Chapter 3

Arbitrariness

We will study in some detail how to define "general solutions" for an involutive equation. For certain closed form representations we will compute possible values for the number of arbitrary functions contained in them. This yields a criterion whether a given solution is a general one.

3.1 General Solution

Intuitively it seems to be obvious what the general solution of a differential equation is: an expression containing some arbitrary elements such that any solution of the equation can be obtained through specialization of these. This is, however, usually not possible, as most differential equations admit so-called "singular solutions" which do not fit into such a general expression. In the last century many studies were devoted to the relationship between "general solutions", "complete solutions", "particular solutions" etc. However, no real definition emerged. Later interest in this question ceased.

The geometric theory provides a framework which allows for an easy definition following an idea of Ampère [2, 159]. He proposed that the only differential equations satisfied by a general solution should be those of the original system and those derivable of them (i.e. algebraic combinations and prolongations). In other words, if we eliminate the arbitrary elements of a general solution, we obtain just the original equations or some combinations of them. We have already employed this idea in Definition 2.20 of a general formal power series solution. Following Hermann [72] we can express it more precisely.

Definition 3.1 *A* general solution *of the differential equation $\mathcal{R}_q$ is a subset $\Theta \subset \Gamma_{loc}(\mathcal{E})$ such that*

(i) $\forall r \geq 0 \;\; j_{q+r}(\Theta) \subseteq \mathcal{R}_{q+r}$.

(ii) There does not exist for any $r \geq 0$ a submanifold $\bar{\mathcal{R}}_{q+r} \subset \mathcal{R}_{q+r}$ of lower dimension such that $j_{q+r}(\Theta) \subseteq \bar{\mathcal{R}}_{q+r}$.

Θ is called a local general solution at $x_0 \in X$, *if all sections $\sigma \in \Theta$ are defined in a neighborhood of x_0.*

Now the question naturally arises how one can describe such subsets $\Theta \subset \Gamma_{loc}(\mathcal{E})$. In Section 2.5 we dealt with power series solutions. In applications one usually wants *closed form solutions*. This is another term notoriously difficult to define. We will use here an idea sketched by Hermann [72].

Definition 3.2 *Let Y_k be manifolds and V_k vector spaces for $k = 1, \dots, p$ and let W be a further manifold. All manifolds and vector spaces are assumed to be finite dimensional. A* (local) closed form representation *at $x_0 \in X$ of a general solution is a smooth map*

$$s : \mathcal{C}^\infty(Y_1, V_1) \times \cdots \times \mathcal{C}^\infty(Y_p, V_p) \times W \mapsto \Gamma_{loc}(\mathcal{E}) \tag{3.1}$$

such that its image provides a (local) general solution at x_0.

This definition is still somewhat vague, as it does not contain any further restriction on the map s. It should of course be in some sense in closed form, too. One could for instance require that in some coordinate system s can be expressed in terms of some "allowed functions". These could be elementary functions or additionally some special functions. Precise definitions of such terms are currently only known within the framework of differential algebra [85, 86] where one builds by purely algebraic means extensions of fields of rational functions. We do not pursue this problem here any further.

$\dim W$ corresponds to the number of arbitrary constants. Each factor $\mathcal{C}^\infty(Y_k, V_k)$ represents $\dim V_k$ arbitrary functions depending on $\dim Y_k$ arguments. The Taylor coefficients of these functions plus the constants correspond to the coordinates of the manifolds W_q in Definition 2.20. There are now two basic ingredients necessary to build a closed form representation in a neighborhood $U(x_0)$ of x_0 : First we need maps $\omega_k : U(x_0) \mapsto Y_k$ to define the arguments of the arbitrary functions; $f \circ \omega_k$ transforms a function f over Y_k into one over $U(x_0)$. Then as a second step we must say how to combine these functions.

Example 3.3 As a simple example we consider again the Wave Equation $u_{xx} - u_{tt} = 0$. To obtain a closed form representation in the sense of Definition 3.2, we take $p = 1$, $Y_1 = \mathbb{R}$ and $V_1 = \mathbb{R}^2$ (there is no need for a W). Then we set

$$s_1 : \begin{pmatrix} f \\ g \end{pmatrix} \mapsto f(x-t) + g(x+t)\,. \tag{3.2}$$

This is the well-known formula for the general solution as superposition of left and right moving waves. Alternatively, we could take the D'Alembert formula

$$s_2 : \begin{pmatrix} f \\ g \end{pmatrix} \mapsto \frac{1}{2}\big[f(x-t) + f(x+t)\big] + \frac{1}{2}\int_{x-t}^{x+t} g(\tau)d\tau\,. \tag{3.3}$$

Restricting our attention to analytic solutions we could say that a Taylor expansion of s at x_0 yields a general power series solution. This is equivalent, as condition (2.37) can be directly translated into the two parts of Definition 3.1. The requirement that every function in $I_\infty(x_0)$ must vanish after substituting the power series corresponds to Part (i) and the converse yields Part (ii), as an additional function vanishing on $\mathcal{R}_\infty$ could be used to define for one r a differential equation $\bar{\mathcal{R}}_{q+r}$ of lower dimension than $\mathcal{R}_{q+r}$ which is also satisfied by s.

3.2 Cartan Characters and Hilbert Polynomial

In Section 2.2 the symbol helped to decide whether or not a differential equation is in involution. The symbol of an involutive equation contains, however, much more information on the solution space. The Cartan characters and the Hilbert polynomial will be our basic tools to extract it.

Definition 3.4 *The* Cartan characters $\alpha_q^{(k)}$ *of the involutive differential equation* $\mathcal{R}_q$ *in* n *independent and* m *dependent variables are*

$$\alpha_q^{(k)} = m\binom{q+n-k-1}{q-1} - \beta_q^{(k)} \quad k = 1, \dots, n\,. \tag{3.4}$$

$\beta_q^{(k)}$ gives the number of principal derivatives of class k and order q, i.e. derivatives which can be computed using $\mathcal{R}_q$ by solving for them; $\alpha_q^{(k)}$ is the number of corresponding parametric derivatives, i.e. derivatives which must be provided by initial or boundary conditions to specify a unique solution.

Proposition 3.5 *The Cartan characters of an involutive differential equation* $\mathcal{R}_q$ *satisfy*

$$\alpha_q^{(1)} \geq \alpha_q^{(2)} \geq \cdots \geq \alpha_q^{(n)} \geq 0\,. \tag{3.5}$$

PROOF. For $q = 1$ this is obvious from Corollary 2.13, as $\alpha_1^{(k)} = m - \beta_1^{(k)}$. For a higher order equation we exploit the fact (shown in Appendix C.2), that it can be transformed into an equivalent first-order equation with exactly the same Cartan characters. •

For an involutive symbol it is possible to compute the Cartan characters of its prolongations without explicitly determining the equations describing these symbols. They are given by a simple recursion relation.

Lemma 3.6 *Let* $\mathcal{M}_q$ *be an involutive symbol with characters* $\alpha_q^{(k)}$ *and* $\beta_q^{(k)}$. *Then the characters of its prolongation* $\mathcal{M}_{q+r}$ *are given by*

$$\alpha_{q+r}^{(k)} = \sum_{i=k}^{n} \binom{r+i-k-1}{r-1} \alpha_q^{(i)}\,, \tag{3.6}$$

$$\beta_{q+r}^{(k)} = \sum_{i=k}^{n} \binom{r+i-k-1}{r-1} \beta_q^{(i)}\,. \tag{3.7}$$

PROOF. It suffices to prove (3.7), as (3.6) is then an immediate consequence of (3.4). We use an induction over r. Prolonging an equation of class k or higher with respect to x^k yields an equation of class k. By Proposition 2.9 these are the only independent equations of order $q+1$ in the prolongation, as $\mathcal{M}_q$ is involutive. Thus $\beta_{q+1}^{(k)} = \beta_q^{(k)} + \cdots + \beta_q^{(n)}$ and we have proven (3.7) for $r = 1$.

Let us now assume that (3.7) holds for $r-1$. Since according to Theorem 2.14 the prolongation of an involutive symbol is again involutive, we can apply the formula for $r = 1$ to get $\beta_{q+r}^{(k)} = \beta_{q+r-1}^{(k)} + \cdots + \beta_{q+r-1}^{(n)}$. Iterating yields

$$\beta_{q+r}^{(k)} = \sum_{i=k}^{n}\sum_{l=k}^{i} \binom{r+l-k-2}{r-2} \beta_q^{(i)} = \sum_{i=k}^{n} \binom{r+i-k-1}{r-1} \beta_q^{(i)} \tag{3.8}$$

where the last equality follows from a well-known identity for binomial coefficients. •

Lemma 3.6 allows us to compute easily the rank and the dimension of a prolonged symbol $\mathcal{M}_{q+r}$. By definition we have $\operatorname{rank} \mathcal{M}_{q+r} = \sum_{k=1}^{n} \beta_{q+r}^{(k)}$ and $\dim \mathcal{M}_{q+r} = \sum_{k=1}^{n} \alpha_{q+r}^{(k)}$. Substituting (3.6) and (3.7) yields

$$\operatorname{rank} \mathcal{M}_{q+r} = \sum_{k=1}^{n} \binom{r+k-1}{r} \beta_q^{(k)}\,, \tag{3.9}$$

$$\dim \mathcal{M}_{q+r} = \sum_{k=1}^{n} \binom{r+k-1}{r} \alpha_q^{(k)}\,. \tag{3.10}$$

Theorem 3.7 *The Hilbert polynomial of an involutive equation $\mathcal{R}_q$ with Cartan characters $\alpha_q^{(k)}$ is given by*

$$H_q(r) = \sum_{i=0}^{n-1} \left(\sum_{k=i}^{n-1} \frac{\alpha_q^{(k+1)}}{k!} s_{k-i}^{(k)}(0) \right) r^i . \tag{3.11}$$

Conversely, if the Hilbert polynomial of $\mathcal{R}_q$ is given by $H_q(r) = \sum h_i r^i$, then its Cartan characters satisfy the recursion relation

$$\alpha_q^{(k)} = (k-1)!\, h_{k-1} - \sum_{i=k+1}^{n} \frac{(k-1)!}{(i-1)!} \alpha_q^{(k)} s_{i-k}^{(i-1)}(0) , \quad k = 1, \dots, n . \tag{3.12}$$

Proof. According to Definition 2.22, $H_q(r) = \dim \mathcal{M}_{q+r}$. In order to express it in terms of the Cartan characters we must write the binomial coefficients explicitly as polynomials in r. This is most easily done using the modified Stirling numbers introduced in Appendix C.1. (C.2) yields then (3.11). There is obviously a one-to-one correspondence between the coefficients of the Hilbert polynomial and the Cartan characters. A simple inversion of (3.11) yields (3.12). •

An equation $\mathcal{R}_q$ with vanishing Cartan characters is said to be of *finite type*, as there are no free Taylor coefficients of order q or higher. In the older literature (e.g. [159]) the name *Mayer system* was often used for such equations. They have a finite dimensional solution space parameterized by $\dim \mathcal{R}_q$ parameters. System (2.19) was an example for a finite type system. We will assume in what follows that at least one character is non-zero, as we will show in Section 7.1 that the integration of any system of finite type can be reduced to solving systems of ordinary differential equations.

3.3 Algebraic Representations

We take now a closer look at those closed form representations of the general solution where the map s in Definition 3.2 involves neither differentiations nor integrations. Such representations are called *algebraic*. Representation s_1 of the general solution of the Wave Equation is a prototypical example. Representation s_2 is, however, not algebraic, as it involves an integration.

The image of an algebraic map s can be expressed as sections of the form

$$u^\alpha = \Psi^\alpha\Big(x^i, \dots, F(\phi_1(x), \dots, \phi_k(x)), \dots\Big) \tag{3.13}$$

where the functions ϕ_i represent the arguments of the arbitrary functions F. Thus they are fixed like Ψ. F stands here generically for all arbitrary functions involved. We denote by f_k the number of arbitrary functions depending on k arguments. Note that they do not need to have the same arguments.

Example 3.8 As a second, simple example we consider with $m = 1$ and $n = 2$ the non-linear equation $uu_{xy} - u_x u_y = 0$. $u = \phi_1(x)\phi_2(y)$ represents a general solution. Thus we have an algebraic representation with two functions of one argument: $\Psi(a, b) = ab$ and $F_1 = \{\phi_1, \phi_2\}, F_0 = F_2 = \{\}$.

Our goal will be to determine the possible values for these numbers f_k. If we introduce, following [103], as short hand

$$\begin{bmatrix} k \\ \ell \end{bmatrix} = \binom{k+\ell-1}{\ell} , \tag{3.14}$$

then an arbitrary function F with k arguments yields $\left[{k \atop q+r}\right]$ arbitrary Taylor coefficients at order $q+r$ (cf. Eq. (2.9)). Expression (3.13) contains thus a total of

$$T_q(r) = \sum_{k=1}^{n} f_k \begin{bmatrix} k \\ q+r \end{bmatrix} \tag{3.15}$$

arbitrary Taylor coefficients of order $q+r$. If we assume that the expansion of (3.13) leads to a power series which can be constructed order by order, then $T_q(r)$ must equal the Hilbert polynomial $H_q(r)$. Expanding both as polynomials in r and equating the coefficients leads to a triangular system of linear equations for the f_k which can be easily solved by back-substitution. Thus we have proved

Theorem 3.9 *If there exists an algebraic representation of the general solution of the involutive differential equation $\mathcal{R}_q$ with Cartan characters $\alpha_q^{(k)}$ whose power series expansion can be constructed order by order, then it contains f_k free functions of k variables, where the f_k are determined by the recursion relation:*

$$\begin{aligned} f_n &= \alpha_q^{(n)}\,, \\ f_k &= \alpha_q^{(k)} + \sum_{i=k+1}^{n} \frac{(k-1)!}{(i-1)!} \left(\alpha_q^{(i)} s_{i-k}^{(i-1)}(0) - f_i s_{i-k}^{(i-1)}(q) \right), \quad 0 < k < n\,. \end{aligned} \tag{3.16}$$

Such a representation can exist only, if the solution of this recursion relation contains only non-negative integers[1].

With the results of Appendix C.1 we can solve the recursion relation (3.16) explicitly for first-order equations and even prove that the solution is non-negative. Like all results in this section, this is a purely formal statement which does not imply anything about the existence of a solution, because we consider only formal power series solutions. Some existence theorems will later be given in the context of initial value problems (cf. Chapter 4).

Corollary 3.10 *For a first-order equation $\mathcal{R}_1$ the solution of (3.16) is given by*

$$\begin{aligned} f_n &= \alpha_1^{(n)} = m - \beta_1^{(n)}\,, \\ f_k &= \alpha_1^{(k)} - \alpha_1^{(k+1)} = \beta_1^{(k+1)} - \beta_1^{(k)}\,. \end{aligned} \tag{3.17}$$

It contains only non-negative integers.

Proof. The proof is by descending induction. The case $k = n-1$ is obvious from (C.7). If we assume that (3.17) holds for $k+1$, we get for k that

$$\begin{aligned} f_k \;=\; & \alpha_1^{(k)} - \alpha_1^{(k+1)} + \\ & (k-1)! \sum_{i=k+2}^{n-1} i!\, \alpha_1^{(i+1)} \left(i\, s_{i-k}^{(i-1)}(1) + s_{i-k+1}^{(i)}(0) - s_{i-k+1}^{(i)}(1) \right). \end{aligned}$$

But Lemma C.3 implies that the expression in the bracket vanishes for all i, k. It follows from Proposition 3.5 that the f_k are always non-negative. •

[1] Using a computer algebra system one can show that even for fairly high values of n (3.16) always yields integer solutions. Thus one is tempted to conjecture that this holds for all values of n. One can, however, also show that this is highly non-trivial, as $s_{i-k}^{(i-1)}(q)$ is in general not divisible by the denominator of (3.16). Thus a proof of this conjecture would probably need a closed-form solution of this recursion relation.

One might think that every equation should have such an algebraic representation of its general solution. But we will see that it is actually a rare phenomenon for higher-order equations and in most of the examples we will treat no such representation will exist. One could consider algebraic representations as a kind of generalization of the so-called *non-linear superposition* found in some ordinary differential equations. These are equations were the general solution can be built from some *fundamental solutions* ρ_k, i.e. it can be written in the form

$$u^\alpha = \Psi^\alpha(\rho_1, \dots, \rho_q, c_1, \dots, c_p) \tag{3.18}$$

with arbitrary constants c_k.

In the case of partial differential equations these constants must be replaced by arbitrary functions. Thus we obtain an expression like (3.13). The explicit dependency of the independent variables is now given by the fundamental solutions. Lie [98] proved that in the case of first-order systems of ordinary differential equations such a non-linear superposition rule exists, if and only if the system is of the special form

$$\frac{\partial u^\alpha}{\partial x} = \sum_{k=1}^{r} X_k(x)\xi_k^\alpha(u^\beta) \tag{3.19}$$

where the X_k are arbitrary functions of x and the vector fields

$$V_k = \sum_\alpha \xi_k^\alpha \frac{\partial}{\partial u^\alpha}\,, \quad k = 1, \dots, r\,, \tag{3.20}$$

span an r-dimensional Lie algebra. Thus it is not surprising that in the case of partial differential equations we will rarely find algebraic representations.

Example 3.11 Consider a normal equation of order $q > 1$. We know from Corollary 2.28 that for such an equation $\beta_q^{(n)} = m$ and all other $\beta_q^{(k)}$ vanish. Thus the Cartan characters of a normal equation are given by

$$\alpha_N^{(n)} = 0\,, \quad \alpha_N^{(k)} = m\binom{q+k-1}{q-1}\,. \tag{3.21}$$

(3.16) yields with these values $f_n = 0$, $f_{n-1} = mq$ and $f_{n-2} = mq(1-q)/2$. Thus for $q > 1$ and $n > 2$ we always get negative values in the solution! $n = 2$ is a special case, because then functions of at most one argument can occur. But for such functions integration or differentiation does not change the number of Taylor coefficients at a given order. Thus from the point of view of our formal analysis such functions can always be considered as algebraic.

We will study in the next section the effect of more general representations than purely algebraic ones. We will, however, always stick to the assumption that a series expansion leads to a power series which can be constructed order by order. This is basically a condition on the functions Ψ^α in (3.13) which is especially then satisfied, if these functions do not depend explicitly on the independent variables x^i as e.g. in the case of the solution (3.2) of the Wave Equation.

3.4 Representations with Integrals

Even if Theorem 3.9 yields for a given equation only non-negative values for the f_k, it is interesting to study the effect of more general expressions than (3.13), as many solution methods for differential equations lead for example to integral representations. Then further terms appear in the definition of $T_q(r)$ [2]

$$T_q(r) = \sum_{j\in J}\sum_{k=1}^{n} f_{k,j}\begin{bmatrix} k \\ q+r+j \end{bmatrix}. \tag{3.22}$$

J is some finite set of integers containing the considered *differentiation orders.* $j = 0$ represents the algebraic case. Positive values of j correspond to derivatives; negative values denote integrals.

The shifts in the lower entry of the bracket symbol reflect the simple fact that if we integrate a power series the coefficients of order q become coefficients of order $q+1$. Differentiation is slightly more complicated, as it affects only power products which contain the variable with respect to which one differentiates. To avoid tedious case distinctions we assume that if an arbitrary function is differentiated then all its derivatives of the considered order appear. Differentiation will mainly occur in the context of gauge systems (cf. Section 6.1), where this restriction is almost always satisfied.

The equality $H_q(r) = T_q(r)$ leads now to the following linear system of n diophantine equations

$$\sum_{j\in J}\sum_{k=i}^{n-1} \frac{n!}{k!} s^{(k)}_{k-i}(q+j) f_{k+1,j} = n!\, h_i\,, \quad i = 0,\dots,n-1 \tag{3.23}$$

where h_i denotes the coefficients of the Hilbert polynomial. We have multiplied each equation by $n!$ to avoid fractions. All coefficients including the right hand side are non-negative integers.

If $|J| > 1$, the system is under-determined, i.e. the number of arbitrary functions is no longer uniquely defined. Even if we disregard how a function enters the general solution and consider only the total number $\tilde f_k = \sum_{j\in J} f_{k,j}$ of functions with k variables, we get in general no unique results, as we will show later in this section. There exists, however, one invariant:

Theorem 3.12 *Let $\alpha_q^{(k_0)}$ be the highest non-vanishing Cartan character of the involutive equation $\mathcal{R}_q$. Then every non-negative solution $\{f_{k,j}\}$ of the diophantine system (3.23) satisfies*

$$f_{k,j} = 0\,, \quad \forall j \in J, k > k_0\,, \tag{3.24}$$

$$\tilde f_{k_0} = \sum_{j\in J} f_{k_0,j} = \alpha_q^{(k_0)}\,. \tag{3.25}$$

PROOF. The proof is simple, as under the assumption of the theorem $H_q(r)$ is a polynomial of degree $k_0 - 1$. Setting $h_i = 0$ for $i = k_0,\dots,n-1$ leads consecutively to the equations $\tilde f_n = 0,\dots,\tilde f_{k_0+1} = 0$. Since only non-negative solutions are allowed, this entails (3.24). The next order provides (3.25). •

Note that it is essential here that we maintain the assumption that the power series expansion can be constructed order by order. Otherwise the ansatz $H_q(r) = T_q(r)$ is no longer correct and we cannot draw any conclusions. We call $\alpha_q^{(k_0)}$ the *degré d'arbitraire* (or *index of generality*) of the differential equation $\mathcal{R}_q$. k_0 is called *Cartan genus* of $\mathcal{R}_q$. No solution of $\mathcal{R}_q$ satisfying the above mentioned assumption can contain an arbitrary function depending on more than k_0 arguments.

[2] In order to get a well-defined expression for $j < -q$, we will adopt the convention $\begin{bmatrix} n \\ k \end{bmatrix} = 0$ for $k < 0$.

Corollary 3.13 *An involutive equation $\mathcal{R}_q$ is under-determined, if and only if its Cartan genus is n.*

PROOF. By Corollary 2.28 an involutive equation is under-determined, if and only if we have $\beta_q^{(n)} < m$. But then $\alpha_q^{(n)} > 0$ and the Cartan genus is n. Note that this implies as predicted in Section 2.6 that the general solution contains arbitrary functions of all n independent variables, hence at least one dependent variable is not constrained at all. •

The condition that all $f_{k,j}$ must be non-negative implies that only a finite number of solutions of (3.23) can exist, as the right hand sides can be seen as bounds on weighted sums of the $f_{k,j}$. One can readily design an algorithm which generates for any given set J all non-negative solutions of (3.23).

Since the number of solutions may be very large, it is nevertheless interesting to study ways to construct special solutions. The easiest possibility is to set $J = \{j_0\}$. The condition $H_q(r) = T_q(r)$ leads then to a well-determined linear system of equations for the f_{k,j_0}. Its solution is given by the recursion relation (3.16), if we replace the argument q by $q+j_0$ in the second modified Stirling number.

For two special values of j_0 this modified recursion relation can easily be solved. Setting $j_0 = -q$ gives at once

$$f_{k,-q} = \alpha_q^{(k)}\,, \qquad k = 1,\dots,n\,. \tag{3.26}$$

Choosing $j_0 = 1-q$, we find a solution with $(q-1)$-dimensional integrals generalizing (3.17):

$$f_{n,1-q} = \alpha_q^{(n)}, \quad f_{k,1-q} = \alpha_q^{(k)} - \alpha_q^{(k+1)}\,. \tag{3.27}$$

Again Proposition 3.5 implies that it is a non-negative solution. Note that these two solutions yield the same values for the $\tilde{f}_k$ only if $\alpha_q^{(n)} = \cdots = \alpha_q^{(2)} = 0$.

Another way to construct non-negative solutions of (3.23) is to exploit the relation

$$\begin{bmatrix} n \\ k \end{bmatrix} = \begin{bmatrix} n \\ k-1 \end{bmatrix} + \begin{bmatrix} n-1 \\ k \end{bmatrix}, \tag{3.28}$$

which is an immediate consequence of the Pascal triangle. If now f_k is a negative value in the solution of (3.16), we can eliminate it through

$$f_{k+1}\begin{bmatrix} k+1 \\ q+r \end{bmatrix} + f_k\begin{bmatrix} k \\ q+r \end{bmatrix} = (f_{k+1}+f_k)\begin{bmatrix} k+1 \\ q+r \end{bmatrix} - f_k\begin{bmatrix} k+1 \\ q+r-1 \end{bmatrix}, \tag{3.29}$$

i.e., we "convert" some functions of $k+1$ variables into integrals! But this may result in a negative value for the new f_{k+1}. Then we can apply the same trick again, if $k+1 < n$. But if f_n becomes negative, the procedure has failed. Thus this method will only work, if $\sum_{i=k}^{n} f_i > 0$. It can be generalized by iteration of (3.28) to include higher-dimensional integrals (see [133]).

Finally, we consider the problem of how to decide whether a solution constructed by some technique is the general solution. Under the assumption that the corresponding power series can be constructed order by order, we must simply count the arbitrary functions and compare the resulting $T_q(r)$ with the Hilbert polynomial $H_q(r)$ of the differential equation.

Corollary 3.14 *Let a given solution of the involutive differential equation $\mathcal{R}_q$ contain $f_{k,j}$ functions of differentiation order j with k arguments. Let its power series be constructible order by order. Then the solution is a general one, if and only if the numbers $f_{k,j}$ solve the diophantine system (3.23) for a set J containing all occurring values of j.*