5. Existence and Uniqueness of Solutions

In order to solve this differential equation you look at it till a solution occurs to you.

George Pólya

A fundamental issue in the theory of differential equations is to prove the existence and uniqueness of solutions. Before we discuss partial differential equations, we analyse in Section 5.1 the situation for general systems of ordinary differential equations, often called differential algebraic equations. For involutive equations it is here straightforward to extend the classical existence and uniqueness theorem.

As the formal theory is closely connected to the study of power series solutions, it is relatively easy to obtain results on *analytic* solutions of partial differential equations. In Section 5.2 we recall the Cauchy–Kovalevskaya Theorem for normal systems. After introducing in Section 5.3 the concept of a formally well-posed initial value problem, we generalise in the following section this famous theorem to arbitrary involutive systems, the Cartan–Kähler Theorem. We see here again why formal integrability is *not* sufficient for analysing partial differential equations but that one indeed needs involution.

From an application point of view the restriction to analytic solutions is very severe. This is the price we have to pay for the Cartan–Kähler Theorem requiring essentially no other assumptions than involution. At such a level of generality no stronger results can be expected. However, the technique of proof we use allows us in certain cases to go beyond analytic solutions. As a first simple example we extend in Section 5.5 the classical uniqueness theorem of Holmgren for C^1 solutions to arbitrary linear involutive systems.

The final section studies what we call "weakly overdetermined" systems and derives under certain structural assumptions some existence and uniqueness results for smooth solutions. This class includes in particular Maxwell's equations. Again we exploit the technique of the proof of the Cartan–Kähler Theorem to "lift" classical results on normal systems to overdetermined ones.

As we present in this chapter only local results, we will work throughout in local coordinates and therefore speak of systems.

5.1 Ordinary Differential Equations

In order to simplify the discussion of the existence and uniqueness of solutions of an involutive first-order ordinary differential equation, we specialise the Cartan normal form (3.10) to the case of only one independent variable. One speaks of a *semi-explicit form*, as it entails a split into algebraic and differential equations and the differential equations are solved for some derivatives. As there is no need to distinguish between derivatives of different class, we divide the vector $\mathbf{u}$ of the unknown functions into two parts $\mathbf{v}$ and $\mathbf{w}$ such that no derivatives $\mathbf{w}'$ appear. Now (3.10) takes the form

$$\mathbf{v}' = \boldsymbol{\phi}(x, \mathbf{v}, \mathbf{w}) \, , \tag{5.1a}$$

$$0 = \boldsymbol{\psi}(x, \mathbf{v}, \mathbf{w}) \, . \tag{5.1b}$$

Involution imposes conditions on the functions $\boldsymbol{\phi}$ and $\boldsymbol{\psi}$. The conditions (3.11) are empty in our case, as they stem from cross-derivatives which are not possible for ordinary differential equations. The conditions (3.12) become

$$\frac{\partial \boldsymbol{\psi}}{\partial \mathbf{w}} = \mathbf{A}\boldsymbol{\psi} \, , \qquad \frac{\partial \boldsymbol{\psi}}{\partial x} + \left(\frac{\partial \boldsymbol{\psi}}{\partial \mathbf{v}}\right)^t \boldsymbol{\phi} = B\boldsymbol{\psi} \, . \tag{5.2}$$

Here $\mathbf{A}(x, \mathbf{v}, \mathbf{w})$ is a vector of matrix valued functions and $B(x, \mathbf{v}, \mathbf{w})$ a matrix of appropriate sizes. If the first condition was violated, the prolongation of (5.1b) would lead to new differential equations. It implies that the unknown functions $\mathbf{w}$ may be considered as parameters. If the differential equation $\mathcal{R}_1$ is not underdetermined, the vector $\mathbf{w}$ is empty. The second condition ensures that the prolongation yields no new algebraic equations.

Proposition 5.1.1. *Assume that the system* (5.1) *is involutive and not underdetermined. Then the function* $\boldsymbol{\psi}$ *is a weak invariant of the normal ordinary differential equation* (5.1a).

Proof. This is a trivial reformulation of (5.2). □

"Weak" invariant refers to the fact that the left hand sides of (5.2) vanish only on the constraint manifold $\mathcal{C} \subset \mathcal{E}$ defined by the equations $\boldsymbol{\psi} = 0$. For a strong invariant one requires that they vanish on the full manifold $\mathcal{E}$. Another way to formulate Proposition 5.1.1 is to say that $\mathcal{C}$ is an *invariant manifold* of the ordinary differential equation (5.1a), i. e. if we choose for (5.1a) initial data on $\mathcal{C}$, then the full solution will stay on $\mathcal{C}$.

Remark 5.1.2. The fact that the constraints yield generally only weak invariants has some implications for the numerical solution of differential algebraic equations. While $\mathcal{C}$ is an invariant manifold for any exact solution of (5.1a), this is not necessarily the case for approximate solutions. If we apply some standard numerical method to (5.1a), we cannot expect that the obtained approximation lies on $\mathcal{C}$.

This problem is known as *drift* off the constraint manifold. How severe it becomes, depends on the stability properties of $\mathcal{C}$, i. e. on the behaviour of (5.1a) in its neighbourhood. If $\mathcal{C}$ is attractive, numerical errors are automatically damped; if $\mathcal{C}$ is repulsive, they are amplified and the computations typically break down rapidly.

As we are always free to modify $\boldsymbol{\phi}$ by adding arbitrary linear combinations of $\boldsymbol{\psi}$, it is a natural thought to choose $\boldsymbol{\phi}$ such that $\mathcal{C}$ becomes attractive for (5.1a). This procedure is known as *stabilisation* [10]. Unfortunately, it is rather difficult to find a stabilisation for an arbitrary ordinary differential equation and it tends to make the evaluation of (5.1a) rather expensive. Thus in practice one prefers other approaches like projections on the constraint manifold or special methods for differential algebraic equations [25, 68]. $\triangleleft$

Proposition 5.1.1 makes the existence and uniqueness theory for involutive ordinary differential equations almost trivial. We can reduce the problem to a normal ordinary differential equation living on some submanifold $\mathcal{C} \subset \mathcal{E}$ and apply the standard existence and uniqueness theorem. For simplicity, we assume that $\mathcal{X} \subseteq \mathbb{R}$ which is not a very restrictive assumption, as any connected one-dimensional manifold is either a submanifold of $\mathbb{R}$ or $\mathbb{S}^1$.

Theorem 5.1.3. *Let $\mathcal{X} \subseteq \mathbb{R}$ and $[a, b] \subset \mathcal{X}$ a closed interval. Assume that $\boldsymbol{\phi}$ is continuous on $\pi^{-1}([a, b]) \cap \mathcal{C}$ and furthermore satisfies there a Lipschitz condition in $\mathbf{u} = (\mathbf{v}, \mathbf{w})$. Then the initial value problem for (5.1) with the initial condition $\mathbf{u}(a) = \mathbf{u}_0 \in \mathcal{C}$ possesses a unique solution $\mathbf{u}(x)$ on the whole interval $[a, b]$.*

Of course, one should note that we obtain such a trivial theory only because of our standard assumption that we are dealing with a *regular* differential equation. This entails that the ranks of certain Jacobians are constant and that we may indeed rewrite any local representation in the form (5.1). Many ordinary differential equations appearing in applications do not satisfy this assumption and their existence and uniqueness is harder. As we are mainly interested in partial differential equations, we do not give any details but refer to the literature, e. g. [144].

For a more geometric approach to an existence and uniqueness theory, we may exploit the contact structure of the jet bundle $J_1\mathcal{E}$. The contact distribution $\mathcal{C}_1 \subset TJ_1\mathcal{E}$ is $(m+1)$-dimensional and locally generated by the vector fields $\{\partial_{p_1^\alpha},\ p_1^\alpha\partial_{u^\alpha} + \partial_x\}$ (this is (1.10) specialised to $q = n = 1$). For a given ordinary differential equation $\mathcal{R}_1 \subseteq J_1\mathcal{E}$ we introduce on it the distribution $\mathcal{D} = T\mathcal{R}_1 \cap \mathcal{C}_1$. As for ordinary differential equations $\dim \mathcal{X} = 1$, we may consider a section $\sigma : \mathcal{X} \to \mathcal{E}$ as a curve.

Lemma 5.1.4. *The section $\sigma \in \Gamma_{loc}(\mathcal{E})$ is a solution of $\mathcal{R}_1$, if and only if at every point x of its domain of definition the tangent vector of the prolonged section $j_1\sigma \in \Gamma_{loc}(J_1\mathcal{E})$ lies in the vector space $\mathcal{D}_{j_1\sigma(x)}$.*

Proof. By the definition of a solution, the curve $j_1\sigma$ lies completely in $\mathcal{R}_1$ and thus the curve $(j_1\sigma)'$ in $T\mathcal{R}_1$. On the other hand, by the definition of the contact structure, the tangent space of any prolonged section $j_1\sigma$ is a subspace of the contact distribution $\mathcal{C}_1$. □

As a corollary of this lemma, we may easily characterise geometrically the existence of unique solutions for initial value problems. The differential equation $\mathcal{R}_1$ possesses a unique local solution passing through a prescribed point $\xi \in \mathcal{R}_0^{(1)} \subseteq \mathcal{E}$, if and only if the distribution $\mathcal{D}$ is one-dimensional everywhere on $\mathcal{R}_1$. Obviously, a necessary condition for unique solutions is that $\mathcal{R}_1$ is not underdetermined. In this case, the manifold $\mathcal{R}_1$ contains exactly one point ρ such that $\pi_0^1(\rho) = \xi$. A one-dimensional distribution is trivially involutive and thus integrable. Hence a unique integral curve to $\mathcal{D}$ exists passing through ρ. By Lemma 5.1.4 this integral curve is the prolongation of a solution of $\mathcal{R}_1$.

Theorem 5.1.5. *Let the regular ordinary differential equation $\mathcal{R}_1$ be involutive and not underdetermined. Then the distribution $\mathcal{D}$ is one-dimensional and the initial value problem for $\mathcal{R}_1$ is uniquely solvable.*

Proof. As the equation $\mathcal{R}_1$ is assumed to be regular and not underdetermined, we can locally transform it into the semi-explicit form $\mathbf{p} = \boldsymbol{\phi}(x, \mathbf{u})$, $\boldsymbol{\psi}(x, \mathbf{u}) = 0$ where $\mathbf{p}$ denotes the vector of all first-order derivatives. Any vector in $T\mathcal{R}_1$ must thus annihilate the following one-forms

$$dp_1^\alpha - \frac{\partial\phi^\alpha}{\partial u^\beta}du^\beta - \frac{\partial\phi^\alpha}{\partial x}dx\,, \qquad \frac{\partial\psi^\gamma}{\partial u^\beta}du^\beta + \frac{\partial\psi^\gamma}{\partial x}dx\,. \tag{5.3}$$

The distribution $\mathcal{D}$ is generated by those vectors $X \in \mathcal{C}_1$ that satisfy this condition. The ansatz $X = a^\alpha\partial_{p_1^\alpha} + b(p_1^\alpha\partial_{u^\alpha} + \partial_x)$ yields the following linear system for the coefficients a^α, b:

$$a^{\alpha} = b\left(\frac{\partial\phi^{\alpha}}{\partial u^{\beta}}p_1^{\beta} + \frac{\partial\phi^{\alpha}}{\partial x}\right)\,, \qquad 0 = b\left(\frac{\partial\psi^{\gamma}}{\partial u^{\beta}}p_1^{\beta} + \frac{\partial\psi^{\gamma}}{\partial x}\right)\,. \tag{5.4}$$

As these equations have to be evaluated *on* $\mathcal{R}_1$, the second set is empty for an involutive equation by (5.2). Thus the linear system (5.4) has a one-dimensional solution space. □

For notational simplicity, we have formulated this theorem and its derivation for first-order equations. But obviously, all the considerations extend straightforwardly to higher-order equations; there is not need to rewrite them as first-order equations, as it is usually done. For a regular, involutive and not underdetermined equation $\mathcal{R}_q \subset J_q\mathcal{E}$ we may always introduce the one-dimensional distribution $\mathcal{D} = T\mathcal{R}_q \cap \mathcal{C}_q$ the integral curves of which are the prolongations of solutions of $\mathcal{R}_q$.

5.2 The Cauchy–Kovalevskaya Theorem

With respect to the class of differential equations to which it can be applied, the Cauchy–Kovalevskaya Theorem represents probably the most general existence and unique result we know for the initial value problem for normal equations. Its only structural restriction is that we are dealing with a non-characteristic problem. However, the downside of this generality is that it applies only to *analytic* differential equations, *analytic* initial data and makes only statements about *analytic* solutions. These restrictions are very severe, in fact, too severe for most applications.

Nevertheless, the Cauchy–Kovalevskaya Theorem is the natural continuation of the work we have done so far. In Section 1.4 we discussed the construction of formal power series solutions. Now we show that under certain conditions these series converge. We formulate the theorem only for quasi-linear first-order equations. As detailed in Appendix A.3, any non-linear first-order differential equation can be reduced to a quasi-linear one via a prolongation and the introduction of additional dependent variables. One easily verifies that this operation maps a normal first-order equation again into a normal first-order equation.

It follows from the discussion of Proposition 3.3.7 that a normal quasi-linear first-order system may always be brought into the form

$$\mathbf{u}_{x^n} = \sum_{i=1}^{n-1} C_i(\mathbf{x}, \mathbf{u})\,\mathbf{u}_{x^i} + \mathbf{c}(\mathbf{x}, \mathbf{u}) \tag{5.5a}$$

where the C_i are $m \times m$ matrices with entries $c^{\alpha}_{i\beta}$ and $\mathbf{c}$ is an m-dimensional vector with components c_{α}. We consider the pure value problem

$$\mathbf{u}(x^1, \dots, x^{n-1}, 0) = 0 \,. \tag{5.5b}$$

The extension to general (analytic) initial data $\mathbf{f}$ is trivial; one only has to perform the coordinate transformation $\mathbf{v} = \mathbf{u} - \mathbf{f}$.

Note that the structure of (5.5a) implies that the hyperplane $x^n = 0$ on which the initial data are prescribed is non-characteristic. In physical applications the "evolution parameter" x^n represents usually the time t.

Theorem 5.2.1 (Cauchy–Kovalevskaya). *Let the functions* C_i *and* $\mathbf{c}$ *be real-analytic at the origin. The initial value problem* (5.5) *possesses a unique solution* $\mathbf{u}(\mathbf{x})$ *that is real-analytic at the origin.*

Proof. As (5.5a) is a normal equation, it is automatically formally integrable. Thus we can apply the order by order construction of a formal power series solution discussed in Section 1.4. Because of the special structure of (5.5) it is easy to distinguish the principal and the parametric coefficients in the series: any coefficient a^α_μ of the series for u^α with $\mu_n > 0$ is principal; all others are parametric.

The initial conditions (5.5b) determine uniquely all parametric coefficients of the series. It is not difficult to see that for our special initial data any Taylor coefficient a^α_μ can be represented as a polynomial with positive coefficients in the Taylor coefficients of the functions $c^\alpha_{i\beta}$ and c_α expanded at the origin.

Thus the initial value problem (5.5) possesses a unique formal power series solution. There remains to show that it converges in a neighbourhood of the origin. As the finitely many functions $c^\alpha_{i\beta}$ and c_α are analytic at the origin, constants M and r exist such that they are all contained in $\mathcal{C}_{M,r}(0)$ (see Appendix A.2). Consider the initial value problem

$$\mathbf{U}_{x^n} = \sum_{i=1}^{n-1} \bar{C}_i(\mathbf{x}, \mathbf{U})\, \mathbf{U}_{x^i} + \bar{\mathbf{c}}(\mathbf{x}, \mathbf{U}) \,, \tag{5.6a}$$

$$\mathbf{U}(x^1, \dots, x^{n-1}, 0) = 0 \tag{5.6b}$$

where all entries $\bar{c}^\alpha_{i\beta}$ of the matrices $\bar{C}_i$ and components $\bar{c}_\alpha$ of the vector $\bar{\mathbf{c}}$ are given by the same function

$$\phi(\mathbf{x}, \mathbf{U}) = \frac{Mr}{r - (x^1 + \cdots + x^n + U^1 + \cdots + U^m)} \tag{5.7}$$

and thus majorise the functions in (5.5a): $\bar{c}^\alpha_{i\beta} \succeq c^\alpha_{i\beta}$ and $\bar{c}_\alpha \succeq c_\alpha$.

The formal power series solution $\mathbf{U}$ of (5.6) majorises the formal power series solution $\mathbf{u}$ of (5.5a) (recall, the Taylor coefficients a^α_μ are determined by polynomials with *positive* coefficients). Hence our proof is completed, if we can show that the series $\mathbf{U}$ converges.

Entering (5.7) into (5.6) results in the differential equation

$$U_{x^n} = \frac{Mr}{r - (x^1 + \cdots + x^n + U^1 + \cdots + U^m)} \left(1 + \sum_{i=1}^{n-1} \sum_{\alpha=1}^{m} U^{\alpha}_{x^i} \right) \tag{5.8}$$

which can be solved in closed form. With the help of the simple ansatz $U^\alpha(\mathbf{x}) = V(x^1 + \cdots + x^{n-1}, x^n)$ for $1 \leq \alpha \leq m$ we can reduce (5.8) to a scalar equation for the single unknown $V(\sigma, \tau)$. It solution, computed for example by the method of characteristics, is given by

$$V(\sigma, \tau) = \frac{1}{nm} \left(r - \sigma - \sqrt{(r - \sigma)^2 - 2nmMr\tau} \right) . \tag{5.9}$$

This function is analytic at $\sigma = \tau = 0$ and (5.8) possesses a unique solution which is analytic at the origin. Thus the above constructed power series solution of (5.6) converges in a neighbourhood of the origin and one easily checks that it also satisfies the initial condition. □

It should be stressed that the uniqueness of the solution is asserted only in the category of functions that are real-analytic at the origin. The Cauchy–Kovalevskaya Theorem does not exclude the existence of further solutions that are not real-analytic. We will see in Section 5.5 that for linear systems a stronger uniqueness statement is possible.

Another important point is that Theorem 5.2.1 gives a purely local result; nothing is asserted about the domain of definition of the solution. A lower bound for its radius of convergence is given by the radius of convergence of the function V defined by (5.9). We do not give a precise formula, but obviously the bound depends only on the number of independent and dependent variables, respectively, n and m and on the numbers M, r determined by the coefficients of the equations.

The assumption of analyticity is crucial in the Cauchy–Kovalevskaya Theorem. In general, it is not possible to relax it, say to smooth functions, as the following classical example of Lewy [106] demonstrates (see also [85]). He proved that there exists a real-valued function $f \in C^\infty(\mathbb{R}^3)$ such that the following linear equation for a complex-valued unknown $u(x, y, z)$

$$u_x + iu_y - 2i(x + iy)u_z = f(x, y, z) \tag{5.10}$$

does not possess any solutions in $C^1(\Omega)$ for any open set $\Omega \subset \mathbb{R}^3$.

The way we have stated the Cauchy–Kovalevskaya Theorem (which is the usual way in textbooks), it gives *sufficient* conditions for the existence of a unique analytic solution. An obvious – but rarely posed – question is whether these are also *necessary*. Kovalevskaya gave in her thesis [95] an affirmative answer, however, without proof. The essential question is whether it is necessary to require that on the right hand side of (5.5a) the derivatives with respect to x^i for $i < n$ are also of first order.

All other conditions are obviously necessary. If the initial conditions are posed on a characteristic surface, it is not possible to construct a unique power series solution; this is the essence of the definition of characteristic. If x^n is non-characteristic and the first order system cannot be brought into the form (5.5a), it is either under- or overdetermined. In the first case we do not get a unique solution, in the latter one the initial data cannot be specified arbitrarily but must satisfy compatibility conditions. The necessity of the analyticity of the involved functions is trivial.

Example 5.2.2. Kovalevskaya [95, p. 22] provided a simple counter example to her theorem, if the right hand side of (5.5a) contains higher-order derivatives. Consider the following initial value problem for the heat equation:

$$u_t = u_{xx} \ , \tag{5.11a}$$

$$u(x,0) = 1/(1-x) \ . \tag{5.11b}$$

It is straightforward to construct the unique formal power series solution of this problem:

$$u(x,t) = \sum_{i,j=0}^{\infty} \frac{(2i)!}{i!j!} x^j t^i \ . \tag{5.12}$$

However, one can show that this series converges for no $t \neq 0$. Indeed, it is well-known that the solution of the pure initial value problem for the heat equation is given by convolution with the heat kernel

$$u(x,t) = \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} \exp\left[-\frac{(x-\xi)^2}{4t}\right] f(\xi)\, d\xi \tag{5.13}$$

for $u(x,0) = f(x)$. Obviously, the integral does not converge for arbitrary analytic initial data f, but we must impose on the function f some decay conditions for $|x| \to \infty$. $\lhd$

Of course, such a counter example does not suffice to prove the necessity of the conditions in the Cauchy–Kovalevskaya Theorem. A rigorous proof was achieved comparatively recently by Mizohata [124] for the case of linear systems. However, the techniques used are beyond the scope of this book.

5.3 Formally Well-Posed Initial Value Problems

In the previous section we discussed the initial value problem for normal differential equation. There it was obvious what initial conditions one should impose. For a non-normal equation this question becomes less trivial. In this section we consider the problem of choosing the right form and number of initial conditions on the level of formal power series, i. e. we are only considering formal power series solutions and the initial data take the form of formal power series, too.

Definition 5.3.1. *An initial value problem for a differential equation $\mathcal{R}_q$ is* formally well-posed*, if it possesses a unique formal power series solution for arbitrary formal power series as initial data.*

Thus in a formally well-posed initial value problem the Taylor coefficients of the initial data are in a one-to-one correspondence with the parametric derivatives of the differential equation. It is easy to see that the initial value problem considered in the Cauchy–Kovalevskaya Theorem 5.2.1 is formally well-posed. Here all derivatives that are only with respect to $x^1, \dots, x^{n-1}$ are parametric. As the initial data prescribed consist of functions depending on these variables, each such derivative is uniquely determined by the corresponding coefficient of the initial data.

Remark 5.3.2. The concept of well-posedness plays an important role in the theory of differential equations. According to the classical definition of Hadamard a problem is well-posed, if (i) it has a solution, (ii) the solution is unique and (iii) it depends continuously on the initial data. Of course, this is only a rough description that must be made rigorous by specifying function spaces and topologies. In Definition 5.3.1 we gave a precise formulation of the first two points by restricting to formal power series, but we completely omitted the third point.

This is a general problem with the Cauchy–Kovalevskaya Theorem. It permits the study of initial value problem that are not well-posed in the classical sense, as a famous example of Hadamard [66, p. 33] shows. Consider the following initial value problem for the Laplace equation:

$$u_{tt} + u_{xx} = 0\,, \tag{5.14a}$$

$$u(x, 0) = 0\,, \tag{5.14b}$$

$$u_t(x, 0) = \sin(nx)/n\,. \tag{5.14c}$$

Its (unique) solution is given by $u(x,t) = \sinh(nt)\sin(nx)/n^2$. If we consider the limit $n \to \infty$, we see that the initial data vanish but that for any $t > 0$ the solution blows up. As $u(x,t) \equiv 0$ is the unique solution for vanishing initial data, solutions of (5.14) cannot depend continuously on the initial data: arbitrarily small changes of the initial data may lead to arbitrarily large changes in the solution.

This is a typical effect in elliptic equations and the reason why one usually studies boundary value problems for such equations. We will ignore in the sequel the question of continuous dependency on the initial data and concentrate solely on the existence and uniqueness of (formal) solutions. ◁

Our goal in this section is to determine in a systematic way formally well-posed initial value problems for involutive systems. We start with linear systems and for simplicity we restrict to the case of one dependent variable; in the general case one must simply perform the same analysis for each dependent variable separately. Taking up again the notations of Example 2.2.4, we study an ideal $\mathcal{I} \subseteq \mathcal{D}$ where $\mathcal{D} = \mathbb{F}[\partial_1, \dots, \partial_n]$ denotes the ring of linear differential operators with coefficients in some function field $\mathbb{F}$.

Example 5.3.3. Before we go into technical details, we demonstrate our basic strategy for the construction of formally well-posed initial value problems at a simple example in one dependent and two independent variables where the right initial conditions are easily found by inspection. Assume that we are given an involutive equation of the form

$$\mathcal{R}_2 : \begin{cases} u_{yy} = au + bu_x + cu_y \\ u_{xy} = du + eu_x + fu_y \end{cases} \tag{5.15}$$

where the coefficients are functions of x and y. Thus principal derivatives are u_{yy}, u_{xy} and all their derivatives. Conversely, parametric derivatives are u, u_y and all pure x-derivatives.

In a formally well-posed initial value problem all parametric coefficients must be prescribed by initial conditions. As all pure x-derivatives are parametric, it is obvious that the initial data must comprise a function of x. Thus we take as one initial condition $u(x, 0) = f(x)$. This determines all parametric coefficients in our system with the exception of the one corresponding to u_y. Hence we impose as second initial condition $u_y(0, 0) = g$. This time it is not possible to prescribe an arbitrary function but only a constant.

It is a typical feature of non-normal systems that even in the non-characteristic initial value problem one cannot prescribe all initial data on one hypersurface as in the Cauchy–Kovalevskaya Theorem. In the case of an underdetermined system the data would not suffice to determine all parametric coefficients, as some unknown functions are completely arbitrary. For an overdetermined system one could not prescribe arbitrary initial data, as they would have to satisfy some consistency conditions. ◁

As a first step towards the construction of formally well-posed initial value problems, we need an algebraic approach to the distinction into principal and parametric derivatives. This is easily achieved with the help of a term order and leads to concepts very familiar from Chapter 2.

Definition 5.3.4. *Let $\mathcal{I} \subseteq \mathcal{D}$ be a (left) ideal. For a given term order $\prec$, we call the derivatives ∂^μ with $\mu \in \Delta_\prec(\mathcal{I}) = \mathrm{le}_\prec \mathcal{I}$* principal. *All remaining derivatives are* parametric *and their exponents form the* complementary set $\Pi_\prec(\mathcal{I}) = \mathbb{N}_0^n \setminus \Delta_\prec(\mathcal{I})$.

Comparing with the introduction of principal and parametric derivatives (or coefficients, respectively) in Section 1.4, we see that there at each order of prolongation the principal derivatives could be chosen completely independent of the choices at the previous orders. Now we select the principal derivatives in a systematic manner based on a term order.

While $\Delta_\prec(\mathcal{I})$ is a monoid ideal, the set $\Pi_\prec(\mathcal{I})$ does in general not possess any algebraic structure. Given an involutive division L and an involutive basis $\mathcal{H}$ of the ideal $\mathcal{I}$ with respect to L and $\prec$, we immediately obtain a disjoint decomposition of $\Delta_\prec(\mathcal{I})$ via the involutive cones of the multi indices in $\mathrm{le}_\prec \mathcal{H}$. The basic idea underlying the construction of formally well-posed initial value problems is to construct a similar disjoint decomposition of $\Pi_\prec(\mathcal{I})$. But this question was already successfully treated in Section 2.5 and we exploit the results for obtaining formally well-posed initial value problems.

Given an ideal $\mathcal{I} \subseteq \mathcal{D}$, we decompose the set $\Pi_\prec(\mathcal{I})$ in the form (2.25). We choose an arbitrary but fixed point $\mathbf{x}_0 \in \mathcal{X}$. For subsets $N \subseteq \{1, \dots, n\}$ we introduce two notations: $\mathbf{y} = N(\mathbf{x})$ where $y^i = x^i$, if $i \in N$, and $y^i = x_0^i$, if $i \notin N$ and $\mathbf{x}_N = (x^{i_1}, \dots, x^{i_k})$, if $N = \{i_1, \dots, i_k\}$. In other words, $N(\mathbf{x})$ denotes a projection of the vector $\mathbf{x}$ where those components of it that do not correspond to indices contained in N obtain the values of the entries of $\mathbf{x}_0$ and $\mathbf{x}_N$ simply collects those components of $\mathbf{x}$ that correspond to the elements of N. The decomposition (2.25) is defined by a set $\bar{\mathcal{N}} \subset \Pi_\prec(\mathcal{I})$ and associated sets $N_\nu \subseteq \{1, \dots, n\}$ for each $\nu \in \bar{\mathcal{N}}$. Based on these data, we prescribe in the neighbourhood of $\mathbf{x}_0$ the following initial conditions for $\mathcal{I}$:

$$D_\nu u\big(N_\nu(\mathbf{x})\big) = f_\nu(\mathbf{x}_{N_\nu}) \,, \qquad \forall \nu \in \bar{\mathcal{N}} \,. \tag{5.16}$$

Here, the right hand sides f_ν are arbitrary formal power series in their arguments. If $N_\nu = \emptyset$, then f_ν is simply an arbitrary constant.

Theorem 5.3.5. *The initial value problem for $\mathcal{I}$ defined by the initial conditions* (5.16) *is formally well-posed.*

Proof. We expand the solution of $\mathcal{I}$ into a formal power series about $\mathbf{x}_0$. By definition, its parametric coefficients are in a one-to-one correspondence with the elements of $\Pi_\prec(\mathcal{I})$. Let $\mu \in \Pi_\prec(\mathcal{I})$ be the multi index corresponding to a given parametric coefficient. Then exactly one multi index $\nu \in \bar{\mathcal{N}}$ exists such that $\mu = \nu + \rho$ with $\rho \in \mathbb{N}^n_{N_\nu}$. Our initial value problem prescribes for this coefficient the unique value $D_\rho f_\nu(\mathbf{x}_{0,N})$. This implies that we obtain for arbitrary initial data a unique formal power series solution and, by definition, our problem is formally well-posed. □

Thus we see that the construction of a formally well-posed initial value problem for an involutive system $\mathcal{F} \subset \mathcal{D}$ is in fact equivalent to the determination of a Stanley decomposition of the $\mathcal{D}$-module $\mathcal{D}/\langle\mathcal{F}\rangle$. Both operations reduce to the same problem, namely to find a decomposition of the form (2.25) for the complementary set of $\mathrm{le}_{\prec}\mathcal{F}$.

Example 5.3.6. Given a system in Cauchy–Kovalevskaya form, we must analyse for each dependent variable the monoid ideal $\langle\{[0,\dots,0,1]\}\rangle$. Its complement is obviously given by $[0,\dots,0]+\mathbb{N}^n_{\{1,\dots,n-1\}}$. Thus we are led to the initial value problem

$$u^\alpha(x^1,\dots,x^{n-1},x^n_0) = f^\alpha(x^1,\dots,x^{n-1})\,, \qquad 1 \leq \alpha \leq m \tag{5.17}$$

which is used in the Cauchy–Kovalevskaya Theorem.

As a simple example of a system not in Cauchy–Kovalevskaya form we consider the wave equation in characteristic coordinates $u_{xy}=0$. The corresponding monoid ideal $\mathcal{I} = \langle\{[1,1]\}\rangle$ was already studied in Example 2.5.5. We obtain the classical characteristic initial value problem

$$u(x,y_0) = f(x)\,, \qquad u_y(x_0,y) = g(y)\,. \tag{5.18}$$

The second condition is often substituted by $u(x_0,y) = g(y)$, but then we must impose the consistency condition $f(0) = g(0)$ so that the initial value problem does not possess a solution for arbitrary initial data and thus is not formally well-posed. Our formulation allows us to choose g arbitrarily. ◁

As mentioned in Section 2.5, a decomposition (2.25) of the complementary set $\bar{\mathcal{I}}$ can be determined algorithmically. Concrete algorithms follow from the proofs of Proposition 2.5.3 or, in the important special case of Janet bases, of Proposition 2.5.4. A related algorithm for the construction of formally well-posed initial value problems was proposed by Reid [147].

Remark 5.3.7. In Section 4.1 we introduced closed form representations of general solutions in which manifolds $\mathcal{Y}_\ell$ appears. Our initial value problems take "planes" for these manifolds, namely, $\mathcal{Y}_\nu = \{\mathbf{x} \in \mathcal{X} \mid x^i = x^i_0\ \forall i \notin N_\nu\}$. If we construct them with the help of Pommaret bases, these submanifolds form a flag $\{\mathbf{x}_0\} = \mathcal{Y}_0 \subset \mathcal{Y}_1 \subset \cdots \subset \mathcal{Y}_n = \mathcal{X}$ where $\mathcal{Y}_k = \{\mathbf{x} \in \mathcal{X} \mid x^i = x^i_0\ \forall i > k\}$. For other decompositions this is not necessarily the case, as we have seen above for characteristic initial value problems. ◁

Finally, we study the extension of these results to systems with arbitrary non-linearities. The only available notion of involution for such systems is the intrinsic Definition 3.2.1 within our geometric approach. Thus we start with an involutive equation $\mathcal{R}_q \subseteq J_q\mathcal{E}$ and construct in local coordinates an initial value problem in the neighbourhood of a point $\rho \in \mathcal{R}_q$. We prepare again a local representation in Cartan normal form: each equation is solved for a different derivative which is the largest derivative in the equation with respect to the degree reverse lexicographic order.

Consider again the symbol module $\mathcal{M}^{\mathcal{P}}$; obviously only the equations of order q contribute to it. As we assume that $\mathcal{R}_q$ is involutive, these equations induce by Definition 3.1.1 of an involutive symbol a Pommaret basis of it and their leading derivatives yield a Pommaret basis of $\mathrm{le}_{\prec}\mathcal{M}^{\mathcal{P}}$. Let p_μ be the leading derivative of an equation whose order is less than q. As $\mathcal{R}_q$ is formally integrable, we may choose our local representation such that $p_{\mu+1_i}$ for any $1 \leq i \leq n$ is the leading derivative of some equation, too.

Let $\Delta_{\prec}(\mathcal{R}_q) \subseteq \mathbb{N}_0^n$ be the monoid ideal generated by the multi indices of the leading derivatives;[1] obviously, $\bigl(\Delta_{\prec}(\mathcal{R}_q)\bigr)_{\geq q} = \mathrm{le}_{\prec}\mathcal{M}^{\mathcal{P}}$. Thus our generating set is a weak Pommaret basis of $\Delta_{\prec}(\mathcal{R}_q)$ and, by Proposition 2.1.10, an involutive autoreduction yields a strong Pommaret basis. But now we are in the same situation as above for linear systems: we have a monoid ideal defined by the principal derivatives and must decompose its complement in the form (2.25). Then we impose the initial conditions (5.16) and show as in Theorem 5.3.5 that this yields a formally well-posed initial value problem in the neighbourhood of ρ.

As a corollary to these considerations, we may characterise all the involutive equations $\mathcal{R}_q$ the symbol module $\mathcal{M}^{\mathcal{P}}$ of which is Cohen–Macaulay. It turns out that this property is in a certain sense equivalent to normality, thus these equations have a particularly simple structure.

Proposition 5.3.8. *Let the symbol module $\mathcal{M}^{\mathcal{P}}$ of the involutive differential equation $\mathcal{R}_q$ be Cohen–Macaulay and let d be its depth. Then $\mathcal{R}_q$ is equivalent to a normal equation in d independent variables in the sense that a bijection between their formal solution spaces exists.*

Proof. Let us first consider the case that $d = 1$. Then $\mathcal{R}_q$ is an equation of finite type, as obviously $\mathcal{M}_q$ vanishes. In Example 1.4.3 we have shown that the integration of such an equation may be reduced to the integration of an involutive distribution which in turn may be reduced to solving a normal system of ordinary differential equations (Example 3.2.5).

If $d > 1$, then a local representation of $\mathcal{R}_q$ cannot contain any equation of order less than q, in other words $\mathcal{R}^{(1)}_{q-1} = J_{q-1}\mathcal{E}$. Indeed, the depth of $\mathcal{M}^{\mathcal{P}}$

[1] Despite our notation this monoid ideal is of course not intrinsically defined but depends on the chosen local representation.

corresponds to the lowest class present in the symbol. If a local representation contains lower-order equations, it must also contain all their formal derivatives, as $\mathcal{R}_q$ is formally integrable. Thus equations of class 1 appear in the symbol and its depth is 1.

Now our proposition is a direct consequence of (our proof of) Hironaka's criterion for Cohen–Macaulay modules (Corollary 2.5.12). It implies that a formally well-posed initial value problem for $\mathcal{R}_q$ can be defined by prescribing all derivatives with respect to $x^d, \dots, x^n$ up to order $q-1$ as functions of $x^1, \dots, x^{d-1}$. But these are precisely the initial data for a normal equation in d variables. □

At the end of the next section we will describe the equivalent normal equation and the bijection between the formal solution spaces in more detail. Note that if $\mathcal{R}_q$ is already a normal equation, then $\mathcal{M}^{\mathcal{P}}$ is even a free module.

Example 5.3.9. Assume that the involutive differential equation $\mathcal{R}_2$ in three independent and one dependent variables has a local representation of the following form:

$$\mathcal{R}_2 : \begin{cases} u_{zz} = \phi_1(x, y, z, u, u_x, u_y, u_z, u_{xx}, u_{xy}, u_{xz}) , \\ u_{yz} = \phi_2(x, y, z, u, u_x, u_y, u_z, u_{xx}, u_{xy}, u_{xz}) , \\ u_{yy} = \phi_3(x, y, z, u, u_x, u_y, u_z, u_{xx}, u_{xy}, u_{xz}) . \end{cases} \tag{5.19}$$

It is not difficult to see that its symbol module $\mathcal{M}^{\mathcal{P}}$ is Cohen–Macaulay and $\operatorname{depth} \mathcal{M}^{\mathcal{P}} = 2$. Applying the algorithm of the proof of Proposition 2.5.4 we find that a formally well-posed initial value problem is given by

$$u_z(x, 0, 0) = f(x) , \qquad u_y(x, 0, 0) = g(x) , \qquad u(x, 0, 0) = g(x) . \tag{5.20}$$

Obviously, the same initial data are prescribed for a normal system in two independent and three dependent variables. For the independent variables we may choose the $\mathcal{M}^{\mathcal{P}}$-regular sequence (x, y). ◁

5.4 The Cartan–Kähler Theorem

The Cartan–Kähler Theorem extends the Cauchy–Kovalevskaya Theorem to arbitrary involutive systems. This implies that it inherits all its restrictions. So again the practical use of this theorem is rather limited. However, the proof is quite enlightening. It does not only highlight the importance of involution in contrast to mere formal integrability; it also shows a fairly general methodology to extend results on normal systems to involutive ones.

For simplicity, we formulate it for an involutive first-order system in Cartan normal form without any algebraic equations. Thus we assume that we are given an involutive system of the following form (cf. (3.10)):

$$p_n^\alpha = \phi_n^\alpha(x^i, u^\beta, p_j^\gamma, p_n^\delta)\,, \qquad \begin{cases} 1\leq\alpha\leq\beta_1^{(n)}\,, \\ 1\leq j<n\,, \\ \beta_1^{(n)}<\delta\leq m\,, \end{cases} \tag{5.21a}$$

$$p_{n-1}^\alpha = \phi_{n-1}^\alpha(x^i, u^\beta, p_j^\gamma, p_{n-1}^\delta)\,, \qquad \begin{cases} 1\leq\alpha\leq\beta_1^{(n-1)}\,, \\ 1\leq j<n-1\,, \\ \beta_1^{(n-1)}<\delta\leq m\,, \end{cases} \tag{5.21b}$$

$$\vdots$$

$$p_1^\alpha = \phi_1^\alpha(x^i, u^\beta, p_1^\delta)\,, \qquad \begin{cases} 1\leq\alpha\leq\beta_1^{(1)}\,, \\ \beta_1^{(1)}<\delta\leq m\,. \end{cases} \tag{5.21c}$$

In order to obtain a unique solution, we prescribe the following initial conditions to the system (5.21)

$$u^\alpha(x^1,\dots,x^n) = f^\alpha(x^1,\dots,x^n)\,, \qquad \beta_1^{(n)} < \alpha \leq m\,, \tag{5.22a}$$

$$u^\alpha(x^1,\dots,x^{n-1},0) = f^\alpha(x^1,\dots,x^{n-1})\,, \qquad \beta_1^{(n-1)} < \alpha \leq \beta_1^{(n)}\,, \tag{5.22b}$$

$$\vdots$$

$$u^\alpha(x^1,0,\dots,0) = f^\alpha(x^1)\,, \qquad \beta_1^{(1)} < \alpha \leq \beta_1^{(2)}\,, \tag{5.22c}$$

$$u^\alpha(0,\dots,0) = f^\alpha\,, \qquad 1 \leq \alpha \leq \beta_1^{(1)}\,. \tag{5.22d}$$

This is a formally well-posed initial value problem. The leading derivatives of the equations in (5.21) define a Pommaret basis (hence by Corollary 2.4.9 also a Janet basis) of monoid module generated by their multi indices. The initial conditions (5.22) are obtained from (5.16), if we decompose this module according to the proof of Proposition 2.5.4.

Again one clearly sees that the system is underdetermined, if $\beta_1^{(n)} < m$, as in this case some of the dependent variables u^α are not restricted at all by the differential equations and must therefore be completely prescribed by the initial conditions. One might say that these variables have more the character of parameters than of unknowns.

Theorem 5.4.1 (Cartan–Kähler). *Let the functions ϕ_k^α and f^α be real-analytic at the origin and let the system* (5.21) *be involutive. Then it possesses one and only one solution that is analytic at the origin and satisfies the initial conditions* (5.22).

We first prove a lemma which is at the very heart of the proof of the Cartan–Kähler Theorem we will give. It is also of independent interest, as it shows an important property of involutive systems.

Lemma 5.4.2. *Under the assumptions of Theorem 5.4.1, consider the subsystem consisting of the equations of class* k *in* (5.21),

$$p^\alpha_k = \phi^\alpha_k(x^i, u^\beta, p^\gamma_j, p^\delta_k)\,, \qquad \begin{cases} 1\leq\alpha\leq\beta_1^{(k)}\,,\\ 1\leq j\leq k\,,\\ \beta_1^{(k)}<\delta\leq m\,, \end{cases} \tag{5.23}$$

and set $x^i = 0$ *for* $i > k$ *in it. Let* $\mathbf{U}(x^1,\dots,x^k)$ *be an analytic solution of the thus restricted system* (5.23) *for the initial conditions*

$$\begin{aligned} &u^\alpha(x^1,\dots,x^{k-1},0) = f^\alpha(x^1,\dots,x^{k-1})\,, && 1\leq\alpha\leq\beta_1^{(k)}\,,\\ &u^\alpha(x^1,\dots,x^{k-1},x^k) = f^\alpha(x^1,\dots,x^{k-1},x^k,0,\dots,0)\,, && \beta_1^{(k)}<\alpha\leq m\,. \end{aligned}$$

(Thus the functions u^α *with* $\alpha > \beta_1^{(k)}$ *appear only as parameters). If the initial data* $\mathbf{f}(\mathbf{x})$ *are such that for* $x^k = \cdots = x^n = 0$ *they satisfy all equations in* (5.21) *of class lower than* k*, then the solution* $\mathbf{U}(x^1,\dots,x^k)$ *satisfies these equations also for* $x^k \neq 0$.

Proof. We must analyse what happens, if we enter the solution $\mathbf{U}(x^1,\dots,x^k)$ into the equations of lower class. So we introduce the residuals stemming from the equations of class ℓ

$$\begin{aligned} \Delta^\alpha_\ell(x^1,\dots,x^k) =& \frac{\partial U^\alpha}{\partial x^\ell}(x^1,\dots,x^k) - \\ & \phi^\alpha_\ell\Big(x^1,\dots,x^k,0,\dots,0,\mathbf{U}(x^1,\dots,x^k),\frac{\partial\mathbf{U}}{\partial\mathbf{x}}(x^1,\dots,x^k)\Big) \end{aligned}$$

for all classes $\ell < k$ and all indices $1 \leq \alpha \leq \beta_1^{(\ell)}$. We exploit the fact that the system (5.21) is involutive in order to derive differential equations for the functions Δ^α_ℓ. Adapting (3.11) to the special case we have got here, we see that involution of (5.21) is equivalent to the existence of functions $A^\beta_{ij}(\mathbf{x},\mathbf{u},\mathbf{p})$ and $B^\beta_i(\mathbf{x},\mathbf{u},\mathbf{p})$ such that whenever $1 \leq \ell < k \leq n$

$$D_k(p^\alpha_\ell - \phi^\alpha_\ell) = \sum_{i=1}^{k}\sum_{\beta=1}^{\beta_1^{(i)}}\left\{\sum_{j=1}^{i} A^\beta_{ij} D_j(p^\beta_i - \phi^\beta_i) + B^\beta_i(p^\beta_i - \phi^\beta_i)\right\}. \tag{5.24}$$

Entering $\mathbf{U}(x^1,\dots,x^k)$ into these relations and setting $x^i = 0$ for $i > k$ implies that the residuals Δ^α_ℓ must satisfy the linear system

$$\frac{\partial \Delta_\ell^\alpha}{\partial x^k} = \sum_{i=1}^{k-1} \sum_{\beta=1}^{\beta_1^{(i)}} \left\{ \sum_{j=1}^{i} \bar{A}_{ij}^{\beta} \frac{\partial \Delta_i^\beta}{\partial x^j} + \bar{B}_i^\beta \Delta_i^\beta \right\} , \quad 1 \leq \alpha \leq \beta_1^{(k)} , \tag{5.25}$$

where

$$\bar{A}_{ij}^{\beta}(x^1, \dots, x^k) = A_{ij}^{\beta}\left(x^1, \dots, x^k, 0, \dots, 0, \mathbf{U}(x^1, \dots, x^k), \frac{\partial \mathbf{U}}{\partial \mathbf{x}}(x^1, \dots, x^k)\right) ,$$
$$\bar{B}_i^{\beta}(x^1, \dots, x^k) = B_i^{\beta}\left(x^1, \dots, x^k, 0, \dots, 0, \mathbf{U}(x^1, \dots, x^k), \frac{\partial \mathbf{U}}{\partial \mathbf{x}}(x^1, \dots, x^k)\right)$$

are analytic functions, as A_{ij}^{β}, B_i^{β} and $\mathbf{U}$ are all analytic and the analytic functions form a monoid under composition. On the right hand side of (5.25) the terms with $i = k$ have disappeared, as by assumption $\mathbf{U}$ is a solution of (5.23). This makes (5.25) an analytic *normal* system in the independent variables $x^1, \dots, x^k$. Obviously, $\Delta_\ell^\alpha \equiv 0$ is an analytic solution of it. By the Cauchy–Kovalevskaya Theorem 5.2.1 this solution is unique. Thus the solution $\mathbf{U}$ also satisfies all the equations of lower class. □

Lemma 5.4.2 expresses a kind of conservation property of involutive equations: if the equations of lower class are satisfied for $x^k = 0$, then they are satisfied at all values of x^k for which a solution exists. For the special case of Maxwell's equations (1.64), this is a very classical result. As already mentioned in Example 1.6.1, if a solution of the evolution part of Maxwell's equations satisfies the Gauss laws at the time $t = 0$, it satisfies them at all times. Using a classical identity from vector analysis, this is proven in almost any textbook on electrodynamics. But as we have seen, it is in fact a very general property of involutive systems.

We may also consider Lemma 5.4.2 as a generalisation of Proposition 5.1.1 to partial differential equations. It states that the subsystems of lower class are a kind of "weak invariants" for the "flow" generated by the subsystem of class k. In fact, this becomes a very useful point of view within the formalism of semigroups [134, 149] where partial differential equations are considered as ordinary differential equations on infinite-dimensional spaces.

Proof (of the Cartan–Kähler Theorem). We follow the proof given by Pommaret [139]. The solution is constructed step by step; at each step a normal system appears to which we apply the Cauchy–Kovalevskaya Theorem 5.2.1. Finally, we use repeatedly Lemma 5.4.2.

We start with the subsystem (5.21c) comprising all equations of class 1:

$$p_1^\alpha = \phi_1^\alpha(x^i, u^\beta, p_1^\gamma) , \qquad \begin{cases} 1 \leq \alpha \leq \beta_1^{(1)} , \\ \beta_1^{(1)} < \gamma \leq m . \end{cases} \tag{5.26a}$$

If we set $x^i = 0$ for $i > 1$, the initial conditions (5.22) allow us to substitute for u^β and p_1^β with $\beta > \beta_1^{(1)}$ the functions $\bar{f}^\beta(x^1) = f^\beta(x^1, 0, \dots, 0)$ and their x^1-derivatives. Now (5.26a) becomes a normal system (even of ordinary differential equations) in the independent variable x^1 and we may apply the Cauchy–Kovalevskaya Theorem for the last set of initial conditions in (5.22)

$$u^\alpha(0, \dots, 0) = f^\alpha \in \mathbb{R}\,, \qquad 1 \leq \alpha \leq \beta_1^{(1)}\,. \tag{5.26b}$$

It guarantees the existence of a unique analytic solution $U_1^\alpha(x^1)$ (where now $1 \leq \alpha \leq \beta_1^{(1)}$) of the initial value problem (5.26).

In the next step, we consider the equations of class 2 in (5.21)

$$p_2^\alpha = \phi_2^\alpha(x^i, u^\beta, p_1^\gamma, p_2^\delta)\,, \qquad \begin{cases} 1 \leq \alpha \leq \beta_1^{(2)}\,, \\ \beta_1^{(2)} < \delta \leq m\,. \end{cases} \tag{5.27a}$$

This time we set $x^i = 0$ only for $i > 2$. Then we substitute u^β, p_1^β and p_2^β with $\beta > \beta_1^{(2)}$ by the functions $\bar{f}^\beta(x^1, x^2) = f^\beta(x^1, x^2, 0, \dots, 0)$ and their derivatives with respect to x^1 and x^2. This makes (5.27a) a normal system in the two independent variables x^1 and x^2. We prescribe the initial conditions

$$u^\alpha(x^1, 0) = \begin{cases} f^\alpha(x^1) & \text{for } \beta_1^{(1)} < \alpha \leq \beta_1^{(2)}\,, \\ U_1^\alpha(x^1) & \text{for } 1 \leq \alpha \leq \beta_1^{(1)}\,. \end{cases} \tag{5.27b}$$

Thus for those components u^α we considered already in the first step, we take as initial data the solution $U_1^\alpha(x^1)$ obtained above. For the remaining components we use the initial data in (5.22).

We may again apply the Cauchy–Kovalevskaya Theorem, as all appearing functions are analytic. It guarantees us the existence of a unique analytic solution $U_2^\alpha(x^1, x^2)$ with $1 \leq \alpha \leq \beta_1^{(2)}$ of the initial value problem (5.27). However, it is not obvious that these functions also satisfy (5.26a) with $x^i = 0$ for $i > 2$; we only know this, if in addition $x^2 = 0$. But this is precisely the situation covered by Lemma 5.4.2: it asserts that the functions $U_2^\alpha(x^1, x^2)$ also satisfy the equations of lower class, i. e. in our case (5.26a), for all values of the variable x^2.

We iterate this process. In step k we consider the equations of class k, i. e. (5.23). Setting $x^i = 0$ for $i > k$, we substitute u^β and p_j^β with $\beta > \beta_1^{(k)}$ by the functions $\bar{f}^\beta(x^1, \dots, x^k) = f^\beta(x^1, \dots, x^k, 0, \dots, 0)$ and their derivatives. Then (5.23) becomes a normal system in the independent variables $x^1, \dots, x^k$. As initial conditions we use

$$u^\alpha(x^1, \dots, x^{k-1}, 0) = \begin{cases} f^\alpha(x^1, \dots, x^{k-1}) & \text{for } \beta_1^{(k-1)} < \alpha \leq \beta_1^{(k)}\,, \\ U_{k-1}^\alpha(x^1, \dots, x^{k-1}) & \text{for } 1 \leq \alpha \leq \beta_1^{(k-1)}\,. \end{cases} \tag{5.28}$$

Here $U^{\alpha}_{k-1}(x^1, \dots, x^{k-1})$ represents the solution of the previous step. By the Cauchy–Kovalevskaya Theorem the initial value problem (5.23, 5.28) possesses a unique analytic solution $U^{\alpha}_{k}(x^1, \dots, x^k)$ with $1 \leq \alpha \leq \beta_1^{(k)}$ which by Lemma 5.4.2 also satisfies all equations of lower class.

The solution $U^{\alpha}_{n}(x^1, \dots, x^n)$ with $1 \leq \alpha \leq \beta_1^{(n)}$ obtained in step n is then the sought for analytic solution of the complete original initial value problem (5.21, 5.22), as the remaining components of $\mathbf{u}$ are given by the initial conditions. The uniqueness of the solution follows from the construction. □

One may wonder whether one really needs the assumption that the system (5.21) is involutive; perhaps formal integrability suffices? In fact, the step by step construction used in the proof above is possible for *any* system in triangular form. But it is unclear whether the thus constructed solution also satisfies all equations of lower class. Our proof of Lemma 5.4.2 depends crucially on the involution of the system and it does not hold for arbitrary non-involutive systems, as the following example demonstrates.

Example 5.4.3. We consider the first-order equation

$$\mathcal{R}_1 \,:\, \begin{cases} v_t = w_x \ , \\ w_t = 0 \ , \\ v_x = 0 \ . \end{cases} \tag{5.29}$$

It arises when we rewrite the second-order equation (3.9) as a first-order equation by setting $w = u_t$ and $v = u_x$ (compare Appendix A.3). Thus we already know that $\mathcal{R}_1$ is formally integrable but not involutive. Indeed, the non-multiplicative prolongation $D_t(v_x)$ cannot be reduced to zero and we obtain the obstruction to involution $w_{xx} = 0$.

The crucial step in the proof above applied to this example would be to show that any solution of the normal subsystem comprising the first two equation of (5.29) that satisfies the third equation for $t = 0$ also satisfies it for all values of t. If (5.29) was involutive, this would indeed be the case by Lemma 5.4.2. But as we are only dealing with a formally integrable equation, it is easy to construct a counter example.

Consider the functions

$$\begin{aligned} v(t,x) &= \alpha'(x)t + c \ , \\ w(t,x) &= \alpha(x) \end{aligned} \tag{5.30}$$

where $c \in \mathbb{R}$ is an arbitrary constant and $\alpha(x)$ an arbitrary real-analytic function. These functions solve the first two equations of (5.29). Furthermore, for $t = 0$, they also solve the last equation. But obviously this is not the case for any other value of t, if we choose $\alpha(x)$ such that $\alpha''(x) \neq 0$. Only for

$\alpha(x) = ax + b$ with constants $a, b \in \mathbb{R}$ we obtain a solution of $\mathcal{R}_1$, as only for this choice of α the equation $w_{xx} = 0$ is satisfied. $\triangleleft$

In fact, this example can easily be generalised to a proof that involution is not only sufficient for Lemma 5.4.2 to be applicable at each step of the proof of Theorem 5.4.1 but also necessary. Formal integrability is trivially necessary; thus we restrict to systems which are formally integrable but not involutive. Without loss of generality, we may further assume that in the Cartan normal form (5.21) the subsystem of class n is normal and that an obstruction to involution arises from the prolongation of an equation of lower class with respect to x^n, as this is precisely the situation in which we apply Lemma 5.4.2 in the proof of Theorem 5.4.1.

As the subsystem of class n is normal, we can eliminate any x^n-derivative in the obstruction to involution. Thus it may be considered as a differential equation in $x^1, \dots, x^{n-1}$ which, by construction, is independent of the equations of lower class present in our system. In our example above this was the equation $w_{xx} = 0$. According to the assumptions of Lemma 5.4.2, we choose initial data for $x^n = 0$ satisfying all equations of lower class. But we can always find initial data that do *not* satisfy our obstruction to involution. If we solve the subsystem of class n for such initial data, for $x^n \neq 0$ the solution will *not* satisfy that equation of lower class that generated the obstruction to involution, as its x^n-prolongation does not vanish.

Thus, if (5.21) is only formally integrable but not involutive, then it does not possess a solution for arbitrary choices of the initial data in (5.22). Hence involution is a necessary condition in Theorem 5.4.1, as otherwise our initial value problem is not even formally well-posed. Furthermore, these considerations show that it is a characteristic property of involutive systems that Lemma 5.4.2 holds for $1 \leq k \leq n$.

The Cartan–Kähler Theorem inherits all the restrictions and disadvantages of the Cauchy–Kovalevskaya Theorem, as its proof is based on the latter. For arbitrary differential equations there seems to be no way to avoid this, as no other existence and uniqueness theorem for arbitrary normal equations exists. But if more is known about the structure of the subsystems considered in the proof above, one might be able to use other existence and uniqueness theorems and to obtain stronger results still following the basic strategy of our proof.

If we want to go beyond analytic solutions, a crucial step will be the extension of Lemma 5.4.2. We used the Cauchy–Kovalevskaya Theorem in its proof. But in fact all we need is to prove the uniqueness of the zero solution of (5.25). Note that this is always a *linear* system, even if (5.21) is non-linear. Thus one may apply the Holmgren Theorem (see the next section) instead of the Cauchy–Kovalevskaya Theorem and obtains that the zero solution is unique even among all C^1 solutions. However, this still requires that we are dealing with analytic equations. In order to be able to relax that condition one needs more information about the structure of the system (5.21). A simple example will be given in Section 5.6.

Finally, we continue the discussion of involutive equations with a Cohen–Macaulay symbol module started at the end of the last section. Assume first that we are dealing with a first-order equation and denote again by d the depth of the symbol module. In our proof of the Cartan–Kähler Theorem we considered the subsystems of constant class. If the symbol module is Cohen–Macaulay, each of them becomes normal, as soon as we set $x^{k+1}, \dots, x^n$ to constant values where k is the class of the subsystem. No further substitutions are necessary, as all initial conditions comprise only functions of $x^1, \dots, x^{d-1}$.

In the first step of the proof we study the subsystem of class d, as no equations of lower class are present in the Cartan normal form. Its initial conditions are identical with the initial conditions for the full equation $\mathcal{R}_q$. Thus we may take it for the normal system referred to in Proposition 5.3.8. The bijection between the formal solution spaces of the full system and this normal system is implicitly contained in the proof of the Cartan–Kähler Theorem. We extend the solution $\mathbf{U}(x^1, \dots, x^d)$ of the subsystem to the solution $\mathbf{u}(x^1, \dots, x^n)$ of the full system following step by step the proof. At each step the initial conditions are just the solutions of the previous step; as the symbol module is Cohen–Macaulay, no additional conditions appear. In other words, in (5.28) no functions f^α appear, as $\beta_1^{(k-1)} = \beta_1^{(k)}$. Thus the extension is uniquely determined by the subsystems of higher class; their solutions define the bijection.

The discussion of higher-order equations is similar but somewhat messy. The basic idea is to rewrite as a first-order equation. But all arising equations of class less than d are imposed only as initial conditions. In order to avoid excessive notations, we simply continue Example 5.3.9 instead of providing the general theory.

Example 5.4.4. Consider the local representation (5.19) of the involutive differential equation $\mathcal{R}_2$. It is already in Cartan normal form and splits into two subsystems of class 2 and 3, respectively. The leading derivatives of the subsystem of class 2 are u_{yy} and u_{yz}. This suggests to consider it as a normal first-order system for u_y and u_z. But unfortunately, the right hand sides are not yet of the right form. Thus we introduce the new dependent variables $r = u_x$, $s = u_y$ and $t = u_z$ and augment our system by the equations $u_y = s$ and $r_y = s_x$. Substituting $z = 0$ yields the normal system

$$\begin{aligned} t_y &= \phi_2(x, y, 0, u, r, s, t, r_x, s_x, t_x) \ , \\ s_y &= \phi_3(x, y, 0, u, r, s, t, r_x, s_x, t_x) \ , \\ r_y &= s_x \ , \quad u_y = s \ . \end{aligned} \tag{5.31a}$$

We prescribe the following initial conditions:

$$t(x, 0) = f(x) \ , \ s(x, 0) = g(x) \ , \ r(x, 0) = h'(x) \ , \ u(x, 0) = h(x) \ . \tag{5.31b}$$

This is just a trivial reformulation of (5.20). The initial data for r and u are not independent but imply $r = u_x$ for $y = z = 0$. By Lemma 5.4.2 the solution of the initial value problem also satisfies this relation.

As in the proof of the Cartan–Kähler Theorem we continue with the equations of class 3. In our case there is only one with leading derivative u_{zz}. We rewrite again as a first-order system

$$\begin{aligned} t_z &= \phi_1(x, y, z, u, r, s, t, r_x, s_x, t_x) \ , \\ s_z &= t_y \ , \quad r_z = t_x \ , \quad u_z = t \ . \end{aligned} \tag{5.32a}$$

As initial conditions we take the solution (T, S, R, U) of (5.31):

$$\begin{aligned} t(x, y, 0) &= T(x, y) \ , \quad s(x, y, 0) = S(x, y) \ , \\ r(x, y, 0) &= R(x, y) \ , \quad u(x, y, 0) = U(x, y) \ . \end{aligned} \tag{5.32b}$$

Note that the initial conditions are completely fixed; all arbitrariness is contained in (5.31b). It is obvious that the u-component of the solution of the initial value problem (5.32) solves the original system (5.19). Conversely, any solution of (5.19) induces one of (5.32). Thus we conclude that a one-to-one correspondence exists between the (formal) solutions of (5.19) and of the normal system (5.31a). The bijection is provided by the definition of the variables r, s, t and by the solution of (5.32a).

Basically, we have just rewritten the original second-order equation $\mathcal{R}_2$ as a first-order equation according to the method outlined in Appendix A.3. The main difference is that the equation $u_x = r$ does not appear in any system we consider. Indeed, it is easy to check that if we straightforwardly transform the original equation to first order, then the arising equation will *not* have a Cohen–Macaulay symbol module. The reason is precisely the appearance of the equation $u_x = r$. We have hidden it in the initial conditions (5.31b) and Lemma 5.4.2 ensures that it holds for all values of (x, y, z). ◁

5.5 The Holmgren Theorem

For linear systems it is possible to eliminate some of the restrictions of the Cauchy–Kovalevskaya and the Cartan–Kähler Theorem, respectively. Namely we can prove the uniqueness of continuous solutions of analytic equations. For normal systems, this is known as Holmgren's Theorem. Based on the proof of the Cartan–Kähler Theorem 5.4.1 presented in the last section, it can be easily extended to arbitrary involutive systems.

The basic idea is very simple; it is a bit more difficult to make it rigorous. We consider the following *normal* linear homogeneous system with analytic coefficients

$$A_i^{\alpha\beta}(\mathbf{x})p_i^\alpha + B^{\alpha\beta}(\mathbf{x})u^\alpha = 0\,, \qquad 1 \leq \beta \leq m \tag{5.33}$$

in a "lens-shaped" domain $\Omega \subset \mathbb{R}^n$ whose boundary consists of two non-characteristic hypersurfaces R and S (see Figure 5.1). We furthermore assume that S is analytic. The claim is that the initial value problem for (5.33) with $\mathbf{u}$ prescribed on R has at most one solution in $C^1(\overline{\Omega})$.

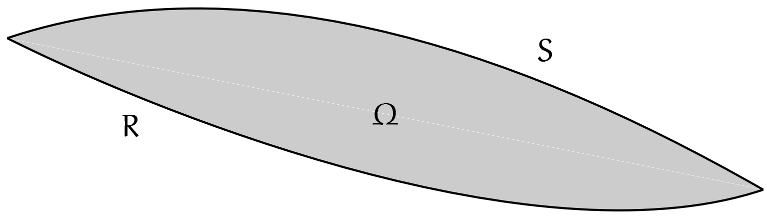

Fig. 5.1. A "lens-shaped" domain

An informal proof can be given as follow. We impose the initial conditions $\mathbf{u}|_R \equiv 0$. Then we add the equations in (5.33) with some yet undetermined functions $\bar{\mathbf{u}} \in C^1(\overline{\Omega})$ as coefficients and integrate over the domain Ω. A partial integration yields

$$\begin{aligned} 0 &= \int_\Omega \big(\bar{u}^\beta A_i^{\alpha\beta}p_i^\alpha + \bar{u}^\beta B^{\alpha\beta}u^\alpha\big)\,dx \\ &= \int_\Omega \Big(\Big[-\bar{p}_i^\beta A_i^{\alpha\beta}u^\alpha + \bar{u}^\beta \frac{\partial A_i^{\alpha\beta}}{\partial x^i}u^\alpha\Big] + \bar{u}^\beta B^{\alpha\beta}u^\alpha\Big)\,dx \,+ \\ &\quad \int_{\partial\Omega} \bar{u}^\beta A_i^{\alpha\beta}u^\alpha n_i\,dA \end{aligned} \tag{5.34}$$

where $\mathbf{n}$ is the outer unit normal to the hypersurface $\partial\Omega = R \cup S$ and $\bar{p}_i^\beta$ is the x^i-derivative of $\bar{u}^\beta$.

Let us assume that $\bar{\mathbf{u}}$ is a solution of the adjoint system

$$-A_i^{\alpha\beta}\bar{p}_i^\beta + \left[\frac{\partial A_i^{\alpha\beta}}{\partial x^i} + B^{\alpha\beta}\right]\bar{u}^\beta = 0\,, \quad 1 \le \alpha \le m \tag{5.35}$$

satisfying the initial conditions $\bar{u}|_S = f$ for some arbitrary continuous function f defined on S. Then (5.34) yields

$$\int_S f^\beta (A_i^{\alpha\beta} n_i) u^\alpha \, dA = 0\,. \tag{5.36}$$

Since we assumed that S is an analytic non-characteristic hypersurface, the term $\sum_{i=1}^n A_i^{\alpha\beta} n_i$ vanishes nowhere on S. Thus (5.36) can only hold for *all* continuous functions f, if $u|_S \equiv 0$.

The problem is to show that we always find such functions $\bar{u}$. As (5.33) is an analytic system, (5.35) is analytic, too. For analytic functions f we can thus apply the Cauchy–Kovalevskaya Theorem. By the Weierstraß Approximation Theorem any continuous function on a compact subset of $\mathbb{R}^n$ can be uniformly approximated by polynomials, so that it indeed suffices to treat analytic initial data, namely polynomials. However, the Cauchy–Kovalevskaya Theorem is a purely local result and we do not know whether the solution of the initial value problem for (5.35) exists in the whole domain Ω.

This forces us to proceed in the following rather technical fashion. We replace the single surface S by a one-parameter family S_λ that sweeps through the whole domain Ω which is now assumed to be of a somewhat special form. Let $\Omega' \subset \mathbb{R}^n$ be a bounded domain such that the coefficients of (5.33) are analytic in it. Let $\Psi(x)$ be an analytic function defined in Ω' such that $\nabla\Psi$ does not vanish anywhere. We set $Z = \{x \in \Omega' \mid x^n = 0\}$ and assume that it is an non-empty and non-characteristic hypersurface.

Now we introduce for $\lambda \in [a, b]$ the parametrised family of hypersurfaces $S_\lambda = \{x \in \Omega' \mid \Psi(x) = \lambda, x^n \ge 0\}$. We make the following assumptions (see Figure 5.2):

1. All $\lambda \in [a, b]$ are regular values of Ψ.
2. $\overline{\Omega} = \bigcup_{\lambda \in [a,b]} S_\lambda$.
3. $S_a \subset Z$ consists of a single point.
4. For each $\lambda \in (a, b]$ the hypersurfaces S_λ are non-characteristic and intersect Z transversally.

Theorem 5.5.1 (Holmgren). *Let $u \in C^1(\overline{\Omega})$ be a solution of* (5.33) *such that $u \equiv 0$ on $Z \cap \partial\Omega$. Then $u \equiv 0$ in $\overline{\Omega}$.*

Proof. We set $\Lambda = \{\lambda \in [a, b] \mid u|_{S_\lambda} \equiv 0\}$. The theorem is equivalent to $\Lambda = [a, b]$. By definition Λ is closed; we will show that it is also open in $[a, b]$, as then $\Lambda = [a, b]$.

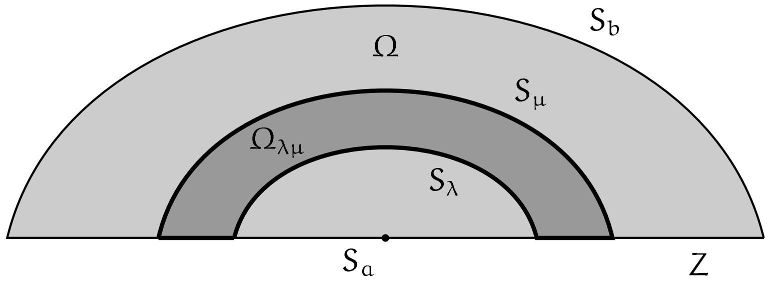

Fig. 5.2. A parametrised family of hypersurfaces

The functions $A_i^{\alpha\beta}$, $B^{\alpha\beta}$, Ψ are analytic and $\overline{\Omega}$ is compact; thus we can find values M, r independent of x such that all of them are in $\mathcal{C}_{M,r}(x)$ at each point $x \in \overline{\Omega}$. If we solve the initial value problem for (5.33) with initial data in $\mathcal{C}_{M,r}$ prescribed on the hypersurface S_λ for $\lambda \in (a, b]$, by the Cauchy–Kovalevskaya Theorem the unique analytic solution will exist in an ϵ-neighbourhood of S_λ where ϵ does not depend on λ.

Polynomials are analytic functions and thus elements of some class $\mathcal{C}_{\bar{M},\bar{r}}$. As polynomials possess only a finite number of non-vanishing derivatives, we can choose an arbitrarily large value for $\bar{r}$ provided we take $\bar{M}$ large enough. As (5.33) is a linear system, we can multiply any solution by a constant and obtain again a solution; this does not change the domain of definition of the solution. By multiplying a polynomial in $\mathcal{C}_{\bar{M},r}$ with $M/\bar{M}$ we obtain a polynomial in $\mathcal{C}_{M,r}$. Thus we can conclude that for any polynomial prescribed as initial data on S_λ the unique analytic solution of (5.35) exist in the whole ϵ-neighbourhood of S_λ.

Let $\lambda \in [a, b]$ and $\epsilon > 0$ be given. In the neighbourhood of some point $y \in S_\lambda$ we can apply the Implicit Function Theorem to solve the parametric equation $\Psi(x) = \mu$ for some coordinate x^i. This yields a function $x^i = \psi(x^1, \dots, x^{i-1}, x^{i+1}, \dots, x^n; \mu)$ which is continuous in μ. So we can choose a $\delta(y) > 0$ such that for all $\mu \in [a, b]$ with $|\lambda - \mu| < \delta(y)$ there exists a point $z \in S_\mu$ with $\|z - y\| < \epsilon$. As S_λ is compact, there exists a constant δ such that $0 < \delta \leq \delta(y)$ for all $y \in S_\lambda$. But this implies that for all $\mu \in [a, b]$ with $|\lambda - \mu| < \delta$ the hypersurface S_μ lies completely in an ϵ-neighbourhood of the hypersurface S_λ.

Now take a $\lambda \in \Lambda$ and a $\mu \in [a, b]$ such that $\mu \neq \lambda$ and $|\lambda - \mu| < \delta$. Then we can apply the argument above to the domain $\Omega_{\lambda\mu}$ bounded by the surfaces S_λ, S_μ and Z (recall that above we did not require that R is analytic, thus we may take S_λ plus parts of Z for the surface R). This time we can guarantee that the solution of the adjoint system (5.35) exists in the whole domain. Thus $\mathbf{u}|_{S_\mu} \equiv 0$ and $\mu \in \Lambda$. But this implies that Λ is open in $[a,b]$, as obviously $a \in \Lambda$ and thus Λ is not empty. □

This is not the strongest possible formulation of the Holmgren Theorem. Using, say, density arguments one can extend the uniqueness to more general spaces; Taylor [176] gives for example a version valid in Sobolev spaces. Note that we are still forced to require that the system is analytic, as we apply the Cauchy–Kovalevskaya Theorem at an intermediate step. But this is much less severe than the restriction to analytic solution which has now been removed.

An interesting aspect for us is that in the proof of Lemma 5.4.2 the decisive normal system (5.25) is always linear. Thus provided that its coefficients $\bar{A}^\beta_{ij}(\mathbf{x})$ and $\bar{B}^\beta_i(\mathbf{x})$ are analytic (for example, if A^β_{ij} and B^β_i in (5.24) do not depend on $\mathbf{u}$ and $\mathbf{p}$), we can prove Lemma 5.4.2 for much larger classes of solutions, as we can replace the use of the Cauchy–Kovalevskaya Theorem by the Holmgren Theorem.

Finally, we proceed to extend the Holmgren Theorem to arbitrary involutive systems. If the system (5.21) is linear, there exist analytic functions $A^{\alpha\delta}_k(\mathbf{x})$, $B^{\alpha\gamma}_{kj}(\mathbf{x})$, $C^{\alpha\beta}_k(\mathbf{x})$, $D^\alpha_k(\mathbf{x})$ such that

$$\begin{aligned}\phi^\alpha_k(\mathbf{x},\mathbf{u},p^\gamma_j,p^\delta_k) = \sum_{\delta=\beta^{(k)}_1+1}^{m} & A^{\alpha\delta}_k(\mathbf{x})p^\delta_k + \sum_{\gamma=1}^{m}\sum_{j=1}^{k-1} B^{\alpha\gamma}_{kj}(\mathbf{x})p^\gamma_j + \\ & + \sum_{\beta=1}^{m} C^{\alpha\beta}_k(\mathbf{x})u^\beta + D^\alpha_k(\mathbf{x})\,.\end{aligned} \tag{5.37}$$

Theorem 5.5.2. *The initial value problem* (5.21), (5.22) *with the right hand sides given by* (5.37) *possesses at most one solution in* $C^1(\overline{\Omega})$.

Proof. As for a normal system, it suffices to prove that in the case of a homogeneous system the only solution for vanishing initial data is the zero solution. Thus we study what happens in our proof of the Cartan–Kähler Theorem in this situation. In the first step we consider the normal system (5.26a) with vanishing initial data. Theorem 5.5.1 implies that the unique solution in $C^1(\overline{\Omega}\cap\{x^2=\cdots=x^n=0\})$ is $U^\alpha_1(x^1)\equiv 0$. Thus in the next step we have again vanishing initial data and can again apply Theorem 5.5.1. In this manner we can go through all steps and obtain finally that $\mathbf{u}\equiv 0$ is the unique solution in $C^1(\overline{\Omega})$. □

5.6 Applications: Weakly Overdetermined Systems

In the last section we only obtained a uniqueness result; in this section we want to prove the existence and uniqueness of *smooth* solutions for a special class of linear systems.

Definition 5.6.1. *An involutive differential equation* $\mathcal{R}_q$ *in* n *independent variables is called* weakly overdetermined, *if* $\beta_q^{(n-1)} > 0$ *and* $\beta_q^{(k)} = 0$ *for* $k < n-1$, *i. e. if any local representation in* δ*-regular coordinates contains only equations of class* n *and* $n-1$.

Note that such a weakly overdetermined equation might well be underdetermined in the sense of Definition 3.3.6. In the sequel, we are only interested in equations that can be interpreted in some sense as evolution equations. We slightly change our notation and denote the independent variables by $(x^1, \dots, x^n, t)$, i. e. we have $n+1$ variables and write x^{n+1} as t. We study now systems with variable coefficients of the following form:

$$E\mathbf{u}_t = A_i \mathbf{u}_{x_i} + B\mathbf{u} \,, \tag{5.38a}$$

$$0 = C_i \mathbf{u}_{x_i} + D\mathbf{u} \,. \tag{5.38b}$$

Here $\mathbf{u}$ is as usual the m-dimensional vector of dependent variables. The matrices $E(\mathbf{x}, t)$, $A_i(\mathbf{x}, t)$ and $B(\mathbf{x}, t)$ have q rows and m columns; the matrices $C_i(\mathbf{x}, t)$ and $D(\mathbf{x}, t)$ have p rows. We make three assumptions that are crucial for our analysis.

(a) The hypersurface $t = 0$ is non-characteristic.
(b) $\operatorname{rank} E(\mathbf{x}, t) \equiv q$.
(c) For at least one i we have $\operatorname{rank} C_i(\mathbf{x}, t) \equiv p$. Without loss of generality, we assume that this is the case for C_n.

All three assumptions together imply that we are dealing with a weakly overdetermined system in a δ-regular coordinate system. Note that (a) excludes parabolic systems. (b) implies that (5.38a) contains all equations of class $n+1$; (c) ensures that all equations in (5.38b) are of class n.

Example 5.6.2. An important example of a weakly overdetermined system are *Maxwell's equations*. It is well-known that their evolution part (1.64a), corresponding to (5.38a) form a hyperbolic system, whereas the constraint equations (1.64b), corresponding to (5.38b), are elliptic. $\triangleleft$

Our first task is to derive the conditions under which (5.38) represents an involutive system. This is a straightforward computation.

Lemma 5.6.3. *The system* (5.38) *is involutive, if and only if there exist* $p \times q$ *matrices* $M_i(x,t)$, $N(x,t)$ *and* $p \times p$ *matrices* $H_i(x,t)$, $K(x,t)$ *such that for all values* $1 \leq i, j \leq n$

$$M_i E = C_i \ , \tag{5.39a}$$
$$H_i C_j + H_j C_i = M_i A_j + M_j A_i \ , \tag{5.39b}$$
$$NE = D - M_k E_{x^k} \ , \tag{5.39c}$$
$$H_i D + K C_i + H_k C_{i,x^k} = M_i B + N A_i + M_k A_{i,x^k} + C_{i,t} \ , \tag{5.39d}$$
$$KD + H_k D_{x^k} = NB + M_k B_{x^k} + D_t \ . \tag{5.39e}$$

Proof. As the system is weakly overdetermined, non-multiplicative variables appear only in (5.38b), namely t is non-multiplicative for each equation. Thus we must try to express the t-prolongation of (5.38b) as linear combination of the original equations and their multiplicative prolongations. The matrices M_i, N, H_i, K are nothing but the coefficients of this linear combination. More precisely, a simple calculation shows that the formal equality

$$\partial_t(5.38b) = M_i \partial_{x^i}(5.38a) + H_i \partial_{x^i}(5.38b) + N(5.38a) + K(5.38b) \tag{5.40}$$

holds, if and only if the matrices satisfy the above conditions. □

The first two conditions are necessary for an involutive symbol; the remaining three conditions ensure the formal integrability of the system. Because of the rank assumptions (b) and (c), it is not difficult to see that if such matrices M_i, N, H_i, K exists, they are uniquely determined by (5.39). For later use, we derive the compatibility conditions of the linear system (5.38) under the assumption that it is involutive. For this purpose, we add on the right hand side of (5.38a) a "perturbation" δ and on the right hand side of (5.38b) a "perturbation" $-\epsilon$.

Lemma 5.6.4. *The perturbed system possesses a formal solution, if and only if the perturbations satisfy the compatibility condition*

$$\epsilon_t - H_i \epsilon_{x_i} - K\epsilon = M_i \delta_{x_i} + N\delta \ . \tag{5.41}$$

Proof. (5.41) follows immediately from (5.40). □

In order to obtain an existence and uniqueness theorem for smooth solutions, we need to make some further assumptions on the structure of (5.38). First of all, we want to avoid underdetermined situations. So we assume that $q = m$. This implies together with Assumption (b) that E is a regular matrix and without loss of generality, we take $E(x,t) = \mathbb{1}_m$, the $m \times m$ identity matrix. Then it follows from (5.39a) that $M_i = C_i$.

Recall the following classical definitions. (5.38a) with $E = \mathbb{1}_m$ is called *hyperbolic* in t-direction, if for any vector $\xi = (\xi^1, \dots, \xi^n) \in \mathbb{R}^n$ the matrix $A_\xi(x,t) = \xi^i A_i(x,t)$ has at any point (x,t) only real eigenvalues and an eigenbasis. It is *strongly hyperbolic*, if there exists for any $A_\xi(x,t)$ a symmetric, positive definite matrix $P_\xi(x,t)$, a *symmetriser*, depending smoothly on x, t and ξ such that $P_\xi A_\xi - A_\xi^t P_\xi = 0$. (5.38b) is *elliptic*, if the matrix $C_\xi(x,t) = \xi^i C_i(x,t)$ defines for all vectors $0 \neq \xi \in \mathbb{R}^n$ a surjective mapping. This condition implies that *all* matrices C_i must possess everywhere maximal rank: $\operatorname{rank} C_i(x,t) \equiv p$.

A classical result in the theory of strongly hyperbolic systems is the following existence and uniqueness theorem (cf. [98, Theorem 6.2.2]) which will form the basis of our existence and uniqueness theorem for (5.38). It provides in addition an estimate on the norm of the solution. In the case of a symmetric hyperbolic problem, i. e. $A = A^t$ and we do not need a symmetriser, we take the usual Sobolev norms $\|\mathbf{u}(x)\|_{H^p}$ defined as square root of the sum of the squares of the L^2 norm of $\mathbf{u}(x)$ and all its derivatives up to order p (note we consider here only functions of the spatial variables x).

In the general case we need a kind of weighted Sobolev norm where the "weight" consists of a pseudodifferential operator constructed out of the symmetriser $P_\xi(x,t)$. More precisely, we introduce the operator

$$\hat{P}(t)v(x) = \int P_{\xi/|\xi|}(x,t)e^{2\pi i \xi \cdot x}\hat{v}(\xi)\, d\xi \tag{5.42}$$

acting on functions depending only on the spatial variables. Here $\hat{v}$ is as usually the Fourier transform of v. Then we define the inner product

$$\big(\mathbf{v}(x), \mathbf{w}(x)\big)_{P(t)} = \int \mathbf{v}(x) \cdot \big(P(t)\mathbf{w}(x)\big)\, dx\ . \tag{5.43}$$

The norm $\|\mathbf{u}(x)\|_{P(t),H^p}$ is then a Sobolev norm where the underlying L^2 norm has been substituted by the norm induced by the inner product (5.43). There are some subtleties involved in the definition of this norm like the domain over which the integrals are taken; in general, one needs a partition of unity on this domain etc. We refer to [98] and references therein.

Theorem 5.6.5. *Let the system*

$$\mathbf{u}_t = A_i(x,t)\mathbf{u}_{x_i} + B(x,t)\mathbf{u} + F(x,t) \tag{5.44}$$

have smooth coefficients A_i, B, F *and be strongly hyperbolic with symmetriser* P_ξ. *Then it possesses a unique smooth solution satisfying periodic boundary conditions and the smooth initial conditions*

$$\mathbf{u}(x,0) = f(x)\ . \tag{5.45}$$

This solution satisfies at any time $t \in [0, T]$ *the following estimate*

$$\|u(\cdot,t)\|_{P(t),H^p} \le K_p \left[\|f\|_{P(t),H^p} + \int_0^t \|F(\cdot,\tau)\|_{P(t),H^p}\, d\tau\right] \tag{5.46}$$

where the constant K_p *depends only on the coefficients of the system, their derivatives up to order* p *and on* T.

Our goal is to extend this theorem to weakly overdetermined systems where (5.38a) is strongly hyperbolic and (5.38b) is elliptic. If we could apply Lemma 5.4.2, this would be straightforward. However, we are not dealing with analytic equations. Thus we must show that the lemma still holds in our more general situation. This can be done with the help of the compatibility condition (5.41) inheriting some properties of (5.38a).

Lemma 5.6.6. *Let* (5.38a) *with* $E = \mathbb{1}_m$ *be hyperbolic in* t*-direction and* (5.38b) *elliptic. Then the compatibility condition* (5.41) *— considered as a system for* ϵ *only — is hyperbolic in* t*-direction, too.*

Proof. As we assume that the system (5.38) is involutive, the matrices A_i, $M_i = C_i$, H_i satisfy the relations (5.39b). This implies (with obvious notation) that $H_\xi C_\xi = C_\xi A_\xi$ for any vector $\xi \in \mathbb{R}^n$. Let v be an eigenvector of A_ξ for the (real) eigenvalue λ. As $H_\xi C_\xi v = C_\xi A_\xi v = \lambda C_\xi v$, we see that $C_\xi v$ is an eigenvector of H_ξ for the eigenvalue λ.

According to our assumptions, all matrices A_ξ possess only real eigenvalues and an eigenbasis and all matrices C_ξ define surjective mappings. Thus there exists an eigenbasis of H_ξ consisting of vectors of the form $C_\xi v$ where v is an eigenvector of A_ξ. This implies furthermore that all eigenvalues of H_ξ are real, as they are also eigenvalues of A_ξ. □

Lemma 5.6.7. *Let the assumptions of Lemma 5.6.6 be satisfied and let furthermore* (5.38a) *be strongly hyperbolic with symmetriser* $P_\xi(x,t)$. *Then the compatibility condition* (5.41) *is strongly hyperbolic, too. As symmetriser we may use the matrix* $Q_\xi(x,t)$ *defined by*

$$Q_\xi^{-1} = C_\xi P_\xi^{-1} C_\xi^t\,. \tag{5.47}$$

Proof. We must show that Q_ξ satisfies the three properties of a symmetriser. Obviously, it inherits the symmetry of P_ξ. For the positive definiteness we note that $v^t Q_\xi^{-1} v = (C_\xi^t v)^t P_\xi^{-1} (C_\xi^t v)$. As C_ξ defines a surjective mapping, its transpose C_ξ^t defines an injective mapping. Thus the expression above vanishes only for $v = 0$. This also implies that Q_ξ^{-1} is indeed invertible. Finally, using again (5.39b)

$$\begin{aligned} H_\xi Q_\xi^{-1} - Q_\xi^{-1} H_\xi^t &= H_\xi C_\xi P_\xi^{-1} C_\xi^t - C_\xi P_\xi^{-1} C_\xi^t H_\xi^t \\ &= C_\xi (A_\xi P_\xi^{-1} - P_\xi^{-1} A_\xi^t) C_\xi^t . \end{aligned} \tag{5.48}$$

But this last expression vanishes, as P_ξ is a symmetriser for A_ξ. □

Based on these technical lemmata, it is straightforward to extend Theorem 5.6.5 to weakly overdetermined equations. The decisive point is that this theorem implies a uniqueness result for the compatibility condition (5.41).

Theorem 5.6.8. *Let* (5.38a) *with* $E = \mathbb{1}_m$ *be strongly hyperbolic and* (5.38b) *elliptic. If the initial data in* (5.45) *satisfies for* $t = 0$ *the equations* (5.38b)*, then the initial value problem* (5.38, 5.45) *with periodic boundary conditions possesses a unique smooth solution.*

Proof. Let us first ignore the constraints (5.38b). Then Theorem 5.6.5 guarantees the existence and uniqueness of a smooth solution. Entering this solution into the equations (5.38b) yields residuals ϵ. These residuals must satisfy the compatibility condition (5.41) with $\delta = 0$. As by assumption the initial data satisfy (5.38b) at $t = 0$, our initial conditions for (5.41) are $\epsilon(x, 0) = 0$. Obviously, $\epsilon \equiv 0$ is one solution of this initial value problem. According to Lemma 5.6.7, we are dealing with a strongly hyperbolic system, so that we can again apply Theorem 5.6.5. It guarantees that the zero solution is in fact the only solution. Hence there exists a unique smooth solution of the overdetermined problem (5.38, 5.45). □

Our method of proving this result has as side effect some implications for the numerical solution of a weakly overdetermined systems. A simple approach consists of considering the equations of lower class (5.38b) only as constraints on the initial data and otherwise ignoring them, i. e. one simply solves the initial value problem for (5.38a) with initial data satisfying (5.38b).

For exact solutions this approach is fine, as we have seen above. For numerical solutions the situation is somewhat different. In general, we must expect a drift off the constraints, i. e. after some time the numerical solution ceases to satisfy the equations of lower class (5.38b) (we noted this problem already for ordinary differential equations in Remark 5.1.2). For us this is easy to understand. A numerical solution is only an approximation of an exact solution, hence there appears a non-vanishing residual δ leading to a non-vanishing right hand side in the compatibility condition (5.41) which determines the evolution of the residual ϵ in the equations (5.38b). So the size of the drift can be estimated by (5.46):

$$\|\epsilon(\cdot, t)\|_{P(t),H^p} \leq K_p \int_0^t \|M_i \delta_{x_i}(\cdot, \tau) + N\delta(\cdot, \tau)\|_{P(t),H^p} \, d\tau \tag{5.49}$$

The problem is aggravated by the fact that the right hand side of the estimate depends not only on the residuals δ but also on their spatial derivatives. While any reasonable numerical method takes care of controlling the size of δ, it is hardly possible to control the size of the derivatives.

Example 5.6.9. The approach outlined above is fairly popular for numerically solving Maxwell's equations. Here the compatibility conditions are the continuity equations (1.65). It follows from their derivation that for Maxwell's equations the estimate (5.46) depends only on the divergence of δ and not on δ itself. Thus a good numerical method for Maxwell's equations should be constructed such that the divergence of the residual vanishes. $\triangleleft$

5.7 Notes

The geometric approach to the initial value problem for ordinary differential equations presented in Section 5.1 is not only of theoretical interest. Tuomela and Arponen [183] use it for the numerical integration of overdetermined equations. One motivation is that the prolonged curve $j_1\sigma$ in $J_1\mathcal{E}$ is often smoother than its projection σ in $\mathcal{E}$. In fact, one can show that many apparent singularities disappear this way [182]. While our proof of Theorem 5.1.5 was based on a local representation in semi-explicit form, this is not necessary for computations. Even if a fully non-linear representation is given, the distribution $\mathcal{D}$ always arises as the solution of a linear system. The only advantage of the semi-linear form is that it yields this system in the special form (5.4); but for numerical computations this is not relevant.

A similar geometric theory exists for partial differential equations and yields an alternative approach to the Cartan–Kähler Theorem. It is known under the name *Vessiot theory*, as it was introduced by Vessiot [189] as a dual form of the Cartan–Kähler theory based on differential forms. A formulation in modern language was given by Stormark [171]; for the special case of hyperbolic equations in the plane see also [187]. The theory becomes considerably more involved for partial differential equations, as the distribution $\mathcal{D}$ itself does not directly yield solutions. One has to search for transverse involutive subdistributions of dimension n.

The proof of the Cauchy–Kovalevskaya Theorem we have used is the one found in most textbooks (see e. g. [85, 149]) and essentially goes back to Goursat [63]. Cauchy gave a first proof in 1842 in a series of notes to the French academy of science [35–37]; later Kovalevskaya studied the problem in her thesis [95]. Note that Cauchy's method of majorants proves the convergence of the formal power series solutions indirectly by comparison with a known series. A direct convergence proof turns out to be surprisingly tricky; Shinbrot and Welland [166] provided a relatively simple one.

Several alternative proofs exist which do not use power series but functional analytic methods. The basic idea is to convert the initial value problem into a fixed point equation in some Banach space; a nice example for this technique can be found in [190]. These approaches have the advantage that they can also handle more general situations leading to "abstract" Cauchy–Kovalevskaya Theorems.

The Janet–Riquier Theory (see the Notes of Chapter 2) also provides a generalisation of the Cauchy–Kovalevskaya Theorem: *Riquier's Theorem*. It considers a formally well-posed initial value problem constructed with the help of an involutive basis. In contrast to our proof of the Cartan–Kähler Theorem, one tries to prove directly the convergence of formal power series. However, so far one could show the convergence only for degree compatible term orders. Thus we meet again the condition that each equation should be solved for a derivative of maximal order. We noted this already for the Cauchy–Kovalevskaya Theorem where it was even a necessary condition. Whether it is also necessary for Riquier's Theorem seems to be an open question.

Our proof of Holmgren's Theorem follows [149]. The basic trick using the adjoint system goes back to Holmgren [79]. Its proper global formulation with the one-parameter family S_λ of analytic surfaces is due to John [85]. The extension to arbitrary involutive systems seems to be new despite its simplicity; it was first presented in [159].

The notion of a weakly overdetermined system was introduced in [165]. That article studies only systems with constant coefficients where the situation is simpler. In this case one does not need the assumption that the constraints are elliptic, as one may simply apply the Holmgren Theorem for proving that the equations of lower class are satisfied.