Chapter 7

Miscellaneous Applications

We will analyze various examples of partial differential equations arising in mathematical or physical applications from a formal point of view. These include complete and Jacobian systems, topologically massive models in three dimensions and two-dimensional dilaton-gravity. Finally we will study an example stemming from variational calculus.

7.1 Complete and Jacobian Systems

As a simple application of the formal theory, we consider complete and Jacobian systems. Besides being an illustrative example this class is quite important, as for instance the construction of invariants leads to complete systems (cf. Section 5.2). Furthermore, as we will see later in this section, the integration of a system of finite type can always be reduced to the integration of a complete system. But any complete system can be integrated by solving only ordinary differential equations. The classical treatment of this class of equations stems mainly from Jacobi and Clebsch [31]. Our presentation follows [22, 97]. We will then show how it can be developed from the point of view of formal theory.

We confine ourselves to linear homogeneous first-order systems with $m = 1$ dependent variables. These equations can be written using vector fields on the base manifold X:

$$V_\tau u = \sum_{i=1}^{n} a_\tau^i(x)\frac{\partial u}{\partial x^i} = 0\,, \qquad \tau = 1,\dots,p\,. \tag{7.1}$$

We assume that the V_τ are linearly independent. System (7.1) is called *complete*, if they form at each point of X a Lie algebra

$$[V_\rho, V_\sigma] = \sum_{\tau=1}^{p} C_{\rho\sigma}^{\tau}(x)V_\tau\,. \tag{7.2}$$

If a commutator is not a linear combination of the vector fields V_τ, we add it to the system. Then we look again whether the resulting system is complete. After a few iteration this process will lead to a complete system, as there exist only n linear independent vector fields on X. Thus we can assume without loss of generality that (7.1) is complete.

Renumbering the independent variables we can always achieve that it is possible to solve the first equation in (7.1) for $\partial u/\partial x^n$, the second one for $\partial u/\partial x^{n-1}$ etc. This leads to an equivalent system generated by vector fields of the form

$$W_\tau = \frac{\partial}{\partial x^{n-\tau+1}} + \sum_{i=1}^{n-p} b^i_\tau(x)\frac{\partial}{\partial x^i}\,. \tag{7.3}$$

These fields obviously satisfy the trivial commutation relations

$$[W_\rho, W_\sigma] = 0\,. \tag{7.4}$$

Systems built with commuting vector fields are called *Jacobian systems.*

Proposition 7.1 *A complete system can be integrated by solving systems of ordinary differential equations. If it is generated by p independent vector fields on an n-dimensional manifold X, then it possesses $n-p$ functionally independent solutions $\psi_k(x)$. Its general solution is given by $\Psi(\psi_k)$ with an arbitrary function Ψ.*

Proof. If our system consists of only one equation

$$Vu = \sum_{i=1}^{n} b^i(x)\frac{\partial u}{\partial x^i} = 0\,, \tag{7.5}$$

it is well-known that its general solution can be obtained by solving its *characteristic system* $dx^i/dt = b^i$ and that it is of the form stated in the proposition.

If we have more than one equation, we have seen above that we can always assume without loss of generality that the system is Jacobian. We start with V_1 and compute its general solution. Let it be spanned by the $n-1$ functions $\psi^j_1(x)$ for $j = 1,\dots,n-1$. Then we transform the independent coordinates according to

$$\bar{x}^n = x^n\,, \quad \bar{x}^j = \psi^j_1(x)\,. \tag{7.6}$$

(Due to the special form (7.3) of the vector fields, this provides indeed an invertible change of coordinates). A short calculation shows that $\bar{V}_1 = \partial/\partial\bar{x}^n$. Since commutators remain invariant under coordinate transformations, the $\bar{V}_\tau$ still commute with $\bar{V}_1$. But this implies that their coefficients are independent of $\bar{x}^n = x^n$. Thus there remains a complete system with one equation less in the independent variables $x^1,\dots,x^{n-1}$.

Repeating this process p times leads to the trivial Jacobian system generated by the vector fields $\bar{V}_\tau = \partial/\partial\bar{x}^{n-\tau+1}$. Obviously its general solution is given by $\Psi(\bar{x}^1,\dots,\bar{x}^{n-p})$. Inverting the performed coordinate changes yields the claim. •

For concrete calculations this proof is not very useful. One can achieve the same with much less ordinary differential equations. One defines the characteristic system of the Jacobian system generated by (7.3) as the following system of $n-p$ total differential equations

$$\omega^k = dx^k - \sum_{\tau=n-p+1}^{n} b^k_\tau dx^\tau = 0\,, \quad k = 1,\dots,n-p\,. \tag{7.7}$$

In the language of Appendix A this represents a Pfaffian system and is nothing else then the annihilator of the Lie algebra spanned by the vector fields W_τ. Note that the special solved form of (7.7) is obtained only for Jacobian systems.

With a trick due to Mayer [107] it can be transformed into a system of ordinary differential equations. One chooses a point $x_0 \in X$ and introduces p parameter λ^τ and a new variable t by writing $x^\tau = x_0^\tau + t\lambda^\tau$ for $\tau = n-p+1,\dots,n$. Since now $dx^\tau = \lambda^\tau dt$, we can rewrite the Pfaffian system (7.7) as a system of ordinary differential equations

$$\frac{dx^i}{dt} = \sum_{\tau=n-p+1}^{n} \lambda^\tau b_\tau^i(x^k, x_0^\sigma + t\lambda^\sigma)\,, \quad \begin{cases} i,k = 1,\dots,n-p\,, \\ \sigma = n-p+1,\dots,n\,. \end{cases} \tag{7.8}$$

The solution of this system will be of the form $x^i = \varphi^i(t, \lambda^\tau, x_0^\tau)$. To obtain a solution of the original system of partial differential equations substituting λ^τ by $(x^\tau - x_0^\tau)/t$ must yield an expression independent of t. A rather lengthy computation proves that this happens, if and only if the original system is complete.

From the point of view of formal theory, we see that the transformation of a complete system into a Jacobian system consists just of the operations needed to put the symbol in solved form. (7.3) yields at once $\beta_{Jac}^{(n)} = \beta_{Jac}^{(n-1)} = \cdots = \beta_{Jac}^{(n-p+1)} = 1$ and all the lower $\beta_{Jac}^{(k)}$ vanish. To check involution of the symbol we must study what happens if we prolong an equation with respect to a non-multiplicative variable (Proposition 2.9). For the equation $W_\rho u = 0$, the variables $x^n, \dots, x^{n-\rho+2}$ are non-multiplicative; let $x^{n-\sigma+1}$ be one of them. Prolongation with respect to it yields an equation with the leading derivative $p_{n-\sigma+1,n-\rho+1}$. The only way to cancel it is to subtract the equation $D_{n-\rho+1}W_\sigma u = 0$. Similarly one can see that it is possible to eliminate all second-order derivatives and that the left hand side of the resulting equation is just $[W_\rho, W_\sigma]u$. Hence prolongations with respect to non-multiplicative variables do not contribute to the prolonged symbol and according to our criterion (2.19) we have an involutive symbol.

In order to have no integrability conditions we must require that the above obtained commutators are all linear combinations of the equations $W_\tau u = 0$, but this means nothing else than that the vector fields W_τ form a Lie algebra. Thus any complete system is involutive. This also explains why in differential geometry a system of vector fields is said to be in involution, if they form a Lie algebra at every point.

We obtain for the Cartan characters

$$\alpha_{Jac}^{(n)} = \cdots = \alpha_{Jac}^{(n-p+1)} = 0\,, \quad \alpha_{Jac}^{(n-p)} = \cdots = \alpha_{Jac}^{(1)} = 1\,. \tag{7.9}$$

According to Corollary 3.10 an algebraic representation of the general solution must contain one arbitrary function of $n-p$ arguments in perfect agreement with Proposition 7.1. Note, however, that this corollary does not prove the existence of a solution in contrast to our analysis here!

Applying Definition 2.24 of a characteristic one-form we find that the forms contained in the characteristic system (7.7) build a basis of the characteristic variety C of the Jacobian system generated by (7.3). Indeed the tableau of this system is given by

$$\mathcal{T}_\tau[\chi] = \chi_{n-\tau+1} + \sum_{i=1}^{n-p} b_\tau^i \chi_i\,. \tag{7.10}$$

Its solution space is obviously spanned by the forms ω_k in (7.7).

One can partially extend these results to non-linear first-order systems using the so-called Jacobi brackets. If F, G are arbitrary functions of x^i, u, p_i, then their Jacobi bracket is defined by

$$[F,G] = \sum_{i=1}^{n} \Big[F_{p_i}(G_{x^i} + p_i G_u) - G_{p_i}(F_{x^i} + p_i F_u)\Big]\,. \tag{7.11}$$

It has similar properties as the commutator of vector fields but obeys a different Jacobi identity. One easily sees that in terms of the formal derivative

$$[F,G] = \sum_{i=1}^{n} \Big(F_{p_i} D_i G - G_{p_i} D_i F \Big) . \tag{7.12}$$

A non-linear system $F_\tau(x^i, u, p_i)$ is involutive, if and only if

$$[F_\rho, F_\sigma] = 0 . \tag{7.13}$$

This can be seen in the same way as before in the linear case. We note further that we get exactly the same Cartan characters and hence the same arbitrariness of the general solution as before.

To conclude this section we show how the integration of a finite type equation, i.e. of an equation with a finite-dimensional solution space, can be reduced to the integration of a complete system. Finite type equations have among others two important applications: They can be used to describe foliations of a manifold and to define Bäcklund transformations.

Proposition 7.2 *Any involutive equation of finite type can be transformed into an equivalent complete system. Thus it can be integrated by solving ordinary differential equations.*

Proof. By Appendix C.2 any equation can be rewritten as a first-order equation. Thus we can assume that our finite type equation is of order one and has the following form

$$\mathcal{R}_1 \,:\, \Big\{ p_i^\alpha = \varphi_i^\alpha(x^j, u^\beta) , \quad \alpha = 1, \dots, m, \; i = 1, \dots, n . \tag{7.14}$$

In general, a finite type system can of course also contain lower order equations. Here these would, however, be algebraic equations, which could be solved for some u^α and thus be used to eliminate these leaving a system of the form (7.14).

Let us now consider the following vector fields

$$X_i = \frac{\partial}{\partial x^i} + \sum_{\alpha=1}^{m} \varphi_i^\alpha \frac{\partial}{\partial u^\alpha} . \tag{7.15}$$

One readily calculates

$$[X_i, X_j] = \sum_{\alpha=1}^{m} (D_i \varphi_j^\alpha - D_j \varphi_i^\alpha) \frac{\partial}{\partial u^\alpha} . \tag{7.16}$$

By assumption $\mathcal{R}_1$ is involutive. This implies that the terms in the parentheses must vanish on $\mathcal{R}_1$, as they represent integrability conditions. Hence the vector fields X_i commute.

We claim now that any solution of the Jacobian system $X_i f = 0$ yields a solution of $\mathcal{R}_1$ and vice versa. $X_i f = 0$ represents a system of n equations with the $n+m$ independent variables x^i and u^α. According to Proposition 7.1 it has m independent solutions $F^\beta(x^i, u^\alpha)$. Introducing m arbitrary constants c^β we solve the equations $F^\beta(x,u) = c^\beta$ for the variables u^α obtaining $u^\alpha = \psi^\alpha(x^i, c^\beta)$. One can see from the proof of Proposition 7.1 that it is indeed always possible to solve for the u^α due to the special form of the vector fields X_i. Since

$$\frac{\partial F^\beta}{\partial x^i} + \sum_{\alpha=1}^{m} \frac{\partial F^\beta}{\partial u^\alpha} \frac{\partial \psi^\alpha}{\partial x^i} = 0 , \tag{7.17}$$

and we can conclude that $u^\alpha - \psi^\alpha(x^i, c^\beta)$ is also a solution of the Jacobian system. But this means nothing else then

$$\frac{\partial \psi^\alpha}{\partial x^i} = \varphi_i^\alpha(x^j, \psi^\beta)\,, \tag{7.18}$$

i.e. ψ^α is a solution of $\mathcal{R}_1$.

Conversely, if $u^\alpha = \psi^\alpha(x^i)$ is a solution of $\mathcal{R}_1$, then

$$X_i\Big(u^\alpha - \psi^\alpha(x^j)\Big) = \varphi_i^\alpha(x^j, \psi^\beta) - \frac{\partial \psi^\alpha}{\partial x^i} = 0\,. \tag{7.19}$$

Thus any solution of $\mathcal{R}_1$ yields a solution of $X_i f = 0$. •

This proof stems from Lie [97]. He formulated it directly for systems of arbitrary order and for systems containing lower order equations. This, however, unnecessarily complicates the notation. The basic idea of the proof is the observation that the vector fields X_i generate symmetries of the equation $\mathcal{R}_1$ (cf. Chapter 5). The solutions F^β of the Jacobian system are differential invariants of this symmetry group. In the theory of Bäcklund transformations the X_i are used to obtain zero-curvature representations of partial differential equations [16].

7.2 Topologically Massive Models in 3 Dimensions

Deser *et al.* [40, 41] proposed the addition of a topologically motivated, gauge invariant term, namely a Chern-Simons form, to the actions of Yang-Mills Theories and general relativity, respectively, in three dimensions (these terms exist only in odd dimensions). This leads to theories with quite distinct properties from their four-dimensional analogues. Another motivation to study such three-dimensional models is the fact that they appear in the high-temperature limit of four-dimensional ones.

We start with Yang-Mills Theory. Compared with Section 6.6 we simplify the notation by suppressing the internal indices corresponding to the gauge group by using a matrix notation. Deser *et al.* studied the action

$$S = \frac{1}{2\kappa^2}\int dx^3\, \mathrm{tr}\,(F_{\mu\nu}F^{\mu\nu}) - \frac{m}{2\kappa^2}\int dx^3\, \epsilon^{\mu\nu\rho}\, \mathrm{tr}\,(F_{\mu\nu}A_\rho - \frac{2}{3}A_\mu A_\nu A_\rho) \tag{7.20}$$

with coupling constants κ, m. Variation leads to the field equations

$$\mathcal{D}_\mu F^{\mu\nu} + \frac{m}{2} \star F^\nu = \mathcal{D}_\mu F^{\mu\nu} + \frac{m}{2}\epsilon^{\nu\rho\sigma}F_{\rho\sigma} = 0 \tag{7.21}$$

where $\mathcal{D}_\mu$ denotes again the covariant derivative. Using the dual equations and the well-known relation $[\mathcal{D}_\mu, \mathcal{D}_\nu] = F_{\mu\nu}$ one can derive the following equation for the dual field strength

$$(D_\nu D^\nu + m^2) \star F_\mu = \epsilon_{\mu\nu\rho}[\star F^\nu, \star F^\rho]\,, \tag{7.22}$$

which in linear approximation becomes a Klein-Gordon Equation with "mass" m.

Comparing (7.21) with the usual Yang-Mills Equations we notice that the additional terms are of lower order. This implies that our analysis of the symbol from Section 6.6 remains still valid and especially that we get the same Cartan characters. Thus to prove involution there remains only to show that there still holds a Noether identity. It is, however, easy to show that due to the definition of $F_{\mu\nu}$

$$\mathcal{D}_\mu \star F^\mu = 0\,. \tag{7.23}$$

Thus from the point of view of a formal analysis there are no changes in the properties of the field equations through the additional terms. Physically, the theories differ nevertheless considerably [41].

General relativity in three dimensions is special in several respects. The Einstein Equations are dynamically trivial: Every Ricci-flat space-time is flat, as Ricci and curvature tensor have the same number of independent components and are equivalent. In the formal analysis one can show that the gauge corrected Cartan characters vanish for $n = 3$. Furthermore the Weyl tensor vanishes. To distinguish conformally flat space-times one must introduce another tensor [48], often called Cotton tensor.

The action proposed by Deser *et al.* is similar to the one used in the Yang-Mills case. One must replace F by the curvature tensor and A by the Levi-Civita connection. This leads to the field equations

$$G^{\mu\nu} + \frac{1}{m\sqrt{g}}C^{\mu\nu} = 0 \tag{7.24}$$

where

$$G^{\mu\nu} = R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R \tag{7.25}$$

denotes the Einstein tensor constructed from the Ricci tensor $R^{\mu\nu}$ and the curvature scalar $R = R^{\mu}_{\mu}$ and

$$C^{\mu\nu} = \epsilon^{\mu\rho\sigma}\mathcal{D}_\rho(R^\nu_\sigma - \frac{1}{4}\delta^\nu_\sigma R) \tag{7.26}$$

is the above mentioned Cotton tensor. We note the fundamental difference to the Yang-Mills case, namely that the Cotton tensor depends on third-order derivatives of the metric, thus it is of higher order than the Einstein tensor.

Alternatively one could ask what is the most general tensor constructible of the metric and its derivatives up to third order that has similar properties as the Einstein tensor, i.e. it is (i) a symmetric second-rank tensor, (ii) divergence-free and (iii) trace-less. Lovelock [100] proved that in three dimensions there exists only one tensor satisfying all these conditions and that is the Cotton tensor. This was the motivation for Dereli and Tucker [39] to study (7.24) (with an additional cosmological term). They constructed wave solutions and showed that there exist no static rotationally symmetric solutions.

Opposed to the Yang-Mills case we can no longer use the results of our previous formal analysis of the Einstein Equations, for the symbol is now determined by the Cotton tensor. In a locally geodesic coordinate system it is given by

$$\begin{aligned} C^{\mu\nu} = \tfrac{1}{2}\epsilon^{\mu\nu\sigma}\eta^{\nu\lambda}\Big[\; & \eta^{o\tau}(\partial_{\sigma o\rho}g_{\tau\lambda} - \partial_{\tau o\rho}g_{\sigma\lambda} - \partial_{\sigma\lambda\rho}g_{\tau o} + \partial_{\tau\lambda\rho}g_{\sigma o}) \\ & - \tfrac{1}{4}\delta_{\sigma\lambda}\eta^{o\pi}\eta^{\tau\omega}(\partial_{o\tau\rho}g_{\omega\pi} - \partial_{\omega\tau\rho}g_{o\pi} - \partial_{o\pi\rho}g_{\omega\tau} + \partial_{\omega\pi\rho}g_{o\tau})\Big] . \end{aligned} \tag{7.27}$$

Since we have only six equations in six dependent variables, it is easier to write explicitly down the matrix of the tableau instead of performing complicated index manipulations. If we label the columns by $g_{33}, g_{32}, g_{22}, g_{31}, g_{21}$ and g_{11} and drop some integer factors, it has the following form

$$\mathcal{T}[\chi] = \left(\begin{array}{ccc}
\chi_1\chi_2\chi_3 & \chi_1(\chi_2^2-\chi_3^2) & -\chi_1\chi_2\chi_3 \\
-\chi_3(3\chi_2^2+\chi_1^2) & 2\chi_2(\chi_1^2-3\chi_3^2) & 3\chi_3(\chi_1^2-\chi_3^2) \\
3\chi_2(\chi_1^2-\chi_2^2) & 2\chi_3(3\chi_2^2-\chi_1^2) & \chi_2(\chi_1^2-3\chi_3^2) \\
\chi_1\chi_2\chi_3 & \chi_1(\chi_1^2-\chi_3^2) & 0 \\
3\chi_1(\chi_1^2-\chi_2^2) & 4\chi_1\chi_2\chi_3 & \chi_1(\chi_1^2-\chi_3^2) \\
0 & \chi_1(\chi_1^2-\chi_2^2) & \chi_1\chi_2\chi_3
\end{array}\right.$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \tag{7.28}$$

$$\left.\begin{array}{ccc}
-\chi_2(\chi_2^2+\chi_3^2) & \chi_3(\chi_2^2+\chi_3^2) & 0 \\
2\chi_1(\chi_2^2-\chi_3^2) & -4\chi_1\chi_2\chi_3 & \chi_3(\chi_2^2+\chi_3^2) \\
-4\chi_1\chi_2\chi_3 & 2\chi_1(\chi_3^2-\chi_2^2) & \chi_2(\chi_2^2+\chi_3^2) \\
-\chi_2(\chi_1^2+\chi_3^2) & \chi_3(\chi_3^2-\chi_1^2) & \chi_1\chi_2\chi_3 \\
2\chi_3(\chi_2^2-3\chi_1^2) & -2\chi_2(\chi_1^2+\chi_3^2) & \chi_1(\chi_2^2+3\chi_3^2) \\
\chi_2(\chi_2^2-\chi_1^2) & -\chi_3(\chi_1^2+\chi_2^2) & \chi_1\chi_2\chi_3
\end{array}\right).$$

Its maximal rank is 3 and can for instance be obtained by choosing $\chi = 2dx^1 + dx^3$. However, the coordinate system is not δ-regular, for all coordinate forms dx^i are systatic (this is not very surprising and basically due to the appearance of the ϵ-tensor in (7.27)). Nevertheless, (7.27) contains only terms of class 2 or 3. Thus without using the 2-tableau we can conclude that $\beta_C^{(3)} = \beta_C^{(2)} = 3$ and $\beta_C^{(1)} = 0$.

Involution of (7.24) comes again from the Bianchi identity. The proof is essentially identical with the one given for the vacuum field equations in Section 6.8. The Bianchi identity ensures that the symbol is involutive and that no integrability conditions arise. Thus for the formal analysis the most important property of the Cotton tensor is that is is divergence-free. The vanishing of the trace implies that without additional fields the curvature scalar vanishes in this modified theory, too.

Although $\beta_C^{(k)} = \beta_{Ein}^{(k)}$, the Cartan characters are different, as we are now dealing with a third-order equation:

$$\alpha_C^{(3)} = 3\,, \quad \alpha_C^{(2)} = 15\,, \quad \alpha_C^{(1)} = 36\,. \tag{7.29}$$

The Hilbert polynomial is given by

$$H_3(r) = 54 + \frac{39}{2}r + \frac{3}{2}r^2\,. \tag{7.30}$$

Of course the theory is still invariant under arbitrary coordinate transformations. Gauge corrections with $\gamma_1 = 3$ gauge functions yield

$$\bar{\alpha}_C^{(3)} = 0\,, \quad \bar{\alpha}_C^{(2)} = 6\,, \quad \bar{\alpha}_C^{(1)} = 18\,. \tag{7.31}$$

The strength of this massive theory is 12, i.e. exactly the value for Einstein and Maxwell Equations, respectively, in four dimensions. Whether this has also some mystical "explanation" is unknown.

7.3 Two-dimensional Dilaton-Gravity

The two-dimensional dilaton-gravity has attracted much attention in recent years. There are several reasons for this. The quantization of gravity still represents a major unsolved problem. Two-dimensional models can serve here as an easier accessible test ground than the full four-dimensional theory. One must, however, add a scalar field to obtain reasonable field equations, as in two dimensions the Einstein tensor vanishes and a Ricci flat space has a zero curvature tensor [48].

There are many other reasons to study these models. One obtains for instance a similar type of action in effective string theory [21]. An interesting feature here is the existence of black hole solutions [164]. This explains the interest in the quantum cosmology of dilaton-gravity.

Under fairly general assumptions the most general coordinate invariant lagrangian density containing the metric and a scalar field is given by

$$\mathcal{L} = \sqrt{-g}\left(\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi) + D(\phi)R\right) \tag{7.32}$$

where R denotes as usual the curvature scalar. Using field redefinitions one can, however, considerably simplify this action [5]. Especially it is possible [99] to eliminate the kinetic term for ϕ by transforming the metric

$$\bar{g}_{\mu\nu} = \Omega^2(\phi)g_{\mu\nu} \tag{7.33}$$

where Ω must satisfy the differential equation

$$4D'(\phi)\frac{d\ln\Omega}{d\phi} = 1\,. \tag{7.34}$$

Furthermore we can eliminate $D(\phi)$ by taking it as a new field $\bar{\phi}$ (one must be careful here, if D has critical points [5]). We will therefore restrict our attention to the model described by the action

$$\mathcal{L} = \sqrt{-g}\left[V(\phi) + \phi R\right]. \tag{7.35}$$

Variation with respect to the metric yields the following "Einstein Equations"

$$\mathcal{D}_\mu\mathcal{D}_\nu\phi - g_{\mu\nu}\left[\mathcal{D}^\rho\mathcal{D}_\rho\phi - V(\phi)\right] = 0\,. \tag{7.36}$$

We see at once that if we consider the metric as a kind of external field, this represents a finite type equation for the dilaton field ϕ. The variation with respect to ϕ leads to the additional equation

$$R - V'(\phi) = 0\,. \tag{7.37}$$

The formal analysis of the combined system of field equations shows that it is in involution. An interesting fact hereby is that (7.36) entails more or less (7.37). The integrability conditions of (7.36) require that either (7.37) is satisfied or ϕ must be constant. Similar effects are known from other theories coupled to gravity. It would be interesting to study whether this stems from the particular way of coupling or whether it is more accidentally.

The Cartan characters of the full system are

$$\alpha^{(2)}_{DG} = 2\,, \quad \alpha^{(1)}_{DG} = 6\,. \tag{7.38}$$

We must, however, adjust for the invariance under coordinate transformations. Since $\gamma_1 = 2$, the gauge corrected Cartan characters vanish. Thus the reduced phase space of this theory is finite-dimensional. This fact was also proven in [5, 99] using an explicit reduction.

We will now perform a similar reduction to a finite type equation in order to be able to apply the results of Section 7.1. This will allow us to consider ordinary differential equations instead of the field equations. First we exploit the well-known fact that every two-dimensional metric is conformally flat [48] and set

$$g_{\mu\nu} = e^{\lambda}\eta_{\mu\nu}\,. \tag{7.39}$$

With this simpler metric the combined field equations can be written in the following form:

$$\mathcal{R}_2 : \begin{cases} \phi_{tt} - \frac{1}{2}(\phi_t\lambda_t + \phi_x\lambda_x) + \frac{1}{2}e^{\lambda}V(\phi) = 0\,, \\ \phi_{xx} - \frac{1}{2}(\phi_t\lambda_t + \phi_x\lambda_x) - \frac{1}{2}e^{\lambda}V(\phi) = 0\,, \\ \phi_{xt} - \frac{1}{2}(\phi_x\lambda_t + \phi_t\lambda_x) = 0\,, \\ \lambda_{tt} - \lambda_{xx} + e^{\lambda}V'(\phi) = 0\,. \end{cases} \tag{7.40}$$

We will show at the end of this section that the conformal gauge (7.39) can be further enhanced by requiring that $\lambda_x = 0$, i.e. it is always possible to find a coordinate system where λ depends only on t. It is easy to see that this gauge fixing condition is compatible with (7.40); there do not arise any integrability conditions. If we additionally rewrite (7.40) as a first-order system by introducing new variables, we obtain with obvious notation the following finite type equation

$$\tilde{\mathcal{R}}_1 : \begin{cases} \dot\phi_t = \frac{1}{2}\left[\dot\phi\dot\lambda - e^{\lambda}V(\phi)\right]\,, & \dot\phi_x = \frac{1}{2}\phi'\dot\lambda\,, \\ \phi'_t = \frac{1}{2}\phi'\dot\lambda\,, & \phi'_x = \frac{1}{2}\left[\dot\phi\dot\lambda + e^{\lambda}V(\phi)\right]\,, \\ \phi_t = \dot\phi\,, & \phi_x = \phi'\,, \\ \dot\lambda_t = -e^{\lambda}V'(\phi)\,, & \dot\lambda_x = 0\,, \\ \lambda_t = \dot\lambda\,, & \lambda_x = 0\,. \end{cases} \tag{7.41}$$

Obviously there is no need to introduce λ'.

Now we construct the two vector fields which define the intermediate Jacobian system. They can easily be read off from (7.41)

$$\begin{aligned} T &= \frac{\partial}{\partial t} + \dot\phi\frac{\partial}{\partial\phi} + \dot\lambda\frac{\partial}{\partial\lambda} + \frac{1}{2}\left[\dot\phi\dot\lambda - e^{\lambda}V(\phi)\right]\frac{\partial}{\partial\dot\phi} + \frac{1}{2}\phi'\dot\lambda\frac{\partial}{\partial\phi'} - e^{\lambda}V'(\phi)\frac{\partial}{\partial\dot\lambda}\,, \\ X &= \frac{\partial}{\partial x} + \phi'\frac{\partial}{\partial\phi} + \frac{1}{2}\phi'\dot\lambda\frac{\partial}{\partial\dot\phi} + \frac{1}{2}\left[\dot\phi\dot\lambda + e^{\lambda}V(\phi)\right]\frac{\partial}{\partial\phi'}\,. \end{aligned} \tag{7.42}$$

We omit the characteristic system and proceed directly to the ordinary differential equations. They are obtained by substituting $t = t_0 + \tau t_1$ and $x = x_0 + \tau x_1$ with fixed but arbitrary constants t_0, t_1, x_0, x_1.

$$\begin{aligned}
2\frac{d\dot{\phi}}{d\tau} &= t_1\left[\dot{\phi}\dot{\lambda} - e^{\lambda}V(\phi)\right] + x_1\phi'\dot{\lambda}\,, \\
2\frac{d\phi'}{d\tau} &= t_1\phi'\dot{\lambda} + x_1\left[\dot{\phi}\dot{\lambda} + e^{\lambda}V(\phi)\right]\,, \\
\frac{d\phi}{d\tau} &= t_1\dot{\phi} + x_1\phi'\,, \\
\frac{d\dot{\lambda}}{d\tau} &= t_1 e^{\lambda}V'(\phi)\,, \\
\frac{d\lambda}{d\tau} &= t_1\dot{\lambda}\,.
\end{aligned} \tag{7.43}$$

(7.43) is a fairly complicated system. It seems hardly feasible to solve it for arbitrary potentials $V(\phi)$. Still it represents some progress compared to the original field equations in so far that we are dealing with ordinary differential equations. This should for instance facilitate quantization, for this is now a problem in quantum mechanics and no longer in quantum field theory. This transformation should also allow for a simpler numerical solution. By choosing specific values for t_0, x_0 and adding the equations $dt_1/d\tau = dx_1/d\tau = 0$ one has arrived at a system which can be treated with any of the many standard methods of numerical analysis like Runge-Kutta etc.

Banks and O'Loughlin [5] also presented reductions of the field equations to ordinary differential equations. They are based on the construction of a Killing vector which is orthogonal to the gradient of ϕ. By using a coordinate system which rectifies this vector field they write a new action in which all fields depend only on the new time variable. Thus the Euler-Lagrange Equations are ordinary differential equations. They call this the homogeneous gauge.

This procedure is somewhat dubious for two reasons. First the Killing vector depends on ϕ. Thus an explicit construction of the rectifying coordinate system is only possible *after* the field equations (in their original form) have been solved. In other words, it is not possible to recover solutions of the original equations from the solutions of the ordinary differential equations, because the transformation is not known! It depends on ϕ given in the *old* coordinates. Furthermore the proof that such a Killing vector always exists needs the original field equations. Thus it is not all clear whether the new and old ones are really equivalent.

In our presentation the ansatz of Banks and O'Loughlin corresponds to the assumption that in the chosen coordinate system $\phi_x = 0$. This implies via the third equation in (7.40) that $\lambda_x = 0$, if ϕ is not to be constant. Thus it looks similar to our gauge fixing but is actually a much stronger condition, for (7.41) does not require $\phi_x = 0$. This reflects the above mentioned fact that there is no way back from their ordinary differential equations to the original field equations. Hence they do not obtain the general solution but only a particular one.

By using only the homogeneous gauge and not the conformal one they also get unnecessarily complicated equations. It is straight-forward to show that the metric they use can be brought into conformal form. Now it is possible to obtain a very simple form of the field equations for arbitrary $V(\phi)$, if one uses their "gauge fixing" $\phi_x = 0$.

For this purpose we note that with this condition the first equation of (7.40) becomes $\phi_{tt} - \phi_t\lambda_t = 0$. This can be integrated once and yields

$$\phi_t = Ae^{\lambda} \tag{7.44}$$

with an integration constant A. Substituting this in the second equation of (7.40) leads to

$$A\lambda_t = -V(\phi)\,. \tag{7.45}$$

Differentiating (7.44) allows to eliminate λ and we arrive finally at the simple equation

$$A\phi_{tt} + V(\phi)\phi_t = 0\,. \tag{7.46}$$

There is no need to consider the fourth equation of (7.40), as it is an integrability condition and thus automatically satisfied.

To conclude the section we give the proof that it is always possible to assume $\lambda_x = 0$ and that one can indeed construct the necessary coordinate transformation.

Lemma 7.3 *There exists a coordinate system such that the conformal factor λ in (7.39) is independent of x.*

PROOF. This proof nicely demonstrates the advantages of the pseudogroup approach using differential equations. We start by analyzing the conditions that a change of coordinates $\bar{x} = f(x,t)$, $\bar{t} = g(x,t)$ respects the conformal form (7.39). It is easy to show that they are

$$f_x f_t = g_x g_t\,, \quad f_x^2 - g_x^2 = g_t^2 - f_t^2\,. \tag{7.47}$$

This represents an involutive system. The condition that the conformal factor is independent of x leads to the additional equation

$$2(f_x f_{xx} - g_x g_{xx})\Lambda + (f_x^2 - g_x^2)\Lambda_x = 0 \tag{7.48}$$

where $\Lambda = e^\lambda$. Hence we must prove that the combined system (7.47,7.48) has a solution for any function Λ. By a simple analysis of the pivots one can show that the symbol of the system obtained after prolonging (7.47) to second order is involutive, if $f_x g_t \neq f_t g_x$, and that furthermore no integrability conditions arise. But this is exactly the condition for f, g to define an invertible change of coordinates. Thus we can always find a solution f, g and construct from it a coordinate system with the required property. •

7.4 A Final Example

In some recent work on variational principles for differential equations Anderson and Pohjanpelto [3] came across the following equation

$$\mathcal{R}_1 : \begin{cases} z^a \dfrac{\partial\lambda}{\partial z^a} + (D+2)\lambda = 0 \\ C^c_{ab} z^b \dfrac{\partial\lambda}{\partial z^c} + C^b_{ab}\lambda = 0\,, \end{cases} \tag{7.49}$$

where all indices run from 1 to D, C^c_{ab} are the structure constants of a D-dimensional Lie algebra and the unknown λ is a function of $z^1, \dots, z^D$. They needed the dimension of the solution space.

Due to the symmetry properties of structure constants the following integrability condition can be derived from the second equation in (7.49)

$$C^c_{ab} z^a z^b \frac{\partial\lambda}{\partial z^c} + C^b_{ab} z^a \lambda = C^b_{ab} z^a \lambda = 0\,. \tag{7.50}$$

Thus if $C^b_{ab} \neq 0$ for some a, then there exists only one solution, namely the trivial one $\lambda(z) \equiv 0$. We will assume in the sequel that $C^b_{ab} = 0$ for all a. This holds for instance for all semi-simple Lie algebras.

Thus we can take from now on the simpler local representation

$$\mathcal{R}_1 : \begin{cases} z^a \dfrac{\partial \lambda}{\partial z^a} + (D+2)\lambda = 0\,, \\ C^c_{ab} z^b \dfrac{\partial \lambda}{\partial z^c} = 0\,. \end{cases} \tag{7.51}$$

It is perhaps worth while noting that this simplification has no real impact on the result. The omitted lower order terms play no role in the analysis of the symbol. They just lead to one additional equation, $\lambda = 0$, which turns a one-dimensional solution space into a zero-dimensional one.

To proceed further we make the assumption that the Lie algebra is semi-simple. The structure theory of semi-simple Lie algebras is well-known [79]. In the Cartan-Weyl basis the structure constants take an especially simple form. Let r be the rank of the algebra. To simplify the notation, we write from now on i, j, k for indices running from 1 to r; l, m, n for indices running from $r + 1$ to D. Indices $a, b, \dots$ run always from 1 to D.

We denote by $\omega^{(l)}$ the roots and by $\omega^{(l)}_i$ the covariant components of the root vectors. Since minus a root is always a root, too, we adopt the convention $\omega^{(l)} = -\omega^{(D+r-l+1)}$. Then we get the following structure constants:

$$\begin{gathered} C^k_{ij} = C^l_{ij} = 0\,, \\ C^j_{il} = 0\,,\ C^m_{il} = \omega^{(l)}_i \delta^m_l\,, \\ C^i_{mn} = \omega^{(m)i} \delta^{-\omega^{(m)}}_{\omega^{(n)}}\,,\ C^l_{mn} = N_{mn} \delta^{\omega^{(l)}}_{\omega^{(m)}+\omega^{(n)}}\,. \end{gathered} \tag{7.52}$$

Substitution yields the system

$$\mathcal{R}_1 : \begin{cases} \displaystyle\sum_{a=1}^{D} z^a \frac{\partial \lambda}{\partial z^a} + (D+2)\lambda = 0\,, \\ \displaystyle\sum_{l=r+1}^{D} \omega^{(l)}_i z^l \frac{\partial \lambda}{\partial z^l} = 0\,, \\ \displaystyle\sum_{i=1}^{D} \left(\omega^{(m)}_i z^i \frac{\partial \lambda}{\partial z^m} + \omega^{(m)i} z^{D+r-m+1} \frac{\partial \lambda}{\partial z^i} \right) + \\ \qquad \displaystyle\sum_{l,n=r+1}^{D} N_{mn} \delta^{\omega^{(l)}}_{\omega^{(m)}+\omega^{(n)}} z^n \frac{\partial \lambda}{\partial z^l} = 0\,, \end{cases} \tag{7.53}$$

For later use, we note that there exist exactly r linearly independent root vectors. We can assume that they are $\omega^{(r+1)}_i, \dots, \omega^{(2r)}_i$. Since the Killing metric κ_{ab} of a semi-simple Lie algebra is non-degenerate, the first r contravariant root vectors are then also linearly independent.

Written as a matrix, the symbol $\mathcal{M}_1$ of $\mathcal{R}_1$ has the following form where the a-th column contains the coefficients of $\partial\lambda/\partial z^a$:

$$
\mathcal{M}_1 = \begin{array}{c}
\begin{array}{cccccc}
z^1 & \cdots & z^r & z^{r+1} & \cdots & z^D
\end{array} \\
\left(\begin{array}{ccc|ccc}
 & & & \omega_1^{(r+1)} z^{r+1} & \cdots & \omega_1^{(D)} z^D \\
 & 0 & & \vdots & & \vdots \\
 & & & \omega_r^{(r+1)} z^{r+1} & \cdots & \omega_r^{(D)} z^D \\
\hline
\omega^{(r+1)1} z^D & \cdots & \omega^{(r+1)r} z^D & -\omega_i^{(r+1)} z^i & & * \\
\vdots & & \vdots & & \ddots & \\
\omega^{(D)1} z^{r+1} & \cdots & \omega^{(D)r} z^d & * & & -\omega_i^{(D)} z^i
\end{array}\right)
\end{array} . \tag{7.54}
$$

The $*$ stands for the terms $N_{mn}\delta^{\omega^{(l)}}_{\omega^{(m)}+\omega^{(n)}} z^n \frac{\partial\lambda}{\partial z^l}$. Note that no such terms occur on the diagonal. We can write the symbol symbolically as

$$
\begin{pmatrix} & z \\ 0 & A \\ B & C \end{pmatrix} . \tag{7.55}
$$

A and B have both maximal rank, namely r, because, according to our assumption on the root vectors, the first r columns of A and the first r rows of B are linearly independent. As we have already seen during the construction of the integrability condition, we can eliminate the last row of the symbol by adding all equations but the first multiplied with the corresponding variable z^a. The remaining rows in C are linearly independent due to the diagonal terms.

We can thus conclude that $\mathcal{M}_1$ has maximal rank, namely D, and consequently dimension 0. A vanishing symbol is, however, trivially involutive. Hence, only the question of further integrability conditions remains. A direct proof that none occur seems rather tedious. There exists, however, a much easier way, for Anderson and Pohjanpelto [3] have constructed a solution of $\mathcal{R}_1$.

A system with vanishing symbol has always a finite dimensional solution space, even if it is not involutive, because all prolongations and projections of it will have again vanishing symbols. Thus an upper bound on the dimension of the solution space is given by the dimension of $\mathcal{R}_1$. As we can see from the analysis of the symbol, $D-1$ of the D equations in the upper line of (7.51) are independent. Together with the remaining one this yields D equations in a $(D+1)$-dimensional space. Thus $\dim \mathcal{R}_1 \leq 1$.

On the other hand, one can verify that

$$
\lambda = (\kappa_{ab} z^a z^b)^{-\frac{D+2}{2}} \tag{7.56}
$$

is a solution. Due to the linearity of the system, all multiples of (7.56) are solutions, too, and we have thus at least a one-dimensional solution space. Together with the result above this implies that the dimension of the solution space is exactly 1 and thus that no integrability conditions can occur. $\mathcal{R}_1$ is consequently involutive.