Analysis and Application of the Formal Theory of Partial Differential Equations

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Abstract

An introduction to the formal theory of partial differential equations is given emphasizing the properties of involutive symbols and equations. An algorithm to complete any differential equation to an involutive one is presented. For an involutive equation possible values for the number of arbitrary functions in its general solution are determined. The existence and uniqueness of solutions for analytic equations is proven. Applications of these results include an analysis of symmetry and reduction methods and a study of gauge systems. It is shown that the Dirac algorithm for systems with constraints is closely related to the completion of the equation of motion to an involutive equation. Specific examples treated comprise the Yang-Mills Equations, Einstein Equations, complete and Jacobian systems, and some special models in two and three dimensions. To facilitate the involved tedious computations an environment for geometric approaches to differential equations has been developed in the computer algebra system AXIOM. The appendices contain among others brief introductions into Cartan-Kähler Theory and Janet-Riquier Theory.

Contents

List of Figures

Chapter 1

Introduction

Though this be madness, yet there is method in't.

W. Shakespeare: Hamlet, II.2.205

Systems of partial differential equations occur in all branches of physics (and many other sciences). Whenever a continuous process must be modeled mathematically, there is a high probability that differential equations appear. Thus it is not surprising that they represent one of the vastest fields in mathematics. Although in the quotation above Polonius was probably thinking more of Hamlet's state of mind than of differential equations, it characterizes nicely the approach we will take.

Systems of partial differential equations display such a variety of phenomena that it may really appear as madness to try to tackle general systems. But the main feature of our approach will be that it provides us with a method, i.e. a *systematic* way, to treat such systems. This has on one side the advantage that we will be able to unify apparently very different computations for which previously individual techniques had to be found. On the other hand it might often be more efficient to employ some special ad hoc tricks.

The main emphasis in this thesis will be put on initial value problems, i.e. on problems where the initial state of a system is given and a system of differential equations describes its dynamics. One of the central questions will be, how much and what data is necessary to specify an initial state such that the subsequent evolution is unique.

Although this represents a really fundamental problem surprisingly little is known. And this knowledge is not very wide spread. It is well-known that an ordinary differential equation of order q possesses q functionally independent solutions and that this implies that q constants can be prescribed as initial data. This generalizes to a *single* partial differential equation where q functions defined on a submanifold of codimension 1 can be given as initial data. In typical physical applications this is a space-like submanifold of space-time described in some local coordinate system by the equation $t = t_0$.

As soon as one proceeds to *systems* of partial differential equations, the situation becomes considerably more complicated. Now only for so-called normal systems, i.e. systems which can be brought into a form satisfying the conditions of the Cauchy-Kowalevsky Theorem, we obtain a direct generalization of the previous cases, namely if the system contains m unknowns or dependent variables, one can impose mq functions defined on a submanifold of codimension 1 as initial conditions.

However, one encounters in physics many non-normal systems; e.g. no gauge systems can be of this form. This implies that if one still tries to prescribe initial data on a submanifold of codimension 1, then this initial data can no longer be chosen arbitrarily. The conventional way to deal with this effect is to say that one has a system with *constraints.* Many papers have been devoted to the problem of solving the constraints for specific systems like Einstein's field equations in general relativity.

One of the main results of this thesis will be to propose another approach. Instead of insisting on prescribing initial data on one submanifold of codimension 1 we will use a flag of nested submanifolds of different dimensions, i.e. some initial conditions are given as usual on a submanifold of codimension 1, others are given on one of codimension 2 and so on. The main advantage of this ansatz is that the initial data can be chosen completely arbitrary, they do not have to satisfy any constraints.

This does not imply that we have implicitly solved the constraints. What happens is that we do not draw the in many respect artificial distinction in evolutionary equations and constraints. All differential equations are treated on the same footing. This approach should also be interesting for numerical calculations, although we will not pursue this point.

Neither will we consider the problem of quantization, but it is worth while pointing out that our approach might be of considerable interest in this context. Since the initial data are not constrained, they can be considered as an explicit parameterization of the state space which could serve as a starting point for a canonical (or geometric) quantization.

It turns out that the notion of an *involutive* differential equation is of central importance for such a study. Although this concept (and related ones) was already introduced early in this century, it is not very well-known. It is very difficult to find a textbook on differential equations mentioning it at all. Only older books often contain a discussion of complete and Jacobian systems. As we will show in Section 7.1, this is equivalent to studying involution restricted to linear, first order equations in one dependent variable.

One of the goals of this thesis is to show that involution is an important and useful concept for mathematical physics. We will for instance see in Section 6.3 that the well-known Dirac algorithm to compute all constraints of a dynamical system described by a Lagrangian makes nothing else than to complete the equations of motion to an involutive system. This has important implications for the generalization to field theories.

In several respects the here presented theory could be considered as "trivial": We will treat only local problems, although every differential geometer knows that the really difficult questions are the global ones; we will almost all the time restrict our attention to analytic functions, although it is well-known that local solvability becomes really challenging only for smooth or worse functions (and we will not mention at all distributions or Sobolev theory!); finally we will not even try to solve differential equations. We hope, however, that we will be able to convince the reader that there still remain many interesting questions which may even be relevant to applications.

Since all analysis will be of a completely local character, we will often be negligent in our notations. Especially, we almost always omit to denote neighborhoods. In many places one should, strictly seen, read "local neighborhood" instead of "manifold". A similar remark holds for the occurring bundles. Their global structure is never considered and we always operate as if they were trivial. Since in many applications all manifolds will be of the form $\mathbb{R}^n$ and the coordinate systems can be used globally, we do not think that this "sloppiness" harms.

In the geometric approach to partial differential equations one does not distinguish between single equations and systems of equations, as one deals with a geometric object, namely a submanifold in a jet bundle. We will try to reflect this definition in our terminology. *Equation* usually refers to such a submanifold, whereas *system* means a local description of it. However, systems usually consist of equations, thus it is not really possible to maintain such a distinction...

The second chapter contains a detailed introduction into the formal approach to partial differential equations. It explains the fundamental notions and theorems centering about the problem of completion to an involutive equation. Comparing with the books of Pommaret [117, 118, 119] we present the theory more from a computational point of view. Thus all our proofs use local coordinates and operate directly on the equations, whereas he prefers the abstract approach of diagram chasing which is much more powerful for proofs but less useful for practical computations. Due to its introductory character most of this chapter repeats well-known results. New ones can be found in the last two sections, especially the k-tableaux seem to be a new tool.

Chapter 3 applies the machinery of formal theory to an analysis of "general solutions". This requires first a definition of this term. We follow here ideas of Ampère and Hermann and translate them into our geometric language. This definition is surely of more theoretical than practical interest. The main part of this chapter consists of analyzing the number of arbitrary functions which can occur in a representation of a general solution. These results have already been published in [138].

As the results of the previous chapter have been completely formal, we turn our attention to initial value problems in Chapter 4. As already mentioned our approach here is fairly different from the standard one. We give a new intrinsic description of formally well-posed problems, where exactly the right amount of Cauchy data is prescribed on a flag of submanifolds. In the analytic category we prove the existence and uniqueness of solutions: the Cartan-Kähler Theorem. This theorem is well-known in the framework of exterior systems; a coordinate dependent version can be found in [117] from where we have taken our proof. For linear systems we extend the classical Holmgren Theorem from normal systems to arbitrary involutive ones.

The application of the machinery developed in the previous chapters starts in Chapter 5 with an analysis of symmetries and different reduction methods for partial differential equations. After a brief introduction into the basic notions of this field we consider the loss of generality occurring during a reduction and the task of finding differential constraints. Whereas we can present some definite results for the former point, published in [137], the latter one remains a big problem.

Chapter 6 is the first one which contains applications to physics, namely to field theories. It compares our approach to the arbitrariness of the general solution with the one taken by Einstein [47] using the strength of a system. We show, how one can subtract the effect of a gauge symmetry on the arbitrariness. The theory of systems with constraints is studied extensively from a formal point of view. As specific examples the Yang-Mills and the Einstein Equations are analyzed. Some results have already been published in [137, 138].

Some miscellaneous applications are collected in Chapter 7. It starts with reviewing the theory of complete and Jacobian systems from a formal point of view. These systems appear in many places in physics and mathematics. Some of the results have already been used in Chapter 5. Then we study the effect of adding some topologically motivated terms to Yang-Mills and Einstein Equations, respectively, in 2+1 dimensions. These models have interesting physical properties, quite different from the standard theories. After a discussion of the two-dimensional dilaton-gravity we finally treat an example stemming from variational calculus.

Since formal computations tend to become very soon very tedious, we have developed in the computer algebra system AXIOM an environment for them. It is described in Chapter 8. After a brief introduction to AXIOM which emphasizes the differences between this system and others, more "standard" ones, a detailed overview of the implementation is given which can also serve as a kind of user's manual. The application of the environment is demonstrated on several examples. An earlier version of it was presented in [132].

Two appendices briefly describe alternative approaches to the formal analysis of partial differential equations, namely the Cartan-Kähler Theory and the Janet-Riquier Theory, as both have decisively influenced the development of the formal theory. Especially our computational point of view draws heavily on the Janet-Riquier approach. A third appendix contains some smaller calculations concerning the so-called modified Stirling numbers which are needed in Chapter 3 and the reduction of arbitrary equations to first order ones. We also provide a kind of reference manual for our AXIOM environment in form of an appendix.

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