Chapter 9

Conclusions

It is too bad that it has to be mathematics...

R.P. Feynman: The Character of Physical Law

Feynman devoted one of his BBC lectures on the character of physical laws to the relation between mathematics and physics. His point of view was that mathematics represents the language of physics. It provides methods to reason about the physical laws and to deduce further statements from them. This thesis should be understood in this spirit. We have presented a mathematical machinery centered around the notion of involution. Then we begun to show how it can be used to obtain physical results.

As already indicated in the introduction it was one of our main goals to demonstrate that the formal theory of partial differential equations provides useful tools for physics. We hope that we have succeeded in this point, even though we restricted our attention mainly to initial value problems[1]. Although the theory is surely non-trivial, it allows for a clear and unifying treatment of many problems.

We believe that especially the treatment of systems with constraints can benefit from the formal theory. First of all it yields a rigorous foundation for the Dirac algorithm to construct the constraints. Completion to involution is just the mathematical formulation of Dirac's starting point, namely to check whether or not the equations of motion are consistent. We have also seen that it yields a unifying framework for Lagrangian and Hamiltonian formulation as well as for the symplectic formalism.

The next steps would be to obtain an intrinsic formulation for the Legendre transformation and an intrinsic characterization of first and second class constraints. One can say that basically the Legendre transformation uses a different embedding of $J_2\mathcal{E}$ into $J_1(J_1\mathcal{E})$ than the one we employ in Appendix C.2. The distinction between first and second class constraints should be visible in projections of the equations of motion to lower order jet bundles.

We have further seen that in the case of field theories the Dirac algorithm must use a kind of involution analysis as soon as there are constraints of lower class which require to take spatial derivatives into account. But then it is conceptually much clearer to start from the very beginning with such an analysis instead of keeping an artificial distinction into temporal prolongations which are done using Poisson brackets and spatial one which are performed in the usual way. One must also keep always in mind that the involution analysis can as well be applied to the Euler-Lagrange Equations, i.e. in a covariant formalism.

[1] Many other applications can be found in the books of Pommaret [117, 118, 119].

We did not attack the fundamental problem of quantizing such systems. One approach could be based on our novel treatment of initial value problems. Even if many different initial value problems might be possible for a given system, each one will lead to an unconstrained parameterization of the state space. It is, however, yet unclear, how this representation could be used to define canonical coordinates and/or canonical commutation relations. Already at classical level one deals now with functions having domains of definition with different dimensions. Thus one needs a new way to introduce Poisson brackets.

From a physical point of view Chapter 3 could be titled "What is a degree of freedom?". For finite-dimensional systems, or in the formal language for equations of finite type, the answer is fairly simple. The general solution depends only on a finite number of constants and this number is completely independent of any chosen representation. In some canonical coordinate system one can identify these constants with initial positions and velocities. Thus one would say that halving the number of constants yields the number of degrees of freedom.

In field theories the situation becomes much more complicated. Usually one says that the constraints "determine" some of the fields and halving the number of remaining fields yields the number of degrees of freedom. We have used here quotation marks, because the constraints are typically differential equations and thus can not completely fix any fields. As already discussed in Chapter 4 the effect of the constraints is that the initial data for some fields cannot be prescribed on submanifolds of codimension 1 but only on lower-dimensional submanifolds. Hence there remains some arbitrariness. It is thus unclear what could be a good definition of degrees of freedom for field theories.

In some sense this problem is closely related to the above mentioned problem of quantization. If one could find a reasonable way to introduce canonical variables, then this would probably entail a reasonable definition of degrees of freedom. But it is still doubtful in the light of the here presented results that one will obtain a *single* number. It will probably be some kind of sequence like the Cartan characters.

Independent of the physical implications one could ask what is the best way to compare the arbitrariness of the general solution of two different differential equations. We have seen many different possibilities: Cartan characters, Hilbert polynomial, index of generality plus Cartan genus, strength and the number of arbitrary functions. In some sense they are of course all equivalent, as each can be used to compute the others (or at least the relevant part of them): one always ends up with the index of generality and the Cartan genus. The only exception is the strength; this was already discussed in Section 6.2.

If a differential equation has more independent variables than another one, it also has more Cartan characters. Relevant is, however, only the highest non-vanishing one and its index, i.e. the index of generality and the Cartan genus. Similarly one can compare the Hilbert polynomials. But of importance is then only its degree and leading coefficients, i.e. again more or less the index of generality and the Cartan genus.

The number of arbitrary functions depends on the choice of a representation for the general solution. The only way it can be used reasonably is to rewrite every equation as a first-order one and to compare then the values for an algebraic representation. Relevant is then of course only the number of functions with the maximal number of arguments — which is again the index of generality and the Cartan genus.

All these tools are intrinsically defined, i.e. they are invariant under coordinate transformations and independent of any specific representation chosen for the submanifold $\mathcal{R}_q$. But one can make more drastic transformations like e.g. the reduction of an equation to a first-order one. This means that one represents the differential equation in another jet bundle! The Cartan characters are not necessarily invariant under such operations but the Hilbert polynomial (adjusted for the new order taken as basis), the index of generality and the Cartan genus.

Our analysis of symmetries and reduction methods in Chapter 5 is still far from satisfactory. The only problems that can be considered as solved are the determination of the size of symmetry groups and the loss of generality during a reduction. The former one is a trivial spin-off of Chapter 3; the latter one omits the real problem. Of course one wants to obtain as general a solution as possible, but the really important point is to find a reduction such that one can actually solve the reduced equations.

Although we have proposed a way how formal theory could help here, we have failed to produce a new reduction for some equation of interest like e.g. the Korteweg-de Vries Equation. Our attempts run aground, because the resulting conditions for a suitable differential constraint turned out to be intractable. Probably this approach is feasible only in combination with some other ideas.

The dilaton-gravity studied in Section 7.3 belongs to a fairly interesting class of systems, namely those with a finite-dimensional reduced phase-space. Since these systems occupy a kind of intermediate position between field theory and point mechanics, they provide an ideal test ground to investigate methods for quantization. By choosing a gauge fixing it is always possible to reduce questions of quantum field theory to quantum mechanical ones.

While this is at least sometimes exploited in concrete examples, it has apparently not yet been noticed that these systems could also be quite useful to study problems of complete integrability. So far this term has a precise definition only for finite-dimensional systems. For infinite-dimensional ones many criteria based on many different approaches have been proposed, but no coherent picture has emerged. Systems of finite type could help to clarify this unsatisfactory situation.

As some of the examples in Sections 8.7 and 8.8 showed the use of a computer algebra system is more or less inevitable for geometric computations. No matter whether one wants to complete a differential equation to an involutive one or to compute its symmetry algebra, as soon as one gets beyond second or perhaps third order the computational effort is so high that it takes at least days to produce a result. Even if somebody possesses the necessary stamina, hand calculations contain usually more errors than computer results (if a program is thoroughly tested).

As already mentioned in Section 8.9, δ-regularity provides the biggest problem for the automatic completion to involution. Our introduction of the k-tableaux allows surely for a more efficient treatment of it than the brute force approach with a general coordinate system. It is, however, still very time-consuming because of the large dense matrices which must be handled. In hand calculations one can often find ways to avoid applying higher tableaux (see e.g. our treatment of three-dimensional gravity in Section 7.2), but a computer algebra system is probably not able to detect such shortcuts.

A big but often overlooked problem in computer algebra nowadays is the fact that all available systems are closed. Thus much work is duplicated in order to make certain algorithms available in several systems. The time spent for this could be used much better for further research. Especially in symmetry analysis one can see this effect very clearly. The number of available packages is difficult to assess. But many are just translations of others from one computer algebra system into another one. On the whole they all use exactly the same methods and their capabilities hardly differ. The main differences are in the quality of the implementation and thus the performance.

A similar problem occurs, if one wants to treat the output of one algorithm with another one. This can be quite useful as we have seen in the example of the symmetry analysis of the Heat Equation. In fact a determining system in "raw" form is usually not a pretty sight. It contains many dependent or unnecessarily complicated equations. In order to extract information from it one must treat it further. But then one finds the same effect: probably the algorithm one needs is implemented by somebody else in some other computer algebra system!

Some people try to counter this effect by developing interfaces, i.e. programs which transform the output of one package in one computer algebra system into a form which can be read by another package in another system. But this approach is feasible only for a small number of involved packages. Otherwise one ends up with a pile of interfaces. Another problem is that this ansatz requires detailed knowledge of the data structures of each package involved.

Often different packages by different authors written for the same computer algebra system use different data structures. Especially for derivatives, or more generally jet variables, one finds often different approaches. This starts already with the question whether one should use multi-index or repeated index notation. Here systems like AXIOM are advantageous. Due to their modular structure every author can use his favorite notation and still share code with others (at least in theory), if there is an agreement about the needed functionality, i.e. if they specify together a category.

We hope that in the future some other authors will also contribute their methods to the environment we have started in this thesis. In principle this is easily possible, if they can accept the "standards" given by the two implemented categories `JetBundleCategory` and `JetBundleFunctionCategory`. Whether they will like the domain `DifferentialEquation` is another question, because it is really adapted to the requirements of the completion procedure in `CartanKuranishi`. For other applications many of the internally computed and stored data like Jacobi matrices are often not necessary and only slow down the computations. Even our own symmetry package is not based on it!

There exist many possible directions for future extensions of the environment. We mentioned already possible (and probably necessary) improvements of `SymmetryAnalysis`. It can also be generalized to further classes of symmetries like generalized symmetries. Of interest would also be a package for the Painlevé analysis of partial differential equations. Currently there does not seem to exist a package for this in any computer algebra system. Speaking of (completely) integrable systems it could be useful to make a connection between this environment and another program of us for handling pseudo differential operators [139] developed independently of this thesis.

Still very primitive are the provisions for differential geometric calculations. The two existing domains for vector fields and one-forms contain just the necessary operations for symmetry analysis and for the determination of tableaux. One can easily imagine implementing more involved operations in that direction, e.g. for computations with the variational bi-complex [111]. Very useful would be programs for determining the Spencer cohomology or some of the differential sequences frequently applied in formal theory.

The formal theory represents of course not the only way to deal with general systems of partial differential equations. We have already mentioned several times the Cartan-Kähler and Janet-Riquier Theory, respectively. There exists furthermore a dual version of the exterior system approach based on vector fields developed by Vessiot [158]. It is the least known of these theories. The Janet-Riquier Theory can in some sense be considered as superseded by the formal approach which provides an intrinsic treatment. Its specific techniques are, however, quite useful in concrete calculations. Our simplification and reduction routines for general expressions are for instance closely related to Reid's standard form algorithm.

It is probably inevitable to comment on whether or not the formal approach to involution working directly with partial differential equations is "better" than the Cartan-Kähler approach using exterior systems, as this becomes sometimes an almost religious question. The easiest answer is to cite [63] which proves rigorously the equivalence of both approaches. Thus none can claim to be more powerful than the other. Indeed one can often see that preference of one stems from personal taste and not actual advantages. Of course there exist problems which can be handled better with one approach but one cannot give a general judgment.

The exterior approach has surely advantages, if one deals with coordinate free problems. An example of such a problem set up in a frame bundle was given by Hartley *et al.* [68], namely finding a time-like curve in a Minkowski space-time satisfying the Euler-Lagrange equations for a Lagrangian depending on the acceleration and the torsion of the curve. Exterior calculus is also advantageous in maintaining manifest covariance.

On the other hand many problems lead to partial differential equations and not to exterior systems. There are in general many ways to rewrite a partial differential equation as an exterior system. To use the pull-back of the contact system is just one and not necessarily the best. The Korteweg-de Vries Equation is often transformed into an ideal generated by two-forms. Here one must ask why transform at all, if it is possible to do the same calculations directly with the original partial differential equation. The jet bundle formalism also provides an intrinsic differential geometric framework with invariant results.

From a computational point of view using partial differential equations has one advantage: the transformation into an exterior system leads usually to considerably more equations (consider e.g. the case of a single, first-order equation). Since both approaches face the same fundamental problem of combinatorial explosion as soon as prolongations are necessary, this effect is not to be underestimated even when using computer algebra.