Chapter 5

Symmetries and Reduction

After a brief introduction into symmetries of differential equations and different reduction methods, we will show how the formal theory can be applied to some problems arising in this context. Especially we will consider the loss of generality through such a reduction and study the question, how one can find "good" differential constraints.

5.1 Introduction

Some of the most important geometric techniques for studying differential equations belong to the realm of symmetry analysis. It was originally developed at the end of the last century by Lie searching for a "Galois Theory" of differential equations. Meanwhile it is nearly impossible to survey the literature on this subject. We refer only to the basic textbooks [15, 111, 146].

Definition 5.1 *A* symmetry *of a differential equation is a transformation that maps solutions into solutions.*

This is the most general way to introduce the notion of a symmetry. It is, however, fairly vague, as it does not specify what kind of transformations are considered. The geometric theory provides a natural setting for specifying this in more detail. We will restrict our attention here to continuous point transformations, i.e. to diffeomorphism of the manifold $\mathcal{E}$ of independent and dependent variables. There exist many generalizations. One can e.g. add discrete transformations, one can consider generalized or Lie-Bäcklund symmetries or diffeomorphisms on covering manifolds leading to non-local symmetries.

Definition 5.2 *A diffeomorphism $\phi : \mathcal{E} \mapsto \mathcal{E}$ is called a* Lie point symmetry *of the differential equation $\mathcal{R}_q$, if $\phi^*\mathcal{R}_q = \mathcal{R}_q$.*

There arises now of course the (often over-looked) question to what extend these two definitions are equivalent. Even if we restrict "transformation" in Definition 5.1 to the diffeomorphisms used in Definition 5.2, the first one yields in general more symmetries than the second one.

Example 2.3 (cont.) Consider again the differential equation given by

$$\mathcal{R}_1 : \begin{cases} u_z + yu_x &= 0\,, \\ u_y &= 0\,. \end{cases}$$

$\mathcal{R}_1$ is obviously not invariant under y-translations, as y occurs explicitly. Adding the integrability condition $u_x = 0$ leads to the trivial system $u_x = u_y = u_z = 0$ which is invariant under y-translation and so is of course every solution of $\mathcal{R}_1$. This symmetry was lost, because it was requested that the manifold $\mathcal{R}_1$ should remain invariant instead of the smaller one $\mathcal{R}_1^{(1)}$.

If a differential equation is not formally integrable, then it is obvious from this example that Definition 5.2 poses a stricter condition than Definition 5.1. There are points in $\mathcal{R}_q$ which are never reached by a solution, i.e. let $\Theta \subset \Gamma_{loc}(\mathcal{E})$ be the solution space of $\mathcal{R}_q$, then the union of the prolongations of all sections in Θ is in general only a subset of $\mathcal{R}_q$

$$\bigcup_{\sigma\in\Theta} j_q(\sigma) \subseteq \mathcal{R}_q\,. \tag{5.1}$$

One can show that for differential equations where the equal sign holds in (5.1) the two definitions of a symmetry are equivalent [111].

Definition 5.3 *A differential equation $\mathcal{R}_q$ is called* locally solvable, *if there exists for every point $P \in \mathcal{R}_q$ a solution σ such that $P \in j_q(\sigma)$, i.e. every point lies on the prolongation of a solution.*

Note that we do not require uniqueness of this solution. But this is obvious, as the prescribed data are of course in general not sufficient. Local solvability becomes a difficult problem, if we consider only smooth equations or solutions, as we have already seen in the Example 4.7 of Lewy. In the analytic category, we get it as a simple corollary of the Cartan-Kähler Theorem.

Corollary 5.4 *Every analytic and involutive differential equation $\mathcal{R}_q$ is locally solvable.*

Proof. Let $P = (x_0^i, u_0^\alpha, p_{\mu,0}^\alpha)$ be a point on $\mathcal{R}_q$. We can choose for the construction of a power series solution the principal derivatives according to the inverse lexicographic ranking. P prescribes values for all parametric derivatives of order less than or equal to q at the point $x_0 \in X$. If we take all other parametric derivatives as zero, we obtain an initial value problem with polynomial Cauchy data for which the Cartan-Kähler Theorem applies. Hence this problem has a unique analytic solution f and by construction $P \in j_q(f)$.

It is of course not the only possibility to take all other parametric derivatives as zero. Any convergent Taylor series with the same coefficients up to order q will do. Thus we get a unique solution through P only for an equation of finite type, where the only free parameters are the parametric derivatives of order less than q. •

Determining all diffeomorphisms satisfying the condition in Definition 5.2 is a difficult task using finite transformations. It was Lie's great insight to discover that it suffices to study infinitesimal transformations. Let

$$\mathbf{v} = \sum_{i=1}^{n} \zeta^i(x,u)\frac{\partial}{\partial x^i} + \sum_{\alpha=1}^{m} \eta^\alpha(x,u)\frac{\partial}{\partial u^\alpha} \tag{5.2}$$

be a vector field on $\mathcal{E}$. Using the chain rule it is straight-forward to define its prolongation $\mathrm{pr}^{(q)}\mathbf{v}$, a vector field acting on $J_q\mathcal{E}$, by

$$\mathrm{pr}^{(q)}\mathbf{v} = \mathbf{v} + \sum_{\alpha,|\mu|\le q} \eta_\mu^\alpha \frac{\partial}{\partial p_\mu^\alpha} \tag{5.3}$$

where the coefficients can be determined recursively

$$\eta^\alpha_{\mu+1_k} = D_k \eta^\alpha_\mu - \sum_{i=1}^n D_k \zeta^i p^\alpha_{\mu+1_i} \,. \tag{5.4}$$

If now a local representation of $\mathcal{R}_q$ is given by $\Phi^\tau = 0$ and

$$\mathrm{pr}^{(q)}\mathbf{v}(\Phi^\tau)\Big|_{\mathcal{R}_q} = 0 \tag{5.5}$$

holds for all functions Φ^τ, then the flow generated by $\mathbf{v}$ is a symmetry in the sense of Definition 5.2.

One usually calls (5.5) the *determining system* of $\mathcal{R}_q$. It is an, in general over-determined, system of linear partial differential equations. This shows the decisive advantage of symmetry analysis: even if $\mathcal{R}_q$ is a non-linear equation, its determining system is always linear and thus usually much easier to tackle. This linearization corresponds to the fact that we use instead of the Lie group of finite transformations its Lie algebra.

From a computational point of view the biggest problem in evaluating (5.5) lies in the restriction to $\mathcal{R}_q$. Usually this is solved by assuming that one is given a local representation of $\mathcal{R}_q$ where each equation is solved for some derivative. Then one must simply perform the corresponding substitution. For more general equation one could use Gröbner basis techniques (cf. Section 8.6).

As a trivial application of the results of Chapter 3 one can calculate the size of the symmetry algebra, if it is not possible to solve explicitly the determining system (5.5). Reid [125, 126] has shown that it is even possible to obtain the structure constants of the algebra without solving (5.5), although there are some problems in the case of an infinite-dimensional algebra.

Symmetries have many applications. They are e.g. connected with the so-called complete integrability of an equation or they help to decide whether or not a system can be transformed into a linear one. They also give a group theoretic fundament to separation ansätze. We will not discuss any of these topics but refer again to the textbooks mentioned at the beginning of this section. But we will consider in some detail in the next section the perhaps most important application of symmetries, namely the construction of special solutions using reduction methods.

5.2 Reduction of Differential Equations

Reduction with respect to a symmetry group provides a standard technique for the construction of solutions. It is well-known that only for ordinary differential equations one can reconstruct the general solution of the original equation from the general solution of the reduced equation by quadratures. In the case of partial differential equations one obtains only special solutions.

The idea behind symmetry reduction is to look for solutions which are invariant under a symmetry group, i.e. the symmetry transformations map such a solution into itself. If the symmetry is generated by a vector field $\mathbf{v}$ of the form (5.2), then a group invariant solution $u^\alpha(x^i)$ satisfies not only the considered differential equation $\mathcal{R}_q$ but in addition the *invariant surface condition*

$$\sum_{i=1}^n \zeta^i(x,u) p^\alpha_i = \eta^\alpha(x,u) \,, \qquad \alpha = 1, \ldots, m \,. \tag{5.6}$$

These are quasi-linear, first-order equations. In the case of a higher-dimensional symmetry algebra we must add one such set of conditions for each generator. If one can compute the general solution of (5.6), one substitutes it into the equations of $\mathcal{R}_q$ to obtain a reduction. Or, if we assume without loss of generality that $\zeta^n \neq 0$, we can solve (5.6) for the derivatives p_n^α and thus eliminate in $\mathcal{R}_q$ all derivatives with respect to x^n. This yields a differential equation in $n-1$ independent variables.

The latter approach is a special case of a more general symmetry reduction method using *invariants*. These are functions $\delta(x,u)$ satisfying $\mathbf{v}\delta = 0$ for every symmetry generator $\mathbf{v}$. Since the generators form a Lie algebra, they represent the solutions of a complete system (cf. Section 7.1).

Let us assume that the symmetry group with respect to which the reduction is to be performed acts regularly on $\mathcal{E}$ with s-dimensional orbits. It is well-known that there exist locally $m+n-s$ functionally independent invariants. We divide them into two classes

$$\begin{aligned} y^1 &= \xi^1(x,u)\,, \ \dots,\ y^{n-s} = \xi^{n-s}(x,u)\,, \\ v^1 &= \vartheta^1(x,u)\,, \ \dots,\ v^m = \vartheta^m(x,u)\,. \end{aligned} \tag{5.7}$$

The y^i will be the new independent variables, the v^α the new dependent variables. The result of the reduction does not really depend on this division, as the different reduced systems are related by some kind of "hodograph" transformations.

Now we must assume that we can solve (5.7) for $n-s$ of the x^i and for all of the u^α, i.e. that the symmetry group acts transversally. The remaining x^i play the role of parameters. Using the chain rule these solved expressions can be substituted in the differential equation. The resulting equation will be equivalent to one without the parametric x^i. Thus we have obtained an equation containing only $n-s$ independent variables.

Bluman and Cole [14] were the first to observe that for this construction it is not necessary that $\mathbf{v}$ generates a symmetry of the differential equation. It suffices, if $\mathbf{v}$ yields a symmetry of $\mathcal{R}_q$ augmented by the invariant surface condition (5.6)[1]. Since the coefficients ζ^i, η^α of $\mathbf{v}$ are now also part of the differential equation, the resulting determining system is non-linear opposed to the situation in the classical Lie theory. This is a serious draw-back in the concrete application of this so-called *non-classical method*.

The transformations generated by such vector fields are sometimes called *conditional symmetries* [95], as in general they do not map all solutions of $\mathcal{R}_q$ into other solutions. They also do not form a Lie algebra, because every conditional symmetry depends on its own invariant surface condition (5.6).

The non-classical method often yields additional reductions: addition of further equations leads to a smaller submanifold in the jet bundle, thus to less restrictions for symmetry generators. Olver and Rosenau [113, 114] generalized this idea to so-called *weak symmetries* or the method of *differential constraints*. There is no reason why one should add only invariant surface conditions. Every additional equation or differential constraint will lead to a lower-dimensional submanifold and thus potentially to further conditional symmetries. Behind this approach lies the simple observation that *any* solution of a differential equation remains invariant under some group.

The choice of suitable constraints represents, however, a big unsolved problem. Most will simply lead to an inconsistent equation, i.e. to an equation without solution. Others will not yield new symmetries, yet others will lead to intractable equations etc. For this reason the method of differential constraints has not yet found much application. It is

[1] This is especially easy to see using exterior systems (cf. Proposition A.5 in Section A.1).

simply too general. In the context of formal theory it is interesting, because it shifts the emphasis from group theory to the theory of over-determined equations.

The *direct reduction method* of Clarkson and Kruskal [30] could be viewed as an answer to the last problem. It was originally developed for an equation with $m = 1$ dependent variable u and $n = 2$ independent variables x, t. It consists of the solution ansatz

$$u(x,t) = U(x,t,w(z))\,, \qquad z = \xi(x,t) \tag{5.8}$$

where one requires that there exist functions $U(x,t,w), \xi(x,t)$ such that $\mathcal{R}_q$ reduces to an ordinary differential equation for w. The extension to more variables is straight-forward.

The direct reduction method was introduced as an ad hoc ansatz without a rigorous theoretical foundation. Levi and Winternitz [95] were the first to point out that it was closely related to the non-classical method; later Arrigo *et al.* [4] gave exact conditions, when the two are equivalent. Olver [112] showed recently that there is a one-to-one correspondence between direct reduction ansätze (5.8) and invariant surface conditions of the restricted form

$$\zeta(x,t)u_x + \tau(x,t)u_t = \eta(x,t,u)\,, \tag{5.9}$$

i.e. the corresponding vector field $\mathbf{v} = \zeta\partial_x + \tau\partial_t + \eta\partial_u$ generates a group of fiber preserving transformations. ξ is the (more or less) unique invariant of this group: $\mathbf{v}(\xi) = 0$.

Galaktionov [55] proposed a generalization of the direct reduction method using the ansatz

$$u(x,t) = U(x,t,w_1(z),w_2(z)) \tag{5.10}$$

and the requirement that the reduction yields a system of ordinary differential equations for w_1 and w_2. This can obviously be extended to an ansatz of *n-th order* using n functions w_i of the same similarity variable z. Olver [112] showed that this ansatz is equivalent to an n-th order differential constraint of the form

$$\mathbf{v}^n(u) = \Phi\Big(x,t,u,\mathbf{v}(u),\ldots,\mathbf{v}^{n-1}(u)\Big) \tag{5.11}$$

with a vector field $\mathbf{v} = \zeta\partial_x + \tau\partial_t$ satisfying $\mathbf{v}(\xi) = 0$. Note that, (5.9) and (5.11) become parameterized ordinary differential equations after transforming to rectifying coordinates for $\mathbf{v}$.

Estévez [49, 50] has combined the direct method with the singular manifold method from Painlevé analysis [162, 163]. This leads basically to the possibility of obtaining reductions to ordinary differential equations of lower order than the original partial differential equation. The method seems, however, to be applicable only to equations which have at least the conditional Painlevé property.

5.3 Loss of Generality

As already mentioned in the last section reduction methods yield only special solutions in the case of partial differential equations. Our goal in this section will be to show how the loss of generality can be quantified using the results of Chapter 3. One must, however, be aware that this loss represents only one side of the problem. The less we reduce, i.e. the smaller the dimension of the used symmetry group, the smaller the loss of generality. But usually the more we reduce the easier the resulting equation. Indeed, as we have seen in the case of the direct reduction method, in applications one tries very often to obtain an ordinary differential equation, although this represents of course the maximal loss of generality.

We will start with the special case of a classical symmetry reduction of a normal equation, as here it is possible to exhibit general results. We assume that we are given a local representation in solved form

$$p^{\beta}_{[0,\dots,q]} = \Phi^{\beta}(x^i, u^{\alpha}, p^{\alpha}_{\mu})\,, \quad \beta = 1,\dots,m \tag{5.12}$$

where the right hand sides do not contain any derivative of order q which is purely with respect to x^n (cf. Section 4.3). We computed already in Example 3.11 the Cartan characters for such an equation:

$$\alpha_N^{(n)} = 0\,,\ \alpha_N^{(n-k)} = m\binom{q+k-1}{q-1}\,. \tag{5.13}$$

Let us further assume that the symmetry group acts regularly with s-dimensional orbits and that after the reduction the equation is still normal. Then we obtain a system with $\bar{n} = n - s$ independent and $\bar{m} = m$ dependent variables of order $\bar{q} = q - \Delta q$. The Cartan characters of the reduced system are $\bar{\alpha}_{\bar{q}}^{(\bar{n})} = 0, \bar{\alpha}_{\bar{q}}^{(\bar{n}-k)} = m\binom{\bar{q}+k-1}{\bar{q}-1}$. Thus if $\Delta q = 0$, then $\bar{\alpha}_q^{(k)} = \alpha_q^{(k+s)}$. If $\Delta q > 0$, we get an additional loss of generality as to be expected.

Since the Cartan characters form a descending sequence (Proposition 3.5), this result means that we do not only loose all arbitrary functions with more than $n - s$ arguments, but that even the number of arbitrary functions with less arguments decreases.

Even in more general situations the loss of arbitrariness can be computed without explicit construction of the reduced equation. One must simply compare the Cartan characters of the original equation alone with the characters of the equation augmented by the invariant surface conditions (5.6) — after completion to an involutive equation of course. It is, however, difficult to obtain this way general results like the above ones for normal equations, as it is unclear, how one can predict the characters of the augmented equation.

An extension of this method to reductions with respect to generalized or to weak symmetries is straight-forward. For instance, the analysis of Lie-Bäcklund symmetries of normal first-order equations leads to the addition of differential constraints of order r [108]

$$\Psi^{\tau}(x^i, u^{\alpha}, p^{\alpha}_{\mu}) = 0\,, \qquad \begin{cases} \tau &= 1,\dots,p\,, \\ |\mu| &\leq r\,, \\ \mu_n &= 0\,. \end{cases} \tag{5.14}$$

Note that x^n occurs here only as parameter, because all derivatives with respect to x^n can be eliminated using (5.12).

We assume that system (5.14) is in involution and comprises $\tilde{\beta}_r^{(k)}$ equations of class k for $k = 1,\dots,n-1$. The combined system (5.12,5.14) has an involutive symbol, if and only if there exist matrices $(M^i)^{\tau}_{\rho}$ such that

$$(M^i)^{\tau}_{\rho}\frac{\partial\Psi^{\rho}}{\partial p^{\alpha}_{\mu}} = -\frac{\partial\Psi^{\tau}}{\partial p^{\beta}_{\mu}}\frac{\partial\Phi^{\beta}}{\partial p^{\alpha}_{i}}\,, \qquad \begin{cases} i=1,\dots,n-1\,, \\ |\mu|=r\,. \end{cases} \tag{5.15}$$

No integrability conditions arise, i.e. the system is involutive, if and only if additionally

$$D_n\Psi^{\tau} - \sum_{\alpha,|\mu|=r}\frac{\partial\Psi^{\tau}}{\partial p^{\alpha}_{\mu}}(p^{\alpha}_{\mu+1_n} - D_{\mu}\Phi^{\alpha}) - \sum_{i=1}^{n-1}(M^i)^{\tau}_{\rho}D_i\Psi^{\rho} = 0\,. \tag{5.16}$$

Since (5.14) contains no derivatives with respect to x^n, we get for the combined system $\beta_r^{(n)} = m$, $\beta_r^{(n-k)} = m\binom{r+k-2}{r-2} + \tilde{\beta}_r^{(n-k)}$. This yields the Cartan characters

$$\alpha_r^{(n)} = 0\,,\ \alpha_r^{(n-k)} = m\binom{r+k-2}{r-1} - \tilde{\beta}_r^{(n-k)}\,. \tag{5.17}$$

This generalizes the results of [108] to non-linear equations and to constraints which depend on x^n and which do not form a normal system.

5.4 Finding Differential Constraints

We have already said that the big problem in the application of the method of differential constraints is the choice of the constraints. Formal theory can help here, if we follow the basic idea behind the direct reduction method. One should, however, stress already now the word "help": we will not obtain an algorithm like in the direct reduction method.

The goal in reduction is usually to obtain ordinary differential equations. We will prove later in Section 7.1 that the general solution of every equation of finite type can be constructed by solving only ordinary differential equations. Lie [97] showed furthermore that conversely every family of functions parameterized by a finite number of constants can be written as the solution of a finite type equation. Thus our goal should be to obtain such an equation after the reduction. This is considerably more general as the direct reduction method.

Unfortunately this approach has two severe draw-backs. We will see later on that for single second-order equations in two independent variables it yields almost no restriction on the form of the constraints. Furthermore the conditions take usually the form of inequalities. Thus there exists no direct method to find constraints satisfying them.

For most of this section we will concentrate on the case of a single second-order equation

$$\Delta(x,t,u,u_x,u_t,u_{xx},u_{xt},u_{tt}) = 0 \tag{5.18}$$

and the addition of one differential constraint. We will assume throughout that the considered constraints are compatible. Conditions for this will become evident during the analysis.

Proposition 5.5 *Adding a compatible first-order differential constraint to (5.18) leads always to an equation of finite type.*

Proof. We can assume without loss of generality that the constraint is solved for u_x

$$u_x - \phi(x,t,u,u_t) = 0\,. \tag{5.19}$$

Hence all x-derivatives in Eq. (5.18) can be eliminated and we obtain

$$\tilde{\Delta}(x,t,u,u_t,u_{tt}) = 0\,. \tag{5.20}$$

We must now distinguish between a degenerate case when $\tilde{\Delta}_{u_{tt}} = 0$ and the generic case where (5.20) is still a second order equation. This distinction can be characterized intrinsically by the rank of a symbol.

Generic case: We have to investigate, when the equation

$$\mathcal{R}_2\ :\ \begin{cases} u_x - \phi(x,t,u,u_t) = 0\,, \\ u_{xx} - \phi_{u_t}u_{xt} - \phi_u u_x - \phi_x = 0\,, \\ u_{xt} - \phi_{u_t}u_{tt} - \phi_u u_t - \phi_t = 0\,, \\ \tilde{\Delta}(x,t,u,u_t,u_{tt}) = 0 \end{cases} \tag{5.21}$$

is involutive. Its symbol $\mathcal{M}_2$ has not full rank, if and only if

$$\tilde{\Delta}_{u_{tt}} = \Delta_{u_{xx}}\phi_{u_t}^2 + \Delta_{u_{xt}}\phi_{u_t} + \Delta_{u_{tt}} = 0\,. \tag{5.22}$$

Then $\operatorname{rank}\mathcal{M}_2 = 2$ and we are in the degenerate case.

Thus we have $\dim\mathcal{M}_2 = 0$ in the generic case and the symbol is trivially involutive. Prolongation to third order shows that there may be an integrability condition of the following form:

$$\tilde{\Delta}_x - \tilde{\Delta}_t\phi_{u_t} + \tilde{\Delta}_u(\phi - \phi_{u_t}u_t) + \tilde{\Delta}_{u_t}(D_t\phi - u_{tt}) + \tilde{\Delta}_{u_{tt}}D_{tt}\phi = 0\,. \tag{5.23}$$

If this equation vanishes identically due to (5.19) and (5.20), then $\mathcal{R}_2$ is an involutive finite type equation with a two-dimensional solution space. A prototypical example for this situation is the direct reduction with the Clarkson-Kruskal ansatz (5.8) yielding a second order ordinary differential equation.

If, however, (5.23) does not vanish identically, then an integrability condition arises. In general it will be second order. Because of the vanishing symbol $\mathcal{M}_2$, one can always rewrite it as a first order (or perhaps even algebraic) equation. If it is a first order equation, then we can solve it for u_t (u_x can again be eliminated using (5.19)). Depending on whether this new constraint reduces (5.20) to zero or to an algebraic equation, we get a zero or a one-dimensional solution (an algebraic equation is of course an implicit solution, as we assumed that (5.19) is a compatible constraint). The case of a one-dimensional solution should correspond to the reductions obtained by Estévez.

Degenerate case: In this case we are left with a first order equation

$$\mathcal{R}_1 : \begin{cases} u_x - \phi(x,t,u,u_t) = 0\,, \\ \tilde{\Delta}(x,t,u,u_t) = 0\,. \end{cases} \tag{5.24}$$

If $\tilde{\Delta}_{u_t} = 0$, we have found a reduction to an algebraic equation, i.e. an implicit solution. Otherwise the symbol vanishes and is thus involutive. If there occurs an integrability condition, it must be an algebraic equation, i.e. it yields again an implicit solution. If $\mathcal{R}_1$ is involutive, then we get a one-dimensional solution. •

It should be noted that it may not always be possible to find constraints such that the degeneracy condition (5.22) is satisfied. Especially for equations which are non-linear in the highest-order derivatives this will usually not be the case. For quasi-linear systems we rediscover the usual distinction in hyperbolic, parabolic and elliptic systems. One must, however, be careful with the coordinate system, as we have solved the constraint for a distinguished derivative.

Proposition 5.6 *Adding a compatible second-order differential constraint to (5.18) leads almost always to an equation of finite type. A positive Cartan genus occurs, if and only if the combined system is involutive and the symbol of its prolongation has rank 3.*

Proof. Again we can assume without loss of generality that the constraint is in solved form

$$u_{xx} - \psi(x,t,u,u_x,u_t,u_{xt},u_{tt}) = 0\,. \tag{5.25}$$

Eliminating u_{xx} from (5.18) leads to three different cases depending on the form of the reduced equation (and in Case B on an additional condition for the constraint):

$$\begin{array}{ll} \text{A)} & u_{tt} - \tilde{\Delta}(x,t,u,u_x,u_t,u_{xt}) = 0\,, \\ \text{B)} & u_{xt} - \tilde{\Delta}(x,t,u,u_x,u_t) = 0\,, \\ \text{C)} & \tilde{\Delta}(x,t,u,u_x,u_t) = 0\,. \end{array} \tag{5.26}$$

Case C is trivial, one just considers the reduced equation as a first order constraint to the second order equation (5.25) and recovers the situation of Proposition 5.5.

The distinction between the three cases is again induced by the rank of symbols. We are interested in the equation

$$\mathcal{R}_2 : \begin{cases} u_{xx} - \psi(x,t,u,u_x,u_t,u_{xt},u_{tt}) = 0\,, \\ \Delta(x,t,u,u_x,u_t,u_{xx},u_{xt},u_{tt}) = 0\,. \end{cases} \tag{5.27}$$

In Case C the symbol $\mathcal{M}_2$ has not maximal rank. Cases A and B differ in the rank of the prolonged symbol $\mathcal{M}_3$.

Generic case: This is Case A. Independent of the way we order the variables, we find that $\mathcal{M}_2$ has three multiplicative variables (one equation of class 2 and one of class 1). To check whether or not $\mathcal{M}_2$ is involutive, we must compute the rank of its prolongation

$$\mathcal{M}_3 : \begin{cases} v_{xxx} - \psi_{u_{xt}} v_{xxt} - \psi_{u_{tt}} v_{xtt} = 0\,, \\ v_{xxt} - \psi_{u_{xt}} v_{xtt} - \psi_{u_{tt}} v_{ttt} = 0\,, \\ v_{xtt} - \tilde{\Delta}_{u_{xt}} v_{xxt} = 0\,, \\ v_{ttt} - \tilde{\Delta}_{u_{xt}} v_{xtt} = 0\,. \end{cases} \tag{5.28}$$

Since it is obviously four, $\mathcal{M}_2$ is not involutive according to (2.19). But $\mathcal{M}_3$ is involutive, as it vanishes. If there arises no integrability condition in the prolongation from $\mathcal{R}_3$ to $\mathcal{R}_4$, then $\mathcal{R}_3$ is an involutive finite type equation with a four-dimensional solution space. The prototypical example for this case is given by direct reduction using Galaktionov's ansatz (5.10) leading to two second order ordinary differential equations.

One can compute the general form of the possible integrability condition, but we omit it here, as it is a fairly lengthy expression. Since $\mathcal{M}_3$ vanishes, we can conclude that the integrability condition is at most of second order. There arise now many case distinctions depending on the order of the condition and on whether or not the projected system is involutive. We omit these, too, and conclude by noting that we have in any case a finite type equation with a maximally three-dimensional solution space.

Degenerate case: In Case B we can write the symbol $\mathcal{M}_3$ in the following form

$$\mathcal{M}_3 : \begin{cases} v_{xxx} - \psi_{u_{xt}} v_{xxt} - \psi_{u_{tt}} v_{xtt} = 0\,, \\ v_{xxt} - \psi_{u_{xt}} v_{xtt} - \psi_{u_{tt}} v_{ttt} = 0\,, \\ v_{xxt} = 0\,, \\ v_{xtt} = 0\,. \end{cases} \tag{5.29}$$

Its rank depends now on $\psi_{u_{tt}}$. If it is not zero, the rank of the symbol is still four and we are in Case A. If it vanishes, however, then $\operatorname{rank} \mathcal{M}_3 = 3$ and according to (2.19) the symbol $\mathcal{M}_2$ is now involutive.

During the prolongation from $\mathcal{R}_2$ to $\mathcal{R}_3$ the following integrability condition arises

$$D_x\tilde{\Delta} - \psi_{u_{xt}} D_t\tilde{\Delta} - D_t^{(1)}\psi = 0\,, \tag{5.30}$$

where the superscript (1) indicates that only the jet variables up to order 1 are taken into account. If it vanishes identically, then $\mathcal{R}_2$ is involutive. However, it is not a finite type equation, as $\dim \mathcal{M}_2 = 1$. Since the Cartan characters are $\alpha_2^{(2)} = 0$ and $\alpha_2^{(1)} = 1$, we find a Cartan genus of 1.

If (5.30) truly provides an integrability condition, then the projected equation $\mathcal{R}_2^{(1)}$ is of finite type. Again we get a flood of case distinctions depending on the order of (5.30) and on whether or not $\mathcal{R}_2^{(1)}$ is involutive. In any case the solution space is at most three-dimensional. •

The two preceding propositions can be extended to equations of higher than second order. We will restrict ourselves mainly to the respective generic cases. There arise now much more case distinctions, as there are more possibilities for the order of the integrability conditions. But we have already seen, that this affects basically only the dimension of the solution space and not the principal results. Furthermore, the generic case is the most interesting one, as it yields the maximal dimension.

The case of first order constraints runs completely analogously to Proposition 5.5. Again all derivatives with respect to x can be replaced by t-derivatives using the constraint; the reduced equation is a parameterized ordinary differential equation. The degeneracy condition (5.22) is replaced by

$$\Delta_{u_{[q,0]}}\phi^q_{u_t} + \Delta_{u_{[q-1,1]}}\phi^{q-1}_{u_t} + \cdots + \Delta_{u_{[0,q]}} = 0 \tag{5.31}$$

where q denotes the order of the differential equation and where we have used the notation

$$u_{[r,s]} = \frac{\partial^{r+s}u}{\partial x^r\,\partial t^s}\,. \tag{5.32}$$

Prolongation of the constraint (5.19) to order q yields q independent equations. One can easily see, that the symbol $\mathcal{M}_q$ of the equation $\mathcal{R}_q$ which arises by adjoining these equations to the original equation always vanishes, if condition (5.31) does not hold. In the case that no integrability conditions occur, $\mathcal{R}_q$ is thus an involutive finite type equation with a q-dimensional solution space. Again the direct reduction with the Clarkson-Kruskal ansatz (5.8) provides the prototypical example for this situation.

Proposition 5.6 is generalized by considering a constraint of order q. For simplicity we assume again that the constraint is in solved form

$$u_{[q,0]} = \psi(x,t,u,u_x,u_t,\ldots,u_{[0,q-1]},u_{[q-1,1]},\ldots,u_{[0,q]})\,. \tag{5.33}$$

Eliminating $u_{[q,0]}$ yields the equation

$$\tilde{\Delta}(x,t,u,u_x,u_t,\ldots,u_{[0,q-1]},u_{[q-1,1]},\ldots,u_{[0,q]}) = 0\,. \tag{5.34}$$

We must now analyze the equation $\mathcal{R}_q$ comprising (5.33) and (5.34).

Omitting the analog of Case C in the proof of Proposition 5.6, $\operatorname{rank}\mathcal{M}_q = 2$ and there are three multiplicative variables. The prolonged symbols $\mathcal{M}_{q+r}$ are given by

$$\mathcal{M}_{q+r}\;:\;\begin{cases} v_{[q+i,r-i]} - \sum\limits_{j=1}^{q}\psi_{u_{[q-j,j]}}v_{[q-j+i,r+j-i]} = 0 \\ \sum\limits_{j=1}^{q}\tilde{\Delta}_{u_{[q-j,j]}}v_{[q-j+i,r+j-i]} = 0 \end{cases} \qquad i = 0,\ldots,r\,. \tag{5.35}$$

As long as these symbols have maximal rank, we get $\operatorname{rank}\mathcal{M}_{q+r} = 2r+2$ and $\mathcal{M}_{q+r}$ has $2r+3$ multiplicative variables. Since this symbol is defined in a vector space of the dimension $q+r+1$ (the number of derivatives of order $q+r$), we will reach a vanishing and thus involutive symbol for $r = q-1$. The corresponding system $\mathcal{R}_{2q-1}$ will comprise $q(q+1)$ equations and thus its solution space has dimension q^2. This case will arise with a direct reduction ansatz of order q leading to q ordinary differential equations of order q.

We see that as long as every symbol has maximal rank

$$\operatorname{rank}\mathcal{M}_{q+r+1} = \#\text{multiplicative variables}(\mathcal{M}_{q+r}) + 1 \tag{5.36}$$

holds. The number of multiplicative variables of a symbol provides a lower bound for the rank of its prolongation (Proposition 2.9). Thus if one of the symbols $\mathcal{M}_{q+r}$ has not maximal rank, then its rank is $2r+1$ and the preceding symbol is involutive! If now no integrability conditions arise, i.e. the corresponding system is involutive, too, then we have not an equation of finite type. This situation will, however, never occur, if $\psi_{[0,q]} \neq 0$, as one can easily see from (5.35). Note that this is only a sufficient and not a necessary criterion.

The analysis becomes much more involved, if we consider differential equations in more than two independent variables. In general, one constraint is now of course not sufficient to obtain an equation of finite type. For first order constraints we can extend Proposition 5.5 by considering $n-1$ constraints of the form

$$u_{x^k} = \phi(x^i, u, u_{x^j})\,, \qquad k = 1, \dots, n-1, \quad j \neq k\,, \tag{5.37}$$

where n denotes the number of independent variables.

The only way to be sure that q-th order constraints lead to an equation of finite type is to add so many constraints that the symbol $\mathcal{M}_q$ vanishes. But sometimes just one constraint may suffice, if it generates enough integrability conditions. If we take the equation

$$u_{zz} + y u_{xx} = 0\,, \tag{5.38}$$

then we know already from Example 2.19 of Janet that addition of the constraint

$$u_{yy} = 0 \tag{5.39}$$

suffices to obtain an equation of finite type.

The principal way to use the formal theory for finding differential constraints would be thus to *choose* a scenario for the completion algorithm, i.e. at which step where an integrability condition shall appear, and an ansatz for the constraint. This yields on one side further hints for the form of the constraints from the integrability conditions one wants to have and on the other side differential equations for the constraint from the ones which should vanish. Unfortunately this procedure contains a lot of guessing and we have so far not been able to produce this way a new non-trivial reduction.