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Involution

The formal theory of differential equations and its applications in computer algebra and numerical analysis

involution: anything internally complex or intricate

Oxford Advanced Learner's Dictionary of Current English

inuoluere, ML *involvere*, to envelop, roll up, wrap up, whence 'to *involve*'; pp *inuolutus*, ML *involutus*, yields both the tech adj, Geom n, *involute* and the L *inuolutio*, ML o/s *involution* whence E *involution*

Eric Partridge: *Origins: an etymological dictionary of modern English*

Preface

Mathematics is being lazy. Mathematics is letting the principles do the work for you so that you do not have to do the work for yourself.

George Pólya

The theory of differential equations is one of the largest fields within mathematics and probably any graduate in mathematics has attended at least one course on differential equations. But differential equations are also of fundamental importance in most applied sciences; whenever a continuous process is modelled mathematically, chances are high that differential equations appear. So it does not surprise that many textbooks exist on both ordinary and partial differential equations. But the huge majority of these books makes an implicit assumption on the structure of the equations: either one deals with scalar equations or with *normal* systems, i. e. with systems in Cauchy–Kovalevskaya form. The main topic of the present work is what happens, if this assumption is dropped.

This is not just an academic exercise; non-normal systems are ubiquitous in applications. Classical examples include the incompressible Navier–Stokes equations of fluid dynamics, Maxwell's equations of electrodynamics, the Yang–Mills equations of the fundamental gauge theories in modern particle physics or Einstein's equations of general relativity. But also the simulation and control of multibody systems, electrical circuits or chemical reactions lead to non-normal systems of ordinary differential equations, often called differential algebraic equations. In fact, most of the differential equations encountered by engineers and scientists are probably not normal.

In view of this great importance of non-normal systems, the lack of literature on their general theory is all the more surprising. Of course, much has been written on specific systems like the Navier–Stokes equations but the existence of general approaches to non-normal systems seems to be hardly known, although some of them were developed about a century ago!

We will study the so-called *formal theory of differential equations*. It merges ideas from the Janet–Riquier Theory of partial differential equations and the Cartan–Kähler Theory of exterior differential systems. This combines geometric and algebraic structures into a powerful framework useful not only for theoretical analysis but also for concrete applications. Compared with other books on the formal theory we put a stronger emphasis on computations. This will become most evident in the last chapter where concrete algorithms for the completion to involution are discussed. But also in the other chapters we try to be as constructive as possible.

The adjective "formal" has at least two different meanings. A simple one is that the origin of the theory was the problem of determining formal power series solutions for general systems of differential equations, i. e. of proving the formal integrability. In a broader sense, the word "formal" refers to the fact that we nowhere use solutions; all analysis is based solely on manipulations of the equations themselves.

As the title clearly indicates, our topic will be the notion of involution. Roughly speaking, the message is that whatever one intends to do with a differential equation, one should first render it involutive: *involution is the central principle in the theory of under- or overdetermined systems.*

Unfortunately, the term "involutive" is used in different approaches with somewhat different meanings. In fact, even within this book we will use it sometimes in a broader and sometimes in a narrower meaning. In the algebraic theory we introduce a fairly general notion of involutive bases. But within the geometric theory only the special case of Pommaret bases appears. In the case of differential equations, involution is often confused with formal integrability, i. e. with the mere absence of integrability conditions. But involution requires more and we will stress at several occasions the great importance of this "more".

The book contains six chapters and three appendices. The first chapter is concerned with the geometry behind the formal theory. It introduces the jet bundle formalism as the geometric foundation of differential equations. In fact, it contains two different introductions of jet bundles. The "pedestrian" one is based on the usual Taylor series approach and puts more emphasis on local coordinates. The "intrinsic" one uses an abstract construction stressing the affine structure of jet bundles. It does not often appear in the literature but turns out to be rather natural for the formal theory.

Then differential equations are intrinsically defined as fibred submanifolds of jet bundles and the basic geometric operations with them, prolongation and projection, are introduced. This leads to an intrinsic picture of integrability conditions and the notion of formal integrability. Finally, the geometric symbol of a differential equation is defined and related to the principal symbol well-known from standard partial differential equation theory.

While formal integrability is a very natural and intuitive concept, for many purposes it does not suffice. The first problem is already that no constructive criterion for formal integrability is known so far. But we will also see at several places that it has further shortcomings; for example even in the analytic category an existence and uniqueness theorem can be proven only for involutive equations.

So the second chapter introduces our main theme: involution. But it does so in a purely algebraic framework which, at first sight, is not at all related to differential equations. Involution is introduced for the Abelian monoid $\mathbb{N}_0^n$ and then extended to a fairly general class of algebras: the polynomial algebras of solvable type. This rather abstract formalism has the advantage that it does not only allow us to extract the simple idea underlying involution in a very clear form, it may also be applied without modifications in many different situations: polynomial ideals, rings of differential operators, universal enveloping algebras of Lie algebras, ...

We show that the fundamental idea behind the theory is the determination of combinatorial decompositions of a polynomial module. Involutive bases unite these decompositions with Gröbner bases, the central algorithmic tool in commutative algebra. Although fairly obvious, this point of view seems to be new. For us, the most important type of involutive bases are Pommaret bases, as these will later reappear in the theory of differential equations. They define a special kind of decomposition, namely Rees decompositions, existing only in generic coordinates. We present a new and very cheap criterion for δ-singularity solving the termination problem in completion algorithms.

The literature on involutive bases is almost exclusively concerned with their efficient construction. They are considered mainly because they offer an interesting alternative to Buchberger's algorithm for the determination of Gröbner bases. But the combinatorial decompositions induced by them contain much information about the structure of the ideal they generate. A trivial application is the determination of the Hilbert function. We will show that in addition one can straightforwardly read off a Pommaret basis the depth, the projective dimension and the Castelnuovo–Mumford regularity of their ideal. They also provide a simple realisation of Hironaka's criterion for Cohen-Macaulay modules. Most of these results follow from an involutive version of Schreyer's syzygy theorem presented here for the first time.

In the third chapter, the relation of the developed algebraic theory to differential equations is exhibited by showing that in local coordinates the (geometric) symbol defines at each point of the differential equation a polynomial module. Thus we finally arrive at the definition of an involutive differential equation where the symbol equations yield a Pommaret basis of this module. An important structural property of such an equation is the existence of the Cartan normal form for local representations.

As a first simple application of involution, we rigorously define with the help of the principal symbol under- and overdetermined equations. Such a definition makes sense only for involutive equations and it turns out that the classical counting rules comparing the number of equations and unknowns may be misleading. Rather surprisingly, we have not been able to find a completely sound definition in the literature. Finally, we discuss the differential version of the syzygy resolution studied in the last chapter, the Janet sequence. In particular, we solve the inverse problem which is of considerable importance in control theory.

The fourth chapter contains some "combinatorial games": abstract measures for the size of the formal solution space like the Hilbert polynomial or the Cartan characters. One may argue about their usefulness, but some classical notions like the number of degrees of freedom of a physical system are in fact based on such "games". We introduce the concept of a differential relation between two differential equations generalising Bäcklund transformations and study the induced relation between the respective Hilbert polynomials. For applications in physics, it is often of interest to formally subtract the effect of gauge symmetries. Within the framework of the formal theory, we present a new and fairly simple pseudogroup approach. Einstein introduced for similar purposes the strength of a differential equation. We relate it to the formal theory and show that it contains less information than the full set of Cartan characters.

As an important application of involution we prove in the fifth chapter some existence and uniqueness results. The jet bundle formalism may be understood as a geometric version of Taylor series, so it is not surprising that the simplest results are obtained for analytic solutions. We recall the Cauchy–Kovalevskaya Theorem, as the most general (with respect to the structure of the equation) existence and uniqueness result for analytic solutions of analytic normal systems.

One could try to directly extend its proof to involutive systems. Indeed, this approach is taken in the Janet–Riquier Theory leading to Riquier's theorem. We will use another approach leading to the Cartan–Kähler Theorem. Our proof is based on considering a sequence of normal systems to which the Cauchy–Kovalevskaya Theorem is applied; thus power series appear only indirectly in contrast to the proof of Riquier's theorem. It clearly demonstrates why formal integrability is not sufficient for the analysis of differential equations but the crucial tool of the Cartan normal form of an involutive equation is necessary.

Because of the use of the Cauchy–Kovalevskaya Theorem, the Cartan–Kähler Theorem inherits the restriction to the analytic category and this seriously limits its practical value. However, our proof provides us with a powerful technique for obtaining stronger results, if more is known about the structure of the equation. In such situations it is sometimes possible to substitute the Cauchy–Kovalevskaya Theorem by other results from the extensive literature on existence and uniqueness for normal systems. As a first example we generalise the Holmgren Theorem to involutive systems; despite its triviality this generalisation seems to be new. We also analyse the class of what we call weakly overdetermined systems; this class includes in particular Maxwell's equations. Here the above mentioned technique is used in order to obtain results on smooth solutions of hyperbolic systems with elliptic constraints.

The last chapter is mainly concerned with computational aspects. It studies the problem of completing an arbitrary differential equation to an involutive one and provides concrete algorithms that can be (and are) implemented in a computer algebra system. For ordinary differential equations, completion is a rather simple operation and the basic algorithm has been rediscovered many times in different fields. For partial differential equations, the Cartan–Kuranishi Theorem provides a very general geometric completion strategy which, however, has yet to be turned into an algorithm.

On the algebraic side we present a simple but nevertheless surprisingly efficient algorithm for the construction of involutive bases (some explanations for its efficiency follow from our structure theory of polynomial modules). For linear systems of partial differential equations, it is possible to combine it with the Cartan–Kuranishi strategy. The result is a new algebraic algorithm that is able to deliver intrinsic geometric results. Compared with earlier realisations of the Cartan–Kuranishi completion it is much faster, as due to a clever book-keeping much less prolongations must be computed.

Finally, as a completely new application of involution, we turn to the numerical treatment of non-normal differential equations. As already mentioned, in the case of ordinary differential equations one often speaks of differential algebraic equations. In their numerical analysis the notion of an index plays a fundamental role; roughly speaking an index measures how far away a differential algebraic equation is from a normal ordinary differential equation. This can be translated into how much trouble one has to expect in a numerical integration.

The literature on differential algebraic equations abounds with index definitions. We show that the formal theory does not only help us to get a better understanding of the meaning of the various index concepts, it even leads naturally to formulations that apply equally well to partial differential equations. In fact, we obtain the first index concepts for partial differential equations that do not require integral transforms, semi-discretisations or any other technique of reduction to a differential algebraic equation.

For the special case of linear systems with constant coefficients we furthermore study semi-discretisations transforming overdetermined partial differential equations into differential algebraic equations. Probably one can nowhere see the importance of involution better than in this application. One might expect that if the original partial differential equation does not possess any hidden integrability conditions, then the same should be true for the arising ordinary differential equations. However, this is not the case: we must require that the partial differential equation is involutive! Thus these abstract combinatorial decompositions of polynomial modules acquire suddenly a very concrete meaning as in semi-discretisations obstructions to involution turn into integrability conditions.

As one can see, we cover rather diverse topics. Somebody reading the whole text needs a certain familiarity with different branches of mathematics: jet bundles and differential geometry, commutative and computer algebra, and finally some numerical analysis. Not all readers will have this background or actually be interested in all these topics. This has been taken into account in a two-fold way.

Three appendices collect some basic results from analysis, algebra and differential geometry that are used in the main text. They try to make the text to a reasonable degree self-contained and fix some notations. But of course they cannot substitute a real introduction into the subjects involved. For this reason, references to standard textbooks are given.

Many chapters and sections are rather independent of each other. Depending on the interests of a reader many parts may be safely ignored. For example, somebody only interested in the algebraic theory of involutive bases needs only Chapters 2 and 6. On the other hand, somebody who does not like commutative algebra may ignore these two chapters. For the benefit of readers not so familiar with differential geometry most geometric constructions are described both in an intrinsic manner and in local coordinates. Furthermore, many concrete examples are included to illustrate the theoretical results.

Each chapter ends with a brief section called *Notes*. It usually gives some pointers to the literature and to alternative approaches. I have also tried to trace a bit the history of the main ideas. But as I am not a historian of science, the results should be more taken as rough indications than as a definite answer. Furthermore, in several places I rely on other sources and I have not always checked the correctness of the references. A valuable source of information on the state of the art of the theory of differential equations at the end of the 19th century is the article of Weber [192] in the *Enzyklopädie der Mathematischen Wissenschaften*.

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