Chapter 6

Field Theories

We will apply the results of the previous chapters to some systems of physical interest. We will start by considering the effect of gauge symmetries; then we will show, how the notion of strength is related to the Cartan characters. Systems with constraints will be analyzed from the point of view of formal theory. As concrete examples the Yang-Mills Equations and the Einstein Equations are treated.

6.1 Gauge Systems

Gauge transformations are a special kind of symmetries; they preserve the fibers of the bundle $\mathcal{E}$. We call a differential equation a *gauge system*, if it is invariant under transformation of the form

$$\begin{aligned} \bar{x}^i &= \Omega^i(x^j)\,, \\ \bar{u}^\alpha &= \Lambda^\alpha(x^i, u^\beta, \lambda_a^{(0)}(x), \partial\lambda_a^{(1)}(x), \dots, \partial^p\lambda_a^{(p)}(x)) \end{aligned} \tag{6.1}$$

where γ_0 gauge functions $\lambda_a^{(0)}$ are entering algebraically, γ_1 gauge functions $\lambda_a^{(1)}$ are entering through their first derivatives etc. We assume that all derivatives of order l of the functions $\lambda^{(l)}$ are explicitly occurring in (6.1) and that the gauge functions depend on all independent variables.

The only reason for these assumptions is simplicity. The formalism can handle more complicated situations, but general results would look rather awkward. In such cases it is better to use the pseudogroup approach presented later in this section. In order to be a gauge system a differential equation must obviously be under-determined, as (6.1) implies that its general solution contains arbitrary functions of all independent variables.

Such systems occur frequently in physics, a typical example being the Yang-Mills Equations which will be studied in Section 6.6. Here the arbitrariness as considered in Chapter 3 is not really interesting, because physically solutions which are connected by a gauge transformation can be identified. They lead to the same values for all observables. Thus we must eliminate the effect of the symmetry (6.1) to obtain the number of physical degrees of freedom. Furthermore we will use our results for the analysis of gauge fixing conditions.

The symmetry (6.1) implies that

$$G_q(r) = \sum_{l=0}^{p} \gamma_l \begin{bmatrix} n \\ q+r+l \end{bmatrix} \tag{6.2}$$

Taylor coefficients of order $q+r$ can be arbitrarily set by a gauge fixing. Thus they must be subtracted from the Hilbert polynomial to obtain the physically relevant number of free coefficients. We introduce the *gauge corrected Hilbert polynomial*

$$\bar{H}_q(r) = H_q(r) - G_q(r)\,. \tag{6.3}$$

Its coefficient of r^k is given by

$$\bar{h}_k = h_k - \frac{1}{(n-1)!}\sum_{l=0}^{p}\gamma_l s^{(n-1)}_{n-k-1}(q+l)\,, \quad k=0,\dots,n-1\,. \tag{6.4}$$

Using (3.12) we can associate with the corrected Hilbert polynomial *gauge corrected Cartan characters*

$$\bar{\alpha}^{(k)}_q = (k-1)!\,\bar{h}_{k-1} - \sum_{i=k+1}^{n}\frac{(k-1)!}{(i-1)!}\,\bar{\alpha}^{(k)}_q s^{(i-1)}_{i-k}(0)\,, \quad k=1,\dots,n\,. \tag{6.5}$$

To obtain now the number of physically relevant free functions in the general solution of our equation, we simply apply the results of Chapter 3 using the gauge corrected Cartan characters and Hilbert polynomial, respectively. This yields for instance the following modification to Theorem 3.9.

Theorem 6.1 *If the physical solution space, i.e. the solution space modulo the gauge symmetry (6.1), has an algebraic representation, then it is spanned by p_k arbitrary functions with k arguments, where $p_k = f_k - g_k$ with*

$$\begin{aligned} g_n &= \sum_{l=0}^{p}\gamma_l\,,\\ g_k &= \frac{(k-1)!}{(n-1)!}\sum_{l=0}^{p}\gamma_l\, s^{(n-1)}_{n-k}(q+l) - \sum_{i=k+1}^{n}\frac{(k-1)!}{(i-1)!}\,g_i\, s^{(i-1)}_{i-k}(q)\,. \end{aligned} \tag{6.6}$$

Our formulation of a gauge symmetry has two disadvantages: It is not intrinsic, as there might be different formulations yielding different values for the γ_l and it does not cover the general case, as we had to make the assumption that all derivatives of the specified order of a gauge function occur and that all gauge functions depend on all independent variables. All these problems can be circumvented by the use of a Lie pseudogroup [117] to describe the gauge transformation.

We will not give here a complete introduction into the theory of Lie pseudogroups. We will take the simple point of view that instead of an explicit expression like (6.1) for the gauge transformation we have a differential equation whose general solution is (6.1). In other words we eliminate all the parameters and gauge functions through differentiation and obtain a differential equation with the old coordinates (x^i, u^α) as independent variables and the new coordinates $(\bar{x}^i, \bar{u}^\alpha)$ as dependent variables. The special form of (6.1) implies that $\partial\bar{x}^i/\partial u^\alpha = 0$.

Example 6.2 As a trivial example of a (finite-dimensional) Lie pseudogroup we consider translations

$$\bar{x}^i = x^i + a^i\,, \quad \bar{u}^\alpha = u^\alpha + b^\alpha\,. \tag{6.7}$$

They can be described by the following differential equation (of finite type)

$$\begin{aligned} \frac{\partial\bar{x}^i}{\partial x^j} &= \delta^i_j\,, \qquad & \frac{\partial\bar{x}^i}{\partial u^\beta} &= 0\,,\\ \frac{\partial\bar{u}^\alpha}{\partial x^j} &= 0\,, \qquad & \frac{\partial\bar{u}^\alpha}{\partial u^\beta} &= \delta^\alpha_\beta\,. \end{aligned} \tag{6.8}$$

An example of a Lie pseudogroup where it is not possible to write down an explicit expression of the form (6.1) for the transformations is given by the group of volume preserving mappings. They satisfy

$$\begin{aligned} \frac{\partial \bar{x}^i}{\partial u^\alpha} &= 0\,, & \frac{\partial \bar{u}^\alpha}{\partial x^j} &= 0\,, \\ \det\left(\frac{\partial \bar{x}^i}{\partial x^j}\right) &= 1\,, & \det\left(\frac{\partial \bar{u}^\alpha}{\partial u^\beta}\right) &= 1\,. \end{aligned} \tag{6.9}$$

Let the Lie pseudogroup be realized as a differential equation $\mathcal{G}_g$ of order g. Its Hilbert polynomial $H_g^{(G)}(r)$ yields the number of arbitrary coefficients at order $g+r$ in the explicit transformation law (6.1). An intrinsic way to compute $G_q(r)$ is given by

$$G_q(r) = H_g^{(G)}(r + q - g)\,. \tag{6.10}$$

Obviously this expression is only valid for $r > \max(g - q, 0)$. Possible values for the numbers γ_l can now be computed using the methods of Chapter 3. The form (6.1) arises, if there exists a solution of the diophantine system (3.23) with $f_{n,l} = \gamma_l$ and all other $f_{k,j}$ vanish.

We can easily compute the remaining gauge freedom after some gauge conditions have been imposed, if they take the form of differential equations. These equations are added to the original system. This yields in general an over-determined system. It should be invariant only under a proper subgroup of the gauge group (It is, however, possible that new symmetries arise!).

A necessary condition for a complete gauge fixing is that the Cartan characters of this augmented equation are the same as the gauge corrected Cartan characters of the original equation (one can of course also use the Hilbert polynomial). Note, however, that this is only a necessary criterion, as there may still remain a finite-dimensional symmetry depending on some arbitrary *constants*. This cannot be checked with the Cartan characters, for they are only related with the number of arbitrary functions.

One should note that the gauge correct Cartan characters are computed in a formal, algebraic manner. Hence one cannot conclude that there exists necessarily a differential equation leading to exactly these characters. Especially, they do not necessarily satisfy Proposition 3.5. If this happens, it is not possible to specify a complete gauge fixing using differential equations (this comprises of course purely algebraic conditions like the Coulomb gauge). One must use other conditions like initial or boundary conditions.

6.2 Strength

Einstein [47] introduced another measure for the arbitrariness, the *strength*. He was lead to such formal considerations during his search for a unified field theory. His philosophy was: if we have two systems of field equations and we cannot decide which is the better one for physical reasons, then we should take the one which restricts the solution stronger. He discussed this problem already earlier with Cartan [28], who tried to explain him his theory of involutive systems. The strength has not found much attention in the literature, only a few papers [20, 74, 103, 105, 106, 110, 115, 133, 147] have been devoted to it.

We introduce

$$Z_q(r) = H_q(r) / \left[{n \atop q+r}\right]. \tag{6.11}$$

Z_q is a rational function of r. The denominator is of degree $n-1$; the numerator has a degree less than or equal to $n-1$. Z_q describes the growth of the number of free Taylor coefficients in the general solution relative to a function of n variables. We expand $Z_q(r)$ in powers of $1/r$

$$Z_q(r) = Z_q^{(0)} + Z_q^{(1)}/r + O(1/r^2)\,. \tag{6.12}$$

Definition 6.3 *We call* $Z_q^{(0)}$ *the* compatibility coefficient *and* $Z_q^{(1)}$ *the* strength *of* $\mathcal{R}_q$. *An equation with* $Z_q^{(0)} = 0$ *is said to be* absolutely compatible.

Einstein introduced the strength only for absolutely compatible equations. It is, however, well-defined for any differential equation. The next theorem relates it to the Cartan characters and thus to the results of Chapter 3.

Theorem 6.4 *The compatibility coefficient and the strength of an involutive equation* $\mathcal{R}_q$ *with Cartan characters* $\alpha_q^{(k)}$ *are given by*

$$Z_q^{(0)} = \alpha_q^{(n)}\,, \tag{6.13}$$

$$Z_q^{(1)} = (n-1)\left(\tfrac{1}{2}n\,\alpha_q^{(n)} + \alpha_q^{(n-1)}\right). \tag{6.14}$$

Proof. Using (3.11) and (C.2), $Z_q^{(0)}$ is readily obtained by taking the limit $r \to \infty$. Subtracting $Z_q^{(0)}$, multiplying (6.12) by r and taking again the limit yields $Z_q^{(1)}$. •

If there exists an algebraic representation of the general solution, then we can express the compatibility coefficient and the strength through the f_k instead of the $\alpha_q^{(k)}$:

$$\begin{aligned} Z_q^{(0)} &= f_n\,, \\ Z_q^{(1)} &= (n-1)\left(\tfrac{1}{2}(n+2q)f_n + f_{n-1}\right). \end{aligned} \tag{6.15}$$

Thus a system is absolutely compatible, if it is not under-determined. For such a system the strength measures the dominant term in the Hilbert polynomial, the coefficient of r^{n-2}.

Since the strength is usually computed for gauge systems, we should correct it for the gauge symmetry. Using the notation of Section 6.1, we obtain:

$$\begin{aligned} \bar{Z}_q^{(0)} &= \alpha_q^{(n)} - \sum_{l=0}^{p} \gamma_l\,, \\ \bar{Z}_q^{(1)} &= (n-1)\Big(\frac{1}{2}n\alpha_q^{(n)} + \alpha_q^{(n-1)} - \sum_{l=0}^{p}(\frac{1}{2}n + q + l)\gamma_l\Big)\,. \end{aligned} \tag{6.16}$$

Alternatively, we could apply Theorem 6.4 using the gauge corrected Cartan characters. Absolute compatible means now that after the gauge correction no arbitrary functions of n arguments are admitted in the general solution.

Finally, we comment on some remarks by Sué [147]. He compared the index of generality (or degré d'arbitraire) (cf. Section 3.4) with the strength and came to the conclusion that the latter one is more useful when dealing with gauge system. He compares, however, a gauge corrected strength with a not gauge corrected index of generality! As soon as we correct both, the result is somewhat different. For most systems of physical interest both contain the same amount of information, as for them $\bar{\alpha}_q^{(n)} = 0$ and $\bar{\alpha}_q^{(n-1)} \neq 0$. But if we encounter a system with $\bar{\alpha}_q^{(n-1)} = 0$, then the index of generality contains further information about the lower Cartan characters which are not available using the strength. Sué did not realize that the reason to consider only the highest non-vanishing Cartan character is not a lack of information about the system but Theorem 3.12. If we know the index of generality, then we always know the Cartan genus, too. The strength, however, gives the Cartan genus only, if it does not vanish.

6.3 Constrained Dynamics I — Point Mechanics

The classical treatment of systems with constraints was developed by Dirac [43, 44, 45] (see also [65, 148]). The purpose of this section is to show that in the case of finite-dimensional systems his algorithm corresponds exactly to rendering the equations of motion involutive. In the language of exterior differential systems this was already noted by Hartley *et al.* [68]. We will, however, see in the next section that in the case of infinite-dimensional systems the Dirac analysis alone is in general not sufficient to prove that the field equations are consistent, as it involves only part of the completion procedure of Section 2.4.

Let the bundle $\mathcal{E}$ be given by $\mathcal{E} = Q \times T$, where Q is the N-dimensional configuration space with coordinates q^i and T the time axis, and the natural projection $\mathcal{E} \to T$. A dynamical system can then be specified by a function on $J_1\mathcal{E}$, its *Lagrangian* $L(q^i, \dot{q}^i)$. For simplicity, we assume that L does not depend explicitly on t. The dynamics of the system is determined by the condition that the action

$$S = \int L(q^i, \dot{q}^i)\, dt \tag{6.17}$$

is stationary along trajectories $q^i(t)$. It is well-known from the calculus of variation that this leads to the *Euler-Lagrange Equations*

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^i}\right) - \frac{\partial L}{\partial q^i} = 0\,, \qquad i = 1, \dots, N\,. \tag{6.18}$$

(6.18) represents a second-order equation whose symbol is determined by the matrix

$$M_{ij} = \frac{\partial^2 L}{\partial \dot{q}^i \partial \dot{q}^j}\,. \tag{6.19}$$

If the symbol has rank N, the Euler-Lagrange Equations are normal and no constraints occur. Its general solution is parameterized by $2N$ arbitrary constants.

If, however, the symbol has lower rank, it is possible to eliminate the second-order derivatives in some of the equations. Now it is no longer obvious, whether or not (6.18) is involutive. Since we are dealing with an ordinary differential equation, the symbol is always involutive. But the prolongation of the lower order equations might lead to integrability conditions, if the resulting equations are independent of the remaining second-order equations in (6.18). Then we have to check whether some of these conditions are again of lower order; in that case we have to repeat the procedure. After a finite number of iterations we will obtain either an involutive equation or an inconsistency.

We have seen in Section 6.1 that a gauge system must be under-determined. If it comprises at least as many equations as dependent variables, this implies that the system cannot be normal, i.e. there are equations of either lower order (that is for ordinary differential equations the only possibility) or of lower class. Thus we see that there is a intimate connection between gauge symmetries and constraints.

To relate this approach to constrained dynamics to the classical one by Dirac we must pass by a Legendre transformation to the Hamiltonian formulation. At the level of the differential equations this means that instead of the configuration space Q the phase space P is used to construct $\mathcal{E}$. In other words, we introduce N additional dependent variables p_i and transform (6.18) into a first-order equation

$$\mathcal{R}_1 \,:\, \begin{cases} \dot{p}_i = \dfrac{\partial L}{\partial q^i}\,, \\[2mm] p_i = \dfrac{\partial L}{\partial \dot{q}^i}\,, \end{cases} \qquad i = 1, \dots, N\,. \tag{6.20}$$

The first set of equations consists of course just of the Euler-Lagrange Equations with the second set of equations plugged in.

If the matrix M_{ij} has full rank, the second set of equations can be solved for the $\dot{q}^i$ and (6.20) is a normal equation. Otherwise we obtain some algebraic equations of the form $\phi_\ell(q,p) = 0$. Dirac called these *primary constraints*. Just as in the Lagrangian formulation, involution of (6.20) depends on the behavior of their prolongation[1]

$$D_t\phi_\ell = \left(\frac{\partial\phi_\ell}{\partial q^i}\dot{q}^i + \frac{\partial\phi_\ell}{\partial p_i}\dot{p}_i\right) = 0\,. \tag{6.21}$$

We are only interested in these equations restricted to $\mathcal{R}_1$. This corresponds to the "weak equalities" considered by Dirac.

For each of the equations in (6.21) there are three possibilities for the result of the restriction: (i) it yields an inconsistency, i.e. an equation of the form $1 = 0$; (ii) it vanishes, i.e. it becomes $0 = 0$; (iii) we obtain a new constraint, a *secondary constraint* in Dirac's terminology. We will exclude (i), as it implies that L is not a valid Lagrangian. (iii) can be further divided into two cases: the secondary constraint is either algebraic or it can be solved for a velocity $\dot{q}^j$ yielding an additional differential equation. Dirac introduced here the notion of *first* and *second class constraints*, respectively.

If secondary constraints appear, we must repeat the procedure to check the consistency of the equations of motion (6.20). After a finite number of steps we will either have found an inconsistency or we will have constructed an involutive equation of the form

$$\mathcal{R}_1^{(s)}\,:\,\begin{cases} \dot{p}_i = \dfrac{\partial L}{\partial q^i}\,, & i = 1,\dots,N\,,\\[2ex] \dot{q}^j = f^j(q,\dot{q},p)\,, & j = 1,\dots,M\,,\\[2ex] \phi_\ell(q,p) = 0\,, & \ell = 1,\dots,K\,. \end{cases} \tag{6.22}$$

To make contact with the classical Dirac algorithm we introduce the so-called *canonical Hamiltonian*

$$H_C = p_i\dot{q}^i - L(q,\dot{q})\,. \tag{6.23}$$

Its differential is given by

$$dH_C = \left[\dot{q}^i dp_i - \frac{\partial L}{\partial q^i}dq^i + \left(p_i - \frac{\partial L}{\partial \dot{q}^i}\right)d\dot{q}^i\right]\,; \tag{6.24}$$

thus on $\mathcal{R}_1$ we obtain

$$dH_C\big|_{\mathcal{R}_1} = \left(\dot{q}^i dp_i - \frac{\partial L}{\partial q^i}dq^i\right)\,. \tag{6.25}$$

This implies that on $\mathcal{R}_1$ we can substitute $\dot{q}^i$ by $\partial H_C/\partial p_i$ and $\dot{p}_i$ by $-\partial H_C/\partial q^i$. This allows to express the prolongation of the constraints in the familiar way through *Poisson brackets*

$$D_t\phi_\ell\big|_{\mathcal{R}_1} = \left(\frac{\partial\phi_\ell}{\partial q^i}\frac{\partial H_c}{\partial p_i} - \frac{\partial\phi_\ell}{\partial p_i}\frac{\partial H_c}{\partial q^i}\right) = \{\phi_\ell, H_C\}\,. \tag{6.26}$$

We see that indeed the Dirac algorithm coincides with the completion algorithm outlined in Section 2.4. Note that it is of no importance here whether we use the canonical Hamiltonian (6.23) or the so-called total one which contains in addition the primary constraints, as we always work on $\mathcal{R}_1$ where both coincide.

[1] We use from now the Einstein convention that a summation is understood over repeated indices, if not explicitly stated otherwise.

One of the fundamental goals in constrained dynamics is to count the number of degrees of freedom of the system. We will give now an alternative method using the formal analysis of (6.22) to express this number by intrinsic quantities. We have $\beta_1^{(1)} = N + M$ and hence $\alpha_1^{(1)} = N - M$. Thus the general solution of (6.22) depends on $N - M$ arbitrary functions.

In the classical language this arbitrariness stems from the gauge transformations generated by the first class constraints. To obtain the number of degrees of freedom we must count the number of constants necessary to characterize a physical state. Since $\dim J_1\mathcal{E} = 4N$, we obtain $\dim \mathcal{R}_1^{(s)} = 3N - M - K$. Thus in a power series expansion of the general solution this number of zeroth and first order coefficients can be chosen arbitrarily.

If we identify the arbitrary functions with $\dot{q}^i$ for $i > M$, as these are not restricted by (6.22), then we must subtract $N - M$ constants coming from the arbitrary functions. Thus an initial state at $t = t_0$ is specified by $2N - K$ constants. Depending on the choice of the arbitrary functions we will, however, obtain different values for $q^i(t_1)$ and $p_i(t_1)$ at some later instant t_1. As these correspond nevertheless, by definition, to the same physical state, we must subtract further $N - M$ constants for the gauge symmetry. Thus a physical state is specified by $N + M - L$ constants and the number of degrees of freedom F is half of this number

$$F = \frac{1}{2}(N + M - K)\,. \tag{6.27}$$

Expressing M by N and $\alpha_1^{(1)}$ and similarly K by N, $\alpha_1^{(1)}$ and $\dim \mathcal{R}_1^{(s)}$, we obtain an intrinsic expression for F independent of any specific representation of the manifold $\mathcal{R}_1^{(s)}$

$$F = \frac{1}{2}\dim \mathcal{R}_1^{(s)} - \alpha_1^{(1)}\,. \tag{6.28}$$

Note that this expression remains valid, if we use instead of the Hamiltonian equations of motion (6.20) the Euler-Lagrange Equations (6.18), as we will obtain exactly the same dimension and Cartan character for the final involutive equation no matter whether we work in first- or second-order because of the different dimensions of the base spaces.

Classically F is determined by the number of first and second class constraints, N_F and N_S, respectively. These numbers can easily be written in intrinsic terms, too. Comparing $F = N - N_F - 1/2N_S$ with (6.28) yields at once

$$N_F = \alpha_1^{(1)}\,, \quad N_S = 2N - \dim \mathcal{R}_1^{(s)}\,. \tag{6.29}$$

By the same argument as above this result remains valid in the Lagrangian formalism.

Example 6.5 Consider the classical problem of a particle whose movement is restricted to the surface of a sphere in a D-dimensional space but otherwise free, often also called the rigid rotator [52]. Without loss of generality we can take the radius of the sphere as one and start with the Lagrangian

$$L = \frac{m}{2}\dot{q}^2 + \lambda(q^2 - 1) \tag{6.30}$$

where λ is a multiplier. The canonically conjugate momenta p_i, π are given by

$$\begin{aligned} p_i &= m\dot{q}_i\,, \qquad i = 1, \dots, D\,, \\ \pi &= 0\,. \end{aligned} \tag{6.31}$$

If we introduce a further multiplier μ, we can write the total Hamiltonian as

$$H_T = \frac{1}{2m}p^2 - \lambda(q^2 - 1) + \mu\pi \tag{6.32}$$

Obviously there is one primary constraint, namely $\pi = 0$. The next steps of the Dirac algorithm lead to the constraints $q^2 = 1$, $pq = 0$, and finally $p^2 = -2m\lambda$. It is easy to see that all these constraints are second class. The system contains thus $D-1$ degrees of freedom which can be formally calculated by subtracting from the dimension of the configuration space — D coordinates q_i plus one coordinate λ — half the number of second class constraints, i.e. 2.

The last constraint plus the primary one allow us to eliminate λ and its conjugate momentum π from the theory. The other two could in principle be solved for some conjugate pair q_i, p_i. We will, however, compute instead for later use the corresponding Dirac brackets [45]. A straightforward calculation yields

$$\begin{aligned} \{q_i, q_j\}_* &= 0\,, \\ \{q_i, p_j\}_* &= \delta_{ij} - q_i q_j\,, \\ \{p_i, p_j\}_* &= q_j p_i - p_j q_i\,. \end{aligned} \tag{6.33}$$

Now we will obtain the number of degrees of freedom using a formal analysis of the Euler-Lagrange Equations

$$\mathcal{R}_2 \,:\, \begin{cases} m\ddot{q}_i - 2\lambda q_i = 0\,, \\ q^2 - 1 = 0\,. \end{cases} \tag{6.34}$$

The completion to an involutive equation requires four projections leading to the integrability conditions $q\dot{q} = 0$, $q\ddot{q} + \dot{q}^2 = 0$, $\dot{\lambda} = 0$ and finally $\ddot{\lambda} = 0$. After some trivial manipulations we have thus the involutive equation

$$\mathcal{R}_2^{(4)} \,:\, \begin{cases} m\ddot{q}_i - 2\lambda q_i = 0\,, \\ \ddot{\lambda} = 0\,, \\ m\dot{q}^2 + 2\lambda = 0\,, \\ q\dot{q} = 0\,, \\ \dot{\lambda} = 0\,, \\ q^2 - 1 = 0\,. \end{cases} \tag{6.35}$$

It is easy to see that this represents a finite type equation and thus there are no first class constraints. Since $\dim \mathcal{R}_2^{(4)} = 2D - 2$, (6.28) yields $F = D - 1$ degrees of freedom. Furthermore we obtain from (6.29) that there are four second class constraints. Both results are in perfect agreement with the Dirac analysis.

6.4 Constrained Dynamics II — Field Theory

In field theories the equations of motion become partial differential equations. Thus involution is now a more complicated concept. The prolongation of lower order equations is no longer the only way to generate integrability conditions. Constraints are now mostly equations of lower class. In a typical field theory the base space X of independent variables is a D-dimensional space-time. We can identify the variable x^D with time. Thus simplified spoken, the equations of class D are the evolutionary ones; the remaining ones are constraints.

The relation between the Dirac algorithm and our completion procedure becomes now much less clear than in the case of point mechanics. Dirac prolongs all constraints only with respect to time. If all constraints are of class $D-1$, this corresponds to our approach, because by Proposition 2.9 it suffices to analyze the prolongations with respect to the non-multiplicative variables and we find in this case only one, namely x^D.

But now the question arises what happens if constraints of lower class appear. Then prolongations with respect to the other multiplicative variables, i.e. with respect to some spatial coordinates, may lead to additional integrability conditions not considered by the Dirac algorithm.

Example 6.6 In order to exhibit this effect in "pure form" we begin with a highly unphysical example without any kinematical term in the Lagrangian density. But we will later show that this is not the important point. Consider the system described by the Lagrangian density

$$\mathcal{L} = \mu[\partial_x\phi - f(\phi)] + \lambda[\partial_y\phi - g(\phi)] \tag{6.36}$$

on a three-dimensional space-time with coordinates x, y, t. μ, λ are two multipliers, f, g are fixed but so far arbitrary functions. Variation with respect to ϕ yields the equation

$$\partial_x\mu + \partial_y\lambda + f'(\phi)\mu + g'(\phi)\lambda = 0\,. \tag{6.37}$$

More interesting equations are obtained from the multipliers. They generate an over-determined system for ϕ

$$\begin{aligned} \partial_x\phi - f(\phi) &= 0\,, \\ \partial_y\phi - g(\phi) &= 0\,. \end{aligned} \tag{6.38}$$

Obviously this system is consistent, if and only if f, g satisfy the equation

$$f'g = fg'\,. \tag{6.39}$$

This requires that f is a multiple of g. It is easy to see that under this condition the Euler-Lagrange Equations are involutive.

Now we look at the outcome of the Dirac algorithm. Obviously all three canonically conjugate momenta $\pi_\phi, \pi_\mu, \pi_\lambda$ vanish and represent thus the primary constraints. The total Hamiltonian density is given by

$$\mathcal{H}_T = -\mu[\partial_x\phi - f(\phi)] - \lambda[\partial_y\phi - g(\phi)] + u\pi_\phi + v\pi_\mu + w\pi_\lambda\,. \tag{6.40}$$

This yields the following secondary constraints

$$\begin{aligned} \{\pi_\phi, \mathcal{H}_T\} &= -[\partial_x\mu + \partial_y\lambda + f'(\phi)\mu + g'(\phi)\lambda]\,, \\ \{\pi_\mu, \mathcal{H}_T\} &= \partial_x\phi - f(\phi)\,, \\ \{\pi_\lambda, \mathcal{H}_T\} &= \partial_y\phi - g(\phi)\,. \end{aligned} \tag{6.41}$$

Thus we obtain exactly the Euler-Lagrange Equations above. But note the crucial difference in the further analysis. Following Dirac we look only whether these equations are *algebraically* related, i.e. whether one vanishes, if we take the others into account. But this does not happen here.

Since all constraints are second class, we continue to compute the tertiary constraints in order to fix the multipliers u, v, w introduced with the total Hamiltonian density. They are

$$\begin{aligned} &\partial_x v + \partial_y w + f''(\phi)\mu u + f'(\phi)v + g''(\phi)\lambda u + g'(\phi)w = 0\,, \\ &\partial_x u - f'(\phi)u = 0\,, \\ &\partial_y u - g'(\phi)u = 0\,. \end{aligned} \tag{6.42}$$

We find now the interesting phenomenon that although there are six second class constraints for three degrees of freedom ϕ, μ, λ, they do not fix all multipliers. v, w appear only in the first equation, hence one of them can be chosen arbitrarily. u is the solution of an over-determined system which is consistent, if f and g satisfy (6.39).

This example shows that the real failure of the Dirac algorithm lies in the purely algebraic treatment of the constraints. In point mechanics we could always substitute one constraint in another one, if both contained the same coordinate q^i or a (time) derivative of it. This is no longer possible in field theories, because there might be derivatives with respect to different coordinates present. To check the consistency of such constraints requires the analysis of integrability conditions. But then the question arise why treat the time derivatives different from the spatial ones? It is conceptually simpler to treat everything on the same footing.

Example 6.6 (cont.) One might argue that (6.36) is a rather peculiar Lagrangian density. But the addition of a kinetic term for ϕ does not really change the outcome, although the computations are slightly more complicated due to the appearance of further integrability conditions. Take

$$\bar{\mathcal{L}} = \frac{1}{2}(\partial_t \phi)^2 + \mathcal{L}\,. \tag{6.43}$$

Its Euler-Lagrange Equations are

$$\begin{aligned} &\partial_{tt}\phi + \partial_x \mu + \partial_y \lambda + f'(\phi)\mu + g'(\phi)\lambda = 0\,, \\ &\partial_x \phi - f(\phi) = 0\,, \\ &\partial_y \phi - g(\phi) = 0\,. \end{aligned} \tag{6.44}$$

Only the first equation has changed. But since the only second-order derivative is one of ϕ, our completion procedure generates besides (6.39) two further integrability conditions involving second-order derivatives of μ, λ. We omit them here, because they are rather complicated.

We also do not show the Dirac analysis of this system. It suffices to note that the two new conditions (in Hamiltonian form) are also found by the Dirac algorithm. But this was to be expected, because their construction requires prolongations with respect to time. Only (6.39) is again overlooked, for no spatial cross-derivatives are performed.

The natural question now is whether systems of this form are unphysical for some reason or whether this effect occurs often and might lead to a wrong analysis. Such purely spatial cross-derivatives (or to be more precise: linear combinations of spatial prolongations) are only necessary, if the field equations form an over-determined system. This might appear at first sight quite unusual but one can find such systems in the literature. A typical example is a gauge fixing condition built into the Lagrangian density.

On the other hand, it is difficult to overlook this effect, for it shows of course in the constraints. It was already mentioned above that in principle one could still use the Dirac algorithm but that one must augment it with a kind of involution analysis, if constraints like in Example 6.6 appears. But again: why not use the involution analysis from the very beginning?

Another point which might be worth while mentioning concerns covariance. It is well-known that the introduction of a Hamiltonian breaks covariance, as a time coordinate must be explicitly chosen. But using our involution based approach it is at least possible to exhibit the constraint structure in a covariant way by completing the Euler-Lagrange Equation.

Given a Lagrangian density $\mathcal{L}(\phi_a, \partial_\mu \phi_a)$ the Legendre transformation is usually performed by introducing the momenta

$$\pi^a = \frac{\partial \mathcal{L}}{\partial \partial_0 \phi_a}\,. \tag{6.45}$$

In general, the Hamiltonian field equations are then still second order because of the spatial derivatives. To obtain a first-order description one must define additional "momenta" for them. But this can be done covariantly by writing

$$\pi^a_\mu = \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi_a}\,. \tag{6.46}$$

This transformation yields a "covariant form" of the Hamiltonian field equations which can then be completed until involution.

6.5 Constrained Dynamics III — Symplectic Formalism

To conclude our analysis of systems with constraints we want to comment on the so-called *Hamiltonian* or *symplectic formalism* of Faddeev and Jackiw [51]. It was first applied by Floreanini and Jackiw [54] for the quantization of self-dual fields in two dimensions. Their circumvention of the classical Dirac algorithm sparked some discussion [12, 32] whether or not it is correct and necessary, respectively. For simplicity we will restrict most of our exposition to finite-dimensional systems.

This formalism is applicable to first-order Lagrangians, i.e. they are linear in the velocities:

$$L = a_i(q)\dot{q}^i - V(q)\,. \tag{6.47}$$

The corresponding Euler-Lagrange Equations are

$$\mathcal{R}_1 \,:\, \left\{ A_{ij}\dot{q}^j - \frac{\partial V}{\partial q^i} = 0\,, \quad i = 1,\dots,N \right. \tag{6.48}$$

where the matrix A is given by cross-derivatives of the a_i

$$A_{ij} = \frac{\partial a_j}{\partial q^i} - \frac{\partial a_i}{\partial q^j}\,. \tag{6.49}$$

Obviously, $\mathcal{R}_1$ is an involutive first-order equation provided this matrix is non-singular. Hence from the formal point of view there is no need to introduce any constraints; the system is unconstrained.

If one follows the Dirac algorithm, one is forced to introduce N constraints, as all canonically conjugate momenta are independent of the velocities. Mathematically this corresponds to insisting on treating (6.48) as a *second-order* differential equation. Since $\mathcal{R}_1$ is involutive, its prolongation is involutive, too, and no secondary constraints arise. Furthermore, all constraints are second class, for (6.48) is of course a finite-type equation.

The decisive observation of Faddeev and Jackiw [51] was that it is unnecessary to introduce the canonically conjugate momenta, because (6.48) has already the structure of a Hamiltonian system with Hamiltonian V provided we define the Poisson bracket by

$$\left\{ q^j, q^i \right\} = (A^{-1})^{ij}\,. \tag{6.50}$$

This definition makes sense as long as the matrix A is non-singular. Cronström and Noga [35] demonstrated recently, how one can explicitly construct the corresponding canonical coordinates using essentially only linear algebra.

We will show later that exactly the same bracket structure arises, if one follows Dirac and eliminates the second class constraints using Dirac brackets. This was already noted by Govaerts [61] and earlier explicitly demonstrated for the special Lagrangian treated in [54] by Costa and Girotti [32].

In the case that the matrix A is singular, we have "true" constraints. Faddeev and Jackiw resort then to Darboux's theorem to eliminate them explicitly. If this elimination is too complicated, they recommend to apply the usual Dirac algorithm. At least for the construction of all constraints, there is no need for this. Since we are dealing with a system of ordinary differential equations, we have always an involutive symbol and our completion algorithm will never require a prolongation. Thus we can always obtain an involutive, *first-order* equation.

The only point is that it is now no longer possible to use the matrix A to define a symplectic structure. But even if one wants to follow Dirac's approach, it is computationally much simpler to complete a first-order system and to prolong it to second-order afterwards than the other way round. The true constraints are algebraic equations even in the Lagrangian formalism, and the other constraints are just prolongations of them.

On the other hand it depends on the constraint structure whether one gains something by transforming a second-order Lagrangian into a first-order one by extending the configuration space. This amounts essentially to transforming the Euler-Lagrange Equations (6.18) into a first-order equation along the lines of Appendix C.2. One obtains of course the same number of degrees of freedom, since this reduction does not change the Cartan characters. Computationally one might save a few steps compared to the Dirac algorithm, especially if second class constraints are present.

For systems having only second class constraints one can use an idea of Barcelos-Neto and Wotzasek [7, 8] to compute the Dirac brackets in a direct manner. Thus we consider now the case that the matrix A has not maximal rank. Let a basis of its null space be given by the vectors v^i_α, i.e.

$$v^i_\alpha(q)A_{ij}(q) = 0\,, \quad \alpha = 1, \ldots, M\,^2\,. \tag{6.51}$$

This implies the existence of M "primary" constraints

$$\phi_\alpha = v^i_\alpha \frac{\partial V}{\partial q^i} = 0\,. \tag{6.52}$$

Following Barcelos-Neto and Wotzasek we introduce M additional coordinates λ^α and continue with the modified Lagrangian

$$\tilde{L} = \left(a_i + \lambda^\alpha \frac{\partial \phi_\alpha}{\partial q^i}\right)\dot{q}^i - V\,. \tag{6.53}$$

The new equations of motion are

$$\tilde{\mathcal{R}}_1 : \begin{cases} A_{ij}\dot{q}^j + \dfrac{\partial \phi_\alpha}{\partial q^i}\dot{\lambda}^\alpha - \dfrac{\partial V}{\partial q^i} = 0\,, \\ \dfrac{\partial \phi_\alpha}{\partial q^i}\dot{q}^i = 0\,. \end{cases} \tag{6.54}$$

Let us assume that the second set of equations is independent of the original equations of motions, i.e. that the prolongation of the constraints leads to new ones. (6.54) can be written again in the form of (6.48) on the enlarged configuration space, if we introduce the "deformed" $(N+M)\times(N+M)$ matrix $\tilde{A}$

$$\begin{aligned} \tilde{A}_{ij} &= \frac{\partial a_i}{\partial q^j} - \frac{\partial a_j}{\partial q^i}\,, \\ \tilde{A}_{i\alpha} &= -\frac{\partial \phi_\alpha}{\partial q^i}\,, \\ \tilde{A}_{\alpha\beta} &= 0\,. \end{aligned} \tag{6.55}$$

If the matrix $\tilde{A}$ is non-singular, Barcelos-Neto and Wotzasek use it to define a Poisson structure. Otherwise they repeat the process. There are two possible outcomes: either one obtains after a finite number of iterations a regular matrix or the process stops, because the constraints do not generate new ones. In the latter case gauge symmetries, i.e. first class

[2] It could of course happen, that M varies on the configuration space. But we will assume that it is constant.

constraints, are present. We will concentrate on the first case and assume for simplicity that already $\tilde{A}$ has maximal rank.

From the point of view of formal theory this assumption implies that the modified Euler-Lagrange Equations $\tilde{\mathcal{R}}_1$ are involutive. Indeed they are now even normal, as contraction of the first set of equations with the vectors v^i_α leads to the equations

$$v^i_\alpha \frac{\partial \phi_\beta}{\partial q^i} \dot{\lambda}_\beta - \phi_\alpha = 0\,. \tag{6.56}$$

The condition on the rank of $\tilde{A}$ implies that the matrix

$$B_{\alpha\beta} = v^i_\alpha \frac{\partial \phi_\beta}{\partial q^i} \tag{6.57}$$

has maximal rank. Thus we can obtain the solved form (4.15) used in the Cauchy-Kowalevsky Theorem by a simple linear transformation.

To prove that the inverse of $\tilde{A}$ yields the correct Dirac brackets we compute them in the standard way. As already mentioned we find N primary constraints

$$\psi_i = p_i - a_i(q) = 0\,. \tag{6.58}$$

The total Hamiltonian of the system is given by

$$H_T = V(q) + \mu^i \psi_i\,. \tag{6.59}$$

The primary constraints lead to the consistency conditions

$$\{\psi_i, H_T\} = A_{ij}\mu^j - \frac{\partial V}{\partial q^i} = 0\,. \tag{6.60}$$

If the matrix A is non-singular, these secondary constraints can be used to determine all multipliers μ^i and the Dirac algorithm stops here. In the case of a singular matrix we can solve only for some of them, as contraction with v^i_α yields the "true" constraints ϕ_α. Their consistency conditions

$$\{\phi_\alpha, H_T\} = \mu^i \frac{\partial \phi_\alpha}{\partial q^i} \tag{6.61}$$

allow us to determine the remaining multipliers under the above made assumptions. Thus tertiary constraints do not appear.

To determine the Dirac bracket we must compute the structure constants of the Poisson algebra generated by the constraints

$$\begin{aligned} \{\psi_i, \psi_j\} &= A_{ij}\,, \\ \{\psi_i, \phi_\alpha\} &= -\frac{\partial \phi_\alpha}{\partial q^i}\,, \\ \{\phi_\alpha, \phi_\beta\} &= 0\,. \end{aligned} \tag{6.62}$$

Note that despite its appearance this is a closed algebra under our assumptions. In the right hand sides we rediscover the matrix $\tilde{A}$ defined above by the modified Lagrangian $\tilde{L}$.

Example 6.5 (cont.) To treat the rigid rotator with this formalism we must first rewrite its Lagrangian as a first order one. This is most easily done by using the momenta as additional "configuration" space coordinates. Furthermore it is simpler to replace the multiplier λ by the time derivative of another one [8] (this corresponds to adding a total derivative). We will, however, continue to call it λ and obtain thus the Lagrangian

$$L = p\dot{q} + \dot{\lambda}(q^2 - 1) - \frac{1}{2}p^2 \,. \tag{6.63}$$

It leads to the constraint $pq = 0$ and to the modified Lagrangian $\tilde{L} = L + \dot{\mu}pq$. The corresponding matrix $\tilde{A}$ is given by (if we label the columns q_i, p_i, λ, μ)

$$\tilde{A} = \begin{pmatrix} 0 & -\delta_{ij} & q_i & p_i \\ \delta_{ij} & 0 & 0 & q_i \\ -q_j & 0 & 0 & 0 \\ -p_j & -q_j & 0 & 0 \end{pmatrix} \tag{6.64}$$

and is obviously non-singular. Thus this approach is computationally much simpler than the Dirac algorithm. Inversion of $\tilde{A}$ yields exactly the Dirac brackets (6.33).

Barcelos-Neto and Wotzasek omit in their papers [7, 8] a discussion of the precise relation between the equations of motion (6.48) derived from the original Lagrangian L and those (6.54) obtained from the modified one $\tilde{L}$. This is, however, quite important for understanding the meaning of the multipliers λ^α.

For this purpose we introduce vectors w^i_ρ, $\rho = M+1, \ldots, N$ such that $\{v^i_\alpha, w^i_\rho\}$ is a linearly independent set. Under the above mentioned assumptions we find the following local representation of the equation $\mathcal{R}_1^{(1)}$ obtained from completing (6.48) consisting only of independent equations

$$\mathcal{R}_1^{(1)} \;:\; \begin{cases} w^i_\rho \left(A_{ij}\dot{q}^j - \dfrac{\partial V}{\partial q^i} \right) = 0 \,, \\ \dfrac{\partial \phi_\alpha}{\partial q^i}\dot{q}^i = 0 \,, \\ \phi_\alpha = 0 \,. \end{cases} \tag{6.65}$$

Similarly we obtain a local description of $\tilde{\mathcal{R}}_1$ containing under our assumptions only independent equations, too,

$$\tilde{\mathcal{R}}_1 \;:\; \begin{cases} w^i_\rho \left(A_{ij}\dot{q}^j + \dfrac{\partial \phi_\alpha}{\partial q^i}\dot{\lambda}^\alpha - \dfrac{\partial V}{\partial q^i} \right) = 0 \,, \\ \dfrac{\partial \phi_\alpha}{\partial q^i}\dot{q}^i = 0 \,, \\ v^i_\beta \dfrac{\partial \phi_\alpha}{\partial q^i}\dot{\lambda}^\alpha + \phi_\beta = 0 \,. \end{cases} \tag{6.66}$$

Although both represent finite type equations note the crucial difference that $\tilde{\mathcal{R}}_1$ is normal and does not contain algebraic equations. This entails especially a relaxation of the constraints: we do no longer require that $\phi_\alpha = 0$ but only $\phi_\alpha = const$! Furthermore while the solution space of $\mathcal{R}_1^{(1)}$ is $(N-M)$-dimensional, $\dim \tilde{\mathcal{R}}_1 = N + M$. But this discrepancy is removed as soon as we require that $\lambda^\alpha(t) \equiv const$, because then both systems are identical and possess of course the same solutions.

Thus we can conclude that the geometric idea behind this approach is the embedding of the original system into another one living in a larger phase space. Projecting the trajectories of the latter one on a leave of the foliation $\lambda^\alpha = const$ recovers the solutions of the former one. The choice of the leave is not important, for $\tilde{\mathcal{R}}_1$ is invariant under translations in λ^α.

Example 6.7 Consider a system with a three-dimensional configuration space whose dynamics is determined by the Lagrangian [6, 91]

$$L = (q_2 + q_3)\dot{q}_1 + k\dot{q}_3 + \frac{1}{2}(k^2 - 2q_2q_3 - q_3^2) \tag{6.67}$$

where k is a constant. In the formal analysis we start with the Euler-Lagrange Equations. After a trivial algebraic manipulation they can be written as

$$\mathcal{R}_1 : \begin{cases} \dot{q}_2 + \dot{q}_3 = 0\,, \\ \dot{q}_1 - q_3 = 0\,, \\ q_2 = 0\,. \end{cases} \tag{6.68}$$

Obviously this equation becomes involutive after one projection adding the integrability condition $\dot{q}_2 = 0$. It is also trivial to integrate it in closed form

$$q_1(t) = at + b\,, \quad q_2(t) = 0\,, \quad q_3(t) = a\,. \tag{6.69}$$

For the Dirac analysis we need the canonically conjugate momenta

$$p_1 = q_2 + q_3\,, \quad p_2 = 0\,, \quad p_3 = k\,. \tag{6.70}$$

These are at the same time the primary constraints. The total Hamiltonian is

$$H_t = \frac{1}{2}(q_3^2 + 2q_2q_3 - k^2) + \mu_1(p_1 - q_2 - q_3) + \mu_2 p_2 + \mu_3(p_3 - k)\,. \tag{6.71}$$

This leads to the secondary constraints $\mu_2 + \mu_3 = 0$, $q_3 = \mu_1$ and $q_2 = 0$. The tertiary constraints are $\mu_2 = \mu_3 = 0$. Hence we have four second class constraints $p_2 = q_2 = 0$ and $p_3 = k, q_3 = p_1$. By direct inspection one sees that there is only one dynamical degree of freedom, namely (q_1, p_1), having as Dirac bracket its standard Poisson bracket.

In the symplectic formalism one finds in the first step the "true" constraint $q_2 = 0$ and continues with the modified Lagrangian $\tilde{L} = L + \lambda\dot{q}_2$. From it we derive the matrix $\tilde{A}$

$$\tilde{A} = \begin{pmatrix} 0 & -1 & -1 & 0 \\ 1 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \tag{6.72}$$

where the columns and rows are labeled by q_1, q_2, q_3, λ. This matrix is obviously invertible and yields the correct bracket, if we identify q_3 with the momentum p_1. If we compute, however, the equations of motion, we obtain after some trivial manipulations

$$\tilde{\mathcal{R}}_1 : \begin{cases} \dot{q}_3 = 0\,, \\ \dot{q}_2 = 0\,, \\ \dot{q}_1 - q_2 - q_3 = 0\,, \\ \dot{\lambda} - q_2 = 0\,. \end{cases} \tag{6.73}$$

As expected the equation $q_2 = 0$ is missing and the solution space is consequently larger

$$q_1(t) = (\bar{a} + \bar{b})t + \bar{c}\,, \quad q_2(t) = \bar{b}\,, \quad q_3(t) = \bar{a}\,, \quad \lambda(t) = \bar{b}t + \bar{d}\,. \tag{6.74}$$

However, as soon as we require that $\lambda(t) \equiv const$ we recover the correct solution, as this implies $\bar{b} = 0$.

This example also serves well to demonstrate the difference between first- and second-order systems. Actually it stems from a slightly more complicated example where k is not treated as constant but as an additional coordinate [75]. But the real starting point is the following system of second-order equations

$$\ddot{x} = -\dot{y}\,, \quad \ddot{y} = -y\,. \tag{6.75}$$

One can prove that these equations neither are nor can be transformed into the Euler-Lagrange Equations of some Lagrangian [76]. But if we rewrite the system as a first-order one

$$\dot{x} = z\,, \quad \dot{y} = w\,, \quad \dot{z} = -w\,, \quad \dot{w} = -y\,, \tag{6.76}$$

then a Lagrangian exist, namely

$$L = (y+z)\dot{x} + w\dot{z} + \frac{1}{2}(w^2 - 2yz - z^2)\,. \tag{6.77}$$

A normal form can be obtained by setting

$$q^1 = x\,, \quad q^2 = z\,, \quad p_1 = y + z\,, \quad p_2 = w\,. \tag{6.78}$$

Note, however, that we are now using as configuration space coordinates x and $\dot{x}$, if we compare with the original second-order system. In first-order systems the distinction between configuration and phase space begins to blur. This is the basis of the symplectic method.

Barcelos-Neto and Wotzasek [7] apply their method also to field theories. But again there are problems with systems like the ones discussed in Example 6.6. Now they are worse, because the algorithm does no longer necessarily terminate even if there are only second class constraints present. The reason is obviously that only the prolongation of the constraints with respect to time is considered in the construction of the "deformed" Lagrangian density, as it contains only the time derivative of the new multiplier appears. However, it might be possible to design a modification of this approach which also uses the spatial derivatives of the multipliers.

Example 6.6 (cont.) Consider the following slight modification of the first-order Lagrangian density (6.36)

$$\mathcal{L} = \nu\partial_t\phi + \mu[\partial_x\phi - f(\phi)] + \lambda[\partial_y\phi - g(\phi)]\,. \tag{6.79}$$

ν is an additional multiplier. If we order the fields ϕ, ν, μ, λ the matrix A is given by

$$A = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}\,. \tag{6.80}$$

Thus there exist two eigenvectors and we obtain the two obvious constraints

$$\Omega_1 = \partial_x\phi - f(\phi)\,, \quad \Omega_2 = \partial_y\phi - g(\phi)\,. \tag{6.81}$$

Following the same steps as in the finite-dimensional case, we are lead to consider the modified Lagrangian density

$$\tilde{\mathcal{L}} = \nu\partial_t\phi + \partial_t\alpha[\partial_x\phi - f(\phi)] + \partial_t\beta[\partial_y\phi - g(\phi)] \tag{6.82}$$

where α, β are the new multipliers. The disappearance of the multipliers μ, λ is not surprising, because they remained completely arbitrary in the previous Euler-Lagrange Equations. But we have gained nothing by this deformation, as the new matrix $\tilde{A}$ still has only rank 2 and the only difference to the previous field equations is that we get now the prolongations of the constraints with respect to time. However, we know from our previous discussion that these are unimportant.

6.6 Yang-Mills Theory

In this and the following sections we will change our notation and use the standard notions of theoretical physics. This includes the Einstein convention that summation over repeated indices is always understood if not stated explicitly otherwise. Indices $\mu, \nu, \dots$ denote now no longer multi-indices but space-time indices; $a, b, c, \dots$ are used for Lie algebra indices. Since the metric is usually not important in the formal analysis, we will sometimes be a bit sloppy with raising and lowering indices. Most of the time we will assume that the metric has the form $\operatorname{diag}(\eta_\mu)$ without fixing the signature.

We consider the field equations of a Yang-Mills theory with a d-dimensional gauge group in an n-dimensional space-time. The dependent variables are the $m = dn$ components A^a_μ of the vector potential. Using the field strengths $F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + C^a_{bc} A^b_\mu A^c_\nu$ where C^a_{bc} are the structure constants of the Lie algebra of the gauge group we get dn field equations

$$\phi^a_\nu[A^b_\mu] = \partial_\mu F^a_{\mu\nu} + C^a_{bc} A^b_\mu F^c_{\mu\nu} = 0\,, \qquad \begin{cases} a = 1, \dots, d\,, \\ \nu = 1, \dots, n\,. \end{cases} \tag{6.83}$$

In order to check for involution and to compute the Cartan characters, we must first analyze the symbol or the principal part of (6.83). It is given by $\partial_{\mu\mu} A^a_\nu - \partial_{\mu\nu} A^a_\mu$. Thus for $\nu \neq n$ we can always take $\partial_{nn} A^a_\nu$ as principal derivative. If however $\nu = n$, then this term disappears and we get $\partial_{n,n-1} A^a_{n-1}$ as principal derivative. This yields the values $\beta^{(n)}_{YM} = d(n-1)$, $\beta^{(n-1)}_{YM} = d$. All other $\beta^{(k)}_{YM}$ vanish. Note that this result is indeed not influenced by the metric.

Since in principle the maximal possible value for $\beta^{(n)}_{YM}$ is dn, we must check carefully whether the lower value obtained is intrinsic or only the consequence of a not δ-regular coordinate system. For this purpose we determine the tableau of (6.83). Following Definition 2.23 it is given by

$$\tau_\chi \,:\, A^a_\mu \mapsto \left(\frac{\partial \phi^b_\nu}{\partial(\partial_{\tau\rho} A^a_\mu)}\right) \chi_\rho \chi_\tau A^a_\mu\,. \tag{6.84}$$

(6.83) yields for its matrix

$$\mathcal{T}^{b\mu}_{a\nu}[\chi] = \delta^b_a \Big[\delta^\mu_\nu \Big(\sum_{\rho \neq \nu} \eta_\rho \chi^2_\rho\Big) + (\delta^\mu_\nu - 1)\eta_\mu \chi_\nu \chi_\mu\Big]\,. \tag{6.85}$$

The indices b, ν label the rows, a, μ the columns. It is easy to see that for any values of a, b, ν

$$\chi_\mu \mathcal{T}^{b\mu}_{a\nu}[\chi] = 0\,. \tag{6.86}$$

This equation yields d identities and we obtain an upper bound for the rank of the matrix $\mathcal{T}[\chi]$, namely $d(n-1)$. But this provides at the same time an upper bound for $\beta^{(n)}_{YM}$ due to (2.27). Since our analysis above yielded this value, our coordinate system is δ-regular and the computed values for $\beta^{(k)}_{YM}$ are correct.

But the identities (6.86) further implies that the prolonged symbol does not have maximal rank. Indeed at the level of the symbol, multiplication with χ_μ corresponds to prolonging with respect to x^μ. Now the prolonged symbol is a matrix with dn^2 rows and thus the maximal value for its rank is given by $d(n^2 - 1)$. On the other hand, we know from the obtained $\beta^{(k)}_{YM}$ that this is also the minimal value for its rank. Thus we can conclude that the symbol of the Yang-Mills Equations is involutive.

For the Cartan characters we get

$$\alpha^{(n)}_{YM} = d, \quad \alpha^{(n-1)}_{YM} = d(2n-1), \quad \alpha^{(n-k)}_{YM} = dn(k+1)\,. \tag{6.87}$$

We recover of course the identities (6.86) also at the level of the differential equations. Since the symbol has not maximal rank, we must show that the corresponding integrability conditions are satisfied as a consequence of the field equations (6.83). After a somewhat lengthy computation one can show that

$$\partial_\nu \phi^a_\nu + C^a_{bc} A^b_\nu \phi^c_\nu = 0 \tag{6.88}$$

holds as a consequence of the properties of the C^a_{bc}. Thus involution of the Yang-Mills Equations (6.83) is equivalent to C^c_{ab} being the structure constants of a Lie algebra. (6.88) are often called *Noether identities*. If we introduce an external current J^a_ν at the right hand side of (6.83), then a solution can only exist, if it satisfies the compatibility condition

$$\mathcal{D}_\nu J^a_\nu = \partial_\nu J^a_\nu + C^a_{bc} A^b_\nu J^c_\nu = 0\,, \tag{6.89}$$

i.e. its covariant derivative must vanish.

Since by (6.86) every one-form χ is characteristic, the Yang-Mills Equations are underdetermined as expected. This stems of course from the invariance of (6.83) under the gauge transformations

$$\bar{A}^a_\mu = A^a_\mu + \partial_\mu \lambda^a + C^a_{bc} A^b_\mu \lambda^c \tag{6.90}$$

with $\gamma_1 = d$ gauge functions λ^a. This yields a gauge corrected strength

$$\bar{Z}^{(1)}_{YM} = 2(n-1)(n-2)d\,. \tag{6.91}$$

For simplicity, we restrict ourselves for the remainder of the section to $n = 4$. (3.7) leads to the Hilbert polynomial

$$H_{YM}(r) = 36d + \frac{73}{3}dr + \frac{9}{2}dr^2 + \frac{1}{6}dr^3\,. \tag{6.92}$$

Adjusting for the gauge symmetry (6.90) yields the corrected polynomial

$$\bar{H}_{YM}(r) = 16d + 12dr + 2dr^2 \tag{6.93}$$

and the corrected Cartan characters

$$\bar{\alpha}^{(4)}_{YM} = 0, \quad \bar{\alpha}^{(3)}_{YM} = 4d, \quad \bar{\alpha}^{(2)}_{YM} = \alpha^{(1)}_{YM} = 6d. \tag{6.94}$$

We see that the dimension d of the gauge group appears everywhere linearly. Thus we get just d-times the values of the Maxwell Equations. This implies that the results are independent of the gauge group, as the structure constants of the algebra do not appear in the symbol. Especially it makes no difference whether we treat an Abelian or a non-Abelian theory.

To conclude the analysis of the Yang-Mills equations, we consider as an example of a gauge fixing the *Lorentz gauge* $\partial_\mu A^a_\mu = 0$. Adding this condition to the field equations (6.83) leads to a simple over-determined system with the Cartan characters

$$\alpha^{(4)}_{Lor} = 0, \quad \alpha^{(3)}_{Lor} = 4d, \quad \alpha^{(2)}_{Lor} = 12d, \quad \alpha^{(1)}_{Lor} = 16d\,. \tag{6.95}$$

A comparison with (6.94) shows that although the first two characters have the same values as the gauge corrected ones, the lower two are too large. Thus the Lorentz condition does not completely fix the gauge.

6.7 Electrodynamics

In the special case of the Maxwell Equations there exists also a first-order formulation using the field strengths as dependent variables. In potential formulation there are 4 dependent variables A_μ and the same number of second-order equations

$$\partial_\mu\partial_\mu A_\nu - \partial_\mu\partial_\nu A_\mu = 0\,; \tag{6.96}$$

in field strength formulation there are 6 dependent variables $F_{\mu\nu}$ and the same number of first-order equations

$$\begin{aligned} \partial_\mu F_{\mu\nu} &= 0\,, \\ \partial_{[\rho}F_{\mu\nu]} &= 0\,. \end{aligned} \tag{6.97}$$

The second set of equations represent the conditions for the existence of a potential A_μ such that

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{6.98}$$

We find $\alpha_P^{(1)} = 16, \alpha_P^{(2)} = 12, \alpha_P^{(3)} = 7, \alpha_P^{(4)} = 1$ for the vector potential formulation (6.96) and $\alpha_F^{(1)} = \alpha_F^{(2)} = 6, \alpha_F^{(3)} = 4, \alpha_F^{(4)} = 0$ for the field strength formulation (6.97). Theorem 3.9 yields for the number of arbitrary functions in an algebraic representation of a general solution $f_1 = f_2 = -1, f_3 = 5, f_4 = 1$ for (6.96) and $f_1 = f_4 = 0, f_2 = 2, f_3 = 4$ for (6.97). Thus only the field strength formulation admits such a representation of its general solution containing four arbitrary functions of three variables and two arbitrary functions of two variables. If we allow one-dimensional integrals, too, we can find for the potential formulation the following solution of the condition $H_q(r) = T_q(r)$

$$\begin{aligned} f_{1,0} = f_{2,0} = f_{3,0} = f_{4,0} = 1\,, \\ f_{2,-1} = 2,\ f_{3,-1} = 4\,. \end{aligned} \tag{6.99}$$

But this is an obvious consequence of the definition of the field strength. If we consider (6.98) as a system of differential equations for the A_μ by given $F_{\mu\nu}$, then its general solution has an algebraic representation consisting of the four functions entering algebraically in (6.99). The other six functions are just the arbitrary functions occurring in the general solution of Equation (6.97).

These four additional arbitrary functions reflect of course the gauge freedom (6.90) present in the vector potential but not in the field strength formulation. Indeed Theorem 6.1 yields with $\gamma_1 = 1$ the gauge corrections $g_1 = g_2 = g_3 = g_4 = 1$. Subtracting them leads to the same result as for the field strength formulation. More generally, we find that correcting the Cartan characters of the potential formulation for the gauge symmetry yields exactly the values $\alpha_F^{(k)}$ showing the equivalence of the two formulations.

To demonstrate the use of Lie pseudogroups, we compute now this correction intrinsically. For $d = 1$ the gauge transformation (6.90) are the general solution of the first order differential equation

$$\mathcal{G}_1 : \left\{ \begin{array}{ll} \dfrac{\partial \bar{x}^\mu}{\partial x^\nu} = \delta^\mu_\nu\,, & \dfrac{\partial \bar{x}^\mu}{\partial A_\nu} = 0\,, \\[2ex] \dfrac{\partial \bar{A}_\mu}{\partial x^\nu} = \dfrac{\partial \bar{A}_\nu}{\partial x^\mu}\,, & \dfrac{\partial \bar{A}_\mu}{\partial A_\nu} = \delta^\nu_\mu\,. \end{array} \right. \tag{6.100}$$

(For simplicity, we have allowed rigid translations of the underlying space-time). It is easy to see that this represents an involutive equation with Cartan characters

$$\alpha_G^{(k)} = \left\{ \begin{array}{ll} 0 & k = 5, \ldots, 8\,, \\ 5 - k & k = 1, \ldots, 4\,. \end{array} \right. \tag{6.101}$$

This yields for the Hilbert polynomial of the Lie pseudogroup

$$H_1^{(G)}(r) = 10 + \frac{47}{6}r + 2r^2 + \frac{1}{6}r^3 \tag{6.102}$$

and hence

$$G_2(r) = H_1^{(G)}(r+1) = 20 + \frac{37}{3}r + \frac{5}{2}r^2 + \frac{1}{6}r^3 . \tag{6.103}$$

Subtracting this from the Hilbert polynomial of the potential formulation yields the gauge corrected Hilbert polynomial (6.93) which equals the Hilbert polynomial of the field strength formulation.

Using the methods outlined in Section 3.4, one could easily produce more solutions of the diophantine system (3.23). A few of them are discussed in [138]. But in the case of the field strength formulation it is easily possible to produce the general solution of (3.23). We consider only the case $J = \{0, -1\}$, i.e. representations of the solutions containing algebraic functions and one-dimensional integrals. After canceling a few unnecessary integer factors we obtain the following system

$$\begin{array}{rcl} f_{3,0} + f_{3,1} &=& 4\,, \\ 2f_{2,0} + 2f_{2,1} + 5f_{3,0} + 3f_{3,1} &=& 24\,, \\ f_{1,0} + f_{1,1} + 2f_{2,0} + f_{2,1} + 3f_{3,0} + f_{3,1} &=& 16\,. \end{array} \tag{6.104}$$

(Here we have already taken into account that the Maxwell Equations have a Cartan genus of 3). All non-negative integer solution of this system are given by

$$\begin{array}{ll} f_{3,0} = 4 - f_{3,1}\,, & f_{3,1} = 0, \dots, 4\,, \\ f_{2,0} = 2 + f_{3,1} - f_{2,1}\,, & f_{2,1} = 0, \dots, f_{3,1} + 2\,, \\ f_{1,0} = f_{2,1} - f_{1,1}\,, & f_{1,1} = 0, \dots, f_{2,1}\,. \end{array} \tag{6.105}$$

One can easily show that there are exactly 80 different solutions. A similar analysis of the vector potential formulation yields more than 30.000 solutions, if we take $J = \{0, -1, -2\}$!

If we are only interested in the total number of arbitrary functions with a given number of arguments, then we obtain

$$\tilde{f}_3 = 4, \quad \tilde{f}_2 = 2 + f_{3,1}, \quad \tilde{f}_1 = f_{2,1}\,. \tag{6.106}$$

Thus there are 25 different sets of values for the $\tilde{f}_k$. For the arbitrary functions of one argument, it does not make much sense to distinguish between different differentiation orders. In any case such a function contributes one arbitrary Taylor coefficient at each order. One only changes the number of arbitrary constants. Adjusting for this, we see that there are essentially only 25 different possibilities for the $f_{k,j}$.

Finally, we give a formally well-posed initial value problem for the field strength formulation. For an easier interpretation we use the electric and the magnetic fields **E** and **B** and we choose $x^4 = t$. Then one has for each field an evolutionary equation

$$\partial_t \mathbf{E} = \operatorname{curl} \mathbf{B}\,, \quad \partial_t \mathbf{B} = -\operatorname{curl} \mathbf{E} \tag{6.107}$$

and additionally the two constraints

$$\operatorname{div} \mathbf{E} = 0\,, \quad \operatorname{div} \mathbf{B} = 0\,. \tag{6.108}$$

We choose a distinguished spatial direction, say the x-direction, and solve the constraints for the corresponding derivatives. Thus there are two principal derivatives belonging to the x-components of the fields and they can be prescribed only on the y-z-plane. All other fields can be given in the whole space.

6.8 Gravity

We will treat Einstein's vacuum field equations for an n-dimensional space-time

$$R_{\mu\nu}[g_{\rho\sigma}] = 0 \tag{6.109}$$

where $R_{\mu\nu}$ denotes the Ricci tensor. If we take the components of the metric tensor $g_{\rho\sigma}$ as dependent variables, then because of the symmetries of Ricci tensor and metric, respectively, (6.109) represents $n(n+1)/2$ equations for as many dependent variables. Since two-dimensional gravity is trivial, we will assume $n > 2$.

Expanding (6.109) in a general coordinate system in terms of $g_{\mu\nu}$ leads to very complicated expressions. It is nearly impossible to check this way, whether or not the Einstein equations form an involutive system. A closer look at Chapter 2 reveals, however, that all definitions there are local, i.e. they can be applied in a point. We will exploit this fact by choosing a coordinate system which yields in one point x_0 of space-time an especially simple form of the metric and its derivatives, namely *locally geodesic coordinates*. It is well-known, that in these coordinates the Christoffel symbols vanish at x_0 and $g_{\mu\nu}(x_0) = \eta_{\mu\nu}$, the constant metric of Minkowski space.

In locally geodesic coordinates, the Einstein Equations (6.109) take in x_0 the form [145]

$$R_{\mu\nu}(x_0) = \frac{1}{2}\eta^{\rho\sigma}\left[\partial_{\mu\rho}g_{\nu\sigma}(x_0) + \partial_{\nu\sigma}g_{\mu\rho}(x_0) - \partial_{\mu\nu}g_{\sigma\rho}(x_0) - \partial_{\sigma\rho}g_{\mu\nu}(x_0)\right] = 0\,. \tag{6.110}$$

Furthermore the first prolongation is also very simple, as the metricity of the Levi-Civita connection $D_\rho g_{\mu\nu} = 0$ implies that $\partial_\rho g_{\mu\nu}(x_0) = 0$.

$$\partial_\tau R_{\mu\nu}(x_0) = \frac{1}{2}\eta^{\rho\sigma}\left[\partial_{\tau\mu\rho}g_{\nu\sigma}(x_0) + \partial_{\tau\nu\sigma}g_{\mu\rho}(x_0) - \partial_{\tau\mu\nu}g_{\sigma\rho}(x_0) - \partial_{\tau\sigma\rho}g_{\mu\nu}(x_0)\right] = 0\,. \tag{6.111}$$

Note, however, that the higher prolongations become more complicated, as then further derivatives of $g_{\mu\nu}$ not vanishing in x_0 must be taken into account.

We will proceed as for the Yang-Mills Equations using the tableau to prove involution of the symbol and δ-regularity of the coordinate system. Because of the symmetry one can assume $\mu \geq \nu$. We distinguish four cases in the choice of the principal derivatives.

(i) $\mu \neq n$ and $\nu \neq n$: The principal derivative is $\partial_{nn}g_{\mu\nu}$.

(ii) $\mu = n$ and $\nu < n-1$: The principal derivative is $\partial_{n,n-1}g_{n-1,\nu}$.

(iii) $\mu = n$ and $\nu = n-1$: The principal derivative is $\partial_{n,n-1}g_{11}$.

(iv) $\mu = n$ and $\nu = n$: The principal derivative is $\partial_{n-1,n-1}g_{11}$.

Besides the last case this can be seen at once. For (iv) several substitutions of other equations are necessary. This yields $\beta_{Ein}^{(n)} = n(n-1)/2$, $\beta_{Ein}^{(n-1)} = n$. The other $\beta_{Ein}^{(k)}$ vanish[3].

The matrix of the tableau of (6.109) is given by

$$\begin{aligned}\mathcal{T}_{\alpha\beta}^{\mu\nu}[\chi] &= 4\frac{\partial R_{\alpha\beta}}{\partial(\partial_{\kappa\lambda}g_{\mu\nu})}\chi_\kappa\chi_\lambda \\ &= \delta_\beta^\mu\chi^\nu\chi_\alpha + \delta_\alpha^\mu\chi^\nu\chi_\beta + \delta_\beta^\nu\chi^\mu\chi_\alpha + \delta_\alpha^\nu\chi^\mu\chi_\beta - 2\eta^{\mu\nu}\chi_\alpha\chi_\beta - \chi^2(\delta_\alpha^\mu\delta_\beta^\nu + \delta_\alpha^\nu\delta_\beta^\mu)\,.\end{aligned} \tag{6.112}$$

[3] If one uses the full Einstein tensor $G_{\mu\nu}$ instead of the Ricci tensor, one obtains the same values for the $\beta_{Ein}^{(k)}$. However, in (iv) another principal derivative must be chosen. A cosmological term does not influence the symbol.

Recalling the symmetries of $R_{\mu\nu}$ and $g_{\mu\nu}$ it is a square matrix with $n(n+1)/2$ rows and columns. α, β label the rows, μ, ν the columns. The matrix has not full rank, as for any values of α, μ, ν the identity

$$\mathcal{T}^{\mu\nu}_{\alpha\beta}[\chi]\chi^\beta - \frac{1}{2}\mathcal{T}^{\mu\nu}_{\gamma\beta}[\chi]\eta^{\gamma\beta}\chi_\alpha = 0 \tag{6.113}$$

holds.

Thus the maximal rank of $\mathcal{T}[\chi]$ cannot exceed $n(n-1)/2$. Comparing with the above obtained value for $\beta^{(n)}_{Ein}$ shows that locally geodesic coordinates are indeed δ-regular and hence that we have got the correct values. Furthermore, we can show as in the case of the Yang-Mills Equations that the symbol is involutive. The Cartan characters are consequently

$$\alpha^{(n)}_{Ein} = n, \quad \alpha^{(n-1)}_{Ein} = n^2, \quad \alpha^{(n-k)}_{Ein} = \frac{n(n+1)(k+1)}{2}\,. \tag{6.114}$$

Since x_0 was an arbitrary point, these results hold on the entire space-time.

At the level of the differential equations we recover (6.113) in the so-called *contracted Bianchi identities*

$$\partial_\nu(R^\nu_\mu) - \partial_\mu(R^\nu_\nu) = \Gamma^\rho_{\nu\mu}R^\nu_\rho - \Gamma^\nu_{\nu\rho}R^\rho_\mu\,, \tag{6.115}$$

where Γ denotes the Christoffel symbols. Here we have fully written out the covariant derivative to see that indeed the lower order terms are algebraically dependent of the field equations. Thus no integrability conditions arise and the Einstein equations form an involutive system. Like the Yang-Mills Equations they are under-determined, as every one-form χ is characteristic.

To prove the involution of the Einstein Equations it is not necessary to use the identities (6.115). Instead, we can even give a simple indirect proof of them, for it suffices to show that the integrability conditions vanish in locally geodesic coordinates. But this is obvious, as the equations in (6.111) do not contain lower order derivatives. Thus the number of independent equations in (6.111) is given by the rank of the prolonged symbol and Theorem 2.7 proves that there arise no integrability conditions. Note that it suffices to calculate the number of independent equations in a point, because the Einstein Equations describe a manifold in the jet bundle, i.e. the dimension is the same at every point.

Again the under-determinacy stems from a gauge invariance. (6.109) is invariant under arbitrary coordinate changes, i.e. under gauge transformations[4]

$$g_{\mu\nu} = \frac{\partial\bar{x}^\rho}{\partial x^\mu}\frac{\partial\bar{x}^\sigma}{\partial x^\nu}\bar{g}_{\rho\sigma}\,, \tag{6.116}$$

with $\gamma_1 = n$ gauge functions $\bar{x}^\mu$. This yields a gauge corrected strength

$$\bar{Z}^{(1)}_{Ein} = n(n-1)(n-3)\,. \tag{6.117}$$

Analogously to our previous analysis of the Lorentz gauge, we can consider *harmonic coordinates*, i.e. we pose the gauge condition

$$\Box x^\mu = \frac{1}{\sqrt{-g}}\partial_\nu\left(\sqrt{-g}g^{\mu\nu}\right) = 0\,. \tag{6.118}$$

[4]Note that "gauge transformations" have been defined in Section 6.1 purely mathematically. Physically we are dealing here with trivial coordinate transformations. To write gravity as a gauge theory requires the introduction of a frame bundle with $SO(n-1,1)$ as structure group.

In locally geodesic coordinates for x_0 this is equivalent to $\partial_\nu g_{\mu\nu}(x_0) = 0$. After prolonging to second order, we get thus n^2 additional equations to (6.110). One sees easily that always n of them are in the same class. Hence, the Cartan characters of the combined system are

$$\alpha_H^{(n)} = 0, \quad \alpha_H^{(n-1)} = n(n-1), \quad \alpha_H^{(n-k)} = \frac{n^2(k+1) + n(k-1)}{2}. \tag{6.119}$$

Thus (6.118) provides no complete gauge fixing, because already $\alpha_H^{(n-1)}$ is too big, as we obtain for the highest gauge corrected Cartan characters

$$\bar{\alpha}_{Ein}^{(n)} = 0, \quad \bar{\alpha}_{Ein}^{(n-1)} = n(n-3). \tag{6.120}$$

Specializing to $n = 4$ we obtain for the Hilbert polynomial

$$H_{Ein}(r) = 90 + \frac{184}{3}r + 12r^2 + \frac{2}{3}r^3. \tag{6.121}$$

Gauge correction yields

$$\bar{H}_{Ein}(r) = 10 + 12r + 2r^2. \tag{6.122}$$

This is almost the same result as for the Maxwell Equations in four dimensions. Only the constant term is larger in (6.93). Correspondingly all the gauge corrected Cartan characters are identical with the exception of $\bar{\alpha}_{Ein}^{(1)} = 0$ instead of 6. This implies especially that both equations have the same strength. The general expressions for the strength, (6.91) and (6.117), respectively, show that this is only the case in four dimensions. This perhaps surprising result has lead to some number mysticism [115] that it "explains" why we live in a four-dimensional world. Taking the complete set of Cartan characters into account the argument looses considerably. Note that we have got here an example of an equation where gauge correction leads to a set of Cartan characters which do not satisfy Proposition 3.5. Thus it is not possible to fix the gauge completely using differential equations.