3. Involution II: Differential Theory

I consider that I understand an equation when I can predict the properties of its solutions, without actually solving it.

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In the previous chapter we introduced involutive bases for polynomial algebras. Now we relate them to the formal theory of differential equations. The key is the (geometric) symbol $\mathcal{M}_q$. In the first section we show that in local coordinates it defines at each point of the differential equation $\mathcal{R}_q$ a homogeneous submodule of a free polynomial module. This permits us to apply our algebraic theory and leads naturally to the notion of an involutive symbol. For easier reference we repeat some results of the previous chapter in the language of differential equations.

In the following section we finally define involutive differential equations and study some of their elementary properties. In particular, we show that, in contrast to formal integrability, involution can be checked by a finite criterion. Furthermore, we introduce the Cartan normal form of an involutive equation, a useful tool in many situations.

As a first application of involution we revisit in the third section the principal symbol and give a rigorous definition of under- and overdetermined equations. It turns out that the classical counting rules are not always valid. Furthermore, we indicate how the principal symbol allows us a partial solution of the problem of δ-regularity.

The last section extends some of the results of Section 2.6 on syzygies of polynomials to linear differential operators. Syzygies correspond to compatibility conditions and the syzygy resolution is dual to the Janet sequence. We study the inverse problem of compatibility and provide an algorithmic solution of it for systems with constant coefficients.

3.1 Involutive Symbols

In Section 1.5 we introduced the (geometric) symbol $\mathcal{M}_q$ of a differential equation $\mathcal{R}_q$. Now we apply the algebraic methods developed in the last chapter to analyse $\mathcal{M}_q$. This analysis is pointwise and takes place in local coordinates. Thus the question arises how intrinsic the results are. In particular, we are using Pommaret bases where we have already seen in Section 2.4 that they only exist in δ-regular coordinates. But there we also showed that generic coordinates are δ-regular and we gave a simple criterion for detecting δ-singularity. Thus we assume throughout this section that we are dealing with δ-regular coordinates.

Let $(\mathbf{x}, \mathbf{p})$ be local coordinates in $J_q\mathcal{E}$ in a neighbourhood of the point $\rho \in \mathcal{R}_q$. The symbol space $(\mathcal{M}_q)_\rho$ consists of all solutions of the real linear system (1.50). If we denote $x = \pi^q(\rho) \in \mathcal{X}$ and $\xi = \pi_0^q(\rho) \in \mathcal{E}$, then $(\mathcal{M}_q)_\rho$ is a subspace of $S_q(T_x^*\mathcal{X}) \otimes V_\xi\mathcal{E}$. Thus the linear equations in (1.50) may be considered as elements of the dual space $S_q(T_x\mathcal{X}) \otimes V_\xi^*\mathcal{E}$. A basis of this space is given by $\{\partial_{x^\mu} \otimes du^\alpha \mid 1 \leq \alpha \leq m, |\mu| = q\}$ where $\partial_{x^\mu} = (\partial_{x^1})^{\mu_1} \cdots (\partial_{x^n})^{\mu_n}$. Our linear system describes then the kernel of the forms

$$\lambda^\tau = \sum_{\substack{1 \leq \alpha \leq m \\ |\mu| = q}} \frac{\partial \Phi^\tau}{\partial p_\mu^\alpha}(\mathbf{x}, \mathbf{p})\, \partial_{x^\mu} \otimes du^\alpha\ , \quad \tau = 1, \dots, p\ . \tag{3.1}$$

Instead of considering only $S_q(T_x\mathcal{X})$, we take the full symmetric algebra $S(T_x\mathcal{X})$ which is isomorphic to the polynomial ring $\mathcal{P} = \Bbbk[x^1, \dots, x^n]$ (see Appendix B.1). So we may identify $S(T_x\mathcal{X}) \otimes V_\xi^*\mathcal{E}$ with a free $\mathcal{P}$-module and the forms λ^τ generate a homogeneous submodule $\mathcal{M}^\mathcal{P}$; comparing with (1.55) we see that the kernel of all the elements of the component $\mathcal{M}_{q+r}^\mathcal{P}$ is the prolonged symbol space $(\mathcal{M}_{q+r})_\rho$.

Definition 3.1.1. *The symbol $\mathcal{M}_q$ of the differential equation $\mathcal{R}_q$ is called* involutive *at the point $\rho \in \mathcal{R}_q$, if local coordinates $\mathbf{x}$ exist on a neighbourhood of $\pi^q(\rho)$ such that an involutive head autoreduction of the set $\Lambda = \{\lambda^\tau \mid 1 \leq \tau \leq p\}$ yields a Pommaret basis of the symbol module $\mathcal{M}^\mathcal{P}$ for a class respecting term order.*

Recall that essentially only one class respecting term order exists: the degree reverse lexicographic order (see Lemma B.2.2). We require here the use of such an order for a simple reason: the involutive size of the autoreduced set $\Lambda^\triangle$ is maximal for a class respecting term order. Thus, if we do not get a Pommaret basis for such an order, we will not get a Pommaret basis for any other term order.

Obviously, it is well possible for non-linear equations that the symbol is involutive at some points $\rho \in \mathcal{R}_q$ but not at other points. Thus involution is a *local* property. For simplicity, we will assume throughout this book, that the properties of $\mathcal{M}_q$ do not vary over the manifold $\mathcal{R}_q$. In the remainder of this section, we translate this definition into the form usually found in books on the formal theory. Thus to a certain extent we just repeat material of the last chapter in a different language.

In practice, the test for involution of a symbol is performed as follows. We take the matrix M_q and sort its columns according to a class respecting term order. In fact, it suffices, if we take care that a column corresponding to an unknown $\dot{p}^\alpha_\mu$ (representing coordinates in $S_q(T^*_x\mathcal{X}) \otimes V_\xi\mathcal{E}$) is always to the left of a column corresponding to the unknown $\dot{p}^\beta_\nu$, if $\operatorname{cls}\mu > \operatorname{cls}\nu$. Then we compute a row echelon form $M^\triangle_q$ of M_q using only row operations. This is equivalent to an involutive head autoreduction of the set Λ. The rows of $M^\triangle_q$ define a new set $\Lambda^\triangle \subset S_q(T_x\mathcal{X}) \otimes V^*_\xi\mathcal{E}$. We are interested in which column the first non-vanishing entry of each row sits; the corresponding unknown $\dot{p}^\alpha_\mu$ is called the *leader* of the row.

Definition 3.1.2. *Let $\beta^{(k)}_q$ be the number of leaders that are of class k. These numbers are the* indices *of the symbol $\mathcal{M}_q$.*

The problem of δ-regularity concerns this definition. The class of a derivative is not invariant under coordinate transformations. In different coordinate systems we may thus obtain different values for the indices. δ-regular coordinates are distinguished by the fact that the sum $\sum_{k=1}^n k\beta^{(k)}_q$ takes its maximal value, as it gives the involutive size of $\Lambda^\triangle$ with respect to the Pommaret division and a class respecting term order.

If we recall the construction of formal power series solutions in Section 1.4, then the computation of the row echelon form corresponds to a special choice of the principal derivatives: the derivative p^α_μ is principal, if and only if $\dot{p}^\alpha_\mu$ is a leader. Thus the index $\beta^{(k)}_q$ determines the number of principal derivatives of order q and class k.

Proposition 3.1.3. *The symbol $\mathcal{M}_q$ with the indices $\beta^{(k)}_q$ is involutive, if and only if the matrix M_{q+1} of the prolonged symbol $\mathcal{M}_{q+1}$ satisfies*

$$\operatorname{rank} M_{q+1} = \sum_{k=1}^{n} k\beta^{(k)}_q \,. \tag{3.2}$$

Proof. One direction is a trivial consequence of the definition of a Pommaret basis. If $\Lambda^\triangle$ is such a basis, all elements of $\mathcal{M}^\mathcal{P}$ of degree $q+1$ are obtained via multiplicative prolongations and their number is given by the right hand side of (3.2). We postpone the proof of the converse to Chapter 6 where the concept of *local involution* will be introduced and where we will show that for the Pommaret division it is equivalent to involution. If the equality (3.2) holds, the basis $\Lambda^\triangle$ is locally involutive. $\square$

Let us consider this result at the level of the differential equation and not only the symbol. In the neighbourhood of a point $\rho \in \mathcal{R}_q$ we solve each equation in a local representation for a derivative. We can do this in such a way that each equation is solved for a different derivative and that this derivative is the leading derivative of the equation with respect to a class respecting term order. If the class of a leading derivative is k, we assign the multiplicative variables $x^1, \dots, x^k$ to the corresponding equation. If we do the same with the prolonged equation $\mathcal{R}_{q+1}$ in the neighbourhood of a point in $\left(\pi^{q+1}_q\right)^{-1}(\rho)$, then all equations of order $q+1$ in the local representation of $\mathcal{R}_{q+1}$ are algebraically dependent of the formal derivatives of the equations of order q in the representation of $\mathcal{R}_q$. If the symbol is involutive, we do not need all formal derivatives of these equations.

Proposition 3.1.4. *The symbol $\mathcal{M}_q$ of the differential equation $\mathcal{R}_q$ is involutive, if and only if all equations of order $q+1$ in the local representation of the prolonged equation $\mathcal{R}_{q+1}$ are algebraically dependent of the formal derivatives of the equations of order q in $\mathcal{R}_q$ with respect to their multiplicative variables only.*

Proof. Obviously, we may prepare our local representation of $\mathcal{R}_q$ in such a way that applying (1.50) yields the symbol directly in row echelon form. All equations obtained by prolonging each equation only with respect to its multiplicative variables are independent, as they possess distinct leading derivatives. Since there are $\beta^{(k)}_q$ equations of order q and class k in our local representation of $\mathcal{R}_q$, this yields at least $\sum_{k=1}^{n} k\beta^{(k)}_q$ independent equations of order $q+1$ in the local representation of $\mathcal{R}_{q+1}$. By assumption, $\operatorname{rank}\mathcal{M}_{q+1}$ equals this number. Hence the local representation of $\mathcal{R}_{q+1}$ cannot contain more independent equations of order $q+1$. Note that it may well contain further independent equations, but these must be of lower order, i. e. they are integrability conditions (cf. the proof of Theorem 1.5.4). □

Example 3.1.5. We reconsider Example 2.1.9 in form of the second-order equation $\mathcal{R}_2$ defined by the system $u_{xx} = u_{yy} = 0$. Its symbol is

$$\mathcal{M}_2 : \begin{cases} \dot{u}_{yy} = 0\,, \\ \dot{u}_{xx} = 0\,. \end{cases} \tag{3.3}$$

If we order $x^1 = x$ and $x^2 = y$, we see that the first equation is of class 2 and the second one of class 1. The symbol matrix is already in row echelon form and for the second equation y is a non-multiplicative variable.

For the indices of $\mathcal{M}_2$ we find $\beta^{(2)}_2 = \beta^{(1)}_2 = 1$. The prolonged symbol $\mathcal{M}_3$ is defined by the equations

$$\mathcal{M}_3 : \begin{cases} \dot{u}_{yyy} = 0\,, & \dot{u}_{xyy} = 0\,, \\ \dot{u}_{xxy} = 0\,, & \dot{u}_{xxx} = 0\,. \end{cases} \tag{3.4}$$

Thus we get that $\operatorname{rank} M_3 = 4 \neq \beta_2^{(1)} + 2\beta_2^{(2)} = 3$ and according to our definition $\mathcal{M}_2$ is *not* involutive. In contrast, it is easy to see that $\mathcal{M}_3$ is involutive, as $\dim \mathcal{M}_3 = 0$: a vanishing symbol is always involutive (exercise!). We will show later in Chapter 6 that this is not accidental; any symbol becomes involutive after a finite number of prolongations. Here already one prolongation suffices.

At the level of the differential equation we get the following picture. Prolonging the equation $u_{xx} = 0$ with respect to it non-multiplicative variable yields the equation $u_{xxy} = 0$. It is easy to see that this equation is algebraically independent of all multiplicative prolongations. Note that the equation $u_{xxy} = 0$ is *not* an integrability condition; one easily checks that $\mathcal{R}_q^{(1)} = \mathcal{R}_q$ for all $q \geq 2$. As in the last chapter we call such an equation an *obstruction to involution.* $\triangleleft$

The leaders of an involutive symbol matrix in row echelon form cannot take arbitrary values. The following result is a reformulation of Lemma 2.4.2 showing that the leading exponents define a stable monoid ideal.

Lemma 3.1.6. *Let $\mathcal{M}_q$ be an involutive symbol and $M_q^{\triangle}$ its matrix in row echelon form where the columns have been ordered according to a class respecting term order. If $\dot{p}^\alpha_\mu$ is a leader of $M_q^{\triangle}$ of class k, then $\dot{p}^\alpha_{\mu - 1_k + 1_j}$ is also a leader for all $j > k$.*

Example 3.1.7. We apply our criterion to the analysis of the symbol of the Einstein equations (1.73). In arbitrary coordinates it is rather unpleasant to write down the defining equations. So we exploit that involution is a local property: if we want to check involution at a point in the fibre over $x_0 \in \mathcal{X}$, we may use the locally geodesic coordinates mentioned in Example 1.6.4.

The form (1.74) permits an easy construction of the symbol at any point in the fibre over x_0. As no lower-order terms are present, it makes no difference whether we consider the symbol equations or the differential equations. Our first task is to determine the leaders. We distinguish four cases.

$i \neq n$ and $j \neq n$: The leader is $\partial_{x^n x^n} g_{ij}$.
$i = n$ and $j < n-1$: The leader is $\partial_{x^n x^{n-1}} g_{n-1,j}$.
$i = n$ and $j = n-1$: The leader is $\partial_{x^n x^{n-1}} g_{11}$.
$i = n$ and $j = n$: The leader is $\partial_{x^{n-1} x^{n-1}} g_{11}$.

In the last case several substitutions of other equations are necessary for obtaining an autoreduced form. The other cases are rather obvious. One easily verifies that the leaders satisfy Lemma 3.1.6.

The indices of the Einstein equations are therefore

$$\beta_2^{(n)} = \frac{n(n-1)}{2}\,, \quad \beta_2^{(n-1)} = n\,, \quad \beta_2^{(n-k)} = 0 \text{ for } 1 < k < n\,. \tag{3.5}$$

We mentioned already in Example 1.6.4 that for the first prolongation one may act as if (1.74) was indeed a differential equation and not just some relations valid only in the fibre over x_0. So the prolonged symbol is again easily constructed and the rank of its matrix evaluated to $\frac{1}{2}n^2(n+1)-n$. But this is just $n\beta_2^{(n)}+(n-1)\beta_2^{(n-1)}$ and, by Proposition 3.1.3, the symbol of the Einstein equations is involutive. Note that this implies that our coordinates are δ-regular, as in δ-singular coordinates the criterion of Proposition 3.1.3 always fails. $\lhd$

For *first-order* symbols this lemma implies that the indices form an ascending sequence. For higher-order symbols this is in general not the case; a trivial counter example is the symbol of the second-order equation defined by $u_{yy} = u_{xy} = u_{xx} = 0$ where $\beta_2^{(1)} = 2 > \beta_2^{(2)} = 1$.

Corollary 3.1.8. *Let $\mathcal{M}_1$ be the involutive symbol of a first order equation in n independent and m dependent variables. Then*

$$0 \le \beta_1^{(1)} \le \beta_1^{(2)} \le \cdots \le \beta_1^{(n)} \le m\,. \tag{3.6}$$

Proof. This is an immediate consequence of Lemma 3.1.6. If $\dot{p}_k^\alpha$ is a leader of class k, then for any $j > k$ we have a leader $\dot{p}_j^\alpha$ of class j. Hence $\beta_1^{(k)} \le \beta_1^{(j)}$ for $k < j$. $\square$

Example 3.1.9. Let us analyse the symbol of Maxwell's equations (1.64). We order the independent variables as (x, y, z, t) so that time derivatives have the highest class. With this ordering (1.64) yields the symbol matrix directly in row echelon form. In fact, as no lower-order terms are present in the equations, it makes no real difference whether we study the symbol or the differential equation. Thus we do not introduce any extra notations for the variables in the symbol equations.

The rows of maximal class, i. e. of class 4, correspond to the six evolution equations in (1.64a); the remaining two constraint equations (1.64b) are of class 3, as each can by solved for a z-derivative. Thus we obtain for the indices

$$\beta_1^{(4)} = 6\,, \quad \beta_1^{(3)} = 2\,, \quad \beta_1^{(2)} = \beta_1^{(1)} = 0\,. \tag{3.7}$$

They form indeed an ascending sequence of the form (3.6). Only for the z-components of the fields $\mathbf{E}$ and $\mathbf{B}$ several leaders appear, namely $E_z^{(z)}$ and $E_t^{(z)}$ (and correspondingly for $B^{(z)}$). This agrees with Lemma 3.1.6.

In order to check whether the symbol is involutive, we must determine what happens in the non-multiplicative prolongations. In our case, this means that we must differentiate the constraint equations (1.64b) with respect to t. We find

$$\partial_t(\nabla \cdot E) = \nabla \cdot E_t = \nabla \cdot (\nabla \times \mathbf{B}) = 0 \tag{3.8}$$

where in the first step we used multiplicative prolongations of equations in (1.64a) and the last step is a well-known result in vector analysis. The same result is obtained with the equations for the magnetic field $\mathbf{B}$. Thus we can rewrite any non-multiplicative prolongation as a linear combination of multiplicative ones and the symbol is involutive. $\triangleleft$

Finally, we prove the following two somewhat technical results. Every prolongation of an involutive symbol is again involutive. In principle, this is obvious from the properties of a Pommaret basis; however, we give here a direct computational proof. The second result concerns a kind of commutativity of projection and prolongation for equations with an involutive symbol. Both results will be needed in the next section for proving a simple criterion for involutive differential equations.

Proposition 3.1.10. *Let the symbol $\mathcal{M}_q$ of the differential equation $\mathcal{R}_q$ be involutive.*

(i) *The prolonged symbol $\mathcal{M}_{q+1}$ is involutive, too.*
(ii) *The equality $(\mathcal{R}_q^{(1)})_{+1} = \mathcal{R}_{q+1}^{(1)}$ holds.*

Proof. We start with Part (i); Part (ii) will be a corollary. Without loss of generality, we assume that the symbol matrix M_q is in row echelon form with the columns ordered according to a class respecting term order. Let $\dot{p}_\mu^\alpha$ be the leaders of M_q. By Proposition 3.1.4, the leaders of M_{q+1} are $\dot{p}_{\mu+1_\ell}^\alpha$ with $1 \leq \ell \leq \operatorname{cls}\mu$. The class of such a leader is ℓ.

To prove Part (i), we must show that prolonging a row in M_{q+1} with respect to a non-multiplicative variable leads to a linearly dependent equation. Thus let $k = \operatorname{cls}\mu$ and let $\ell \leq k$. Prolonging the rows of M_{q+1} with respect to all independent variables leads to rows with leaders of the form $\dot{p}_{\mu+1_\ell+1_j}^\alpha$ for $1 \leq j \leq n$. Three cases must be distinguished:

$j \leq \ell$: These are the prolongations with respect to multiplicative variables. All obtained rows are linearly independent, as their leaders are different.

$\ell < j \leq k$: Since $j \leq k$, the variable x^j is multiplicative for the row in M_q with the leader $\dot{p}_\mu^\alpha$. Thus we can first prolong this row with respect to x^j and then the result with respect to x^ℓ and obtain the same result. As now each prolongation is only with respect to a multiplicative variable, the resulting row is already contained in the rows of the previous case.

$j > k$: Let us denote the row in M_{q+2} by r. As in the previous case we interchange the prolongations, thus prolong first with respect to x^j and then with respect to x^ℓ to obtain r. Since $j > k$, $\dot{p}^\alpha_{\mu-1_k+1_j}$ is by Lemma 3.1.6 also a leader in M_q. Its class is greater than or equal k; thus x^k is a multiplicative variable for the corresponding row. By prolonging it with respect to x^k we see that $\dot{p}^\alpha_{\mu+1_j}$ is among the leaders of M_{q+1}. Since x^ℓ is multiplicative for this equation, we have found a row contained in the first case with leader $\dot{p}^\alpha_{\mu+1_\ell+1_j}$.
If we compare the two rows with leader $\dot{p}^\alpha_{\mu+1_\ell+1_j}$, we see that they are both prolongations with respect to x^ℓ of rows with the leader $\dot{p}^\alpha_{\mu+1_j}$. One of these was obtained by prolonging a row of M_q with respect to a non-multiplicative variable. Since $\mathcal{M}_q$ is involutive, this row is not contained in the set of independent rows we use for $\mathcal{M}_{q+1}$ but is a linear combination of those. Thus r can be written as x^ℓ-prolongation of this linear combination. If x^ℓ is a non-multiplicative variable for some of the rows involved, we apply the same analysis again. Since the considered leaders become smaller and smaller, this process eventually terminates and then we have expressed r as a linear combination of rows contained in the first case.

Hence we have shown that all independent rows in M_{q+2} can be obtained by prolonging the rows of M_{q+1} only with respect to their multiplicative variables. But this means nothing else than that $\mathcal{M}_{q+1}$ is involutive.

Part (ii) may look trivial at first sight, but it is not. On the left hand side only the integrability conditions that arise during the projection from order $q+1$ to order q are taken into account; then one prolongs again to order $q+1$. On the right hand side, however, we project from order $q+2$ to order $q+1$. Hence we have to show that all integrability conditions which occur during this projection are prolongations of integrability conditions of the projection of order $q+1$ to q.

In order to achieve this we make basically the same considerations as in the proof of Part (i) but now with the full equations instead of only with their highest-order part. Let $\Phi^\tau = 0$ be an equation of class k in a local description of $\mathcal{R}_q$. We have to analyse all equations of the form $D_\ell D_j \Phi^\tau = 0$. We distinguish two cases assuming always $\ell \leq j$:

$j \leq k$: In this case $D_\ell D_j \Phi^\tau = 0$ contains only prolongations with respect to multiplicative variables. By Proposition 3.1.4, these are the only independent equations of order $q+2$ in our local description of $\mathcal{R}_{q+2}$.

$k < j$: This implies that $D_j \Phi^\tau = 0$ was a prolongation with respect to a non-multiplicative variable. By Proposition 3.1.4 we can express it as a linear combination of prolongations of equations of $\mathcal{R}_q$ with respect to multiplicative variables and perhaps some integrability conditions. Following again the argument of the third case above, we find that $D_\ell D_j \Phi^\tau = 0$ is a linear combination of the equations contained in the first case and of prolonged integrability conditions.

Note that it is not necessary here to distinguish between $\ell \leq k$ and $\ell > k$, as we are allowed to use all prolongations to construct $(\mathcal{R}_q^{(1)})_{+1}$. □

Part (ii) says that for a differential equation with involutive symbol we obtain the same, if we either first prolong it twice and then project once or alternately prolong, project and prolong. Denoting prolongations by one order by ρ and projections by one order by π, this may concisely be written as $\rho \circ \pi \circ \rho = \rho^2 \circ \pi$. Note that it is crucial that we start on both sides with a prolongation; we may not omit the leading ρ.

3.2 Involutive Differential Equations

Based on the results in the last section, we finally introduce the notion of an involutive differential equation. It combines the geometric concept of formal integrability with the combinatorial ideas underlying involutive bases. We will see in the subsequent chapters that this indeed makes sense. In fact, it will turn out that involution provides a very powerful framework for analysing many different aspects of differential equations.

Definition 3.2.1. *The differential equation* $\mathcal{R}_q$ *is called* involutive*, if it is formally integrable and if its symbol* $\mathcal{M}_q$ *is involutive.*

In the literature, the term "involution" is often used in a rather imprecise manner. In particular, involution is sometimes taken as a synonym for formal integrability. While Definition 3.2.1 obviously implies that an involutive equation is also formally integrable, the converse is generally not true: involution is a stronger concept than formal integrability.

Example 3.2.2. The probably simplest example of a differential equation which is formally integrable but *not* involutive is the following second-order equation in two independent and one dependent variable:

$$\mathcal{R}_2 : \begin{cases} u_{yy} = 0 \, , \\ u_{xx} = 0 \, . \end{cases} \tag{3.9}$$

We have already shown in Example 3.1.5 that its symbol $\mathcal{M}_2$ is not involutive. But $\mathcal{R}_2$ is trivially formally integrable, as all right hand sides are zero. Since the prolonged symbol $\mathcal{M}_3$ is involutive, $\mathcal{R}_3$ is an involutive equation. ◁

The first important point about involution is that it is much easier to verify than formal integrability. Recall that Definition 1.4.2 requires to check infinitely many conditions, namely that no integrability conditions arise at any order of prolongation. Up to now, no finite criterion for formal integrability is known. But although formal integrability is a necessary condition for involution, the latter one can be checked in a finite manner.

Theorem 3.2.3. *$\mathcal{R}_q$ is an involutive differential equation, if and only if its symbol $\mathcal{M}_q$ is involutive and $\mathcal{R}_q^{(1)} = \mathcal{R}_q$.*

Proof. The necessity of the two given conditions is obvious from Definition 3.2.1, as they are weaker than the conditions there. The interesting part is their sufficiency and represents largely a simple corollary to Proposition 3.1.10 which can be applied since the symbol $\mathcal{M}_q$ is assumed to be involutive. By Part (ii) of it and our assumption on $\mathcal{R}_q^{(1)}$, we obtain $\mathcal{R}_{q+1}^{(1)} = (\mathcal{R}_q^{(1)})_{+1} = \mathcal{R}_{q+1}$. Furthermore, Part (i) implies that all prolonged symbols $\mathcal{M}_{q+r}$ are involutive, too, so that we can use this argument inductively at any order. Hence $\mathcal{R}_{q+r}^{(1)} = \mathcal{R}_{q+r}$ for all $r \geq 0$ and $\mathcal{R}_q$ is formally integrable. □

Now we understand the meaning of the at first sight rather technical Proposition 3.1.10. If the symbol $\mathcal{M}_q$ is involutive and at some order of prolongation integrability conditions are hidden, then at least some of these conditions must arise already during the *first* prolongation (here we meet again the idea of local involution to be discussed in Section 6.3); the remaining ones are prolongations of these. This is obviously a very strong result and forms the basis of any concrete computation with involutive equations. In particular, the Cartan–Kuranishi completion, which we will discuss in Section 6.2, depends crucially on it.

Example 3.2.4. Theorem 3.2.3 permits us to conclude that the Einstein equations are involutive. We showed already in Example 1.6.4 that $\mathcal{R}_2^{(1)} = \mathcal{R}_2$ and in Example 3.1.7 that the symbol $\mathcal{M}_2$ is involutive. Thus all conditions of the theorem are satisfied. $\triangleleft$

Involutive differential equations possess local representations of a special form: the *Cartan normal form.* In order to avoid excessive notation we present it only for a first-order equation $\mathcal{R}_1$. This poses no real restriction, as every differential equation can be transformed into an equivalent first-order one (see Appendix A.3). The first two main points about the Cartan normal form are that it yields the symbol matrix immediately in row echelon form and that each equation in the system is solved for its principal derivative. Thus it looks as follows:

$$p_n^\alpha = \phi_n^\alpha(x^i, u^\beta, p_j^\gamma, p_n^\delta)\,, \qquad \begin{cases} 1 \le \alpha \le \beta_1^{(n)}\,, \\ 1 \le j < n\,, \\ \beta_1^{(n)} < \delta \le m\,, \end{cases} \tag{3.10a}$$

$$p_{n-1}^\alpha = \phi_{n-1}^\alpha(x^i, u^\beta, p_j^\gamma, p_{n-1}^\delta)\,, \qquad \begin{cases} 1 \le \alpha \le \beta_1^{(n-1)}\,, \\ 1 \le j < n-1\,, \\ \beta_1^{(n-1)} < \delta \le m\,, \end{cases} \tag{3.10b}$$

$$\vdots$$

$$p_1^\alpha = \phi_1^\alpha(x^i, u^\beta, p_1^\delta)\,, \qquad \begin{cases} 1 \le \alpha \le \beta_1^{(1)}\,, \\ \beta_1^{(1)} < \delta \le m\,, \end{cases} \tag{3.10c}$$

$$u^\alpha = \phi^\alpha(x^i, u^\beta)\,, \qquad \begin{cases} 1 \le \alpha \le \beta_0 \le m;, \\ \beta_0 < \beta \le m\,. \end{cases} \tag{3.10d}$$

The subsystem in the first line comprises all equations of class n, the one in the second line all of class $n-1$ and so on. The derivatives on the right hand side have always a class lower than or equal to the one on the left hand side. The subsystem in the last line collects all algebraic constraints; their number is denoted by β_0. The subsystems may be empty from a certain class on, as the indices satisfy $0 \le \beta_1^{(1)} \le \cdots \le \beta_1^{(n)} \le m$ (Corollary 3.1.8).

These purely structural aspects of the normal form (3.10) do not yet capture that our differential equation is involutive; they only express that we have chosen a local representation in triangular form. Any differential equation can be brought into such a form. The most important point about the Cartan normal form is that involution implies a number of relations between prolongations of the individual equations in (3.10). First of all, involution of the symbol $\mathcal{M}_1$ requires by Proposition 3.1.4 that any non-multiplicative prolongation can be written as a linear combination of multiplicative ones. Thus functions $A_{ij}^\beta(\mathbf{x}, \mathbf{u}, \mathbf{p})$, $B_i^\beta(\mathbf{x}, \mathbf{u}, \mathbf{p})$, and $C^\beta(\mathbf{x}, \mathbf{u})$ must exist such that whenever $1 \le \ell < k \le n$

$$\begin{aligned} D_k(p_\ell^\alpha - \phi_\ell^\alpha) = \sum_{i=1}^{k} \sum_{\beta=1}^{\beta_1^{(i)}} \left\{ \sum_{j=1}^{i} A_{ij}^\beta D_j(p_i^\beta - \phi_i^\beta) + B_i^\beta(p_i^\beta - \phi_i^\beta) \right\} + \\ + \sum_{\beta=1}^{\beta_0} C^\beta(u^\beta - \phi^\beta)\,, \end{aligned} \tag{3.11}$$

as x^k is non-multiplicative for all equations solved for an x^ℓ-derivative. It follows easily from an analysis of the appearing derivatives that no equations of class higher than k are needed on the right hand side. Strictly speaking, involution of the symbol requires only that in (3.11) the second-order derivatives cancel. The functions B_i^β, and C^β take already care that (3.11) does not lead to integrability conditions.

Formal integrability of $\mathcal{R}_1$ requires furthermore that the prolongation of the algebraic equations in (3.10d) does not lead to new equations. This implies the existence of functions $\bar{C}^\beta(\mathbf{x}, \mathbf{u})$ such that

$$\phi_k^\alpha - \sum_{\beta=\beta_0+1}^{m} \frac{\partial \phi^\alpha}{\partial u^\beta} \phi_k^\beta = \frac{\partial \phi^\alpha}{\partial x^k} + \sum_{\beta=1}^{\beta_0} \bar{C}^\beta \left(u^\beta - \phi^\beta\right) . \tag{3.12}$$

Note that involution of (3.10) requires not only that $\beta_0 \leq \beta_1^{(1)}$ but in fact that even on the right hand side of (3.10d) always $\beta \leq \beta_1^{(1)}$. Otherwise, some derivatives in the prolonged equations could not be eliminated by the differential equations in (3.10).

Example 3.2.5. As a larger example we study linear homogeneous first-order equations in one dependent variable. This is a very classical problem which found much interest in the late nineteenth century. Most textbooks on differential equations in the early twentieth century discussed it at length (see e. g. [32]). It leads to what is nowadays called the *Frobenius Theorem* in differential geometry (see Appendix C.3). In local coordinates the differential equations we are considering take the form

$$\mathcal{R}_1 \,:\, \left\{ \sum_{i=1}^{n} a_\tau^i(\mathbf{x}) p_i = 0 \,, \qquad 1 \leq \tau \leq t \,. \right. \tag{3.13}$$

Note that no terms of order zero appear.

We want to study under what conditions $\mathcal{R}_1$ is involutive. In order to facilitate the analysis of the symbol, we transform the system into a triangular form. By taking appropriate linear combinations we solve for some derivatives. We assume that these are those of maximal class; otherwise we change the independent variables. So we arrive at a system of the form

$$\mathcal{R}_1 \,:\, \left\{ p_{n+1-\tau} + \sum_{i=1}^{n-t} \bar{a}_\tau^i(\mathbf{x}) p_i = 0 \,, \qquad 1 \leq \tau \leq t \,. \right. \tag{3.14}$$

If (3.13) is involutive, this is its Cartan normal form.

Now we must check whether relations of the form (3.11) hold; so we consider the non-multiplicative prolongations of the equations in (3.14). For the equation with index τ these are the prolongations with respect to the variables $x^{n+1-\sigma}$ for $1 \leq \sigma < \tau$. They lead to second-order equations

$$p_{n+1-\sigma,n+1-\tau} + \sum_{i=1}^{n-t} \bar{a}^i_\tau(\mathbf{x}) p_{i,n+1-\sigma} + \sum_{i=1}^{n-t} \frac{\partial \bar{a}^i_\tau(\mathbf{x})}{\partial x^{n+1-\sigma}} p_i = 0 \,. \tag{3.15}$$

In order for $\mathcal{R}_1$ to be involutive, we must be able to express (3.15) as a linear combination of the remaining prolongations. The first term can be cancelled by subtracting the prolongation of the equation with index σ with respect to $x^{n+1-\tau}$: it is obtained by interchanging σ and τ in (3.15).

Next we study the remaining second-order derivatives in the first sum in (3.15) (and in the equation with σ and τ interchanged, respectively). Each of them can be cancelled individually by subtracting (respectively adding) the prolongation of the equation with index τ (respectively σ) with respect to x^i multiplied by $\bar{a}^i_\tau(\mathbf{x})$ (respectively $\bar{a}^i_\sigma(\mathbf{x})$). One easily checks that hereby all appearing derivatives p_{ij} with $1 \leq i, j \leq n - t$ cancel.

Thus we are able to eliminate all second-order derivatives implying that the symbol $\mathcal{M}_1$ is involutive. But if we perform all these eliminations, we obtain as integrability condition for each pair $1 \leq \sigma < \tau \leq t$ a first-order differential equation:

$$\begin{gathered} \sum_{i=1}^{n-t} \left[\frac{\partial \bar{a}^i_\tau(\mathbf{x})}{\partial x^{n+1-\sigma}} - \frac{\partial \bar{a}^i_\sigma(\mathbf{x})}{\partial x^{n+1-\tau}} + \right. \\ \left. \sum_{j=1}^{n-t} \bar{a}^j_\sigma(\mathbf{x}) \frac{\partial \bar{a}^i_\tau(\mathbf{x})}{\partial x^j} - \sum_{j=1}^{n-t} \bar{a}^j_\tau(\mathbf{x}) \frac{\partial \bar{a}^i_\sigma(\mathbf{x})}{\partial x^j} \right] p_i = 0 \,. \end{gathered} \tag{3.16}$$

As it does not contain any derivative p_i with $i > n-t$, it cannot be expressed as a linear combination of equations contained in (3.14): $\mathcal{R}_1$ is formally integrable and thus involutive, if and only if in (3.16) the coefficients in the bracket vanishes.

For a more geometric version of this result, we consider the following set of vector fields

$$\bar{X}_\tau = \partial_{x^{n+1-\tau}} + \sum_{i=1}^{n-t} \bar{a}^i_\tau(\mathbf{x}) \partial_{x^i} \,, \qquad 1 \leq \tau \leq t \,. \tag{3.17}$$

They allow us to express (3.14) succinctly in the form $\bar{X}_\tau u = 0$. Comparing with (C.6) it is not difficult to see that the above conditions for involution are equivalent to the vanishing of the Lie brackets of the fields $\bar{X}_\tau$:

$$[\bar{X}_\tau, \bar{X}_\sigma] = 0 \,, \qquad 1 \leq \sigma < \tau \leq t \,. \tag{3.18}$$

Thus we are in the same situation as in Example 1.4.3 where also the integrability conditions could be expressed as the vanishing of Lie brackets.

We may write the original system (3.13) in terms of the vector fields

$$X_\tau = \sum_{i=1}^{n} a^i_\tau(\mathbf{x})\partial_{x^i}\ , \qquad 1 \le \tau \le t\ . \tag{3.19}$$

They are linear combinations of the vector fields $\bar{X}_\tau$ – but with $\mathfrak{X}(M)$ considered as $\mathcal{F}(M)$-module (see Appendix C.3). The Lie bracket (C.6) is bilinear only with respect to linear combinations with real coefficients. Thus we cannot conclude that for an involutive system (3.13) the vector fields X_τ must commute. But it is easy to see that the commutator of two of the vector fields must be an $\mathcal{F}(M)$-linear combination of the X_τ, i. e. functions $C^\rho_{\sigma\tau}(\mathbf{x})$ must exist such that

$$[X_\sigma, X_\tau] = \sum_{\rho=1}^{t} C^\rho_{\sigma\tau}(\mathbf{x})X_\rho\ , \tag{3.20}$$

i. e. the distribution $\mathcal{D}$ spanned by these vector fields is closed under the Lie bracket. Thus the equation $\mathcal{R}_1$ is involutive, if and only if the distribution $\mathcal{D}$ is involutive.

In the classical language, a system (3.13) such that the vector fields X_τ satisfy (3.20) is called *complete*; this terminology goes back at least to Clebsch. If it is in the triangular form (3.14), one speaks of a *Jacobian system*.

The results above may be partially extended to non-linear systems using the so-called *Jacobi bracket* [32]. If F and G are two arbitrary functions defined on the first-order jet bundle $J_1\mathcal{E}$ where the fibre dimension of $\mathcal{E}$ is one, then the Jacobi bracket is given in local coordinates by

$$[F, G] = \frac{\partial F}{\partial p_i}\left(\frac{\partial G}{\partial x^i} + p_i\frac{\partial G}{\partial u}\right) - \frac{\partial G}{\partial p_i}\left(\frac{\partial F}{\partial x^i} + p_i\frac{\partial F}{\partial u}\right)\ . \tag{3.21}$$

It possesses similar properties as the commutator of two vector fields but it obeys a different Jacobi identity. Using the formal derivative (1.11) it is more concisely written as $[F, G] = F_{p_i}D_iG - G_{p_i}D_iF$. In a straightforward extension of the computations above one shows that the non-linear first-order system $\Phi^\tau(\mathbf{x}, u, \mathbf{p}) = 0$ is involutive, if and only if all Jacobi brackets $[\Phi^\sigma, \Phi^\tau]$ vanish modulo the system.

Finally, let us note that any complete system can be integrated solving only *ordinary* differential equations. If it consists of t equations, then it possesses $n - t$ functionally independent solutions $\psi_k(\mathbf{x})$ with $1 \le k \le n - t$ and its general solution is given by $\Psi\bigl(\psi_1(\mathbf{x}), \dots, \psi_{n-t}(\mathbf{x})\bigr)$ where Ψ is an arbitrary function.

Indeed, if $t = 1$, this is a simple consequence of the classical method of characteristics discussed in any introductory textbook on partial differential equations. For $t > 1$ we use an extension of it due to Mayer [117]. For simplicity, we formulate it in local coordinates and assume without loss of generality that we are dealing with the Jacobian system defined by the vector fields (3.17). Consider the one-forms

$$\omega^k = dx^k - \sum_{\tau=n-t+1}^{n} \bar{a}^k_\tau(\mathbf{x})dx^\tau\,, \qquad 1 \leq k \leq n-t\,. \tag{3.22}$$

They generate a Pfaffian system which is the annihilator $\mathcal{D}^0$ of the distribution $\mathcal{D}$ spanned by vector fields (3.17).

We choose a point $x_0 \in \mathcal{X}$ and introduce t parameters λ^τ and one new variable s by setting $x^\tau(s) = x_0^\tau + \lambda^\tau s$ for $n-t < \tau \leq n$. This defines a curve on a t-dimensional coordinate plane through x_0. On it $dx^\tau = \lambda ds$ and the Pfaffian system (3.22) becomes a system of $n-t$ ordinary differential equations

$$\frac{dx^k}{ds} = \sum_{\tau=n-t+1}^{n} \lambda^\tau \bar{a}^k_\tau(x^\ell, x_0^\sigma + \lambda^\sigma s)\,, \qquad \begin{cases} 1\leq k,\ell\leq n-t\,, \\ n-t<\sigma\leq n\,. \end{cases} \tag{3.23}$$

Its solutions are of the form $x^k(s) = \phi^k(s, x_0, \lambda^\sigma)$. If $u(\mathbf{x})$ is a solution of our Jacobian system, then one easily shows that $(d/ds)u\bigl(x^i(s)\bigr) = 0$. As in the classical theory of characteristics, solutions of the original system are constant along the solution of the (generalised) characteristic system (3.23). The parameters λ^τ can be eliminated by setting $\lambda^\tau = (x^\tau - x_0^\tau)/s$. Then as claimed above, $n-t$ functionally independent functions ψ_k exist such that $(d/ds)\psi_k\bigl(x^i(s)\bigr) = 0$.

This relation of the Pfaffian system (3.22) to the method of characteristics is not surprising, as the forms $\omega^k\big|_{x_0}$ generate the characteristic variety C_{x_0} of our equation at the point $x_0 \in \mathcal{X}$. $\triangleleft$

The final proposition in this section should be rather obvious to any reader who has understood the meaning of involution. We mention it only for historical reasons, as it generalises an old result of Finzi [51] (cf. [129]) on systems with the same number of equations and dependent variables.

Proposition 3.2.6. *A differential equation $\mathcal{R}_q$ possesses identities or integrability conditions, if and only if either $\beta_q^{(n-1)} > 0$ or $\mathcal{R}^{(1)}_{q-1} \subset J_{q-1}\mathcal{E}$.*

Proof. The second case, $\mathcal{R}^{(1)}_{q-1} \subset J_{q-1}\mathcal{E}$, is trivial. It means that any local representation of $\mathcal{R}_q$ contains not only equations of order q but also of lower order. Obviously, this implies the existence of either identities or integrability conditions arising from the prolongation of these lower order equations. Thus we exclude this possibility and concentrate on the first case.

If $\beta_q^{(n-1)} > 0$, any local representation of $\mathcal{R}_q$ contains equations with non-multiplicative variables, as only equations of class n do not possess any.

Thus according to Proposition 3.1.4 the system must have either identities or integrability conditions. For the converse we note that $\beta_q^{(n-1)} = 0$ implies that all indices $\beta_q^{(k)}$ with $k < n$ must vanish; this can be shown analogously to Corollary 3.1.8. Thus there exists a local representation of $\mathcal{R}_q$ where all equations are of class n. Hence all their prolongations are algebraically independent and neither identities nor integrability conditions can arise from cross-derivatives. □

3.3 The Principal Symbol Revisited

In this section we prove a result on the relation of the highest index $\beta_q^{(n)}$ and the principal symbol of a differential equation. In principle, this result can be extended to give a fully constructive solution of the problem of δ-regularity (see [159]). However, this solution is computationally rather expensive; the approach presented in Section 2.4 is simpler and cheaper. So instead we use the mentioned relation only for a rigorous definition of under- and overdetermined equation.

Theorem 3.3.1. *Let $\mathcal{R}_q$ be a differential equation in n independent variables with principal symbol τ_χ. In different coordinate systems we may obtain different values for the index $\beta_q^{(n)}$ of the geometric symbol $\mathcal{M}_q$ at the point $\rho \in \mathcal{R}_q$ with $\pi(\rho) = x$; let $\tilde{\beta}_q^{(n)}$ be the maximum of these values. Then at the point ρ*

$$\max_{\chi \in T_x^*\mathcal{X}} \operatorname{rank} \tau_\chi = \tilde{\beta}_q^{(n)} . \tag{3.24}$$

Proof. By the definition of $\tilde{\beta}_q^{(n)}$, we find in a neighbourhood of ρ a local description of $\mathcal{R}_q$ where the first $\tilde{\beta}_q^{(n)}$ equations $\Phi^\tau(\mathbf{x}, \mathbf{p}) = 0$ satisfy

$$\frac{\partial \Phi^\tau}{\partial p^\alpha_{[0,\dots,0,q]}} = \delta^\tau_\alpha . \tag{3.25}$$

Setting $\chi = \chi_i dx^i\big|_x$ has the following effect on the matrix of the principal symbol τ_χ at the point ρ: the diagonal elements of the top left $\tilde{\beta}_q^{(n)} \times \tilde{\beta}_q^{(n)}$ submatrix of $T[\chi]$ are of the form $(\chi_n)^q + \dots$ and the monomial $(\chi_n)^q$ appears nowhere else in the matrix. If we choose $\chi = dx^n\big|_x$, all entries vanish with the exception of these diagonal elements which become one. Hence the rank of τ_χ at ρ is $\tilde{\beta}_q^{(n)}$ for this particular choice of χ.

Assume that for some $\chi \in T_x^*\mathcal{X}$ the rank of τ_χ was greater than $\tilde{\beta}_q^{(n)}$. We choose in a neighbourhood of x a new coordinate system $\tilde{\mathbf{x}}$ given by $\tilde{\mathbf{x}} = \boldsymbol{\psi}(\mathbf{x})$

where the function ψ^n satisfies $d\psi^n\big|_x = \chi$. Obviously, such coordinates always exist, as this is only a condition at a single point. For the dependency on the derivatives of class n we obtain after this transformation

$$\frac{\partial\tilde{\Phi}^\tau}{\partial\tilde{p}^\alpha_{[0,\dots,0,q]}} = \frac{\partial\Phi^\tau}{\partial p^\alpha_\nu}\prod_{i=1}^{n}\left(\frac{\partial\psi^n}{\partial x^i}\right)^{\nu_i} . \tag{3.26}$$

At the point x we can evaluate the derivatives of ψ^n and the right hand side becomes the matrix $T[\chi]$ of the principal symbol τ_χ. By assumption, its rank is greater than $\tilde{\beta}^{(n)}_q$. Thus we have found a coordinate system where more than $\tilde{\beta}^{(n)}_q$ equations are of class n contradicting the definition of $\tilde{\beta}^{(n)}_q$. □

Recall that by definition of δ-regular coordinates, they must always yield the value $\tilde{\beta}^{(n)}_q$ for the index $\beta^{(n)}_q$. Thus for such coordinates the one-form $\chi = dx^n$ must lead to a principal symbol of maximal rank. One-forms that lead to a lower rank are sometimes called *systatic*.

Corollary 3.3.2. *If a one-form χ exists such that the principal symbol τ_χ of the differential equation $\mathcal{R}_q$ is surjective, then $\mathcal{R}_q$ is involutive.*

Proof. The principal symbol τ_χ can only be surjective, if the number t of (independent) equations in a local representation of $\mathcal{R}_q$ is less than or equal to the number m of dependent variables and if all these equations are of order q. The rank of τ_χ is then t and by Theorem 3.3.1 all equations are of class n. But this implies that $\mathcal{M}_q$ is involutive and that no integrability conditions arise; hence $\mathcal{R}_q$ is involutive. □

Example 3.3.3. Despite the computational complexity of this approach, Theorem 3.3.1 is sometimes quite useful for checking the δ-regularity of a coordinate system. We demonstrate this for the Yang–Mills equations (1.70). Their principal part is linear and given by $\eta^{ij}(\partial_{x^ix^j}A^a_k - \partial_{x^ix^k}A^a_j)$. Thus we obtain in these coordinates $\beta^{(n)}_2 = d(n-1)$, as for all equations with an index $k \neq n$ we may take $\partial_{x^nx^n}A^a_k$ as principal derivative of class n. In the d remaining equations with $k = n$ we choose $\partial_{x^{n-1}x^n}A^a_{n-1}$ as principal derivative of class $n-1$. Thus $\beta^{(n-1)}_2 = d$ and all other indices vanish.

The question is whether some of the equations we classified as being of class $n-1$ could be "promoted" to class n by a coordinate transformation, i.e. whether our coordinates are δ-regular. As only $\beta^{(n)}_2$ and $\beta^{(n-1)}_2$ are greater than zero, we are done once we know whether $\beta^{(n)}_2 = \tilde{\beta}^{(n)}_2$ and by Theorem 3.3.1 this can be decided with the help of the principal symbol. For the entries of its matrix we obtain

$$T^{a\ell}_{bk}[\chi] = \delta^a_b\eta^{ij}\chi_i\left(\delta^\ell_k\chi_j - \delta^\ell_j\chi_k\right) \tag{3.27}$$

with $1 \leq a, b \leq d$ and $1 \leq k, \ell \leq n$. The indices a, k label the rows (i. e. the equations in the system) and the indices b, ℓ the columns (i. e. the fields).

It is easy to see that these entries satisfy for any one-form χ the d identities $\chi_\ell T^{a\ell}_{bk}[\chi] = 0$, as then the expression in the parentheses vanishes. This implies that the rank of the matrix $T[\chi]$ can never be larger than $d(n-1)$. But from our analysis above, we know already that for $\chi = dx^n$ we obtain this rank. Thus this is indeed the maximal rank and our coordinates are δ-regular.

The special form of these identities implies immediately that the symbol $\mathcal{M}_2$ of the Yang–Mills equations is involutive, as a multiplication by χ corresponds to a differentiation. The principal part of the prolongation $\mathcal{R}_3$ has the form $\eta^{ij}(\partial_{x^i x^j x^\ell} A^a_k - \partial_{x^i x^k x^\ell} A^a_j)$ and thus consists of dn^2 equations. Contraction with $\eta^{k\ell}$ yields obviously zero and we have at least d dependencies between rows of the symbol matrix M_3 (corresponding to the d identities above) and its maximal rank is $d(n^2-1)$. On the other hand, the number of multiplicative variables of the symbol $\mathcal{M}_2$ is a lower bound for this rank and in our case $n\beta_2^{(n)} + (n-1)\beta_2^{(n-1)} = d(n^2-1)$. Hence $\mathcal{M}_2$ is involutive by Proposition 3.1.3. We determined already in Example 1.6.3 that $\mathcal{R}_2^{(1)} = \mathcal{R}_2$. Hence the Yang–Mills equations are involutive.

Note that the concrete form of neither the metric η^{ij} nor the structure coefficients C^a_{bc} plays a role for this result! In particular, it is independent of the signature of the metric and remains valid in curved space-times. In the analysis of the symbol the structure constants do not appear at all; the formal integrability requires only that they satisfy the Jacobi identity. ◁

Example 3.3.4. In the same manner one may analyse the Einstein equations and give an alternative proof that their symbol is involutive. In locally geodesic coordinates the matrix of the principal symbol has the entries

$$\begin{aligned} T^{ij}_{k\ell}[\chi] = \chi_m \Big[&\eta^{jm}(\delta^i_\ell \chi_k + \delta^i_k \chi_\ell) + \eta^{im}(\delta^j_\ell \chi_k + \delta^j_k \chi_\ell) - \\ &2\eta^{ij}\chi_k\chi_\ell - \eta^{mn}\chi_m\chi_n(\delta^i_k\delta^j_\ell + \delta^i_\ell\delta^j_k)\Big] \,. \end{aligned} \tag{3.28}$$

Due to the symmetries of the metric and the Ricci tensor, T is a square matrix with $n(n+1)/2$ rows and columns; the former ones are labelled by the indices k, ℓ, the latter ones by i, j. One easily verifies the identities

$$\eta^{m\ell}\Big(T^{ij}_{km}[\chi]\chi_\ell - \frac{1}{2}T^{ij}_{m\ell}[\chi]\chi_k\Big) = 0\,. \tag{3.29}$$

They imply that the maximal rank of T cannot exceed $n(n-1)/2$. But we know already from Example 3.1.7 that in locally geodesic coordinates the index $\beta_2^{(n)}$ has this value. Thus no further identities exist and the coordinates are δ-regular.

We may again interpret the identities (3.29) as relations between the rows of the prolonged symbol matrix and a count similar to the previous example proves involution of the symbol. Note that (3.29) represents the principal symbol of the contracted Bianchi identity. Thus our calculations in locally geodesic coordinates yield an indirect proof of them. ◁

It is really surprising that one has difficulties to find in the literature rigorous definitions for such basic notions as under- and overdeterminacy. Often one uses the simple counting familiar from linear algebra: how many equations and how many unknown functions are there. Unfortunately, this may easily lead to wrong results, as the next example shows. The main point is that here a definition and a criterion are confused. In linear algebra, a linear system of equations is called underdetermined, if some components of the solution vector are not restricted by the system, i. e. may be chosen arbitrarily. It is trivial to show that this happens, if and only if the system comprises less independent equations than unknowns. For differential equations the situation is more complicated, as the "only if" is no longer true.

Example 3.3.5. Consider the $U(1)$-Yang–Mills equations on a two-dimensional space-time (i. e. electrodynamics in one space dimension):

$$\mathcal{R}_2 : \begin{cases} u_{tt} - v_{xt} = 0\,, \\ u_{xt} - v_{xx} = 0\,. \end{cases} \tag{3.30}$$

Obviously, we have a system with two equations and two unknown functions $u(x,t)$ and $v(x,t)$ (corresponding to A_2 and A_1). Thus one might expect that this should be a well-determined system; however, (3.30) is underdetermined! This can be seen as follows. One easily checks that (3.30) is invariant under the transformation[1] $u \to u + \partial_x \Lambda$ and $v \to v + \partial_t \Lambda$ where Λ is an arbitrary function of the independent variable x and t. Thus we may choose either u or v arbitrarily and still find a solution of (3.30). This is precisely the behaviour we expect of an under- and not of a well-determined system.

In fact, all Yang–Mills equations in any dimension of space-time are underdetermined, as they remain invariant under the transformation

$$\bar{A}^a_i = A^a_i + \partial_{x^i}\Lambda^a + C^a_{bc}A^b_i\Lambda^c \tag{3.31}$$

where the number of arbitrary functions $\Lambda^a(\mathbf{x})$ equals the dimension of the underlying Lie group $\mathcal{G}$. In differential forms language, $\bar{\mathbf{A}} = \mathbf{A} + d\mathbf{\Lambda}$ where the function $\mathbf{\Lambda}$ takes its values in the Lie algebra $\mathfrak{g}$ of $\mathcal{G}$. ◁

[1] This is a simple example of a *gauge symmetry*; more on this topic is contained in Section 4.3.

With the help of the principal symbol we can give rigorous definitions of under-, over-, and well-determined equations. They reduce the problem to linear algebra. In concrete computations with local coordinates the classification requires only the analysis of a parametrised matrix, essentially the determination of its rank.

Definition 3.3.6. *An involutive differential equation $\mathcal{R}_q$ is* underdetermined *at a point $\rho \in \mathcal{R}_q$, if no one-form $\chi \in T^*_{\pi^q(\rho)}\mathcal{X}$ exists such that the principal symbol τ_χ of $\mathcal{R}_q$ is injective at ρ. If a one-form χ exists such that τ_χ is bijective at ρ, the differential equation $\mathcal{R}_q$ is* well-determined *or* normal. *In any other case $\mathcal{R}_q$ is* overdetermined.

These definitions are rather abstract but have simple interpretations in terms of the highest index $\beta_q^{(n)}$ of the symbol $\mathcal{M}_q$ (of course computed in δ-regular coordinates). We will see in Section 4.2 that they indeed capture the intuitive idea of an underdetermined equation.

Proposition 3.3.7. *An involutive differential equation $\mathcal{R}_q$ in n independent and m dependent variables is underdetermined, if and only if $\beta_q^{(n)} < m$. It is normal, if and only if $\beta_q^{(n)} = m$ and all other indices vanish.*

Proof. By definition, $\mathcal{R}_q$ is underdetermined, if and only if no one-form χ exists such that $\operatorname{rank}\tau_\chi = m$. By Theorem 3.3.1 this happens if and only if $\beta_q^{(n)} < m$. Similarly, we get the value of $\beta_q^{(n)}$ for a normal equation. It follows from the bijectivity of the principal symbol that in this case all other indices vanish, as a local representation cannot contain any further equations. Indeed, they would be of lower order and thus contribute zero rows to the matrix of the principal symbol. □

Based on this result it is very easy to characterise the local form of normal equations. In the case of an ordinary differential equation, normality excludes the presence of lower-order equations. Thus a first-order normal ordinary differential equation possesses local representations of the form $\dot{\mathbf{u}} = \mathbf{f}(x, \mathbf{u})$, i. e. each equation is solved for a derivative and we have one equation for each derivative. In the case of a partial differential equation we rename x^n as t. Then a normal first-order equation takes locally the *Cauchy–Kovalevskaya form* $\mathbf{u}_t = \mathbf{f}(\mathbf{x}, t, \mathbf{u}, \mathbf{u}_\mathbf{x})$: each equation is solved for a t-derivative and we have one equation for each t-derivative. Thus, as expected, normality requires more than just the same number of equations and unknown functions; a distinguished independent variable t must exist such that we can write the system in the above special form.

Recalling Definition 1.5.7 of a characteristic one-form, we see that for an underdetermined equation any one-form $\chi \in T^*\mathcal{X}$ is characteristic. For any other equation characteristic and systatic are equivalent and we may say that a necessary condition for a δ-regular coordinate system is that dx^n is not characteristic. Indeed, if δ-singular coordinates appear in applications, this is mostly due to the use of characteristic coordinates. A trivial example is the wave equation in the form $u_{xy} = 0$; this is just the differential equation version of Example 2.1.12.

Example 3.3.8. We continue with Example 3.3.5. The matrix of the principal symbol is

$$T[\chi] = \begin{pmatrix} (\chi_1)^2 & -\chi_1\chi_2 \\ \chi_1\chi_2 & -(\chi_2)^2 \end{pmatrix} . \tag{3.32}$$

If we multiply the first row by χ_2 and the second by χ_1, we obtain the same. Thus an upper bound for the rank of T is 1 and it takes indeed this value for every non-vanishing one-form χ. This yields $\tilde{\beta}_2^{(2)} = 1$ and it is not possible to find a coordinate system in which both equations of (3.30) are of class 2. The principal symbol is never injective and every one-form is characteristic. Hence $\mathcal{R}_2$ is indeed underdetermined, although there are as many equations as dependent variables. $\triangleleft$

Note that Definition 3.3.6 applies only to *involutive* equations. Otherwise, no reasonable definitions of under- or overdeterminacy can be given, as hidden integrability conditions may change the character of the equation. We must be sure that we know all equations that possibly affect the principal symbol are exhibited.

Example 3.3.9. We augment (3.30) by the first-order equation $v_t - u_x = 0$ (this corresponds to the so-called *Lorentz gauge*). The arising equation $\bar{\mathcal{R}}_2$ is no longer formally integrable. If we nevertheless applied Definition 3.3.6, it would still classify the equation as underdetermined: the addition of a lower-order equation only adds a row of zeros to the matrix of the principal symbol. A local representation of $\bar{\mathcal{R}}_2^{(1)}$ is

$$\bar{\mathcal{R}}_2^{(1)} : \begin{cases} u_{tt} - u_{xx} = 0 \,, \quad u_{xt} - v_{xx} = 0 \,, \\ v_{tt} - v_{xx} = 0 \,, \quad v_{xt} - u_{xx} = 0 \,, \\ \qquad\qquad v_t - u_x = 0 \,. \end{cases} \tag{3.33}$$

It is not difficult to show that this equation is involutive. The matrix of its principal symbol is

$$T[\chi] = \begin{pmatrix} (\chi_1)^2 - (\chi_2)^2 & 0 \\ \chi_1\chi_2 & -(\chi_2)^2 \\ 0 & (\chi_1)^2 - (\chi_2)^2 \\ -(\chi_2)^2 & \chi_1\chi_2 \\ 0 & 0 \end{pmatrix} \tag{3.34}$$

and it is obviously injective for any one-form χ with $\chi_1 \neq \chi_2$. Thus the principal symbol is never bijective and we are dealing with an overdetermined equation. In fact, the basic idea behind such a *gauge fixing* consists of reducing the arbitrariness in the solutions of a differential equation. ◁

3.4 Applications: The Janet Sequence

Let $\pi_0 : \mathcal{E}_0 \to \mathcal{X}$ and $\pi_1 : \mathcal{E}_1 \to \mathcal{X}$ be two fibred manifolds over the same base space $\mathcal{X}$. A differential operator Δ_0 of order q is a map $\Gamma_{loc}(\mathcal{E}_0) \to \Gamma_{loc}(\mathcal{E}_1)$ that can be "factored" in the form $\Delta_0 = \Phi_0 \circ j_q$ with a fibred morphism $\Phi_0 : J_q\mathcal{E}_0 \to \mathcal{E}_1$ (cf. Section 1.3). If $\mathcal{E}_0$ and $\mathcal{E}_1$ are vector bundles over $\mathcal{X}$ and Φ_0 is a vector bundle morphism, then Δ_0 is a linear differential operator. In this section we are exclusively concerned with linear operators.

Let $\pi_2 : \mathcal{E}_2 \to \mathcal{X}$ be a further vector bundle and $\Delta_1 : \Gamma_{loc}(\mathcal{E}_1) \to \Gamma_{loc}(\mathcal{E}_2)$ a linear differential operator. It represents the *compatibility conditions* of the operator Δ_0, if $\operatorname{im}\Delta_0 = \ker\Delta_1$. In other words, the inhomogeneous linear system $\Delta_0\sigma_0 = \sigma_1$ for a given right hand side $\sigma_1 \in \Gamma_{loc}(\mathcal{E}_1)$ has a solution, if and only if $\Delta_1\sigma_1 = 0$. Obviously, we may iterate this construction and search for the compatibility conditions of Δ_1 and so on. Denoting by $\Theta \subseteq \Gamma_{loc}(\mathcal{E}_0)$ the solution space of Δ_0, this leads to an exact sequence

$$0 \longrightarrow \Theta \longrightarrow \Gamma_{loc}(\mathcal{E}_0) \xrightarrow{\Delta_0} \Gamma_{loc}(\mathcal{E}_1) \xrightarrow{\Delta_1} \Gamma_{loc}(\mathcal{E}_2) \longrightarrow \cdots , \qquad (3.35)$$

the *Janet sequence* of the linear operator Δ_0. By the definition of exactness, the equality $\operatorname{im}\Delta_i = \ker\Delta_{i+1}$ holds at every step. The sequence terminates as soon as a surjective operator is reached.

Example 3.4.1. A simple but classical example of a Janet sequence is formed by the basic operators of vector analysis: grad, curl, and div. Let us start with the differential equation $\operatorname{grad} u = \nabla u = 0$. If we add a right hand side, we obtain the system

$$u_x = v^1 , \quad u_y = v^2 , \quad u_z = v^3 . \qquad (3.36)$$

It is trivial to check that this an involutive system. It possesses solutions, if and only if the three functions v^α appearing on the right hand side satisfy the differential equation

$$v^2_z - v^3_y = 0 , \quad v^3_x - v^1_z = 0 , \quad v^1_y - v^2_x = 0 , \qquad (3.37)$$

in other words, if $\operatorname{curl}\mathbf{v} = \nabla \times \mathbf{v} = 0$. This corresponds to the well-known identity $\operatorname{curl} \circ \operatorname{grad} = 0$.

Now we may add a right hand side to (3.37), i. e. we consider $\operatorname{curl}\boldsymbol{v} = \boldsymbol{w}$. This equation is solvable, if and only if $\boldsymbol{w}$ satisfies

$$w^1_x + w^2_y + w^3_z = 0 \,. \tag{3.38}$$

Of course, this means nothing but $\operatorname{div}\boldsymbol{w} = \boldsymbol{\nabla}\cdot\boldsymbol{w} = 0$ and we recover the next classical identity $\operatorname{div}\circ\operatorname{curl} = 0$.

We may represent these identities in the following exact sequence

$$\begin{array}{c} 0 \longrightarrow \Gamma_{\text{loc}}(\mathbb{R}^3,\mathbb{R}) \xrightarrow{\text{grad}} \Gamma_{\text{loc}}(\mathbb{R}^3,\mathbb{R}^3) \xrightarrow{\text{curl}} \\ \Gamma_{\text{loc}}(\mathbb{R}^3,\mathbb{R}^3) \xrightarrow{\text{div}} \Gamma_{\text{loc}}(\mathbb{R}^3,\mathbb{R}) \longrightarrow 0 \,. \end{array} \tag{3.39}$$

Readers familiar with differential geometry know of course that this is nothing but a special case of the *de Rham complex*. Let $\mathcal{M}$ be an n-dimensional manifold. We introduced in Appendix C.3 the vector bundles $\Omega^k(\mathcal{M})$ of differential k-forms. Together with the exterior derivative d as differential, they form the complex

$$0 \longrightarrow \Omega^0(M) \xrightarrow{d} \Omega^1(M) \xrightarrow{d} \cdots \xrightarrow{d} \Omega^n(M) \longrightarrow 0 \,. \tag{3.40}$$

If we consider global forms, it is in general not exact and its cohomology contains important topological information on the manifold $\mathcal{M}$. Locally, (3.40) is exact by the Poincaré Lemma (see Remark C.3.6). ◁

Above we have defined differential operators between spaces of smooth sections. Of course, one could consider instead, say, distributional solutions. In principle, the exactness of the Janet sequence depends on the chosen function spaces and its verification requires the corresponding existence theory. Even for linear operators, this is a rather hard subject. In Section 5.4 we will discuss the existence and uniqueness of analytic solutions for arbitrary involutive equations. But already the extension to smooth solutions is only possible under additional assumptions on the structure of the equation (an example of such an extension will be given in Section 5.6). For this reason, we consider the *formal* Janet sequence, i. e. we are only concerned with formal power series solutions.

This restriction permits us to take a purely algebraic approach to the problem of compatibility. The basic idea is to ignore the function spaces on which our differential operators act and to consider instead free modules over the ring of linear differential operators. We must use *right* modules, as otherwise we do not obtain module homomorphisms. This inverts the arrows in the sequence (3.35) and we obtain a free resolution in the form (B.17).

Formal series are local objects, so we go into a coordinate chart where the linear differential operator $\Delta_0 : \Gamma_{\text{loc}}(\mathcal{E}_0) \to \Gamma_{\text{loc}}(\mathcal{E}_1)$ takes the form

$$v^\beta(\mathbf{x}) = \sum_{\alpha=1}^{m_0} \sum_{0\le|\mu|\le q} A^\beta_{\alpha\mu}(\mathbf{x}) D^\mu u^\alpha(\mathbf{x}) \,. \tag{3.41}$$

Here m_i denotes the fibre dimension of the vector bundle $\mathcal{E}_i$. We assume that the coefficients $A^\beta_{\alpha\mu}$ are elements of some function field $\mathbb{F}$ and introduce as in Example 2.2.4 the ring $\mathcal{D} = \mathbb{F}[\partial_1, \dots, \partial_n]$ with $\partial_i = \partial/\partial x^i$. We associate with Δ_0 the (left) $\mathcal{D}$-module homomorphism

$$D_0 : \begin{cases} \mathcal{D}^{m_1} & \longrightarrow \qquad \mathcal{D}^{m_0} \\ (P_\beta) & \longmapsto \Big(\sum_{\beta=1}^{m_1} \sum_{0\le|\mu|\le q} P_\beta \cdot A^\beta_{\alpha\mu}\partial^\mu\Big) \end{cases} . \tag{3.42}$$

Note that we act on the vector $\mathbf{P}$ from the right; thus we should think of it as a row vector. If we represent the homomorphism D_0 by the $m_1 \times m_0$ matrix $(\hat{D}_0)^\beta_\alpha = \sum_{0\le|\mu|\le q} A^\beta_{\alpha\mu}\partial^\mu$, we may write compactly $D_0(\mathbf{P}) = \mathbf{P}\cdot\hat{D}_0$. It is essential that we multiply from the right, as otherwise we would not obtain a left $\mathcal{D}$-module homomorphism (obviously $\hat{D}_0 \cdot (Q\mathbf{P}) \neq Q(\hat{D}_0 \cdot \mathbf{P})$ but $(Q\mathbf{P})\cdot\hat{D}_0 = Q(\mathbf{P}\cdot\hat{D}_0)$ for arbitrary operators $Q \in \mathcal{D}$).

Lemma 3.4.2. *The sequence*

$$\cdots \longrightarrow \mathcal{D}^{m_2} \xrightarrow{D_1} \mathcal{D}^{m_1} \xrightarrow{D_0} \mathcal{D}^{m_0} \longrightarrow \mathcal{M} \longrightarrow 0 \tag{3.43}$$

with $\mathcal{M} = \operatorname{coker} D_0 = \mathcal{D}^{m_0}/\operatorname{im} D_0$ *is exact, if and only if the Janet sequence* (3.35) *is formally exact.*

Proof. We have $(D_i \circ D_{i+1})(\mathbf{P}) = \mathbf{P}\cdot\hat{D}_{i+1}\cdot\hat{D}_i = 0$ for all $\mathbf{P} \in \mathcal{D}^{m_{i+2}}$, if and only if $\Delta_{i+1} \circ \Delta_i = 0$. Furthermore, $D_i(\mathbf{P}) = \mathbf{P}\cdot\hat{D}_i = 0$ implies that $\mathbf{P}$ induces a compatibility condition of Δ_i and vice versa. All such conditions are dependent of Δ_{i+1}, if and only if $\mathbf{P} = \mathbf{Q}\cdot\hat{D}_{i+1}$ for some $\mathbf{Q} \in \mathcal{D}^{m_{i+2}}$. □

Thus determining the formal Janet sequence of Δ_0 is equivalent to computing the free resolution (3.43) of $\mathcal{M} = \operatorname{coker} D_0$ which amounts to syzygy computations in free $\mathcal{D}$-modules. As discussed in the Notes of the last chapter, the proof of Theorem 2.6.3 remains valid, if we substitute the commutative ring $\mathcal{P}$ by the non-commutative ring $\mathcal{D}$. Now we may reformulate it in the following form.

Theorem 3.4.3. *Assume that the rows of the matrix* $\hat{D}_0$ *form a Pommaret basis of the submodule* $\operatorname{im} D_0 \subseteq \mathcal{D}^{m_0}$ *for a class respecting term order and let* $n-d$ *be the lowest class of a row. Then the Janet sequence* (3.35) *terminates with the operator* Δ_d.

Proof. By Theorem 2.6.3 the free resolution (3.43) terminates under our assumptions with D_d. The claim follows now from Lemma 3.4.2. □

Example 3.4.4. We return to Example 3.4.1 above. Here $\Delta_0 = \text{grad}$ and thus $\hat{D}_0 = (\partial_x\ \partial_y\ \partial_z)^t$. Obviously, the rows of the matrix form a Pommaret basis and we have $d = 2$. Thus the Janet sequence (3.39) has the maximally possible length for an operator in $n = 3$ independent variables and stops only with the operator $\Delta_2 = \text{div}$. ◁

Remark 3.4.5. Let $\Delta_0 = \Phi_0 \circ j_q$ be a differential operator. We associate with it the submanifold $\mathcal{R}_q \subseteq J_q\mathcal{E}_0$ defined by the zeros of Φ_0. One may think that under the assumptions of Theorem 3.4.3 the differential equation $\mathcal{R}_q$ is involutive, but generally this is not the case. However, if q' is the minimal order of a row in $\hat{D}_0$, then $\mathcal{R}_q^{(q-q')}$ is involutive. This can be shown in the same manner as Lemma 2.4.1: we simply add the multiplicative prolongations of all lower-order equations. In the language of Section 6.5, the rows of $\hat{D}_0$ form a "skeleton" of the differential equation $\mathcal{R}_q^{(q-q')}$. ◁

For the remainder of the section we turn to the inverse problem: given a linear differential operator Δ_1, does another differential operator Δ_0 exist such that $\operatorname{im}\Delta_0 = \ker\Delta_1$? If yes, then we call Δ_0 a *parametrisation* of Δ_1. Somewhat surprisingly, this problem permits a constructive solution.

We present here only the theory for systems with constant coefficients. In this case $\mathcal{D}$ is commutative, i. e. the ordinary polynomial ring, and we may write the maps D_i as *left* multiplication of *column* vectors with the *transposed* matrix $\hat{D}_i^t$. Now our problem can be formulated as follows: given a module homomorphism $\beta : \mathcal{D}^q \to \mathcal{D}^r$, we would like to find a further module homomorphism $\gamma : \mathcal{D}^r \to \mathcal{D}^s$ such that $\operatorname{im}\beta = \ker\gamma$. We develop an algorithm that decides the existence of such a map and, if the answer is affirmative, explicitly determines it.

The basic observation is that a syzygy computation "goes in the wrong direction": it determines a map $\alpha : \mathcal{D}^p \to \mathcal{D}^q$ with $\operatorname{im}\alpha = \ker\beta$. Thus we must revert the directions of the arrows. This is achieved by taking the *dual*; in our commutative situation this means applying the Hom functor. For notational simplicity, we denote the module $\operatorname{Hom}_{\mathcal{D}}(\mathcal{D}^q, \mathcal{D})$ of all $\mathcal{D}$-linear maps from $\mathcal{D}^q$ to $\mathcal{D}$ by $(\mathcal{D}^q)^*$. It is trivially isomorphic to $\mathcal{D}^q$. Now β induces the dual map $\beta^* : (\mathcal{D}^r)^* \to (\mathcal{D}^q)^*$ defined by $\beta(\phi) = \phi \circ \beta$. If B is a matrix representation of the map β (with respect to some bases of the free modules $\mathcal{D}^q$ and $\mathcal{D}^r$), then a matrix representation of β^* (with respect to the dual bases) is simply the transposed matrix B^t.

The algorithm proceeds in five steps:

1. Compute β^* (as mentioned above, at the level of the matrices, this simply means to transpose the matrix of β).

2. Determine a map $\gamma^* : (\mathcal{D}^s)^* \to (\mathcal{D}^r)^*$ such that $\operatorname{im}\gamma^* = \ker\beta^*$. This corresponds to a syzygy computation.
3. Compute γ (transpose again).
4. Determine a map $\beta_c : \mathcal{D}^{q_c} \to \mathcal{D}^r$ such that $\operatorname{im}\beta_c = \ker\gamma$. This requires another syzygy computation.
5. Check whether $\operatorname{im}\beta_c = \operatorname{im}\beta$. If yes, then γ is a map with the required properties. Otherwise, no such map exists.

All maps and modules appearing in the course of the algorithm are sketched in the following diagram:

$$\begin{tikzcd}
\mathcal{D}^{q_c} \arrow[dr, "\beta_c"] & & \\
& \mathcal{D}^r \arrow[r, "\gamma"] \arrow[dd, <->, dotted] & \mathcal{D}^s \arrow[dd, <->, dotted] \\
\mathcal{D}^q \arrow[ur, "\beta"'] \arrow[d, <->, dotted] & & \\
(\mathcal{D}^q)^* & (\mathcal{D}^r)^* \arrow[l, "\beta^*"'] & (\mathcal{D}^s)^* \arrow[l, "\gamma^*"']
\end{tikzcd} \tag{3.44}$$

Let B and C be the matrices of the maps β and γ, respectively. Then the condition in the second step implies that $B^tC^t = 0$ and thus $CB = 0$ or equivalently $\operatorname{im}\beta \subseteq \ker\gamma$. But although $\operatorname{im}\gamma^* = \ker\beta^*$, we cannot conclude that $\operatorname{im}\beta = \ker\gamma$ (the functor $\operatorname{Hom}_{\mathcal{D}}(\cdot, \mathcal{D})$ is not exact, as free $\mathcal{D}$-modules are not injective [131, Section 2.4]).

Example 3.4.6. A special case arises in our algorithm, if the map β^* is injective. Then $s = 0$, as no syzygies exist, and the algorithm stops already in the second step. In this situation the map β obviously does not define the compatibility conditions of any other system.

As a concrete example we consider the *Poincaré–Steklov problem* in three dimensions. Given a vector field $\mathbf{v}$ and a function w, it looks for a vector field $\mathbf{u}$ satisfying $\nabla \times \mathbf{u} = \mathbf{v}$ and $\nabla \cdot \mathbf{u} = w$. These equations appear for instance in electrostatics (if we study only time-independent solutions, Maxwell's equations decouple and the electrical field $\mathbf{E}$ solves a Poincaré–Steklov problem) but also in the theory of the Navier–Stokes equations.

One can show that only a single compatibility condition exists for this problem: $\nabla \cdot \mathbf{v} = 0$. If we study the inverse problem via our algorithm, the matrix of the map β is

$$B = \begin{pmatrix} 0 & \partial_z & -\partial_y & \partial_x \\ -\partial_z & 0 & \partial_x & \partial_y \\ \partial_y & -\partial_x & 0 & \partial_z \end{pmatrix} . \tag{3.45}$$

It is easy to see that the rows of this matrix are linearly independent; thus the map β^* (with matrix B^t) is injective. At the level of the differential operators this means that the *adjoint* Poincaré–Steklov problem is underdetermined, as it comprises less equations than unknowns. $\triangleleft$

We still have to prove the correctness of our algorithm. One direction is trivial: if $\operatorname{im}\beta = \operatorname{im}\beta_c$, then γ is a map with the wanted properties. The difficult part is to show that no such map exists, if $\operatorname{im}\beta \neq \operatorname{im}\beta_c$. Recall that a *torsion element* of a $\mathcal{D}$-module $\mathcal{M}$ is an $\mathfrak{m} \in \mathcal{M}$ such that for at least one $p \in \mathcal{D} \setminus \{0\}$ the product $p\mathfrak{m}$ is zero; a module without torsion elements is *torsion free*. Now the correctness is implied by the following result.

Theorem 3.4.7. *Let $\beta : \mathcal{D}^q \to \mathcal{D}^r$ be a module homomorphism and the map $\beta_c : \mathcal{D}^q \to \mathcal{D}^r$ be constructed by the algorithm above. Then the following three statements are equivalent.*

(i) *The $\mathcal{D}$-module* $\operatorname{coker}\beta = \mathcal{D}^r/\operatorname{im}\beta$ *is torsion free.*
(ii) *A module homomorphism $\bar{\gamma} : \mathcal{D}^r \to \mathcal{D}^{\bar{s}}$ exists with* $\operatorname{im}\beta = \ker\bar{\gamma}$.
(iii) *The equality* $\operatorname{im}\beta = \operatorname{im}\beta_c$ *holds.*

Proof. (iii) trivially implies (ii); we simply choose $\bar{\gamma} = \gamma$. Assume that (ii) holds and consider the module homomorphism $\phi : \operatorname{coker}\beta \to \mathcal{D}^{\bar{s}}$ given by $\phi([\mathbf{f}]) = \bar{\gamma}(\mathbf{f})$ where $[\mathbf{f}]$ denotes the equivalence class of $\mathbf{f} \in \mathcal{D}^r$ in the factor space $\mathcal{D}^r/\operatorname{im}\beta$. The map ϕ is well-defined as $\operatorname{im}\beta = \ker\bar{\gamma}$ and obviously injective. Thus $\operatorname{coker}\beta$ is isomorphic to a submodule of a free module implying that it is torsion free and (ii) implies (i).

There remains to show that (i) implies (iii). We given an indirect proof. If $\mathcal{F}$ is the quotient field of $\mathcal{D}$, i.e. the field of rational functions in $x^1, \dots, x^n$, then β trivially extends to a vector space homomorphism $\tilde{\beta} : \mathcal{F}^q \to \mathcal{F}^r$ and similarly for the maps β_c and γ. We claim that $\operatorname{im}\tilde{\beta} = \operatorname{im}\tilde{\beta}_c$. Indeed, by construction, $\operatorname{im}\gamma^* = \ker\beta^*$ and thus also $\operatorname{im}\tilde{\gamma}^* = \ker\tilde{\beta}^*$; as $\mathcal{F}$ is a field, the functor $\operatorname{Hom}_{\mathcal{F}}(\cdot, \mathcal{F})$ is exact[2] and $\operatorname{im}\tilde{\beta} = \ker\tilde{\gamma} = \operatorname{im}\tilde{\beta}_c$.

Let us now assume that (iii) does *not* hold. This means $\operatorname{im}\beta \subset \operatorname{im}\beta_c$, as by construction $\operatorname{im}\beta \subseteq \operatorname{im}\beta_c$. Consider an element $\mathbf{f} \in \operatorname{im}\beta_c \setminus \operatorname{im}\beta$. We may identify it with an element of $\operatorname{im}\tilde{\beta}_c$. Since $\operatorname{im}\tilde{\beta} = \operatorname{im}\tilde{\beta}_c$, rational functions $a_\alpha \in \mathcal{F}$ exist such that $\mathbf{f} = \sum_{\alpha=1}^{q} a_\alpha \tilde{\beta}(\boldsymbol{e}_\alpha)$ (here $\boldsymbol{e}_\alpha$ denotes again the elements of the standard basis of a free module). We can choose a polynomial $h \in \mathcal{D}$ such that $h_\alpha = h a_\alpha \in \mathcal{D}$. This implies the existence of a syzygy over the ring $\mathcal{D}$,

$$h\mathbf{f} = \sum_{\alpha=1}^{q} h_\alpha \beta(\boldsymbol{e}_\alpha) , \tag{3.46}$$

[2] For arbitrary rings $\mathcal{R}$ the functor $\operatorname{Hom}_{\mathcal{R}}(\cdot, \mathcal{R})$ is only left exact; full exactness requires that $\mathcal{R}$ is a field [102, pp. 97–98].

so that $f \notin \operatorname{im}\beta$ but $hf \in \operatorname{im}\beta$. Thus $[f]$ is a torsion element in $\operatorname{coker}\beta$ and (i) cannot hold. □

Example 3.4.8. Our algorithm permits us to derive the relation between the $U(1)$ Yang–Mills equations and the Maxwell equations discussed in Example 1.6.3. For this purpose we take the four equations $\boldsymbol{\nabla}\cdot\mathbf{B} = 0$ and $\mathbf{B}_t = -\boldsymbol{\nabla}\times\mathbf{E}$. Rewriting them as a linear differential operator yields

$$\Delta_0 = \begin{pmatrix} \partial_t & 0 & 0 & 0 & -\partial_z & \partial_y \\ 0 & \partial_t & 0 & \partial_z & 0 & -\partial_x \\ 0 & 0 & \partial_t & -\partial_y & \partial_x & 0 \\ \partial_x & \partial_y & \partial_z & 0 & 0 & 0 \end{pmatrix} . \tag{3.47}$$

As β is defined by the transposed matrix, this is simultaneously a matrix representation of the map β^* in (3.44). Following our algorithm yields in a straightforward computation the parametrisation

$$\Delta_1 = \begin{pmatrix} 0 & -\partial_z & \partial_y & 0 \\ \partial_z & 0 & -\partial_x & 0 \\ -\partial_y & \partial_x & 0 & 0 \\ -\partial_t & 0 & 0 & -\partial_x \\ 0 & -\partial_t & 0 & -\partial_y \\ 0 & 0 & -\partial_t & -\partial_z \end{pmatrix} . \tag{3.48}$$

(This matrix also represents the map γ^*). This result does not come as a surprise, as Δ_1 obviously encodes the familiar relations $\mathbf{B} = \boldsymbol{\nabla}\times\mathbf{A}$ and $\mathbf{E} = -\boldsymbol{\nabla}\cdot A_4 - \partial_t\mathbf{A}$. Thus we have rediscovered the vector potential.

It is interesting to note that if we omit in Δ_0 one of the first three rows, we still obtain the same operator Δ_1. But now $\operatorname{im}\beta \neq \operatorname{im}\beta_c$ and β_c is given by the original operator Δ_0. Thus the compatibility analysis automatically recovers the dropped equation. ◁

Example 3.4.9. As a further quite famous example we consider the question whether the linearised Einstein equations are the compatibility conditions of some other system. It was posed by Wheeler about 1970 and much later solved by Pommaret [141].

The linearisation of the Einstein equations (1.73) around the Minkowski metric, i. e. the ansatz[3] $g_{ij} = \eta_{ij} + h_{ij}$, yields the following linear system

$$\eta^{k\ell}\left[\partial_{x^i x^j} h_{k\ell} + \partial_{x^k x^\ell} h_{ij} - \partial_{x^k x^i} h_{\ell j} - \partial_{x^k x^j} h_{\ell i}\right] = 0 . \tag{3.49}$$

[3] Recall from Example 1.6.3 that $\eta^{ii} = 1$ for $1 \leq i < n$, $\eta^{nn} = -1$ and $\eta^{ij} = 0$ in all other cases. η_{ij} are the entries of the inverse matrix.

Because of the symmetry we may restrict to $1 \leq i \leq j \leq n$ so that these are $n(n+1)/2$ equations for the same number of unknowns but the system is still underdetermined. Thus for $n = 4$ we have 10 equations.

For space reasons we omit to write down the matrix corresponding to the map β; its entries follow straightforwardly from (3.49). The matrix representation of the map γ is

$$C = \begin{pmatrix} 2\partial_x & 0 & 0 & 0 & \partial_y & 0 & 0 & \partial_z & 0 & \partial_t \\ 0 & 2\partial_y & 0 & 0 & \partial_x & \partial_z & 0 & 0 & \partial_t & 0 \\ 0 & 0 & 2\partial_z & 0 & 0 & \partial_y & \partial_t & \partial_x & 0 & 0 \\ 0 & 0 & 0 & -2\partial_t & 0 & 0 & -\partial_z & 0 & -\partial_y & -\partial_x \end{pmatrix} . \tag{3.50}$$

Computing its syzygies yields a system of 20 independent equations. Thus $\operatorname{im}\beta \neq \operatorname{im}\beta_c$ and (3.49) does not admit a parametrisation.

One can obtain further interesting results in differential geometry; for lack of space we refer again to [141]. We mention here only that the differential operator corresponding to γ^* has a simple geometric interpretation. We may write it as $h_{ij} = \eta_{kj}\partial_{x^i}L^k + \eta_{ik}\partial_{x^j}L^k$ or $h = \mathcal{L}_L\eta$, i. e. h is the Lie derivative of the Minkowski metric with respect to the vector field $L = L^k\partial_{x^k}$. ◁

3.5 Notes

We remarked already at the end of the last chapter that many of the ideas around involution have their origin in the theory of differential equations (although we presented the theory first in algebraic form). But the Janet–Riquier theory mentioned there was not the only approach to deal with general systems of differential equations. A completely intrinsic theory based on a representation by an exterior ideal of differential forms was developed by Cartan [33] and Kähler [87]; a modern presentation of their work is contained in [27]. The Cartan–Kähler Theory is in particular very natural for many problems in differential geometry. A dual version was developed by Vessiot [189] (see also [171])

We used a combinatorial approach to involution; in the formal theory it is traditionally introduced as a cohomological concept. We only briefly sketch this theory here; more details can be found in [27, 42, 113, 139] (and references therein). For notational simplicity, we omit throughout the explicit reference to a base point, although all constructions are to be understood pointwise.

Consider the map $\delta : S_{r+1}(T^*\mathcal{X}) \to T^*\mathcal{X} \otimes S_r(T^*\mathcal{X})$ defined by the composition of the natural inclusion $S_{r+1}(T^*\mathcal{X}) \hookrightarrow T^*\mathcal{X} \otimes \bigotimes^r T^*\mathcal{X}$ with the canonical projection $T^*\mathcal{X} \otimes \bigotimes^r T^*\mathcal{X} \to T^*\mathcal{X} \otimes S_r(T^*\mathcal{X})$. By wedging from the left with $\Lambda_s(T^*\mathcal{X})$ and tensoring from the right with $V\mathcal{E}$ we can extend the map δ to another map $\Lambda_s(T^*\mathcal{X}) \otimes S_{r+1}(T^*\mathcal{X}) \otimes V\mathcal{E} \to \Lambda_{s+1}(T^*\mathcal{X}) \otimes S_r(T^*\mathcal{X}) \otimes V\mathcal{E}$. This leads to the so-called *δ-sequences*

$$\begin{aligned} 0 \longrightarrow S_r(T^*\mathcal{X}) \otimes V\mathcal{E} \longrightarrow T^*\mathcal{X} \otimes S_{r-1}(T^*\mathcal{X}) \otimes V\mathcal{E} \longrightarrow \cdots \\ \longrightarrow \Lambda_s(T^*\mathcal{X}) \otimes S_{r-s}(T^*\mathcal{X}) \otimes V\mathcal{E} \longrightarrow \cdots \\ \longrightarrow \Lambda_n(T^*\mathcal{X}) \otimes S_{r-n}(T^*\mathcal{X}) \otimes V\mathcal{E} \longrightarrow 0 \end{aligned} \tag{3.51}$$

where we set $S_i(T^*\mathcal{X}) = 0$ for $i < 0$. The *formal Poincaré Lemma* states that these sequences are exact for all $r > 0$.

Setting $\mathcal{M}_i = 0$ for $i < 0$ and $\mathcal{M}_i = S_i(T^*\mathcal{X}) \otimes V\mathcal{E}$ for $0 \leq i < q$, the δ-sequence (3.51) can be restricted to a sequence

$$0 \longrightarrow \mathcal{M}_{q+r} \longrightarrow T^*\mathcal{X} \otimes \mathcal{M}_{q+r-1} \longrightarrow \cdots \longrightarrow \Lambda_n(T^*)\mathcal{X} \otimes \mathcal{M}_{q+r-n} \longrightarrow 0 \tag{3.52}$$

which is still a complex but in general no longer exact. Its (bigraded) cohomology is called the *Spencer cohomology* of the symbol $\mathcal{M}_q$. We denote by $H^{s,r}(\mathcal{M}_q)$ the cohomology group at $\Lambda_s(T^*\mathcal{X}) \otimes \mathcal{M}_r$ and define the symbol $\mathcal{M}_q$ to be involutive, if the groups $H^{s,q+r}(\mathcal{M}_q)$ vanish for all $0 \leq s \leq n$ and all $r \geq 0$.

This approach can be connected to ours as follows. Let $\{e_1, \dots, e_n\}$ be an ordered basis of $T^*\mathcal{X}$. For a given symbol $\mathcal{M}_q$ we define for $1 \leq k \leq n$ the subspaces

$$\mathcal{M}_{q,k} = \{\rho \in \mathcal{M}_q \mid \rho(e_i, v_1, \dots, v_{q-1}) = 0\,,\ \forall 1 \leq i \leq k\,, \forall v_1, \dots, v_{q-1} \in T^*\mathcal{X}\} \tag{3.53}$$

and $\mathcal{M}_{q,0} = \mathcal{M}_q$. We call the basis *quasi-regular* (for $\mathcal{M}_q$), if

$$\dim \mathcal{M}_{q+1} = \sum_{k=0}^{n-1} \dim \mathcal{M}_{q,k}\,. \tag{3.54}$$

One can show that a symbol is involutive, if and only if quasi-regular bases exist for it. In fact, coordinates $(x^1, \dots, x^n)$ on $\mathcal{X}$ are δ-regular, if and only if the corresponding basis $\{\partial/\partial x^1, \dots, \partial/\partial x^n\}$ of $T^*\mathcal{X}$ is quasi-regular. The linear equations defining $\mathcal{M}_{q,k}$ correspond to the symbol equations of class greater than or equal to k and thus (3.54) is equivalent to our criterion (3.2).

Our definition of an underdetermined equation is more precise than the one given by Pommaret [139, Definition 6.6]. He defines an equation to be underdetermined, if a one-form χ exists such that the principal symbol τ_χ is surjective. While this condition covers the elementary case that any local representation comprises less equations than unknown functions, it fails for Example 3.3.5: the rank of the matrix $T[\chi]$ given by (3.32) is at most one, thus τ_χ can never be surjective.

Our approach generalises the definitions of Olver [129, Definition 2.86] for systems with the same number of equations and unknown functions. They are in so far not satisfactory, as he does not require the equation to be involutive. Thus it may happen that the addition of an integrability condition renders an underdetermined equation overdetermined or vice versa. It should be stressed again that this classification makes sense only for involutive equations!

The satisfaction of the compatibility conditions of a linear differential equation is an obvious necessary condition for the existence of solutions for the inhomogeneous equation. Whether it is also sufficient depends of course on in what kind of solutions one is interested. For the simple case of formal solutions the sufficiency is obvious. The results of Section 5.4 will imply the sufficiency for analytic solutions of analytic equations. In the case of constant coefficients stronger results have been obtained in [133] (and references therein).

The inverse problem of syzygies has not found much interest in commutative algebra; we are not aware of any work discussing it in the context of polynomials. The situation is different for differential equations where it is of considerable interest for applications in control theory [128, 140, 194]. A central concept here is the *controllability* of a system. Roughly it means that we can "steer" it from any point to any other point by a suitable choice of the controls.

Within the behavioural approach [138] a continuous system (or *behaviour*) is modelled by the solution space of a differential equation. In contrast to the traditional approach, no distinction into input, output and state variables is made. Somewhat surprisingly it turns out that controllability is equivalent to the existence of a parametrisation, i. e. a solution to the inverse problem.

Apparently, the presented algorithm for the construction of a parametrisation is due to Oberst [128]; our proof of Theorem 3.4.7 is a mixture of results found in [142] and [194, Section 1.2]. Pommaret [140] showed that the theory remains valid for linear systems with variable coefficients, if we use the formally adjoint operator instead of the transposed (the meaning of the $*$ in (3.44) must be changed). The extension of our proof to this more general situation will appear elsewhere.

Finally, we mention that the described algorithm for the solution of the inverse problem can be easily extended to compute the extension modules $\mathrm{Ext}^i_{\mathcal{D}}(\mathcal{M}, \mathcal{D})$. In fact, as the proof of Theorem 3.4.7 shows, $\mathrm{Ext}^1_{\mathcal{D}}(\mathcal{M}, \mathcal{D})$ is generated precisely by the constructed torsion elements. For the higher extension modules we first follow a few steps the free resolution (3.43) and then apply the same algorithm [143].