $f^i = t^{m_p i}\bar{f}$, where the constant coefficient of $\bar{f} \in \mathbb{C}[[t]]$ is non zero, simply yields $e_1 = m_p \nu$.

Similarly, for the second independent local solution of L_B at $x = 0$, which has exponent $-\nu$, we obtain the generalized exponent $e_2 = -m_p \nu$. Hence, the singularity p is regular and $\Delta(\mathrm{M}, p) = \pm(e_1 - e_2) = \pm 2 m_p \nu$.

If $\nu \in \mathbb{Z}$ the second independent solution contains a logarithmic $\ln(x)$. However, we can still do the same computation. The solution z would then involve $\ln(t)$ and the result for the exponent is still true.

(ii) A similar approach work in second case. Let p be a pole of f with multiplicity $m_p \in \mathbb{N}$. Then, f can also be written as $f = t^{-m_p} \sum_{i=0}^{\infty} f_{i-m_p} t^i$ with $f_i \in \mathbb{C}, f_{-m_p} \neq 0$.

We start with the local solution y of L_B at $x = \infty$ corresponding to the exponent $e := \frac{1}{t_\infty} + \frac{1}{2}$. There exists a series $S \in \mathbb{C}[[t_\infty]]$ such that

$$\begin{aligned} y &= \exp\left(\int \frac{e}{t_\infty}\, dt_\infty\right) S \\ &= \exp\left(\int \left(\frac{1}{t_\infty^2} + \frac{1}{2t_\infty}\right) dt_\infty\right) S \end{aligned}$$

so

$$y = \exp\left(-\frac{1}{t_\infty}\right) t_\infty^{1/2} S \tag{2.3}$$

is a solution of L_B. In order to get a solution z of M we have to replace x by f, i.e. $t_\infty = \frac{1}{x}$ by $\frac{1}{f}$. Hence, we do the following substitutions:

$$\begin{cases} t_\infty & \longrightarrow \quad \dfrac{1}{f} = t^{m_p} \displaystyle\sum_{i=0}^{+\infty} \tilde{f}_i t^i, \quad \tilde{f}_i \in \mathbb{C},\ \tilde{f}_0 \neq 0 \\ \dfrac{1}{t_\infty} & \longrightarrow \quad f, \\ \textit{and}\ \ t_\infty^{1/2} & \longrightarrow \quad \dfrac{1}{f^{1/2}} = t^{m_p/2} \displaystyle\sum_{i=0}^{+\infty} \bar{f}_i t^i, \quad \bar{f}_i \in \mathbb{C},\ \bar{f}_0 \neq 0. \end{cases} \tag{2.4}$$

We apply these substitutions to (2.3) and get a local solution z of M at $x = p$:

$$z = \exp(-f) t^{m_p/2} \tilde{S}, \quad \tilde{S} \in \mathbb{C}[[t]],$$

where $\tilde{S}$ combines all the new series that we obtain from (2.4). In order to determine the exponent at p we have to rewrite this expression into the form (2.1). We have to handle the positive and negative power of t in f separately. For the power series part $f^+ = \sum_{i=0}^{+\infty} f_i t^i$ we get

$$\exp\left(-f^+\right) = \exp\left(-\sum_{i=0}^{+\infty} f_i t^i\right).$$

With $\exp(x) = \sum_{i=0}^{+\infty} \frac{x^i}{i!}$ we can rewrite this as a power series in t:

$$\begin{aligned}\exp\left(-f^+\right) &= \sum_{j=0}^{+\infty} \frac{1}{j!}\left(-\sum_{i=0}^{+\infty} f_i t^i\right)^j \\ &= \sum_{i=0}^{\infty} a_i t^i \quad \textit{with} \quad a_i \in \mathbb{C},\ a_0 = 1\ .\end{aligned}$$

The negative powers of t remain in the exponential part, which then becomes

$$\begin{aligned}\exp\left(-\sum_{i=-m_p}^{-1} f_i t^i\right) t^{m_p/2} &= \exp\left(-\sum_{i=-m_p}^{-1} f_i t^i + \frac{m_p}{2}\ln(t)\right) \\ &= \exp\left(\int\left(\sum_{i=-m_p}^{-1} (-i) f_i t^{i-1} + \frac{m_p}{2t}\right) dt\right) \\ &= \exp\left(\int \frac{1}{t}\left(\sum_{i=-m_p}^{-1} (-i) f_i t^i + \frac{m_p}{2}\right) dt\right).\end{aligned}$$

Combining the two results we get

$$z = \exp\left(\int \frac{1}{t}\left(\sum_{i=-m_p}^{-1} (-i) f_i t^i + \frac{m_p}{2}\right) dt\right)\overline{S},$$

where $\overline{S} \in k((t^{\frac{1}{n}}))[\ln(t)]$ has a non-zero constant term. Thus, z has the generalized exponent $-\left(\sum_{i=-m_p}^{-1} i f_i t^i\right) + \frac{m_p}{2}$.

If we start with the second independent solution with generalized exponent $-\frac{1}{t_\infty}+\frac{1}{2}$ we similarly get $\left(\sum_{i=-m_p}^{-1} if_i t^i\right)+\frac{m_p}{2}$. Hence, p is an irregular singularity of M and $\Delta(\mathrm{L},p)=\pm 2\sum_{i=-m_p}^{-1} if_i t^i$.

□

By this theorem, we will now distinguish between two more separated cases which will correspond to zeros and poles of f.

Definition 2.3.1. *A point p of $\mathrm{L}\in\mathbb{C}(x)[\partial]$ for which $\Delta(\mathrm{L},p)\in\mathbb{Z}$ and L is not logarithmic at p is called an exp-apparent point. If p is not exp- apparent, p is called*

(i) exp-regular $\Longleftrightarrow$ $\Delta(\mathrm{L},p)\in\mathbb{C}\backslash\mathbb{Z}$ *or* L *is logarithmic at* p,

(ii) exp-irregular $\Longleftrightarrow$ $\Delta(\mathrm{L},p)\in\mathbb{C}[1/t_p]\backslash\mathbb{C}$.

We denote the set of singularities that are exp-regular by S_{reg} and those that are exp-irregular by S_{irr}.

Remarks 2.3.1. *For an operator* $\mathrm{L}\in\mathbb{C}(x)[\partial]$ *of order two*

1. *Regular points are also exp-apparent, and singular points which are exp-apparent are called exp-apparent singularities. Hence, every point which is not exp-apparent must be a singularity.*
2. *If* L *is such that* $\mathrm{L}_B \xrightarrow{f}_C \mathrm{M} \longrightarrow_{EG} \mathrm{L}$
   - *a- exp-irregular singularities of* L *are irregular singularities of* M *and correspond exactly to the poles of* f.
   - *b- Since we will look at the exponent difference modulo* $\mathbb{Z}$ *we might lose some information about the zeros of* f. *Depending on* ν *and the multiplicity of the zero, their exponent difference can be an integer. Hence, the zeros of* f *can become either exp-regular or exp-apparent singularities of* L. *So regular singularities points of* M *which are exactly zeros of* f *become by exp-product and gauge transformation exp-regular or exp-apparent singularities of* L. *Those hold when* $\nu\in\mathbb{Q}$. *If* $\nu\notin\mathbb{Q}$, *the zeros of* f *are exactly exp-regular points of* L.

We combine these important results in the following corollary

Corollary 2.3.1. *If* $\mathrm{L}_B \xrightarrow{f}_C \mathrm{M} \longrightarrow_{EG} \mathrm{L}$, *the following holds:*

(i) for L*,* $p \in S_{irr} \iff p$ *is a pole of* f*.*

(ii) for L*,* $p \in S_{reg} \implies p$ *is a zero of* f*.*

Proof: Obvious (we just use theorem 2.3.1 and the definition of S_{reg} and S_{irr}). □

By the equivalence in (i) and theorem 2.3.1 we can compute every polar parts of f using S_{irr} and find candidates for the parameter f up to a constant by summing all those polar parts. The polar part of f at a point p is the negative power part of the series representation of f at p. It is non-zero if and only if p is a pole of f. So if $f = \sum_{i=-m}^{+\infty} f_i t_p^i,\ m \in \mathbb{Z}$, the polar part of f at p is $\sum_{i=-m}^{-1} f_i t_p^i$. Theorem 2.3.1 show how to compute those polar parts from the exponent difference $\Delta(\mathrm{L}, p)$ at the exp-irregular points. Since the $\Delta(\mathrm{L}, p)$ are defined up to $\pm$ signs, we obtain each polar part up to a sign as well. We denoted by $\mathfrak{F}$ the set of those candidates of f up to a constant.

We will now summarize this in the following algorithm:

Algorithm 1:

Input: A differential operator $\mathrm{L} \in \mathbb{C}(x)[\partial]$

Output: A list $\mathfrak{F}$ for which the following holds: If $\mathrm{L}_B \xrightarrow{f}_C \mathrm{M} \longrightarrow_{EG} \mathrm{L}$ for some $\nu \in \mathbb{C},\ f \in \mathbb{C}(x)$ and $\mathrm{M} \in \mathbb{C}(x)[\partial]$, there exists a constant $c \in \mathbb{C}$ such that $f - c \in \mathfrak{F}$.

compute singularities S of L and extract S_{irr}

foreach $s \in S_{irr}$

  $d_s := \Delta(\mathrm{L}, s)$

  let $d_s = \sum_{i=-m}^{-1} a_i t_s^i$

  $p_s := \frac{1}{2} \sum_{i=-m}^{-1} \frac{a_i}{i} t_s^i$

$\mathfrak{F} = \left\{ \sum_{s \in S_{irr}} \pm p_s \right\}$

return $\mathfrak{F}$

Since we have no equivalence in (ii) we might not see all zeros of f. If $S_{reg} \neq \emptyset$, one use elements of S_{reg} to compute the constant c for each candidate of f and reduce the set $\mathfrak{F}$. But if $S_{reg} = \emptyset$, then we will need an additional method that is also giving in this subsection.

The Bessel Parameter ν

Let consider $\mathrm{L}_B \xrightarrow{f}_C \mathrm{M} \longrightarrow_{EG} \mathrm{L}$. Since $\nu \in \mathbb{C}$, we have many cases for ν : $\nu \in \mathbb{Z}$ or $\nu \in \mathbb{Q}\backslash\mathbb{Z}$ or $\nu \notin \mathbb{Q}\backslash\mathbb{Z}$. It follows from that:

1- The Bessel parameter is an integer if and only if there exists an exp-regular singularity p of L such that L is logarithmic at p. Hence, if there is one exp-regular point p at which L is logarithmic, then L is logarithmic at every exp-regular point; So, if $\nu \in \mathbb{Z}$ the zeros of f are exactly the singularities S_{reg}.

2- A regular singular point of M can become an exp-apparent point of L only if $\nu \in \mathbb{Q}\backslash\mathbb{Z}$. Hence, if $\nu \in \mathbb{Q}\backslash\mathbb{Z}$, Then for all $s \notin S_{irr}$ we have :

$$\Delta(\mathrm{L}, s) \in \mathbb{Z} \Longleftrightarrow S_{reg} = \emptyset.$$

3- For $\nu \notin \mathbb{Q}\backslash\mathbb{Z}$, the zeros of f are exactly the singularities S_{reg}.

The exponent difference is also associated with the Bessel parameter and we can still distinguish between different cases for ν.

Theorem 2.3.2. *Consider* $\mathrm{L}_B \xrightarrow{f}_C \mathrm{M} \longrightarrow_{EG} \mathrm{L}$

(i) Integer case: if $S_{reg} = \emptyset$ *then* $\nu \in \mathbb{Q}\backslash\mathbb{Z}$.
The following hold for any $p \in S_{reg}$*:*

(ii) Logarithmic case: L *logarithmic at* p *if and only if* $\nu \in \mathbb{Z}$,

(iii) Rational case: if $\Delta(\mathrm{L}, p) \in \mathbb{Q}\backslash\mathbb{Z}$ *then* $\nu \in \mathbb{Q}\backslash\mathbb{Z}$,

(iv) Base field case: $\Delta(\mathrm{L}, p) \in \mathbb{C}\backslash\mathbb{Q}$ *if and only if* $\nu \in \mathbb{C}\backslash\mathbb{Q}$.

Proof: Obvious. □

The following definition will be very important in order to find the Bessel parameter.

Definition 2.3.2. *Consider* $\mathrm{L}_B \xrightarrow{f}_C \mathrm{M} \longrightarrow_{EG} \mathrm{L}$ *and let* $s \in S_{reg}$ *be a zero of* f. *Let* m_s *be the multiplicity of* s *and* $m \in \mathbb{N}^*$. *We define*

$$\begin{aligned} \mathcal{N}_s &= \left\{ \frac{\Delta(\mathrm{L}, s) + i}{2m_s} \middle| 0 \le i \le 2m_s - 1 \right\}, \\ \mathcal{N} &= \left\{ \pm\nu \bmod \mathbb{Z} \middle| \forall s \in S_{reg},\ \exists z_s \in \mathbb{Z} : \nu + z_s \in \mathcal{N}_s \ or -\nu + z_s \in \mathcal{N}_s \right\}, \\ and\ \ \mathcal{N}(m) &= \left\{ \frac{i}{2m},\ i = 1, ..., 2m - 1 \right\}. \end{aligned}$$

Both set $\mathcal{N}_s$ and $\mathcal{N}$ are finite in the case $S_{reg} \neq \emptyset$. In the integer case we have an infinitely large set $\mathcal{N} = \mathbb{C}$.

If s is a zero of f with multiplicity m_s and $\Delta(\mathrm{L}, s) \in \mathbb{Z}$, then $\mathcal{N}(m_s) = \mathcal{N}_s$ *modulo* $\mathbb{Z}$.

Remarks 2.3.2.

1. If $S_{reg} \neq \emptyset$

- *for every $s \in S_{reg}$, the Bessel parameter ν appears in $\mathcal{N}_s$ modulo some integer.*
- *the set $\mathcal{N}$ can be regarded as the intersection of all $\mathcal{N}_s$ modulo $\mathbb{Z}$, $s \in S_{reg}$.*
- *by the invariance of $\mathbb{C}(x)\mathrm{B}_\nu(x) + \mathbb{C}(x)\mathrm{B}_\nu(x)'$ under $\nu \longrightarrow \nu + 1$ and $\nu \longrightarrow -\nu$, we only need to find ν modulo an integer. Hence, we can regarde $\mathcal{N}$ as a set of candidates for ν.*

2. If $S_{reg} = \emptyset$

- *there exists $p, l \in \mathbb{Z}$ such that $p \mid n$ and $\nu + l \in \mathcal{N}(p)$ where n is the degree of the numerator of f.*
- *similarly by the invariance of $\mathbb{C}(x)\mathrm{B}_\nu(x) + \mathbb{C}(x)\mathrm{B}_\nu(x)'$ under $\nu \longrightarrow \nu + 1$ and $\nu \longrightarrow -\nu$, we can regarde the set $\{\bigcup \mathcal{N}(p) \mid p \text{ divides } n\}$ as a set (large) of candidates for ν.*

The Algorithm

The input of our algorithm is a differential operator L and we want to know whether the solutions can be expressed in terms of Bessel functions. We assume that $\mathrm{L}_B \longrightarrow \mathrm{L}$ for some transformations. If we find a solution to that problem, then we also find the solution space of L. If we do not succeed, we know that the solutions of L can not be expressed with Bessel functions.

Let L be a differential operator of degree two with coefficients in $\mathbb{C}(x)$ and let's summarize the steps of the algorithm that we know from previous results:

A **(Singularities)** We can compute the singularities S of L by factoring the leading coefficient of L and the denominators of the other coefficients into linear factors.

B **(Generalized exponents)** For each $s \in S$ we compute $d_s = \Delta(\mathrm{L}, s)$, isolate exp-apparent points with $d_s \in \mathbb{Z}$, and differ between exp-regular singularities S_{reg} with $d_s \in \mathbb{C}$ and exp-irregular singularities S_{irr} with $d_s \in \mathbb{C}[t_s^{-1}]\backslash\mathbb{C}$.

C **(Polar parts)**We can use the exponent differences d_s for $s \in S_{irr}$ to compute candidates $\mathfrak{F}$ for the parameter f up to a constant $c \in \mathbb{C}$.

D **(Constant term of f)**In all cases but the integer case we know at least one zero of f by picking some $s_0 \in S_{reg}$. So we can also compute the missing constant c for each $\widetilde{f} \in \mathfrak{F}$.

E **(The set $\mathcal{N}$)** The set $\mathcal{N}$ is the set of candidates for ν. When not being in the integer case, this set is finite. But the set might depend on the candidates $f \in \mathfrak{F}$.

F **(Compute** M**)** For each pair (ν, f) *in* $\mathcal{N} \times \mathfrak{F}$ we can compute an operator $\mathrm{M} = \mathrm{M}_{(\nu,f)} \in \mathbb{C}(x)[\partial]$ such that $\mathrm{L}_B \xrightarrow{f}_C \mathrm{M}$.

G **(Exp-product and gauge transformation)** For each M we can decide whether M $\longrightarrow_{EG}$ L, and if so, we compute the transformations.

Steps D and E have to be done by cases differentiation. If we call "$findBessel\nu f$" the algorithm which treats Steps D and E then the basic procedure will look like this:

Algorithm 2:

Input: An operator $\mathrm{L} \in \mathbb{C}(x)[\partial]$.

Output: V(L) when it can be represented in terms of Bessel functions and FAIL otherwise.

compute singularities S of L and S_{reg}, S_{irr}
use algorithm 1 to get $\mathfrak{F}$
$P = \{\ \}$
for each $f \in \mathfrak{F}$
    $P = P \cup$ **findBessel**ν **f**(f, S_{reg})
for each $(\nu, f) \in P$
    M = **changeOfVar**(L_B, f)
    if $\exists r_0, r_1, r_2 \in \mathbb{C}(x)$: $\mathrm{M} \xrightarrow{r_0}_E \widetilde{\mathrm{M}} \xrightarrow{r_1, r_2}_G \mathrm{L}$ for some $\widetilde{\mathrm{M}} \in \mathbb{C}(x)[\partial]$ **then**
        return V(L)
return FAIL

The only case in which this algorithm does not yet work is when $S_{reg} = \emptyset$, which we will study in the next part.

Note that one can also use the cases separation of theorem 2.3.2 to reduce the number of candidates that we obtain from steps D and E and this will be the next part.

The Cases Separation

For each case we extend the algorithm "$findBessel\nu f$" which will take candidates $f \in \mathfrak{F}$ and the exp-regular points S_{reg}, and will return a set of pairs (ν, f).

Logarithmic Case

Since $\nu \in \mathbb{Z}$ the zeros of f are exactly the singularities S_{reg}.

$S_{reg} \neq \emptyset$, there exists at least one $s_0 \in S_{reg}$ which must be a zero of f. We can use it to compute the constant part c of each candidate in $\mathfrak{F}$. With the other points in S_{reg} we verify the constant or exclude some candidates.

In general, we assume $\nu = 0$. By lemma 1.4 taking $\nu = 0$ when $\nu \neq 0$ can generate gauge transformations which may not be needed and the result will not be much simpler than the result obtained by taking $\nu \neq 0$ which is computed much faster. If there is no gauge transformations needed to transform L_B into L, we compute ν as follow: take the average of exponent differences each divided by the corresponding multiplicity.

Algorithm 3:

pick one $s_0 \in S_{reg}$
$c :=$ **solve** $(f|_{x=s} + c = 0, c)$
if $f|_{x=s} + c = 0$ for all $s \in S_{reg}$ **then**
    return $\{(0, f + c)\}$
else
    return $\{\ \}$

Integer Case

Here all zeros of f are exp-apparent singularities of L.

Let $p, l \in \mathbb{Z}$ be as in remarks 2.3.2. Since $\nu + l \in \mathcal{N}(p)$, p cannot be one, otherwise $\nu \in \frac{1}{2}$ modulo $\mathbb{Z}$ and the operator L_B is reducible.

We denoted by $deg(numer(f))$ the degree of the numerator of f and $deg(denom(f))$ the degree of the denominator of f.

For each $\tilde{f} \in \mathfrak{F}$ and $n = deg(numer(f))$ we take a divisor $p \in \mathbb{N}\backslash\{0,1\}$ of n and check whether for certain constants c the monic part of the numerator of f become a $p-th$ power. Hence, we get a finite set $\mathfrak{C}_p$ of possible values for c. We denoted by $\mathcal{N}$ and $\mathfrak{C}$ the union of the sets $\mathcal{N}(p)$ and $\mathfrak{C}_p$ respectively. So if the pair (ν, c) exists, it must be in the set $\mathcal{N} \times \mathfrak{C}$.

The problem now is how to find the degree n of the numerator of f without knowing the constant part c.

Lemma 2.7. *Consider* $L_B \xrightarrow{f}_C M \longrightarrow_{EG} L$. *Let us be in the integer case and* $\nu = \frac{\nu_1}{2p}$ *for some* $\nu_1 \in \mathbb{Z}$, $p \in \mathbb{N}\backslash\{0,1\}$ *and* $gcd(\nu_1, p) = 1$.

(i) If $\infty \in S_{irr}$, *then* $deg(numer(f)) = deg(numer(f + c))$ *for all* $c \in \mathbb{C}$.

(ii) If $\infty \notin S_{irr}$, *then* $p \mid deg(numer(f)) \Longleftrightarrow p \mid deg(denom(f))$.

<u>**Proof:**</u> After using the exp-irregular points S_{irr} to find the polar parts, f has the form

$$f = \frac{f_1}{f_2} + c + f_3, \tag{2.5}$$

where $f_1, f_2, f_3 \in \mathbb{C}[x]$ and $deg(f_1) < deg(f_2)$ or $f_1 = 0$. The polar parts for $s \in S_{irr}\backslash\{\infty\}$ are combined in $\dfrac{f_1}{f_2}$. The polynomial f_3 is the polar part of $\infty \in S_{irr}$.

(i) In this case $\infty \in S_{irr}$ and hence $f_3 \neq 0$. So c does not effect the degree of the numerator of f.

(ii) Since $\infty \notin S_{irr}$, $\quad f_3 = 0$ in equation (2.5) and $f = \dfrac{f_1}{f_2} + c$ with $deg(f_1) < deg(f_2)$.

- If $c \neq 0$, then $deg(numer(f)) = deg(denom(f))$ and nothing remains to be proven.
- If $c = 0$, ∞ is a zero of f. The multiplicity m must be a multiple of p. Otherwise $\Delta(\mathrm{L}, \infty) \notin \mathbb{Z}$ since $\Delta(\mathrm{L}, \infty) = 2m\nu + z$ for some $z \in \mathbb{Z}$ and $2\nu = \dfrac{\nu_1}{p}$. Hence, $m = kp$ for some $k \in \mathbb{N}$.

  This multiplicity of the point ∞ is $m = deg(denom(f)) - deg(numer(f))$. This can be seen if $f(\frac{1}{x})$ is written as power series at the points 0. In total we get $deg(numer(f)) = deg(denom(f)) - kp$ for some $k \in \mathbb{N}$ and this proves (ii).

□

Algorithm 4:

P:= { }
if $\infty \in S_{irr}$ **then**
  $n := deg(numer(f))$
else
  $n := deg(denom(f))$
for each $p \mid n$
  $g :=$ **ispower**$(numer(f + c), x, p)$ (here c is a variable)
  $\mathfrak{C} :=$ **solve** $(numer(f + c) - g^p = 0, c)$ (find solution $c \in \mathbb{C}$)
  for each $c \in \mathfrak{C}$
    $P := P \cup \{(\nu, f + c) \mid \nu \in \mathcal{N}(p)\}$
return P

Rational Case

In this case a zero of f can also be an exp-apparent point of L because $\nu \in \mathbb{Q}\backslash\mathbb{Z}$.

Since $S_{reg} \neq \emptyset$, we define the set $\mathcal{N} = \bigcap_{s \in S_{reg}} \mathcal{N}_s \bmod \mathbb{Z}$ which is the set of possible candidates for ν.

With each candidate of f in $\mathfrak{F}$ we do the sames computations and verifications using S_{reg} as in the logarithmic case. For each $s \in S_{reg}$ we can compute the multiplicity m_s.

Let $g = numer(f)/\prod_{s \in S_{reg}}(x - s)^{m_s}$, be a polynomial function. If $deg(g) > 0$ then all zero of g are exactly exp-apparent zeros of f and similar arguments as in the integer case are used.

For all zeros s of g, we have $\Delta(\mathrm{L}, s) = 2m_s\nu + z \in \mathbb{Z}$ for some $z \in \mathbb{Z}$. So m_s must be a multiple of $p = denom(2\nu)$. Hence, the monic part of g must be a $p - th$ power.

Algorithm 5:

$P := \{ \ \}$
$c :=$ **solve** $(f|_{x=s} + c = 0, c)$

---

if $f|_{x=s} + c = 0$ for all $s \in S_{reg}$ **then**
  $g := numer(f + c)$
  for each $s \in S_{reg}$
    $g := g/(x - s)^{m_s}$
  $\mathcal{N} := \bigcap_{s \in S_{reg}} \mathcal{N}_s \ \ mod\ \mathbb{Z}$
  for each $\nu \in \mathcal{N}$
    $p := denom(2\nu)$
    if $g = h^p$ for some $h \in \mathbb{C}[x]$ **then**
      $P := P \cup \{(\nu, f + c)\}$
return P

Base Field Case

In this case all the zeros of f are exactly exp-regular points of L since $\nu \notin \mathbb{Q}$.

We define the set of possible candidates of ν as in the rational case.

With each candidate in $\mathfrak{F}$ we do the sames computations and verifications using S_{reg} as in the logarithmic case.

Now we will see by the following lemma that each candidate in $\mathfrak{F}$ must satisfy another property which is satisfying by f and if not we will exclude it.

Lemma 2.8. *Consider* $\mathrm{L}_B \xrightarrow{f}_C \mathrm{M} \longrightarrow_{EG} \mathrm{L}$. *Let* $\nu \in \mathbb{C} \backslash \mathbb{Q}$, $S_{reg} = \{s_1, ..., s_n\}$ *and* $d_i = \Delta(\mathrm{L}, s_i)$, $i = 1, ..., n$. *Then we can do the following steps:*

1. *Compute* $r_i, t_i \in \mathbb{Q}$ *such that* $d_i = r_i d_1 + t_i$.
2. *Let* $a_i, b_i \in \mathbb{Z}$ *be such that* $r_i = \dfrac{a_i}{b_i}$ *and* $gcd(a_i, b_i) = 1$.
3. *Let* $l = lcm(b_i, 1 \leq i \leq n)$.

Then the monic part of the numerator of f *is a power of* $h \in \mathbb{C}[x]$ *where*

$$h = \prod_{i=1}^{n} (x - s_i)^{l r_i}. \tag{2.6}$$

<u>Proof</u>: Let m_i be the multiplicity of s_i as a zero of f. Since a gauge transformation can change the exponent difference by an integer we know

$$d_i = 2 m_i \nu + z_i \quad for \quad some \quad z_i \in \mathbb{Z}. \tag{2.7}$$

This equation yields for $i = 1$ the equation

$$\nu = \frac{d_1 - z_1}{2 m_1}.$$

Plugging this into (2.7) we get

$$d_i = \frac{m_i}{m_1} d_1 + z_i - \frac{m_i z_1}{m_1}.$$

So the numbers

$$r_i = \frac{m_i}{m_1} \quad and \quad t_i = z_i - \frac{m_i z_1}{m_1} \tag{2.8}$$

satisfy the equation in step 1. Since $d_i \notin \mathbb{Q}$ the rational factor r_i is unique.

Now Let $a_i, b_i \in \mathbb{Z}$ be such that $r_i = \frac{a_i}{b_i}$ and $gcd(a_i, b_i) = 1$. Then $m_i = \frac{a_i}{b_i} m_1$. Since $m_i \in \mathbb{Z}$ and $b_i \nmid a_i$ we obtain $b_i | \, m_1$. Then also $l := lcm(b_i, 1 \leq i \leq n) | \, m_1$. We use $m_i = r_i m_1$ and finally get $lr_i | \, m_i$. So the exponents in (2.6) each divide the multiplicity in the numerator of f.

Let $p_i \in \mathbb{N}$ be such that $lr_i p_i = m_i$. To prove (2.6) we have to see that all p_i are equal. Using the equation for r_i in (2.8) yields $lp_i = m_1$. So $p = p_i = \frac{m_1}{l}$ is independent of i and the numerator of f must be a scalar multiple of a $p - th$ power of h. $\square$

Algorithm 6:

$P := \{ \ \}$
let $S_{reg} = \{s_1, ..., s_n\}$
for $i = 1, ..., n$
  $d_i := \Delta(\mathrm{L}, s_i)$
  compute r_i, t_i, such that $d_i = r_i d_1 + t_i$
$l := lcm\,(denom(r_i), i = 1, ..., n)$
$h := \prod_{i=1}^{n} (x - s_i)^{lr_i}$
pick one $a \in S_{reg}$
$c :=$ **solve** $(f|_{x=a} + c = 0, c)$
if $f|_{x=s} + c = 0$ for all $s \in S_{reg}$
  and $numer(f + c) = h^p$ for some $p \in \mathbb{N}$ **then**
    $\mathcal{N} := \bigcap_{s \in S_{reg}} \mathcal{N}_s \ \ mod \ \mathbb{Z}$
    for each $\nu \in \mathcal{N}$
      $P := P \cup \{(\nu, f)\}$
return P

2.3.2 The Parameter of the change of variables is not in $\mathbb{C}(x)$

Note that $\mathrm{L}_{\check{B}} \xrightarrow{g}_C \mathrm{M}$ is the same as $\mathrm{L}_B \xrightarrow{f}_C \mathrm{M}$, where $f^2 = g \in \mathbb{C}(x)$. In order to use the same notation as in the case $f \in \mathbb{C}(x)$, we will use the second form, i.e $\mathrm{L}_{\check{B}} \xrightarrow{g}_C \mathrm{M}$.

Since we now work with the operator $\mathrm{L}_{\check{B}}$, we will take $m = 2$.

The main problem in the case $f \notin \mathbb{C}(x)$ is to construct a finite set of candidates for (f, ν) from $\Delta(\mathrm{L}, p)$.

The Exponent Difference

We also define:

Definition 2.3.3. *A singularity p of $\mathrm{L} \in \mathbb{C}(x)[\partial]$ is called:*

(i) apparent singularity if and only if $\Delta(\mathrm{L}, p) \in \mathbb{Z}$ and L is not logarithmic at p.

(ii) regular singularity if and only if $\Delta(\mathrm{L}, p) \in \mathbb{C}\backslash\mathbb{Z}$ or L is logarithmic at p.

(iii) irregular singularity if and only if $\Delta(\mathrm{L}, p) \in \mathbb{C}[t_p^{-\frac{1}{2}}]\backslash\mathbb{C}$.

We also denote the set of regular singularities and irregular singularities by S_{reg} and S_{irr} respectively. As in the case $f \in \mathbb{C}(x)$, the sets of those three kind of singularities are separated.

Theorem 2.3.3. *Let $\mathrm{L}_B \xrightarrow{f}_C \mathrm{M} \longrightarrow_{EG} \mathrm{L}$, where $f^2 \in \mathbb{C}(x)$.*

(i) if p is a zero of f with multiplicity $m_p \in \frac{1}{2}\mathbb{Z}^+$, then p is an apparent singularity or $p \in S_{reg}$, and $\Delta(\mathrm{M}, p) = 2m_p\nu$.

(ii) p is a pole of f with pole order $m_p \in \frac{1}{2}\mathbb{Z}^+$ such that $f = \sum_{i=-m_p}^{\infty} f_i t_p^i$, if and only if $p \in S_{irr}$ and $\Delta(\mathrm{M}, p) = 2\sum_{i=-m_p}^{-1} i f_i t_p^i$.

<u>Proof</u>: We can use the same proof as in the case $f \in \mathbb{C}(x)$. □

Remarks 2.3.3.

1. *if $p \in S_{reg}$, then $\Delta(\mathrm{L}, p) \equiv \Delta(\mathrm{M}, p) \bmod \mathbb{Z}$ which means that we can compute $2m_p\nu \bmod \mathbb{Z}$.*

2. *If $p \in S_{irr}$, then $\Delta(\mathrm{L}, p) \equiv \Delta(\mathrm{M}, p) \bmod \mathbb{Z}$. Then $\sum_{i=-m_p}^{-1} f_i t_p^i$ can be computed from $\Delta(\mathrm{L}, p)$ by dividing coefficients by $2i$ (the congruence only affects the t_p^0- term of Δ, but that term is not used when $p \in S_{irr}$).*

Definition 2.3.4. *Let $f = \sum_{i=N}^{\infty} f_i t_p^i$, $N \in \mathbb{Z}$, $f_N \neq 0$. We say that we have a k-term truncated power series for f when the coefficients of $t^N, ..., t^{N+k-1}$ are known.*

Remark 2.3.1. *If a k-term truncated series for f is known, then we can compute a k-term truncated series for f^2.*

According to theorem 2.3.3 (ii), from $\Delta(\mathrm{M}, p)$, we can get a $[m_p]$ term truncated series of f at p. Since we have to compute $g = f^2 \in \mathbb{C}(x)$, we square it to obtain a truncated series of g which also have $[m_p]$ terms (see the previous remark). So we have the following corollary:

Corollary 2.3.2. *If $\mathrm{L}_B \xrightarrow{f}_C \mathrm{M} \longrightarrow_{EG} \mathrm{L}$ and $g = f^2$ then we have:*

(i) if $p \in S_{reg}$ then p is a zero of g.

(ii) $p \in S_{irr}$ if and only if p is a pole of g. We can also get a $[m_p]$-term truncated series of g from $\Delta(\mathrm{L}, p)$, where m_p is the pole order of f.

The Bessel Parameter ν and The Parameter $g = f^2$

We assume $\mathrm{L}_B \xrightarrow{f}_C \mathrm{M} \longrightarrow_{EG} \mathrm{L}$, we want to get information about f from L and the Bessel parameter ν. Since f might not in $\mathbb{C}(x)$, but $g = f^2$ is in, we can assume $g = \frac{A}{B}$, $A, B \in \mathbb{C}[x]$, B is monic and $gcd(A, B) = 1$.

Now, we want to get information about A, B and ν from L.

The Bessel Parameter ν

All things that we have said about the Bessel parameter in the case $f \in \mathbb{C}(x)$ hold here, but with small extension that we will give.

We will divide into three cases by different situations in theorem 2.3.2. We call (ii) logarithmic case, (i) and (iii) rational case, and (iv) base field case. We have to introduce another case when the number of linear equations for coefficients of A is greater than $deg(A)$. This case is called the " easy case ".

For the rational case we have:

Remark 2.3.2. *Since $\mathbb{C}(x)B_\nu(x)+\mathbb{C}(x)B'_\nu(x)$ is invariant under $\nu \to \nu+1$ and $\nu \to -\nu$, for $\nu \in \mathbb{Q}$, we can just focus on $\nu \in [0, \frac{1}{2}]$. Also since when $\nu = \frac{1}{2}$ the operator will be reducible, it is easy to solve the operator by factoring. So we just consider $\nu \in [0, \frac{1}{2})$*

If we fix f, then we have:

Lemma 2.9. *Let p be a zero of f with multiplicity m_p and let*

$$\mathcal{N}'_p := \left\{ \frac{\Delta(\mathrm{L}, p) + i}{2m_p} \middle| 0 \leq i \leq 2m_p - 1, i \in \mathbb{Z} \right\}.$$

We can make the rational part of each element in $\mathcal{N}'_p$ belong to $[0, \frac{1}{2}]$. Let the new set be $\mathcal{N}_p$. Then $\nu \in \mathcal{N} := \bigcap_{p \in S_{reg}} \mathcal{N}_p$.

<u>Proof:</u> The lemma follows from the fact that we know the number $\Delta(\mathrm{L}, p) \equiv 2m_p\nu \mod \mathbb{Z}$, and the fact that $\mathbb{C}(x)B_\nu(x) + \mathbb{C}(x)B'_\nu(x)$ is invariant under $\nu \to \nu + 1$ and $\nu \to -\nu$. □

The Parameter $g = f^2$

The following lemma help us about B:

Lemma 2.10. *We can retrieve B from S_{irr}.*

<u>Proof:</u> According to theorem 2.3.3, if $p \in S_{irr}$ then p is a pole of f. Let $m_p \in \frac{1}{2}\mathbb{Z}$ be pole order of $\Delta(\mathrm{M}, p)$. g has a pole order $2m_p$ at p. The theorem implies $B = \prod_{p \in S_{irr} \setminus \{\infty\}} (x-p)^{2m_p}$.
□

The following lemma help us about the degree of A:

Lemma 2.11. *Let*
$$d_A = \begin{cases} deg(B) + 2m_\infty & if \quad \infty \in S_{irr} \\ deg(B) & otherwise \end{cases}$$

(i) If $\infty \in S_{reg}$ then $deg(A) < d_A$;

(ii) If $\infty \in S_{irr}$ then $deg(A) = d_A$;

(iii) otherwise $deg(A) \leq d_A$.

In all cases, we can write $A = \sum_{i=0}^{d_A} a_i x^i$, so we have $d_A + 1$ unknowns.

Proof:

(i) According to corollary 2.3.2 (i), if $\infty \in S_{reg}$ then ∞ is a zero of g. So we have $deg(A) < deg(B)$. Hence, $deg(A) < d_A$.

(ii) According to corollary 2.3.2 (ii), if $\infty \in S_{irr}$ with pole order m_∞ then ∞ is a pole of g with pole order $2m_\infty$. So $deg(A) = deg(B) + 2m_\infty$. Hence, $deg(A) = d_A$.

(iii) If $\infty \notin S_{irr}$ then f does not have a pole at ∞ , so that $deg(A) \leq deg(B)$. Hence, $deg(A) \leq d_A$.

□

Once we have an idea about degree of A, we will now focus about the coefficients of A.

A- **Easy, Logarithmic and Base Field Cases**

Lemma 2.12. *If $p \in S_{reg}$, we will get one linear equation for the coefficients of A. If $p \in S_{irr}$ with m_p as pole order of $\Delta(\mathrm{L}, p)$, we will get $[m_p]$- linear equations.*

Proof: According to corollary 2.3.2 (ii), if $p \in S_{reg}$, p is a zero of A. Then we will get a linear equation of $\{a_i\}_{i=0,\ldots,d_A}$ by setting $rem(A, x - p) = 0$.

For each $p \in S_{irr}$ with pole order m_p, by corollary 2.3.2 (ii) we will have a $[m_p]$-term truncated series of g at p. Then we can get the truncated series of $A = gB$. On the other hand, we can rewrite $A = \sum_{i=0}^{d_A} a_i x^i$ as a truncated series at p (by Taylor or Laurent series). Since the terms in a Taylor series or Laurent series depend linearly on the coefficients of A, by comparing the coefficients, each term will give a linear equation of a_i. □

Remarks 2.3.4.

- *In the easy case, since the number of linear equations for coefficients of A is greater than $deg(A)$, then we can solve them and find A.*

- *In both logarithmic and base field case, we know all zeros of A. In the base field case we know their multiplicities as well, but in the logarithmic case, we have to do a combinatorial search: try all possible combinations of multiplicities of zeros of A. So in both logarithmic and base field case, there is only one unknown coefficient, the leading coefficient of A. To get this coefficient, we only need one equation. But we have $\frac{1}{2}d_A$ linear equations, enough to get A.*

B- **Rational Case**

Let p be a root of A, i.e a zero of f with multiplicity $m_p \in \mathbb{Z}$. Since $\nu \in \mathbb{Q}\backslash\mathbb{Z}$, there exists a, $d \in \mathbb{Z}$ with $gcd(a,d) = 1$ such that $\nu = \dfrac{a}{d}$. We will have $d > 2$ because $\nu \notin \mathbb{Z}$ and if $\nu \in \frac{1}{2}\mathbb{Z}$ then L_B will be reducible. By remark 2.3.2, if we fix d, we get a set of candidates for ν :

$$\left\{ \frac{a}{d} \;\middle|\; gcd(a,d) = 1,\ 1 \leq a \leq \tfrac{1}{2}d \right\}.$$

By theorem 2.3.3 (i), $\Delta(\mathrm{L}, p) = 2m_p\nu \bmod \mathbb{Z} = \dfrac{2m_p}{d}a \bmod \mathbb{Z}$. If $d \mid 2m_p$, change of variables $x \to f$ will send p to an apparent singularity and then $p \notin S_{reg}$, which means that not all roots of A are known (not all roots of A are in S_{reg}). So the multiplicities of all zeros of A which are apparent singularities must be a multiple of d. Thus, we can rewrite $A = CA_1A_2^d$, where $A_1, A_2 \in \mathbb{C}[x]$ and $C \in \mathbb{C}$, A_1 is monic and the roots of A_1 are the known roots of A (the elements of S_{reg}).

Despite we know S_{reg}, we can not say that A_1 is getting because we also have to know the multiplicity order of each element in S_{reg}. Hence, the problem in this case is to compute A_1, A_2 and d.

The rational case include two cases for S_{reg} : $S_{reg} = \emptyset$ and $S_{reg} \neq \emptyset$.

For $S_{reg} = \emptyset$ we can let $A_1 = 1$ and fix d by the following lemma:

Lemma 2.13. *If $S_{reg} = \emptyset$, then $d \mid d_A$.*

<u>Proof</u>: Since $A = CA_1A_2^d$, if $S_{reg} = \emptyset$, then $A_1 = 1$ and $A = CA_2^d$. So $d \mid d_A$. □

For $S_{reg} \neq \emptyset$, we have:

Lemma 2.14. *If $S_{reg} \neq \emptyset$, we can find a list of candidate pairs (d, A_1) by solving an equation.*

<u>Proof</u>: See [5] Lemma 10 . □

Now the only remaind problem is the computation of A_2. Here S_{reg} can not help us because A_2 use apparent singularities.

Lemma 2.15. *Let $p \in S_{irr}$. We can choose C, such that the d-th root of the coefficient of the initial term of the truncated series of $A/(CA_1)$ at p is in $\mathbb{C}$.*

Proof: From $\Delta(\mathrm{L}, p)$ we can compute a truncated series for $f^2 = \frac{CA_1A_2^d}{B}$. Let C be the coefficient of the first term of this series which will done the proof $\big($*note that* $f^2B/A_1 = CA_2^d\big)$. □

Assume $A_2 = \sum_{i=0}^{deg(A_2)} b_i x^i$. Since $deg(A_2) \leq \frac{1}{d}d_A \leq \frac{1}{3}d_A$, we have:

Lemma 2.16. *For the rational case, we only need $\frac{1}{3}d_A + 1$ equations to recover A.*

Proof: Since $deg(A_2) \leq \frac{1}{3}d_A$, if we have only $\frac{1}{3}d_A + 1$ equations, we can completely get A_2. Hence we have A. □

Theorem 2.3.4. *In the rational case, for $A = CA_1A_2^d$, and $A_2 = \sum_{i=0}^{deg(A_2)} b_i x^i$, for $p \in S_{irr}$ with m_p as pole order of exponent difference, we will get $[m_p]$-linear equations of $\{b_i\}$.*

Proof: Since the exponent difference at p will give a $[m_p]$-truncated series of $g = \frac{A}{B}$ at $x = p$, we can also write B and CA_1 as a series at p. Then we can get the $[m_p]$-truncated series of $A_2^d = \frac{gB}{CA_1}$. We assume the series is $\sum_{m_p < i \leq 2m_p} c_i t_p^{-i}$ where t_p is the local parameter at p. We can rewrite the series as $c_{2m_p} t_p^{-2m_p} S$, where S is a power series with the initial term 1. Let $S_{1/d}$ be a power series with first term 1 such that $S_{1/d}^d = S$. Write $S_{1/d} = 1 + \sum_{i>0} a_i t_p^i$ where $a_1, ..., a_{[m_p]-1}$ are computed by Hensel lifting. Let $\mu_d = \big\{ r | \, r \in \mathbb{C}, r^d = 1 \big\}$. By lemma 2.15 there should be a $d-th$ root of c_{2m_p} in $\mathbb{C}$. Let c be such a root. Then for each $r \in \mu_d$, let $S_r = ct_p^{-2m_p/d} r S_{1/d}$. Then S_r is a truncated series at p whose $d-th$ power is the truncated series of $\frac{gB}{CA_1}$ at p. Then we can also rewrite $A_2 = \sum_{i=0}^{deg(A_2)} b_i x^i$ as a truncated series at p. By comparing the coefficients of S_r and A_2, we will get $[m_p]$-linear equations. Doing this for every $p \in S_{irr}$ provides enough linear equations to find A.
Note that we have to try all combinations of $r \in \mu_d$ at every $p \in S_{irr}$. □

We can use the results from lemma 2.12 to get equations. So we can always obtain $\geq \frac{1}{2}d_A$ linear equations, while $[\frac{1}{3}d_A] + 1$ equations are sufficient. So we always get enough linear equations.

To sum up, for all different cases, we have:

Theorem 2.3.5. *From $\Delta(\mathrm{L}, p)$, we can always get a list of candidates for (f, ν).*

Proof: We always have at least $\#S_{reg} + \frac{1}{2}d_A$ linear equations for the coefficients of A. But we may have enough equations (easy case), or only need either 1 (logarithmic case and base field case) or $\frac{1}{3}d_A + 1$ equations (rational case) to get g. By remark 2.3.2 and lemma 2.9, we can also get a finite list of ν . □

Algorithms

The input of the algorithm is a differential operator L of order 2. We want to find whether there exists solutions which can be represented in terms of Bessel functions. If they exist, then find the solutions. Otherwise the algorithm output $\emptyset$.

Algorithm 7: Main Algorithm

Input: an irreducible differential operator L
Output: solutions represented in terms of Bessel functions if they exist
Find all singularities by factoring the leading coefficient of L over $\mathbb{C}$
foreach *singularity* p **do**
  compute the generalized exponents at p, then ;
  compute the exponent differences and then;
  the truncated series of g;
end
Get S_{reg} and S_{irr} according to the generalized exponent differences;
Compute B, d_A (lemma 2.10 and 2.11) and the number of linear equation N $(N \geq S_{reg} + \frac{1}{2}d_A)$;
if $N > d_A$ **then**
  go to easy case;
else if L *logarithmic at some* $p \in S_{reg}$ **then**
  go to logarithmic case;
else if *there is* $p \in S_{reg}$ *with* $\Delta(\mathrm{L}, p) \notin \mathbb{Q}$ *(i.e* $\nu \notin \mathbb{Q}$*)* **then**
  go to base field case;
else
  go to rational case;
end
/* It will give us a list of candidates for (f, ν), where f is the function of the change of variables, and ν is the parameter of Bessel functions */
foreach (f, ν), *in list of candidates* **do**
  compute an operator $\mathrm{M}_{(f,\nu)}$ such that;
  $\mathrm{L}_B \xrightarrow{f}_C \mathrm{M}_{(f,\nu)}$;
  if $\exists r_0, r_1, r_2 \in \mathbb{C}(x)$: $\mathrm{M}_{(f,\nu)} \xrightarrow{r_0}_E \widetilde{\mathrm{M}} \xrightarrow{r_1,r_2}_G$ *for some* $\widetilde{\mathrm{M}} \in \mathbb{C}(x)[\partial]$ **then**
    Add the solution to solutions list;
  end
end
Output the solutions list;

Now we will explain by algorithms the detail how to retrieve f, ν in different cases.

A- **(Easy Case)**

In this case, we have enough linear equations from lemma 2.12 to recover g. After that, we can use lemma 2.9 to get ν.

Algorithm 8:

Input: S_{reg}, S_{irr} with truncated series, B, d_A

Output: potential list of (f, ν)

Find all linear equations described in lemma 2.12;

Solve linear equations to find f;

if *there is no solution* **then**

  output $\emptyset$

else

  Use lemma 2.9 to get a list $\mathcal{N}$ of candidates for $\nu' s$

end

foreach $\nu \in \mathcal{N}$ **do**

  Add (f, ν) to output list

end

B- **(Logarithmic Case)**

We can let $\nu = 0$. By remarks 2.3.4, we know all the zeroes of g. We do not yet know the leading coefficient and the multiplicity of each zero. So we can try all the combinations of possible multiplicities.

Algorithm 9: Logarithmic Case

Input: S_{reg}, S_{irr} with truncated series, B, d_A

Output: list of (f, ν)

if *not every singularity* $p \in S_{reg}$ *is logarithmic* **then**

  output $\emptyset$

else

  Let $\nu = 0$, $A = a \prod_{p \in S_{reg} \setminus \{\infty\}} (x - p)^{a_p}$;

  if $\infty \in S_{reg}$ **then**

    $a_\infty \geq 1$ is an integer;

  else

    $a_\infty = 0$;

  end

  foreach $\{a_p\}$ *such that* $\sum_{p \in S_{reg}} a_p = d_A - a_\infty$, $a_p \geq 1$ *are integers* **do**

    Use linear equations described in lemma 2.12 to solve a;

    if *the solution exists* **then**

      Add $(\frac{A}{B}, 0)$ to output list

    end

  end

end

C- **(Base Field Case)**

In this case, by remarks 2.3.4 we have all the zeroes with multiplicities of g. The only unknown part should be the leading coefficient. But we have at least one linear equations.

Algorithm 10:

Input: S_{reg}, S_{irr} with truncated series, B, d_A
Output: list of (f, ν)
Use remarks 2.3.4 to find all zeroes and multiplicities;
Use linear equations given by lemma 2.12 to get the leading coefficient;
Use lemma 2.9 to get a list of candidates for $\nu's$;
Add solutions to output list;

D- **(Rational Case)**

This case is the most complicated case. Let $d = denom(\nu)$ and $f^2 = g = \frac{CA_1A_2^d}{B}$.

Algorithm 11:

Input: S_{reg}, S_{irr} with truncated series, B, d_A
Output: list of (f, ν)
if $S_{reg} = \emptyset$ **then**
  Let the list of candidates for d be the set of factors of d_A;
  Let $A_1 = 1$;
else
  Use lemma 2.14 to get a list of candidates for d and A_1
end
foreach *candidate* (d, A_1) **do**
  Fix C by lemma 2.15;
  Use linear equations given by theorem 2.3.4 to compute A_2;
  If a solution exists, add
  $\{f\} \times \left\{\frac{a}{d} \mid gcd(a,d) = 1,\ 1 \leq a < \frac{1}{2}d\right\}$ to output list
end

Chapter three

Illustrations

3.1 Maple Commands

All of our algorithm are developed with Maple and our examples are illustrated with it too. So in this section we want to introduce some commands we need in Maple.

In Maple, the DEtools package contains commands that help us to work with differential equations. Every command inside has short version or long version.

- For the version, we always need to tell Maple the variable and the derivation.
- For the short version, one needs to tell Maple the symbol for the variable x and the derivation D by using the command:

```
>_Envdiffopdomain:=[D,x]:
```

This command tell Maple that we use x as variable and D as derivation.

In this Master thesis, we always assume that the DEtools package is loaded and the differential domain is defined by $[D, x]$.

We use Bessel or Modified Bessel operator as example.

3.1.1 Generalized Exponents

Generalized exponents can be computed in Maple with the command gen_exp, which belongs to the package DEtools. The input is a operator L, a variable t to express the generalized exponent and a point at which we want to compute the generalized exponent. The output is a list of pairs $[g, eq]$ which each represents a generalized exponent at the given point. In this pair the equation eq describes the variable t which is used to express the generalized exponent g.

```
>gen_exp(LB1,t,x=0);
```

$$[[\nu, t = x], [-\nu, t = x]]$$

```
>gen_exp(LB1,t,x=infinity);
```

$$\left[\left[\frac{RootOf(1+z^2)}{t} + \frac{1}{2}, t = \frac{1}{x}\right]\right]$$

```
>gen_exp(LB2,t,x=0);
```

$$[[\nu, t = x], [-\nu, t = x]]$$

```
>gen_exp(LB2,t,x=infinity);
```

$$\left[\left[\frac{1}{t} + \frac{1}{2}, t = \frac{1}{x}\right], \left[-\frac{1}{t} + \frac{1}{2}, t = \frac{1}{x}\right]\right]$$

The equation $t = \frac{1}{x}$, $t = x$ indicate that t is the local parameter. The algorithm computes two generalized exponents at each point.

3.1.2 Logarithmic Solution

If we want to know whether the operator L has logarithmic solutions at a point $x = p$, we can use the command formal_sol. Let $\nu \in \mathbb{Z}$

```
>formal_sol(LB1,'has logarithm?',x=0);
```

$$true$$

```
>formal_sol(LB2,'has logarithm?',x=0);
```

$$true$$

formal_sol will make sure that enough terms of the Puiseux series are computed such that we know whether a logarithm appears in the formal solution.

3.1.3 Parameter of Transformations

The commands are called changeOfVars, expProduct and gauge, and take an operator L and the parameters, respectively, f or r or r_0, r_1. For example ($\nu \notin \frac{1}{2} + \mathbb{Z}$)

```
>L:=changeOfVars(LB2,sqrt(x));
```

$$\mathrm{L} = x^2\partial^2 + x\partial - \frac{1}{4}(x + \nu^2)$$

3.1.4 Polar Parts of f

First we compute the irregular singularities and their exponent differences with the function irregularSing. It takes the operator L, a variable t and a list of roots that generate the field of constants. The output is a list of elements of the form (p, t_p, d_p), where each element contains a irregular singularity p, the local parameter t_p, and the exponent difference $d_p = \Delta(\mathrm{L}, p)$.

```
>Sirr:=irregularSing(LB2,t,{});
```

$$\left[\left[\infty, \frac{1}{x}, \frac{2}{x}\right]\right]$$

The list of polar part of f at each irregular singularity of L can be compute with besselsubst.

```
>F:=besselsubst(Sirr,t,{});
```

$$\left[\frac{1}{x}\right]$$

3.1.5 Factorization of Linear Differential Operators

For reducible operator, we can use DFactor to favtor it. Let $\nu = \frac{1}{2}$

```
>DFactor(LB1);
```

$$\left[x^2\left(\partial - 1 + \frac{1}{2x}\right), \left(\partial + 1 + \frac{1}{2x}\right)\right]$$

```
>DFactor(LB2);
```

$$\left[x^2\left(\partial + \sqrt{-1} + \frac{1}{2x}\right), \left(\partial - \sqrt{-1} + \frac{1}{2x}\right)\right]$$

3.1.6 Equivalence of Linear Differential Operators

We use equiv command to find the parameters of exp-product and gauge transformation between two operator. The input is two operator and the output is r, G where $\mathrm{G} = r_0 + r_1\partial$. For example

```
>LB:=x^2*D^2+x*D-(x^2+nu^2):
>L:=gauge(subs(nu=0,LB),0,1):
>r,G:=equiv(L,subs(nu=0,LB));
```

$$-\frac{1}{x}, 1 + x\partial$$

3.2 Examples

3.2.1 $f \in \mathbb{C}(x)$

We consider the operator L that is obtained from L_B with $\nu = \frac{2}{3}$ and a change of variables $x \to f = (x-2)^2(x-3)^3(x-5)$:

```
>LB:=x^2*D^2+x*D-(x^2+nu^2):
>f:=(x-2)^2*(x-3)^3*(x-5):
>M:=changeOfVars(subs(nu=2/3,LB),f):
>L:=M:
```

$$\begin{aligned} \mathrm{L} := {} & (x^2-7x+11)(x-5)^2(x-2)^3(x-3)^3D^2 + (x^4-14x^3+72x^2-160x \\ & +131)(x-5)(x-3)^2(x-2)^2D - 4(x-2)(x-3)(9x^{12}-324x^{11} \\ & +5292x^{10}-51876x^9+340038x^8-1570644x^7+5243652x^6 \\ & -12752316x^5+22426713x^4-27820584x^3+23112216x^2-11547360x \\ & +2624404)(x^2-7x+11)^3. \end{aligned}$$

Resolution

The zeros of the leading coefficient of L are: $2, 3, 5, \frac{7-\sqrt{5}}{2}, \frac{7+\sqrt{5}}{2}, \infty$.

```
>gen_exp(L,t,x=2);
```

$$\left[\left[\frac{4}{3}, t = x-2\right], \left[-\frac{4}{3}, t = x-2\right]\right]$$

```
>gen_exp(L,t,x=3);
```

$$[[-2, 2, t = x-3]]$$

```
>gen_exp(L,t,x=5);
```

$$\left[\left[-\frac{2}{3}, t = x-5\right], \left[\frac{2}{3}, t = x-5\right]\right]$$

```
>gen_exp(L,t,x=RootOf(z^2-7*z+11));
```

$$\left[\left[0, 2, t = x - RootOf(Z^2-7Z+11)\right]\right]$$

```
>gen_exp(L,t,x=infinity);
```

$$\left[\left[\frac{6}{t^6}-\frac{90}{t^5}+\frac{528}{t^4}-\frac{1518}{t^3}+\frac{2142}{t^2}-\frac{1188}{t}+3, t=\frac{1}{x}\right], \left[-\frac{6}{t^6}+\frac{90}{t^5}-\frac{528}{t^4}+\frac{1518}{t^3}-\frac{2142}{t^2}+\frac{1188}{t}+3, t=\frac{1}{x}\right]\right]$$

```
>formal_sol(L,'has logarithm?',x=2);
```

$$false$$

```
>formal_sol(L,'has logarithm?',x=3);
```

$$false$$

```
>formal_sol(L,'has logarithm?',x=5);
```

$$false$$

```
>formal_sol(L,'has logarithm?',x=RootOf(z^2-7*z+11));
```

$$false$$

We have $S_{reg} = \{2, 5\}$ and $S_{irr} = \{\infty\}$.

```
>Sirr:=irregularSing(L,t,{});
```

$$\left[\left[\infty, \frac{1}{x}, -\frac{12}{t^6}+\frac{180}{t^5}-\frac{1056}{t^4}+\frac{3036}{t^3}-\frac{4284}{t^2}+\frac{2376}{t}\right]\right]$$

```
>F:=besselsubst(Sirr,t,{});
```

$$\left[x^6-18x^5+132x^4-506x^3+1071x^2-1188x\right]$$

```
>eq1:=x^6-18*x^5+132*x^4-506*x^3+1071*x^2-1188*x+a:
>eq2:=x^6-18*x^5+132*x^4-506*x^3+1071*x^2-1188*x+b:
>solve(sub(x=2,eq1),a);
```

$$540$$

```
>solve(sub(x=5,eq2),b);
```

$$540$$

```
>f1:=factor(x^6-18*x^5+132*x^4-506*x^3+1071*x^2-1188*x+540);
```

$$f_1 = (x-2)^2(x-3)^3(x-5)$$

```
>h:=normal(f1/((x-2)^2*(x-5)));
```

$$h := (x-3)^3.$$

We have: $\mathcal{N}_2 = \left\{\frac{2}{3}, \frac{11}{12}, \frac{7}{6}, \frac{17}{12}\right\}$ and $\mathcal{N}_5 = \left\{\frac{2}{3}, \frac{7}{6}\right\}$. The intersection is $\mathcal{N} = \left\{\frac{2}{3}, \frac{7}{6}\right\}$ which is the set of candidates for ν. In both cases the denominator of 2ν is 3. Since $h = (x-3)^3$ we have two pairs: $(\frac{2}{3}, f_1)$ and $(\frac{7}{6}, f_1)$.

```
>M :=changeOfVars(subs(nu=2/3,LB),f1);
```

$$\begin{aligned} \mathrm{M} := {} & (x^2-7x+11)(x-5)^2(x-2)^3(x-3)^3D^2 + (x^4-14x^3+72x^2-160x \\ & + 131)(x-5)(x-3)^2(x-2)^2D - 4(x-2)(x-3)(9x^{12}-324x^{11} \\ & + 5292x^{10} - 51876x^9 + 340038x^8 - 1570644x^7 + 5243652x^6 \\ & - 12752316x^5 + 22426713x^4 - 27820584x^3 + 23112216x^2 - 11547360x \\ & + 2624404)(x^2-7x+11)^3 \end{aligned}$$

```
>r,G:=equiv(M,L);
```

$$0, 1$$

Therefore,

$$\mathrm{V(L)} = \left\{ \alpha I_{\frac{2}{3}}\left((x-2)^2(x-3)^3(x-5)\right) + \beta K_{\frac{2}{3}}\left((x-2)^2(x-3)^3(x-5)\right) \middle| \alpha, \beta \in \mathbb{C} \right\}.$$

```
>M :=changeOfVars(subs(nu=7/6,LB),f1);
```

$$\begin{aligned} \mathrm{M} := {} & (x^2-7x+11)(x-5)^2(x-2)^3(x-3)^3D^2 + (x^4-14x^3+72x^2-160x \\ & + 131)(x-5)(x-3)^2(x-2)^2D - (x-2)(x-3)(36x^{12}-1296x^{11} \\ & + 21168x^{10} - 207504x^9 + 1360152x^8 - 6282576x^7 + 20974608x^6 \\ & - 51009264x^5 + 89706852x^4 - 111282336x^3 + 92448864x^2 - 46189440x \\ & + 10497649)(x^2-7x+11)^3 \end{aligned}$$

```
>r,G:=equiv(M,L);
```

$$0.$$

So for $\nu = \frac{7}{6}$ the operator M is not equivalent to L.

3.2.2 $f \notin \mathbb{C}(x)$

We consider the operator L that is obtained from L_B with $\nu = \frac{1}{3}$, a change of variables $x \to f = \frac{2}{3}\sqrt{(x^2-1)^3}$, exp-product $r = -\frac{2x^2-1}{x(x+1)(x-1)}$, and a gauge transformation $G = (2x+1)(x+1)(x-1)\partial + x(4x^4+2x^3-6x^2+3)$:

```
>M:=changeOfVars(x^2*D^2+x*D-(x^2+1/9),f):
>M1:=expProduct(M,-(2*x^2-1)/(x*(x-1)*(x+1))):
>L:=gauge(M1,x*(4*x^4+2*x^3-6*x^2+3),(1+2*x)*(x+1)*(x-1)):
```

$$\begin{aligned} \mathrm{L} := {} & x(x-1)(x+1)(32x^9+32x^8-50x^7-56x^6+16x^5+28x^4+2x^3+4x+1)D^2 \\ & + (24x^3-32x^{10}+104x^8+28x^4-84x^6-64x^{11}-3+192x^9+40x^5-8x+7x^2 \\ & - 168x^7)D - 2(64x^{15}-228x^{13}+84x^{12}+308x^{11}-182x^{10}-168x^9+194x^8+54x^7 \\ & - 190x^6+67x^5+87x^4-57x^3-23x^2+5x-3)(x+1). \end{aligned}$$

Resolution

The zeros of the leading coefficient of L are: $-1, 0, 1, \infty$ and the roots of $32x^9+32x^8-50x^7-56x^6+16x^5+28x^4+2x^3+4x+1$.

```
>gen_exp(L,t,x=-1);
```

$$[[-1, 0, t = x+1]]$$

```
>gen_exp(L,t,x=0);
```

$$[[-2, 0, t = x]]$$

```
>gen_exp(L,t,x=1);
```

$$[[-1, 0, t = x-1]]$$

```
>gen_exp(L,t,x=RootOf(1+4*x+28*x^4-56*x^6+2*x^3+32*x^9-50*x^7+16*x^5
                       +32*x^8));
```

$$\left[\left[0, 2, t = x - \mathit{RootOf}(1+4Z+28Z^4-56Z^6+2Z^3+32Z^9-50Z^7+16Z^5+32Z^8)\right]\right]$$

```
>gen_exp(L,t,x=infinity);
```

$$\left[\left[-\frac{2}{t^3}+\frac{1}{t}-\frac{3}{2}, t = \frac{1}{x}\right], \left[\frac{2}{t^3}-\frac{1}{t}+\frac{3}{2}, t = \frac{1}{x}\right]\right]$$

```
>formal_sol(L,‘has logarithm?‘,x=-1);
```

$$false$$

```
>formal_sol(L,‘has logarithm?‘,x=0);
```

$$false$$

```
>formal_sol(L,‘has logarithm?‘,x=1);
```

$$false$$

```
>formal_sol(L,‘has logarithm?‘,x=RootOf(1+4*x+28*x^4-56*x^6+2*x^3+32*x^9
                                         -50*x^7+16*x^5+32*x^8));
```

$$false$$

We have $S_{reg} = \{\emptyset\}$ and $S_{irr} = \{\infty\}$. Let $g = f^2$. Since $g \in \mathbb{C}(x)$, we assume $g = \frac{A}{B}$ where $A, B \in \mathbb{C}[x]$, B is monic and $gcd(A, B) = 1$. The truncated series of f at $x = \infty$ is $\frac{2}{3}x^3 - x$, So the truncated series of g at $x = \infty$ is $\frac{4}{9}x^6 - \frac{4}{3}x^4$, $d_A = 6$ and $B = 1$.
It is the rational case with $A = CA_1A_2^d$. d can only be the factor of d_A, so $d \in \{3, 6\}$. Since we do not know zeroes for A, let $A_1 = 1$. We can write $A = CA_2^d$.

1. If $d = 3$ then $A = CA_2^3$, $A_2 = a_0 + a_1x + a_2x^2$. Since $B = 1$, then the truncated series of gB is the same as g.
   The coefficient of the term with highest degree is $\frac{4}{9}$. So we can let $C = \frac{4}{9}$. Since ∞ is the only singularity, so C is the leading coefficient of A.
   The truncated series of $A_2^3 = \frac{gB}{C}$ is $x^6 + 3x^4 + O(x^3) = x^6(1 - 3x^{-2} + O(x^{-3}))$. Since the only $3rd$ root of 1 in $\mathbb{C}$ is 1, then the only $3rd$ root of $1 - 3x^{-2} + O(x^{-3})$ is $1 - x^{-2} + O(x^{-3})$. So by comparing coefficients of $x^2(1 - x^{-2} + O(x^{-3}))$ and $A_2 = a_0 + a_1x + a_2x^2$, we can get $A_2 = x^2 - 1$ and then $g = \frac{4}{9}(x^2 - 1)^3$. Let $a \in \mathbb{N}$, then
   $\left\{\frac{a}{3}\middle|\, gcd(a, 3) = 1, 1 \le \frac{a}{3} < \frac{1}{2}\right\} = \left\{\frac{a}{3}\middle|\, gcd(a, 3) = 1, 1 \le a < \frac{3}{2}\right\} = \left\{\frac{1}{3}\right\}$ is the set of candidates for ν.

2. We can do this process for $d = 6$; in this case, there are no solutions.

So we have $\left(\frac{2}{3}\sqrt{(x^2-1)^3}, \frac{1}{3}\right)$ as the only possible candidate for (f, ν).

```
>M:=changeOfVars(x^2*D^2+x*D-(x^2+1/9),f);
```

M $:= x(x-1)^2(x+1)^2D^2 + (x-1)(x+1)(x^2+1)D - (4x^6 - 12x^4 + 12x^2 - 3)x^3$

```
>r,G:=equiv(M,L);
```

$$-\frac{2x^2-1}{x(x+1)(x-1)}, (2x+1)(x+1)(x-1)D + x(4x^4+2x^3-6x^2+3)$$

Therefore, we have the solutions of the form:

$$\alpha \exp\left(\int r dx\right) I_{\frac{1}{3}}(f) + \beta \exp\left(\int r dx\right) K_{\frac{1}{3}}(f), \quad \alpha, \beta \in \mathbb{C}.$$

Conclusion

In this work, we have studied the properties of Bessel functions and show how some second-order differential equations can be solved by means of the Bessel functions. This has been done basically by searching for appropriate transformations, namely the change of variables, the exp-product and the gauge transformation which allow the transformation of the Bessel operators into some specific second-order operators.

the next steps of this work would be:

1)- First to relax conditions imposed when searching the transformation parameters and see if the operator obtained is defined over $\mathbb{C}(x)$ (in this case, our algorithm is complete).

2)- Apply this formalism to solve difference equations with the Bessel functions replaced by the Meixner, the Charlier functions.

Appendix

A.1 Transformations

When we apply to an operator $\mathrm{L} \in \mathbb{C}(x)[\partial]$ of order two the parameter of the transformations that we use, we can always find $\widetilde{\mathrm{L}} \in \mathbb{C}(x)[\partial]$ with $deg(\widetilde{\mathrm{L}}) = 2$ that satisfies the conditions. From this fact, we can derive algorithms that apply a changes of variables, an exp-product or a gauge transformation to a differential operator.

Algorithm 12: **ChangeOfVars**

Input: operator $\mathrm{L} \in \mathbb{C}(x)[\partial]$ of degree two and rational function f.

Output: operator $\widetilde{\mathrm{L}} \in \mathbb{C}(x)[\partial]$ of degree two such that $y(f) \in \mathrm{V}(\widetilde{\mathrm{L}})$ for every $y \in \mathrm{V}(\mathrm{L})$.

$l :=$ **lcoeff**(L, ∂)

$a_0, a_1 :=$ **coeffs**$(\mathrm{L}, \partial)/l$

$$b_0 := \left.\frac{a_0}{a_2}\right|_{x=f} (f')^2$$

$$b_1 := \frac{1}{f'}\left(\left.\frac{a_0}{a_2}\right|_{x=f} (f')^2 - f''\right)$$

return(**collect**(**numer**$(\partial^2 + b_1\partial + b_0), \partial$)).

Algorithm 13: **expProduct**

Input: operator $\mathrm{L} \in \mathbb{C}(x)[\partial]$ of degree two and rational function r.

Output: operator $\widetilde{\mathrm{L}} \in \mathbb{C}(x)[\partial]$ of degree two such that $\exp\left(\int r dx\right) y \in \mathrm{V}(\widetilde{\mathrm{L}})$ for every $y \in \mathrm{V}(\mathrm{L})$.

$l :=$ **lcoeff**(L, ∂)

$a_0, a_1 :=$ **coeffs**$(\mathrm{L}, \partial)/l$

$b_1 := -2r + \frac{a_1}{a_2}$

$b_0 := r^2 - r' - \frac{a_1}{a_2} r + \frac{a_0}{a_2}$

return(**collect**(**numer**$(\partial^2 + b_1\partial + b_0), \partial$)).

Algorithm 14: **gauge**

Input: operator $\mathrm{L} \in \mathbb{C}(x)[\partial]$ of degree two and two rational functions r_0, r_1.

Output: operator $\widetilde{\mathrm{L}} \in \mathbb{C}(x)[\partial]$ of degree two such that $r_0 y + r_1 y' \in \mathrm{V}(\widetilde{\mathrm{L}})$ for every $y \in \mathrm{V(L)}$.

$l :=$ **lcoeff**(L, ∂)

$a_0, a_1 :=$ **coeffs**$(\mathrm{L}, \partial)/l$

$c_0 := \left(r_0' - \frac{a_0}{a_2} r_1\right)' - \frac{a_0}{a_2}\left(r_0 + r_1' - \frac{a_1}{a_2} r_1\right)$

$c_1 := r_0' - \frac{a_0}{a_2} r_1 + \left(r_0 + r_1' - \frac{a_1}{a_2} r_1\right)' - \frac{a_1}{a_2}\left(r_0 + r_1' - \frac{a_1}{a_2} r_1\right)$

$c_2 := -\frac{r_1}{r_0}\left(r_0' - \frac{a_0}{a_2} r_1\right)' + \frac{r_1}{r_0}\frac{a_0}{a_2}\left(r_0 + r_1' - \frac{a_1}{a_2} r_1\right)$

$c_3 := \frac{a_1}{a_2} r_1 + \frac{r_1}{r_0}\left(r_0' - \frac{a_0}{a_2} r_1\right) - r_0 - r_1'$

$b_0 := -\dfrac{1}{r_0}\left[c_0 + \left(r_0' - \dfrac{a_0}{a_2} r_1\right)\dfrac{c_1 + c_2}{c_3}\right]$

$b_1 := \frac{c_1 + c_2}{c_3}$

return(**collect**(**numer**$(\partial^2 + b_1\partial + b_0)$,$\partial$))

A.2 IsPower

In the integer case of the algorithm 11, we had to determine whether a monic polynomial is a p-th power of another polynomial.

Algorithm 15: **ispower**

KwIna monic polynomial $f \in \mathrm{K}[x]$ and $p \in \mathbb{N}$. KwOut $g \in \mathrm{K}[x]$ with the following property: if $y^p = f$ exists, then g is a solution. if $p = 1$ then return f

$d := degree(f, x)$

$n := d/p$

if $n \notin \mathbb{Z}$ then return FAIL

$A := x^n + \sum_{k=0}^{n} a_k x^k$

for $k = 1...n$

$a_{n-k} := solve(coeff(A^p, x, d-k) - coeff(f, x, d-k), a_{n-k})$

Return: A

A.3 Program Description

We will give an overview over the important functions implemented in the programme. We will indicate for some the corresponded Algorithm in the thesis.

besselequiv

Input: An operator $\mathrm{L} \in \mathrm{K}[\partial]$, a rational function $f \in \mathrm{K}$, and a constant $\nu \in \mathbb{C}$.

Output: A sequence $\mathrm{M} \in \mathrm{K}[\partial]$, $[y_1, y_2]$ such that y_1 and y_2 are the (modified) Bessel function of the first and second kind and $\mathrm{M}(y)$ is a solution of L. If such a solution does not exist 0 is returned.

BesselSqrtequiv
Input: An operator $\mathrm{L} \in \mathrm{K}[\partial]$, a rational function $f \in \mathrm{K}$, and a constant $\nu \in \mathbb{C}$.
Output: A sequence $\mathrm{M} \in \mathrm{K}[\partial]$, $[y_1, y_2]$ such that y_1 and y_2 are the (modified) Bessel function of the first and second kind with change of variable $x \mapsto \sqrt{f}$ and $\mathrm{M}(y)$ is a solution of L. If such a solution does not exist 0 is returned.

besselsubst
(implementation of Algorithm 1)
Input: S_{irr} and their exponent differences, local parameter t.
Output: A list $[f_1, ..., f_n]$ that corresponds to possibilities $\sum_{k=1}^{n} \pm f_k$.

changeconstant
Input: A rational function $f \in \mathrm{K} = \mathbb{C}(x)$, a point p.
Output: A rational function $g = g(x) \in \mathrm{K}$ such that $g = f + c$ for some $c \in \mathbb{C}$ and $g(p) = 0$. If $p = \infty, g = f$ is returned.

compare
Input: Two constants $a, b \in \mathbb{C}$.
Output: Two rational number $r, s \in \mathbb{Q}$ such that $a = rb + s$.

dsolve_bessel
Input:(i) A differential operator $\mathrm{L} \in \mathrm{K}[\partial]$ and optionally the domain
(ii)A differential equation and the dependent variable.
Output: The solution space if it can be expressed by Bessel funstions.

equiv
Input: Two operators $\mathrm{L}_1, \mathrm{L}_2 \in \mathrm{K}[\partial]$ of degree two.
Output: An operator M such that $\mathrm{M}y \in \mathrm{V}(\mathrm{L}_2)$ for every $y \in \mathrm{V}(\mathrm{L}_1)$. If a solution $\mathrm{M} \neq 0$ was found, a sequence $r \in \mathrm{K}$, $\mathrm{G} \in \mathrm{K}[\partial]$ which satisfies $\mathrm{M} = \exp\left(\int r dx\right) \mathrm{G}$ would be returned.

findbesselvf
Input: An integer that indicates the case we are in, S_{irr}, S_{reg}, and the variable t for the local parameter.
Output: A list of pairs (ν, f).

findbesselvfln \findbesselvfint \findbesselvfrat \findbesselvfK
Those are the implementations of different cases of the Bessel square-root and non-square-root case.
Input: S_{reg} and S_{irr}.
Output: A list of pairs (ν, f).

ispower
(implementation of Algorithm 15)
Input: A monic polynomial $f \in \mathbb{C}[x]$ and $p \in \mathbb{N}$.
Output: $g \in \mathbb{C}[x]$ with the following property: if a solution for $y^p = f$ exists, then g is a solution.

SimplifyAnswer
Input: $h \in \mathbb{C}(x)$, L and a list of functions F
Output: A list of function obtained by applying the operator $\exp\left(\int hdx\right)$ L to the functions in F.

singgenexp
Input: $L \in \mathbb{C}(x)[\partial]$, a variable t, and an optional parameter to pass some information about singularities.
Output: A list of elements of the form $[p, t, D, q, n]$ such that: p is a singularity of L, q is a polynomial over $\mathbb{C}$ with zero p, $n = deg(p)$, and D is the exponent difference $d = \Delta(\mathrm{L}, p)$.

singInfo
Collect information of Singularities.
Input: Differential Operator L.
Output:S_{reg}, S_{irr} with exponent differences, and determine if it is logarithmic, rational or base field.

singSeries
Input: S_{reg} and S_{irr} with exponent differences.
Output: S_{reg} and S_{irr} with truncated power series, denominator of possible change of variable B, the boundary of degree of numerator d_A and a boolean indicate if it is easy case.

sqrtEasy
This is an implementation of Algorithm 8, find solutions for easy case.
Input: S_{reg}, S_{irr} with truncated series, B, d_A.
Output: List of pairs (f, ν).

findnueasyIrrat
given f and the condition $\nu \notin \mathbb{Q}$, find all possible ν for easy case.
Input: f, S_{reg}.
Output: List of pairs (f, ν).

findnueasyrat
given f and the condition $\nu \in \mathbb{Q}$, find all possible ν for easy case.
Input: f, S_{reg}.
Output: List of pairs (f, ν).

findnuLog
given f and we have logarithmic solutions, find all possible ν for easy case and logarithmic

case.
Input: f, S_{reg}.
Output: List of pairs (f, ν).

sqrtLog
This is an implementation of Algorithm 9, find solutions for logarithmic case.
Input: S_{irr}, S_{reg} with truncated series, B, d_A.
Output: List of pairs (f, ν).

searchKnlog
For logarithm case, try all possible multiplicities for zeroes.
Input: S_{reg}, d_A
Output: List of f.

sqrtIrrat
This is an implementation of Algorithm 10, find solutions for irrational case.
Input: S_{reg}, S_{irr} with truncated series, B, d_A.
Output: List of pairs (f, ν).

sqrtRat
This is an implementation of Algorithm 11, find solutions for rational case.
Input: S_{reg}, S_{irr} with truncated series, B, d_A
Output: List of pairs (f, ν).

findnuRat
Given f, find ν for rational case.
Input: f, up to d disappearing Singularities, S_{reg}, B.
Output: List of pairs (f, ν).

findSqrtf
find possible f for rational case.
Input: S_{irr}, B, up to d disappearing singularities, possible list of A_1.
Output: List of f.

testzeros
Input: $f \in \mathbb{C}(x)$ and a set of points.
Output: True if all points are zeros of f and false otherwise.

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