OPEN COVERS AND LEX POINTS OF HILBERT SCHEMES OVER QUOTIENT RINGS VIA RELATIVE MARKED BASES

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Abstract. We introduce the notion of a *relative marked basis* over quasi-stable ideals, together with constructive methods and a functorial interpretation, developing computational methods for the study of Hilbert schemes over quotients of polynomial rings. Then we focus on two applications.

The first has a theoretical flavour and produces an explicit open cover of the Hilbert scheme when the quotient ring is Cohen-Macaulay on quasi-stable ideals. Together with relative marked bases, we use suitable *general* changes of variables which preserve the structure of the quasi-stable ideal, against the expectations.

The second application has a computational flavour. When the quotient rings are Macaulay-Lex on quasi-stable ideals, we investigate the lex-point of the Hilbert schemes and find examples of both smooth and singular lex-points.

Introduction

Let $R = \mathbb{K}[x_0, \dots, x_n]$ be the polynomial ring over a field $\mathbb{K}$ in $n+1$ variables, endowed with the order $x_0 < \cdots < x_n$, and I be an ideal of R.

We provide a way to analyse Hilbert schemes over a quotient ring R/I using a computer algebra system. The tools that we develop are based on the theory of marked bases over quasi-stable ideals, together with their properties and functorial features [6, 4, 3]. However, the different setting that is considered in this paper presents new problems to solve.

We apply our tools to achieve two different tasks under the hypothesis that I is a monomial quasi-stable ideal. The first one concerns the study of a suitable open cover of such a Hilbert scheme when R/I is Cohen-Macaulay. The second one regards the study of lex-points when R/I is a Macaulay-Lex ring, i.e. a ring in which an analogue of Macaulay's Theorem characterizing the Hilbert functions of homogeneous ideals in a polynomial ring holds.

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We start giving a first non-obvious insight in the application of marked bases to quotient polynomial rings (Section 2). Then we push forward our investigation and look for results analogous to those for relative Gröbner bases and relative involutive bases of ideals in quotient rings that have been recently developed in [13]. Hence, we define *relative marked bases*, which turn out to be suitable to work in R/I, in particular when I is a quasi-stable ideal, and study their functorial interpretation (Sections 3 and 4).

An immediate consequence of this study is that we can construct some open subschemes of the Hilbert scheme over R/I by means of relative marked bases, too (see Proposition 2.6 and Theorem 4.2).

When the field $\mathbb{K}$ is infinite and the quotient ring on a quasi-stable ideal is Cohen-Macaulay, we develop this feature to describe an open cover of Hilbert schemes over such a quotient ring, which we obtain thanks to suitable changes of variables applied on open subsets which parameterise relative marked bases (see Theorem 5.11). This result is achieved generalising the method which is described in [3] and which is based on deterministically computable suitable linear changes of variables.

The novelty of this approach consists in the fact that we show that there are computable general linear changes of variables of the quotient ring that by definition preserve the complete structure of the ideal on which the quotient is performed, instead of destroying it, as it could be expected (e.g., see [21, Introduction]). Hence, this result is not obvious and, together with the underlying idea, is new in the context of the present paper.

Even when $\mathbb{K}$ is not infinite or the quotient ring is not Cohen-Macaulay, the availability of open subschemes described by means of the relative marked bases encourages the study of local properties. For example, when the quotient ring is Macaulay-Lex, the explicit computation of relative marked schemes can be useful in the investigation of the properties of the lex-point of Hilbert schemes over such quotient rings.

It is indeed very well-known that every non-empty Hilbert scheme over a polynomial ring on a field has a unique point, called the *lex-point*, that is defined by a lex-ideal, and which is smooth (see [24]) and characterized by the property that its defining saturated lex-ideal has the minimal possible Hilbert function among the points of the same Hilbert scheme.

It is even true that every non-empty Hilbert scheme over a Macaulay-Lex ring on a quasi-stable ideal has the lex-point, which moreover has the minimal Hilbert function (Theorem 6.6). However, in Section 6 we give classes of examples both of smooth and singular lex-points in Macaulay-Lex rings over quasi-stable ideals.

The problem of the smoothness of the lex-point is studied also in other Hilbert schemes, see for instance [23]. In fact, it is not possible to extend the proof for the smoothness of the lex-point of the Hilbert scheme $\mathbf{Hilb}^{p(z)}_{\mathbb{P}^n}$ given in [24] to other cases because the Zariski tangent space is not the same (see [10, Section 1.7] and [22, Proposition 2.1]).

To the best of our knowledge, analogous examples are not yet available in the literature. Moreover, the benefits obtained by the use of relative marked bases are evident when we count the number of parameters involved in the computations (see Remark 4.4), as we highlight throughout the descriptions of some of our examples.

1. Preliminaries

Let $\mathbb{K}$ be a field and A any Noetherian $\mathbb{K}$-algebra with $1_A = 1_{\mathbb{K}}$. Take the polynomial ring $R = \mathbb{K}[x_0, \dots, x_n]$ endowed with the order $x_0 < \cdots < x_n$, and $R_A := R \otimes_A A = A[x_0, \dots, x_n]$, so that $R = R_{\mathbb{K}}$. A *term* is a power product $x^\alpha = x_0^{\alpha_0} \cdots x_n^{\alpha_n}$. We denote by $\mathbb{T}$ the set of terms. For every $x^\alpha \in \mathbb{T}$, we denote by $\min(x^\alpha)$ the smallest variable dividing x^α and by $\mathcal{X}(x^\alpha)$ the set of the variables smaller than or equal to $\min(x^\alpha)$ which is called the set of *multiplicative variables* of x^α. If N is a finite set of polynomials, we denote by $\langle N \rangle_A$ the A-module generated by N, and by (N) the ideal generated by N in R_A.

We use the standard grading on R_A, that is $\deg(x_j) = 1$ for all $j \in \{0, \dots, n\}$ and $\deg(a) = 0$ for all $a \in A$. Hence we have $\deg(x^\alpha) = |\alpha| = \sum \alpha_i$. We assume that the polynomials, the ideals and A-modules involved in our definitions, statements and arguments are *homogeneous* with respect to this standard grading on R_A.

For an ideal I, we denote by I_t the vector space of the homogeneous polynomial of I of a given degree t and set $I_{\geq t} := \bigoplus_{s \geq t} I_s$.

When we write an equality of the kind $I = B_1 \oplus B_2$, where I is an ideal and B_1, B_2 are A-modules or ideals, we also mean $I_s = (B_1)_s \oplus (B_2)_s$ for every $s \geq 0$. In such situations, we will say that the equality is *graded*.

An ideal $\tilde{I}$ is *monomial* if it is generated by a set of terms. A monomial ideal $\tilde{I}$ has a unique minimal set of generators consisting of terms and we call it the *monomial basis* of $\tilde{I}$, denoted by $\mathcal{B}_{\tilde{I}}$. We define $\mathcal{N}(\tilde{I}) \subseteq \mathbb{T}$ as the set of terms in $\mathbb{T}$ not belonging to $\tilde{I}$. For every polynomial $f \in R_A$, $\mathrm{supp}(f)$ is the set of terms appearing in f with a non-zero coefficient. For every polynomial $f \in R_A$, an *x-coefficient* of f is the coefficient in A of a term in $\mathbb{T} \cap \mathrm{supp}(f)$.

Definition 1.1. For every x^α in $\mathbb{T}$, we define the *Pommaret cone* of x^α as

$$\mathcal{C}_{\mathcal{P}}(x^\alpha) := \{x^\delta x^\alpha \mid \delta_i = 0, \forall x_i \notin \mathcal{X}(x^\alpha)\} \subset \mathbb{T}.$$

A finite set U of terms generating an ideal $\tilde{I}$ is called a *Pommaret basis* of $\tilde{I}$ if

$$\tilde{I} \cap \mathbb{T} = \bigsqcup_{x^\alpha \in U} \mathcal{C}_{\mathcal{P}}(x^\alpha). \tag{1.1}$$

A *quasi-stable* ideal is a monomial ideal having a Pommaret basis. If $\tilde{I}$ is a quasi-stable ideal, we denote by $\mathcal{P}_{\tilde{I}}$ its Pommaret basis.

Proposition 1.2 ([26, Theorem 9.2], [27, Theorem 5.5.15])**.** *If $\tilde{I}$ is a quasi-stable ideal, then $\mathcal{B}_{\tilde{I}} \subseteq \mathcal{P}_{\tilde{I}}$ holds and the regularity* $\mathrm{reg}(\tilde{I})$ *coincides with the maximum degree of a term in $\mathcal{P}_{\tilde{I}}$.*

2. Marked bases and marked functors

2.1. **Marked bases.** A *marked polynomial* is a polynomial $f \in R_A$ together with a fixed term $x^\alpha \in \mathrm{supp}(f)$ whose coefficient is equal to 1_A (see [25]). This term is called *head term* of f and denoted by $\mathrm{Ht}(f)$.

From now, let $\tilde{I}$ denote a quasi-stable ideal.

Definition 2.1. [6, Definition 5.1] A $\mathcal{P}_{\tilde{I}}$*-marked set* is a finite set $F \subset R_A$ of exactly $|\mathcal{P}_{\tilde{I}}|$ marked homogeneous polynomials f_α with pairwise distinct head terms $\mathrm{Ht}(f_\alpha) = x^\alpha \in \mathcal{P}_{\tilde{I}}$ and $\mathrm{supp}(f_\alpha - x^\alpha) \subset \langle \mathcal{N}(\tilde{I})\rangle_A$. A $\mathcal{P}_{\tilde{I}}$-marked set F is a $\mathcal{P}_{\tilde{I}}$*-marked basis* of the ideal (F) if the graded decomposition $(R_A) = (F) \oplus \langle \mathcal{N}(\tilde{I})\rangle_A$ holds.

Definition 2.2. [6, Definition 5.3] Given a $\mathcal{P}_{\tilde{I}}$-marked set $F = \{f_\alpha\}_{x^\alpha \in \mathcal{P}_{\tilde{I}}}$, the set $F^* := \{x^\eta f_\alpha \mid x^\eta x^\alpha \in \mathcal{C}_{\mathcal{P}}(x^\alpha)\} \subseteq (F)$ is made of homogeneous polynomials which are marked on the terms of $\tilde{I}$ in the natural way $\mathrm{Ht}(x^\eta f_\alpha) = x^\eta \mathrm{Ht}(f_\alpha)$.

We denote by $\longrightarrow_{F^*}$ the reflexive and transitive closure of the following reduction relation on R_A: f is in relation with f' if $f' = f - \lambda x^\eta f_\alpha$, where $x^\eta f_\alpha \in F^*$ and $\lambda \neq 0_A$ is the coefficient of the term $x^{\eta+\alpha}$ in f.

We will write $f \longrightarrow^+_{F^*} f_0$ if $f \in R_A$, $f \longrightarrow_{F^*} f_0$ and $f_0 \in \langle \mathcal{N}(\tilde{I})\rangle_A$. In this case we say that "f is reduced to f_0 by F^*", and that "f_0 is reduced with respect to F^*".

It is noteworthy that the reduction relation $\longrightarrow_{F^*}$ is Noetherian and confluent (see [7, Theorem 5.9 and and Corollary 5.11]).

Lemma 2.3. *Let E be any subset of a $\mathcal{P}_{\tilde{I}}$-marked set F. Then, letting $E^* = \{x^\eta p_\beta \mid x^\eta x^\beta \in \mathcal{C}_{\mathcal{P}}(x^\beta), p_\beta \in E\}$, the subreduction relation $\longrightarrow_{E^*}$ is Noetherian and confluent.*

Proof. The reduction relation $\longrightarrow_{E^*}$ is obviously defined as a subreduction of $\longrightarrow_{F^*}$, like it is suggested in [7, Definition 3.4]. Then, the reduction relation $\longrightarrow_{E^*}$ is Noetherian because it is a subreduction of $\longrightarrow_{F^*}$, which is Noetherian. Moreover, $\longrightarrow_{E^*}$ is confluent because it is Noetherian and has disjoint cones (see also [7, Remark 7.2]). ☐

Remark 2.4. Observe that, in the hypotheses of Lemma 2.3, we write $p \longrightarrow^{+}_{E^*} h$ when $\mathrm{supp}(h)$ is included in $\mathbb{T} \setminus \left(\cup_{x^\beta \in \{\mathrm{Ht}(p) | p \in E\}} \mathcal{C}_{\mathcal{P}}(x^\beta)\right)$, according to Definition 2.2.

2.2. **Marked functors.** It is possible to parameterise the set of ideals I having a $\mathcal{P}_{\tilde{I}}$-marked basis by means of a functor from the category of Noetherian $\mathbb{K}$-Algebras to that of Sets, which turns out to be represented by an affine scheme. We briefly recall the definition of this functor and the construction of this affine scheme.

The *marked functor* from the category of Noetherian $\mathbb{K}$-algebras to the category of sets

$$\underline{\mathbf{Mf}}_{\tilde{I}} : \underline{\text{Noeth } \mathbb{K}\text{–Alg}} \longrightarrow \underline{\text{Sets}}$$

associates to any Noetherian $\mathbb{K}$-algebra A the set

$$\underline{\mathbf{Mf}}_{\tilde{I}}(A) := \{(G) \subset R_A \mid G \text{ is a } \mathcal{P}_{\tilde{I}}\text{-marked basis}\}$$

and to any morphism of $\mathbb{K}$-algebras $\sigma : A \to A'$ the map

$$\begin{array}{rccc} \underline{\mathbf{Mf}}_{\tilde{I}}(\sigma) : & \underline{\mathbf{Mf}}_{\tilde{I}}(A) & \longrightarrow & \underline{\mathbf{Mf}}_{\tilde{I}}(A') \\ & (G) & \longmapsto & (\sigma(G)) . \end{array}$$

Note that the image $\sigma(G)$ under this map is indeed again a $\mathcal{P}_{\tilde{I}}$-marked basis, as we are applying the functor $- \otimes_A A'$ to the decomposition $(R_A)_s = (G)_s \oplus \langle \mathcal{N}(\tilde{I})_s \rangle_A$ for every degree s.

Remark 2.5. Generalising [16, Proposition 2.1] to quasi-stable ideals, we obtain

$$\{(G) \subset R_A \mid G \text{ is a } \mathcal{P}_{\tilde{I}}\text{-marked basis}\} = \{I \subset R_A \text{ ideal } \mid R_A = I \oplus \langle \mathcal{N}(\tilde{I}) \rangle_A\}.$$

The functor $\underline{\mathbf{Mf}}_{\tilde{I}}$ is represented by the affine scheme $\mathbf{Mf}_{\tilde{I}}$ that can be explicitly constructed by the following procedure. We consider the $\mathbb{K}$-algebra $\mathbb{K}[C]$, where C denotes the finite set of variables $\left\{C_{\alpha\eta} \mid x^\alpha \in \mathcal{P}_{\tilde{I}}, x^\eta \in \mathcal{N}(\tilde{I}), \deg(x^\eta) = \deg(x^\alpha)\right\}$, and construct the $\mathcal{P}_{\tilde{I}}$-marked set $\mathscr{G} \subset R_{\mathbb{K}[C]}$ consisting of the following marked polynomials

$$g_\alpha = x^\alpha - \sum_{x^\eta \in \mathcal{N}(\tilde{I})_{|\alpha|}} C_{\alpha\eta} x^\eta \tag{2.1}$$

with $x^\alpha \in \mathcal{P}_{\tilde{I}}$. According to Definition 2.2, we consider

$$\mathscr{G}^* = \{x^\delta g_\alpha \mid g_\alpha \in \mathscr{G}, x^\delta x^\alpha \in \mathcal{C}_{\mathcal{P}}(x^\alpha)\}.$$

Then, by the Noetherian and the confluent reduction procedure given in Definition 2.2, for every term $x^\alpha \in \mathcal{P}_{\tilde{I}}$ and every variable x_i not belonging to $\mathcal{X}(x^\alpha)$, we compute a polynomial $p_{\alpha,i} \in \langle \mathcal{N}(I)_{|\alpha|+1}\rangle_A$ such that $x_i g_\alpha - p_{\alpha,i} \in \langle \mathscr{G}^*\rangle_A$. We then denote by $\mathscr{U}$ the ideal generated in $\mathbb{K}[C]$ by the x-coefficients of the polynomials $p_{\alpha,i}$. Then, we have $\mathbf{Mf}_{\tilde{I}} = \mathrm{Spec}(\mathbb{K}[C]/\mathscr{U})$ ([6, Remark 6.3],[3, Theorem 5.1]).

From now we assume that $\tilde{I}$ is in particular a *saturated* quasi-stable ideal. This implies that x_0 does not divide any term of $\mathcal{B}_{\tilde{I}}$ and $R/\tilde{I}$ has positive Krull dimension. Then $\tilde{I}_{\geq t}$ is quasi-stable too, for every integer t, so that we can consider $\mathbf{Mf}_{\tilde{I}_{\geq t}}$. Let $\tilde{p}(z)$ be the Hilbert polynomial of $R/\tilde{I}$ and $\mathbf{Hilb}^{\tilde{p}(z)}_{\mathbb{P}^n}$ be the Hilbert scheme that parameterises the closed subschemes of $\mathbb{P}^n$ having Hilbert polynomial $\tilde{p}(z)$. Then, $\mathbf{Mf}_{\tilde{I}_{\geq t}}$ embeds in $\mathbf{Hilb}^{\tilde{p}(z)}_{\mathbb{P}^n}$, for every integer t (see [4, Proposition 6.13]).

2.3. **Parametrisation of saturated ideals in quotient rings.** Let I now be a *saturated* ideal with a $\mathcal{P}_{\tilde{I}}$-marked basis and take $X := \mathrm{Proj}(R/I)$ together with its Hilbert polynomial $p_X(z)$, which is equal to $\tilde{p}(z)$.

Let $\tilde{J}$ be a *saturated* quasi-stable ideal containing $\tilde{I}$ and let $p(z)$ be the Hilbert polynomial of $R/\tilde{J}$. Then, $p(z)$ is *smaller* than $p_X(z)$, in the sense that $p(t) \leq p_X(t)$, for $t \gg 0$. Let $\mathbf{Hilb}^{p(z)}_X$ be the Hilbert scheme that parameterises the closed subschemes of $X = \mathrm{Proj}(R/I)$ with Hilbert polynomial $p(z)$ and represents the Hilbert functor $\underline{\mathbf{Hilb}}^{p(z)}_X$.

If $\mathcal{P}_{\tilde{J}}$ does not contain any term divisible by x_1, we set $\rho_{\tilde{J}} := 1$. Otherwise, we set $\rho_{\tilde{J}} := \max\{\deg(x^\alpha) \mid x^\alpha \in \mathcal{P}_{\tilde{J}} \text{ is divisible by } x_1\}$.

Proposition 2.6. *With the above notations,*

(1) for every $t \geq \rho_{\tilde{J}} - 1$, $\mathbf{Mf}_{\tilde{J}_{\geq t}} \cap \mathbf{Hilb}^{p(z)}_X \cong \mathbf{Mf}_{\tilde{J}_{\geq t+1}} \cap \mathbf{Hilb}^{p(z)}_X$;

(2) for every $t \geq \rho_{\tilde{J}} - 1$, $\mathbf{Mf}_{\tilde{J}_{\geq t}} \cap \mathbf{Hilb}^{p(z)}_X$ *is an open subscheme of* $\mathbf{Hilb}^{p(z)}_X$.

Proof. We recall that $\mathbf{Hilb}^{p(z)}_X$ is a (closed) subscheme of $\mathbf{Hilb}^{p(z)}_{\mathbb{P}^n}$ (e.g. [9, Exercise VI-26]) and that $\mathbf{Mf}_{\tilde{J}_{\geq t}}$ is a (locally closed) subscheme of $\mathbf{Hilb}^{p(z)}_X$, for every integer t (see [4, Proposition 6.13(iii)]). Then, the first item is a consequence of [4, Corollary 6.11] which states $\mathbf{Mf}_{\tilde{J}_{\geq t}} \cong \mathbf{Mf}_{\tilde{J}_{\geq t+1}}$, for every $t \geq \rho_{\tilde{J}} - 1$. The second item is a consequence of the fact that, for every $t \geq \rho_{\tilde{J}} - 1$, $\mathbf{Mf}_{\tilde{J}_{\geq t}}$ is even an open subscheme of the Hilbert scheme $\mathbf{Hilb}^{p(z)}_{\mathbb{P}^n}$. Indeed, $\underline{\mathbf{Mf}}_{\tilde{J}_{\geq t}}$ is an open subfunctor of $\underline{\mathbf{Hilb}}^{p(z)}_X$, like it is stated and proved in [4, Proposition 6.13(ii)]. ∎

Example 2.7. If we consider the ideal $\tilde{J} := (x_3^2, x_3x_2)$, then $\rho_{\tilde{J}} = 1$, hence $\tilde{J} = \tilde{J}_{\geq \rho_{\tilde{J}}-1}$ and $\mathbf{Mf}_{\tilde{J}} \cong \mathbf{Mf}_{\tilde{J}_{\geq t}}$, for every $t \geq 0$. Starting from the following two marked polynomials:

$$x_2x_3 + x_0^2c_1 + x_0x_1c_2 + x_1^2c_3 + x_0x_2c_4 + x_1x_2c_5 + x_2^2c_6 + x_0x_3c_7 + x_1x_3c_8,$$
$$x_3^2 + x_0^2c_9 + x_0x_1c_{10} + x_1^2c_{11} + x_0x_2c_{12} + x_1x_2c_{13} + x_2^2c_{14} + x_0x_3c_{15} + x_1x_3c_{16},$$

the defining ideal $\mathcal{U} \subset \mathbb{Q}[c_1, \dots, c_{16}]$ of $\mathbf{Mf}_{\tilde{J}}$ is computed obtaining

$\mathcal{U} = (c_1c_6c_7 - c_1c_4 - c_7c_9 + c_1c_{15}, c_2c_6c_7 + c_1c_6c_8 - c_2c_4 - c_1c_5 - c_8c_9 - c_7c_{10} + c_2c_{15} + c_1c_{16}, c_3c_6c_7 + c_2c_6c_8 - c_3c_4 - c_2c_5 - c_8c_10 - c_7c_{11} + c_3c_{15} + c_2c_{16}, c_4c_6c_7 - c_4^2 - c_1c_6 - c_7c_{12} + c_4c_{15} - c_9, c_5c_6c_7 + c_4c_6c_8 - 2c_4c_5 - c_2c_6 - c_8c_{12} - c_7c_{13} + c_5c_{15} + c_4c_{16} - c_{10}, c_6^2c_7 - 2c_4c_6 - c_7c_{14} + c_6c_{15} - c_{12}, c_6c_7^2 - c_4c_7 + c_1, c_3c_6c_8 - c_3c_5 - c_8c_{11} + c_3c_{16}, c_5c_6c_8 - c_5^2 - c_3c_6 - c_8c_{13} + c_5c_{16} - c_{11}, c_6^2c_8 - 2c_5c_6 - c_8c_{14} + c_6c_{16} - c_{13}, 2c_6c_7c_8 - c_5c_7 - c_4c_8 + c_2, c_6c_8^2 - c_5c_8 + c_3, -c_6^2 - c_{14})$.

Although $\tilde{J} \neq \tilde{J}_{\geq 3}$ and the defining ideal $\mathcal{U}'$ of $\mathbf{Mf}_{\tilde{J}_{\geq 3}}$ is contained in the ring $\mathbb{Q}[c_1, \dots, c_{91}]$ and so it is different from $\mathcal{U}$, the ring $\mathbb{Q}[c_1, \dots, c_{16}]/\mathcal{U}$ is isomorphic to the ring $\mathbb{Q}[c_1, \dots, c_{91}]/\mathcal{U}'$ by [4, Corollary 6.11]. If we consider instead the ideal $\tilde{J} := (x_2^3, x_1)$ with Pommaret basis $\mathcal{P}_{\tilde{J}} = \{x_2^3, x_1, x_1x_2, x_1x_2^2\}$, then $\rho_{\tilde{J}} = 1$ like before. Also in this case we have $\tilde{J} = \tilde{J}_{\geq \rho_{\tilde{J}}-1}$ and $\mathbf{Mf}_{\tilde{J}} \cong \mathbf{Mf}_{\tilde{J}_{\geq t}}$, for every $t \geq \rho_{\tilde{J}} - 1 = 0$.

Lemma 2.8. *With the notation above, let F be any set of polynomials generating I. If G is a $\mathcal{P}_{\tilde{J}_{\geq t}}$-marked basis for some $t \geq \rho_{\tilde{J}} - 1$, then the following statements are equivalent:*

(i) $(G)^{sat} \supseteq I$;

(ii) $(G) \supseteq I_{\geq t}$;

(iii) $(G) \supseteq \{x_0^{\max\{0, t-\deg(f)\}} f \mid f \in F\}$.

Proof. By [4, Theorem 3,5 and Corollary 3.7] we have $(G) = {(G)^{sat}}_{\geq t}$ and $(G)^{sat} = ((G) : x_0^\infty)$. It is immediate that (i) implies (ii) and (ii) implies (iii).

We now prove that item (iii) implies item (i). By hypothesis, for every $f \in F$, we have that either f belongs to $(G) \subseteq (G)^{sat}$, if $\deg(f) \geq t$, or $x_0^{t-\deg(f)} f$ belongs to (G), if $\deg(f) < t$. In this latter case, $f \in ((G) : x_0^\infty) = (G)^{sat}$. □

Let Z be a set of polynomials generating I (like, for example, the marked basis F). For every $f \in Z$, we take an integer d in the following way: if $\deg(f) \geq t$, then $d := 0$, otherwise $d := t - \deg(f)$. By a reduction relation like in Definition 2.2, we compute a polynomial $p_f \in \langle \mathcal{N}(\tilde{J}_{\geq t})_d \rangle_A$ such that $x_0^d f - p_f \in \langle \mathscr{G}^* \rangle_A$. We then denote by $\mathscr{V}_Z$ the ideal generated in $\mathbb{K}[C]$ by the x-coefficients of the polynomials p_f.

Theorem 2.9. *With the above notations,* $\mathbf{Mf}_{\tilde{J}_{\geq t}} \cap \mathbf{Hilb}_X^{p(z)}$ *is isomorphic to the affine scheme* $\mathrm{Spec}(\mathbb{K}[C]/(\mathscr{U} + \mathscr{V}_Z))$*, for every* $t \geq \rho_{\tilde{J}} - 1$*.*

Proof. Consider the $\mathcal{P}_{\tilde{J}_{\geq t}}$-marked set $\mathscr{G} \subset R_{\mathbb{K}[C]}$ made of the polynomials in (2.1). For every $\mathbb{K}$-algebra A, a $\mathcal{P}_{\tilde{J}}$-marked set in R_A is uniquely and completely given by a $\mathbb{K}$-algebra morphism $\varphi : \mathbb{K}[C] \to A$ defined by $\varphi(C_{\alpha\gamma}) = c_{\alpha\gamma} \in A$, for every $x^\alpha \in \mathcal{P}_{\tilde{J}_{\geq t}}, x^\gamma \in \mathcal{N}(\tilde{J}_{\geq t})_{|\alpha|}$. We extend φ to a morphism from $R_{\mathbb{K}[C]}$ to R_A in the obvious way.

It is sufficient to observe that $\varphi(\mathscr{G}) \subset R_A$ is a $\mathcal{P}_{\tilde{J}_{\geq t}}$-marked basis if and only if the generators of $\mathscr{U}$ vanish at $c_{\alpha\gamma} \in A$. Furthermore, by Lemma 2.8, the saturation of the ideal generated by $\varphi(\mathscr{G})$ in R_A contains I if and only if the generators of $\mathscr{V}_Z$ vanish at $c_{\alpha\gamma} \in A$.

Hence, $\varphi(\mathscr{G})$ is a $\mathcal{P}_{\tilde{J}_{\geq t}}$-marked basis in R_A and the saturation of the ideal it generates in R_A contains I if only if $\ker(\varphi) \supseteq \mathscr{U} + \mathscr{V}_Z$. In this case, φ factors through $\mathbb{K}[C]/(\mathscr{U} + \mathscr{V}_Z)$. The induced $\mathbb{K}$-algebra morphism from $\mathbb{K}[C]/(\mathscr{U} + \mathscr{V}_Z)$ to A defines a scheme morphism $\mathrm{Spec}(A) \to \mathrm{Spec}(\mathbb{K}[C]/(\mathscr{U} + \mathscr{V}_Z))$. Therefore, the scheme $\mathrm{Spec}(\mathbb{K}[C]/(\mathscr{U} + \mathscr{V}_Z))$ is isomorphic to $\mathbf{Mf}_{\tilde{J}_{\geq t}} \cap \mathbf{Hilb}_X^{p(z)}$. □

Remark 2.10. Observe that the ideal $\mathscr{V}_Z \subseteq \mathbb{K}[C]$ depends on the chosen generating set Z of I. However, if $Z' \subseteq \mathbb{K}[C]$ is another set of polynomials generating I, by Yoneda's Lemma we have that $\mathrm{Spec}(\mathbb{K}[C]/(\mathscr{U} + \mathscr{V}_Z)) \simeq \mathrm{Spec}(\mathbb{K}[C]/(\mathscr{U} + \mathscr{V}_{Z'}))$.

In the following sections we will give an alternative construction of the affine scheme $\mathbf{Mf}_{\tilde{J}_{\geq t}} \cap \mathbf{Hilb}_X^{p(z)}$, for some ideals I.

3. Relative marked bases and reduction relations

Let $\tilde{J} \supseteq \tilde{I}$ be quasi-stable ideals in R. With an abuse of notation, we keep on writing $\tilde{J}$ (resp. $\tilde{I}$) for $\tilde{J} \cdot R_A$ (resp. $\tilde{I} \cdot R_A$).

Definition 3.1. A subset H of a $\mathcal{P}_{\tilde{J}}$-marked set is called a *$\mathcal{P}_{\tilde{J}}$-marked set relative to $\tilde{I}$* if the head terms of the marked polynomials in H are the terms in $\mathcal{P}_{\tilde{J}} \setminus \mathcal{P}_{\tilde{I}}$.

Let I be an ideal belonging to $\underline{\mathbf{Mf}}_{\tilde{I}}(\mathbb{K})$, i.e. I is generated by a $\mathcal{P}_{\tilde{I}}$-marked basis $F \subseteq R$ and, equivalently, the graded decomposition $R_A = I \oplus \langle \mathcal{N}(\tilde{I}) \rangle_A$ holds (with an abuse of notation, we keep on writing I for $I \cdot R_A$).

For every polynomial $p \in R_A$, we denote by $\mathrm{Nf}_I(p)$ the normal form of p modulo I, which is the unique polynomial in $\langle \mathcal{N}(\tilde{I}) \rangle_A$ such that $p - \mathrm{Nf}_I(p) \in I$. Recall that $\mathrm{Nf}_I(p)$ is explicitely computed by the reduction relation of Definition 2.2.

Lemma 3.2. *With the notation above, let $J \subseteq R_A$ be an ideal containing I.*

(i) *For every polynomial $p \in J$, $\mathrm{Nf}_I(p)$ belongs to $\langle \mathcal{N}(\tilde{I})\rangle_A \cap J$. In particular, $\mathrm{Nf}_I(p)$ belongs to $J \setminus I$, unless it is null.*

(ii) *The graded decomposition $J = I \oplus (\langle \mathcal{N}(\tilde{I})\rangle_A \cap J)$ holds.*

Proof. The proof follows from standard arguments. □

Definition 3.3. With the notation above, a $\mathcal{P}_{\tilde{J}}$-marked set H relative to $\tilde{I}$ is called a *$\mathcal{P}_{\tilde{J}}$-marked basis relative to I* if the following graded decomposition holds, where $J \subseteq R_A$ is the ideal generated by $F \cup H$:

$$R_A = I \oplus (\langle \mathcal{N}(\tilde{I})\rangle_A \cap J) \oplus \langle \mathcal{N}(\tilde{J})\rangle_A. \tag{3.1}$$

Theorem 3.4. *With the notation above, let $H \subset R_A$ be a $\mathcal{P}_{\tilde{J}}$-marked set relative to $\tilde{I}$ and J be the ideal generated by $F \cup H$. Then, H is a $\mathcal{P}_{\tilde{J}}$-marked basis relative to I if and only if J is generated by a $\mathcal{P}_{\tilde{J}}$-marked basis containing H.*

Proof. If J is generated by a $\mathcal{P}_{\tilde{J}}$-marked basis G containing H then $R_A = J \oplus \langle \mathcal{N}(\tilde{J})\rangle_A$. Since J contains I by construction, by Lemma 3.2(ii) we have that $J = I \oplus (\langle \mathcal{N}(\tilde{I})\rangle_A \cap J)$, and so H is a $\mathcal{P}_{\tilde{J}}$-marked basis relative to I.

If H is a $\mathcal{P}_{\tilde{J}}$-marked basis relative to I, the decomposition (3.1) holds. Since $J = I \oplus (\langle \mathcal{N}(\tilde{I})\rangle_A \cap J)$ by Lemma 3.2(ii), from decomposition (3.1) we obtain $R_A = J \oplus \langle \mathcal{N}(\tilde{J})\rangle_A$, which means that J is generated by a $\mathcal{P}_{\tilde{J}}$-marked basis G (see Remark 2.5). It remains to show that $H \subseteq G$. By the hypotheses, for every $h \in H$ there is $g \in G$ with $\mathrm{Ht}(h) = \mathrm{Ht}(g)$. Hence, by construction, the polynomial $h - g$ belongs to $J \cap \langle \mathcal{N}(\tilde{J})\rangle_A = \{0\}$ and then $h = g$. □

If H is a $\mathcal{P}_{\tilde{J}}$-marked set relative to $\tilde{I}$, hence contained in a $\mathcal{P}_{\tilde{J}}$-marked set G, we have the reduction relation $\longrightarrow_{H^*}$ thanks to Lemma 2.3.

Despite the nice properties which the reduction relation $\longrightarrow_{H^*}$ inherits from $\longrightarrow_{G^*}$, the following example shows that it is not always true that, for a given polynomial p, $p \longrightarrow^+_{G^*} 0$ is equivalent either to $p \longrightarrow^+_{H^*} p_{H^*}$ with $p_{H^*} \in \langle F^*\rangle_A$, or to $p \longrightarrow^+_{F^*} p_{F^*}$ with $p_{F^*} \in \langle H^*\rangle_A$, for a given $\mathcal{P}_{\tilde{I}}$-marked set F.

Hence, in general $(p_{H^*})_{F^*}$ and $(p_{F^*})_{H^*}$ are not the same polynomial. Moreover, the following example also shows that we might have $p = (p_{F^*})_{H^*}$ (resp. $p = (p_{H^*})_{F^*}$). This means that, although $\longrightarrow_{F^*}$ and $\longrightarrow_{H^*}$ are both Noetherian, the reduction process obtained by first computing the complete reduction of a polynomial by $\longrightarrow_{F^*}$ (resp. by $\longrightarrow_{H^*}$) and successively the complete reduction by $\longrightarrow_{H^*}$ (resp. by $\longrightarrow_{F^*}$) is not Noetherian.

Example 3.5. In the polynomial ring $\mathbb{K}[x_0, x_1, x_2, x_3]$, consider the quasi-stable ideal $\tilde{I} = (x_3^2, x_2^3)$, with Pommaret basis $\mathcal{P}_{\tilde{I}} = \{x_3^2, x_3x_2^3, x_2^3\}$, and the $\mathcal{P}_{\tilde{I}}$-marked basis $F = \{x_3^2, x_3x_2^3, x_2^3 - 3x_3x_2^2\}$. Let I be the ideal generated by F. Then take the quasi-stable ideal $\tilde{J} = (x_3^2, x_3x_2, x_2^3)$, which contains $\tilde{I}$ and has Pommaret basis $\mathcal{P}_{\tilde{J}} = \{x_3^2, x_3x_2, x_2^3\}$. Moreover, take the $\mathcal{P}_{\tilde{J}}$-marked set $H = \{x_3x_2 - 4x_2^2\}$ relative to $\tilde{I}$. In particular, H *is a* $\mathcal{P}_{\tilde{J}}$*-marked basis relative to* I because it is contained in the $\mathcal{P}_{\tilde{J}}$-marked basis $G = \{x_3^2, x_3x_2 - 4x_2^2, x_2^3\}$ and $J = (G) = (H \cup F)$.

Let $h = x_3x_2 - 4x_2^2$ be the unique polynomial of H. Then $x_3h \longrightarrow^+_{H^*} f := x_3^2x_2 - 16x_2^3$, where f does not belong to $I = \langle F^* \rangle_A$. Moreover, $x_3h \longrightarrow^+_{F^*} g := -4x_2^2x_3$ where g does not belong to $\langle H^* \rangle_A$.

Let us now consider x_2^3, which is reduced with respect to $\longrightarrow_{H^*}$. If we rewrite x_2^3 first by $\longrightarrow_{F^*}$ and then by $\longrightarrow_{H^*}$ we find $12x_2^3$, obtaining a loop, unless the characteristic of the field is 2 or 3.

If we consider $x_2^2x_3$ and rewrite it first by $\longrightarrow_{H^*}$ and then by $\longrightarrow_{F^*}$ we find $12x_2^2x_3$, obtaining a loop again, unless the characteristic of the field is 2 or 3.

The problems that Example 3.5 highlights are new with respect to both the theory of marked bases and the theory of relative Gröbner bases and involutive bases.

However, there is a relevant case in which the successive application of the relations $\longrightarrow_{H^*}$ and $\longrightarrow_{F^*}$ has a good behaviour.

Proposition 3.6. *With the notation above, let* H *be a* $\mathcal{P}_{\tilde{J}}$*-marked set relative to* $\tilde{I}$ *and* $F' := \{f \in F \mid \mathrm{Ht}(f) \in \mathcal{P}_{\tilde{I}} \cap \mathcal{P}_{\tilde{J}}\}$*. If* $G := H \cup F'$ *is a* $\mathcal{P}_{\tilde{J}}$*-marked set, then*

(i) For every $p \in R_A$*,* $p \longrightarrow^+_{G^*} 0$ *is equivalent to* $p \longrightarrow^+_{H^*} r$ *with* $r \in \langle F'^* \rangle$*.*

(ii) H *is a* $\mathcal{P}_{\tilde{J}}$*-marked basis relative to* I *if and only if* $H \cup F'$ *is the* $\mathcal{P}_{\tilde{J}}$*-marked basis of* $J = (H \cup F)$*.*

Proof. For item (i), assume that p reduces to 0 by G^*. Since G is equal to $H \cup F'$, this means that we have the following expression:

$$p = \sum_{H^*} c_{\beta\eta} x^\eta h_\beta + \sum_{F'^*} c_{\alpha\gamma} x^\gamma f_\alpha.$$

The above equality gives us that $p \longrightarrow^+_{H^*} \sum_{F'^*} c_{\alpha\gamma} x^\gamma f_\alpha$, and the latter is obviously an element in $\langle F'^* \rangle_A$. Vice versa, if p reduces to an element $r \in \langle F'^* \rangle_A$ by H^*, then $r \longrightarrow^+_{G^*} 0$, being $F' \subset G$. Item (ii) now follows from Theorem 3.4. $\square$

If $I = \tilde{I}$, the hypotheses of Proposition 3.6 are satisfied because $F' = \mathcal{P}_{\tilde{I}} \cap \mathcal{P}_{\tilde{J}}$.

Corollary 3.7. *Let* $\tilde{J} \supseteq \tilde{I}$ *be quasi-stable ideals,* H *a* $\mathcal{P}_{\tilde{J}}$*-marked set relative to* $\tilde{I}$*, and* J *the ideal generated by* $\mathcal{P}_{\tilde{I}} \cup H$*. Then* H *is a* $\mathcal{P}_{\tilde{J}}$*-marked basis relative to* $\tilde{I}$ *if and only if:*

(i) $\forall h_\beta \in H$, $\forall x_i > \min(\mathrm{Ht}(h_\beta))$, $x_i h_\beta \longrightarrow^+_{H^*} r_{\beta,i} \in \tilde{I}$
(ii) $\forall x^\alpha \in \mathcal{P}_{\tilde{I}} \cap \mathcal{P}_{\tilde{J}}$, $\forall x_i > \min(x^\alpha)$, $x_i x^\alpha \longrightarrow^+_{H^*} r_{\alpha,i} \in \tilde{I}$
(iii) $\forall x^\gamma \in \mathcal{B}_{\tilde{I}} \setminus \mathcal{P}_{\tilde{J}}$, $x^\gamma \longrightarrow^+_{H^*} r_\gamma \in \tilde{I}$.

Proof. Denote by $\mathcal{T}_{\tilde{I},\tilde{J}}$ the set $\mathcal{P}_{\tilde{I}} \cap \mathcal{P}_{\tilde{J}}$. Observe that if $p \longrightarrow^+_{H^*} r \in \tilde{I}$, then r belongs to $\langle \mathcal{T}^*_{\tilde{I},\tilde{J}} \rangle \subseteq \tilde{I}$, as pointed out in Remark 2.4. Thanks to Proposition 3.6, H is a $\mathcal{P}_{\tilde{J}}$-marked basis relative to $\tilde{I}$ if and only if $H \cup \mathcal{T}_{\tilde{I},\tilde{J}}$ is the $\mathcal{P}_{\tilde{J}}$-marked basis of $J = (H \cup \mathcal{B}_{\tilde{I}})$. So now it is sufficient to apply [6, Theorem 5.13] to obtain items (i) and (ii) and observe that item (iii) guarantees that $\tilde{I}$ is contained in J. □

4. Relative marked functor

We continue to use the same notation of Section 3, in particular $\tilde{J} \supseteq \tilde{I}$ are quasi-stable ideals in R and the ideal $I \subseteq R_A$ is generated by a $\mathcal{P}_{\tilde{I}}$-marked basis F.

Definition 4.1. The *marked functor on* $\tilde{J}$ *relative to* I, or, when $\tilde{J}$ and I are well-understood, simply the *relative marked functor*, is the functor

$$\underline{\mathbf{Mf}}_{I,\tilde{J}} : \underline{\text{Noeth } \mathbb{K}\text{–Alg}} \longrightarrow \underline{\text{Sets}}$$

such that

$$\underline{\mathbf{Mf}}_{I,\tilde{J}}(A) := \{(H \cup F) \subseteq R_A \mid H \subseteq R_A \text{ is a } \mathcal{P}_{\tilde{J}}\text{-marked basis relative to } I\}$$

and, if $\phi : A \to B$ is a morphism of Noetherian $\mathbb{K}$-algebras (with $\phi(1_\mathbb{K}) = 1_\mathbb{K} \in B$), then the map $\underline{\mathbf{Mf}}_{I,\tilde{J}}(\phi)$ associates to every $H \in \underline{\mathbf{Mf}}_{I,\tilde{J}}(A)$ the $\mathcal{P}_{\tilde{J}}$-marked basis $H \otimes_A B \in \underline{\mathbf{Mf}}_{I,\tilde{J}}(B) \subset R_B$ relative to I.

As the reader might expect, $\underline{\mathbf{Mf}}_{I,\tilde{J}}$ is strictly related to $\underline{\mathbf{Mf}}_{\tilde{J}}$ and to $\mathbf{Hilb}^{p(z)}_X$, with $X = \mathrm{Proj}(R/I)$, when the hypotheses of Proposition 3.6 hold.

Theorem 4.2. *With the notation above and under the hypotheses of Proposition 3.6, the following statements hold:*

(i) the relative marked functor $\underline{\mathbf{Mf}}_{I,\tilde{J}}$ *is a closed subfunctor of* $\underline{\mathbf{Mf}}_{\tilde{J}}$*;*
(ii) if the ideals $\tilde{I}$ *and* $\tilde{J}$ *are both saturated, then for every integer* $t \geq \rho_{\tilde{J}} - 1$ *the relative marked functor* $\underline{\mathbf{Mf}}_{I_{\geq t},\tilde{J}_{\geq t}}$ *is an open subfunctor of* $\underline{\mathbf{Hilb}}^{p(z)}_X$ *and it is represented by* $\mathbf{Mf}_{\tilde{J}_{\geq t}} \cap \mathbf{Hilb}^{p(z)}_X$*.*

Proof. For what concerns item (i), thanks to Theorem 3.4, for every Noetherian $\mathbb{K}$-algebra A there is a bijection between the set $\underline{\mathbf{Mf}}_{I,\tilde{J}}(A)$ and the set

$$\underline{\mathbf{Mf}}_{\tilde{J}}(A) \cap \{J \subseteq R_A : J \text{ is a homogeneous ideal containing } I\}.$$

As we already showed in Section 2, the condition "J contains I" can be imposed on the ideals in $\underline{\mathbf{Mf}}_{\tilde{J}}(A)$ by further closed conditions on the polynomials generating the ideal that defines the scheme which represents the functor $\underline{\mathbf{Mf}}_{\tilde{J}}$. So, we obtain item (i).

Thanks to [4, Theorem 3.5 and Corollary 3.7], item (i), and Lemma 2.8(ii), for every integer t, $\underline{\mathbf{Mf}}_{I_{\geq t},\tilde{J}_{\geq t}}$ is a closed subfunctor of $\underline{\mathbf{Mf}}_{\tilde{J}_{\geq t}}$. Moreover, if $t \geq \rho_{\tilde{J}} - 1$, we have $\underline{\mathbf{Mf}}_{I_{\geq t},\tilde{J}_{\geq t}}(A) = \underline{\mathbf{Mf}}_{\tilde{J}_{\geq t}}(A) \cap \underline{\mathbf{Hilb}}_X^{p(z)}(A)$, and item (ii) holds thanks to Proposition 2.6. $\square$

4.1. **Case $I = \tilde{I}$.** In the particular case $I = \tilde{I}$, we now give a construction of the scheme representing $\underline{\mathbf{Mf}}_{\tilde{I},\tilde{J}}$ which is alternative to the construction that is described in Section 2 for $\mathbf{Mf}_{\tilde{J}_{\geq t}} \cap \mathbf{Hilb}_X^{p(z)}$. We use the computational method that arises from Corollary 3.7 in order to characterize relative marked bases.

Let C' denote the finite set of variables

$$\left\{C_{\beta\eta} \mid x^\beta \in \mathcal{P}_{\tilde{J}} \setminus \mathcal{P}_{\tilde{I}}, x^\eta \in \mathcal{N}(\tilde{J}), \deg(x^\eta) = \deg(x^\beta)\right\}$$

and consider the $\mathbb{K}$-algebra $\mathbb{K}[C']$. Then, we construct the set $\mathscr{H} \subset R_{\mathbb{K}[C']}$ consisting of the following marked polynomials

$$h_\beta = x^\beta - \sum_{x^\eta \in \mathcal{N}(\tilde{J})_{|\beta|}} C_{\beta\eta} x^\eta \tag{4.1}$$

with $x^\beta \in \mathcal{P}_{\tilde{J}} \setminus \mathcal{P}_{\tilde{I}}$. Moreover, we consider

$$\mathscr{H}^* := \{x^\delta h_\beta \mid h_\beta \in \mathscr{H}, x^\delta \in \mathcal{X}(h_\beta)\}.$$

We highlight that the set C' can be identified to a subset of the set C given in Section 2, and up to this identification we can consider $\mathscr{H}$ as a subset of $\mathscr{G}$.

Then, we explicitly compute the following polynomials in $R_{\mathbb{K}[C']}$ by $\longrightarrow_{\mathscr{H}^*}$:

- $\forall h_\beta \in \mathscr{H}$, $\forall x_i > \min(\mathrm{Ht}(h_\beta))$, let $r_{\beta,i}$ be such that $x_i h_\beta \longrightarrow^+_{\mathscr{H}^*} r_{\beta,i}$;
- $\forall x^\alpha \in \mathcal{T}_{\tilde{I},\tilde{J}}$, $\forall x_i > \min(x^\alpha)$, let $r_{\alpha,i}$ be such that $x_i x^\alpha \longrightarrow^+_{\mathscr{H}^*} r_{\alpha,i}$;
- $\forall x^\gamma \in \mathcal{B}_{\tilde{I}} \setminus \mathcal{P}_{\tilde{J}}$, let r_γ be such that $f \longrightarrow^+_{\mathscr{H}^*} r_\gamma$.

For every $h_\beta \in \mathscr{H}$, and for every $x_i > \min(\mathrm{Ht}(h_\beta))$, we collect the coefficients in $\mathbb{K}[C']$ of the terms in $\mathrm{supp}(r_{\beta,i})$ not belonging to $\tilde{I}$, and the same for all the polynomials $r_{\alpha,i}$ and r_γ. Let $\mathscr{R} \subset \mathbb{K}[C']$ be the ideal generated by these coefficients.

Theorem 4.3. *The functor* $\underline{\mathbf{Mf}}_{\tilde{I},\tilde{J}}$ *is the functor of points of* $\mathrm{Spec}(\mathbb{K}[C']/\mathscr{R})$*, which we denote by* $\mathbf{Mf}_{\tilde{I},\tilde{J}}$*.*

Proof. Let $H \subseteq R_A$ be a $\mathcal{P}_{\tilde{J}}$-marked basis relative to $\tilde{I}$ and denote by ϕ_H the evaluation morphism $\phi_H : \mathbb{K}[C'] \to A$ that associates to every variable in C' the corresponding coefficient in the polynomials of H. It is sufficient to observe that H is a $\mathcal{P}_{\tilde{J}}$-marked basis relative to $\tilde{I}$ if and only if ϕ_H factors through $\mathbb{K}[C']/\mathscr{R}$, or in other words if and only if the following diagram commutes

$$\begin{array}{ccc} \mathbb{K}[C'] & \xrightarrow{\phi_H} & A \\ & \searrow \quad \nearrow & \\ & \mathbb{K}[C']/\mathscr{R} & \end{array}\ .$$

Equivalently, H is a $\mathcal{P}_{\tilde{J}}$-marked basis relative to $\tilde{I}$ if and only if $\mathscr{R}$ is contained in $\ker(\phi_H)$, which is true thanks to Corollary 3.7. □

The scheme $\mathbf{Mf}_{\tilde{I},\tilde{J}}$ which has been introduced in the statement of Theorem 4.3 is called a *relative marked scheme*.

Remark 4.4. The scheme $\mathrm{Spec}(\mathbb{K}[C']/\mathscr{R})$ is computationally more advantageous compared with $\mathrm{Spec}(\mathbb{K}[C]/(\mathscr{U} + \mathscr{V}_F))$ considered in Theorem 2.9. Indeed, if $\mathcal{P}_{\tilde{I}} \cap \mathcal{P}_{\tilde{J}} \neq \emptyset$, then $|C'| < |C|$ and the reduction $\longrightarrow_{\mathscr{H}^*}$ involves the relative marked set $\mathscr{H}$, which contains less polynomials than $\mathscr{G}$. Actually, in principle we perform less reduction steps using $\longrightarrow_{\mathscr{H}^*}$.

If the ideals $\tilde{I}$ and $\tilde{J}$ are both saturated and we consider $\underline{\mathbf{Mf}}_{\tilde{I}_{\geq t},\tilde{J}_{\geq t}}$ for some t, we give a further different presentation of the scheme representing it. Take the following polynomials in $\mathbb{K}[C']$:

$$(\star) \qquad \forall x^\gamma \in \mathcal{P}_{\tilde{I}} \setminus \mathcal{P}_{\tilde{J}},\ \text{let } r_\gamma \text{ be such that } x_0^{\max\{0,t-\deg(x^\gamma)\}} x^\gamma \longrightarrow^+_{\mathscr{H}^*} r_\gamma$$

Observe that if t is strictly bigger than the initial degree of $\tilde{I}$, then $|\mathcal{P}_{\tilde{I}} \setminus \mathcal{P}_{\tilde{J}}|$ is strictly smaller than $|\mathcal{P}_{\tilde{I}_{\geq t}} \cap \mathcal{P}_{\tilde{J}_{\geq t}}|$.

Let $\mathscr{R}' \subset \mathbb{K}[C']$ be the ideal generated by the coefficients in $\mathbb{K}[C']$ of the terms not belonging to $\tilde{I}$ of the polynomials $r_{\beta,i}$, $r_{\alpha,i}$ considered for $\mathscr{R}$, and by the coefficients in $\mathbb{K}[C']$ of the terms not belonging to $\tilde{I}$ of the polynomials r_γ in ($\star$).

Theorem 4.5. *The functor* $\underline{\mathbf{Mf}}_{\tilde{I}_{\geq t},\tilde{J}_{\geq t}}$ *is the functor of points of* $\mathrm{Spec}(\mathbb{K}[C']/\mathscr{R}')$*.*

Proof. The proof is analogous to that of Theorem 4.3 thanks to Lemma 2.8. □

Algorithm 1 Algorithm for computing the defining ideal $\mathscr{R}'$ representing the relative marked functor $\underline{\mathbf{Mf}}_{\tilde{I}_{\geq t},\tilde{J}_{\geq t}}$

MarkedFunctor$(\tilde{J}, \tilde{I}, t)$
Input: saturated quasi-stable ideals $\tilde{J} \supseteq \tilde{I}$ and a non-negative integer t
Output: generators of the ideal $\mathscr{R}'$ representing the relative functor $\underline{\mathbf{Mf}}_{\tilde{I}_{\geq t},\tilde{J}_{\geq t}}$
let $\mathscr{H} \subseteq \mathbb{K}[C']$ be the set of the polynomials defined in (4.1) with respect to the quasi-stable ideals $\tilde{J}_{\geq t}$ and $\tilde{I}_{\geq t}$
$\mathscr{R}' := (0)$
for $h_\beta \in \mathscr{H}$ **do**
  for $x_i > \min(\mathrm{Ht}(h_\beta))$ **do**
    compute $r_{\beta,i}$ such that $x_i h_\beta \longrightarrow^{+}_{\mathscr{H}^*} r_{\beta,i}$
    $\mathscr{R}' := \mathscr{R}' +$ (coefficients in $r_{\beta,i}$ of the terms not belonging to $\tilde{I}$)
  end for
end for
for $x^\alpha \in \mathcal{P}_{\tilde{J}_{\geq t}} \cap \mathcal{P}_{\tilde{I}_{\geq t}}$ **do**
  for $x_i > \min(x^\alpha)$ **do**
    compute $r_{\alpha,i}$ such that $x_i x^\alpha \longrightarrow^{+}_{\mathscr{H}^*} r_{\alpha,i}$
    $\mathscr{R}' := \mathscr{R}' +$ (coefficients in $r_{\alpha,i}$ of the terms not belonging to $\tilde{I}$)
  end for
end for
for $x^\gamma \in \mathcal{B}_{\tilde{I}} \setminus \mathcal{P}_{\tilde{J}}$ **do**
  compute r_γ such that $x_0^{\max\{0,t-\deg(x^\gamma)\}} x^\gamma \longrightarrow^{+}_{\mathscr{H}^*} r_\gamma$
  $\mathscr{R}' := \mathscr{R}' +$ (coefficients in r_γ of the terms not belonging to $\tilde{I}$)
end for
return $\mathscr{R}'$

Algorithm 1 collects the instructions to compute the ideal $\mathscr{R}'$.

In the following examples we take into account the fact that, thanks to Theorem 4.2(ii), the relative marked scheme $\mathbf{Mf}_{\tilde{I}_{\geq t},\tilde{J}_{\geq t}}$ defined by the ideal $\mathscr{R}'$ that has just introduced is an open subscheme of $\mathbf{Hilb}_X^{p(z)}$. Hence, the Zariski tangent space to this open subscheme at one of its points is equal to the Zariski tangent space to $\mathbf{Hilb}_X^{p(z)}$ at the same point (see also [5, Corollary 1.9]).

Example 4.6. This is an example of scheme $\mathbf{Mf}_{\tilde{I}_{\geq t},\tilde{J}_{\geq t}}$ which is neither irreducible nor reduced. Take the quasi-stable ideals $\tilde{I} = (x_3^2, x_2^5) \subset \tilde{J} = (x_3^2, x_3x_2, x_3x_1^2, x_2^5) \subset R := \mathbb{K}[x_0, \dots, x_3]$, with $\mathcal{P}_{\tilde{J}} = \{x_3^2, x_3x_2, x_3x_1^2, x_2^5\}$ and $\mathcal{P}_{\tilde{I}} = \{x_3^2, x_3x_2^5, x_2^5\}$. Following Algorithm 1 and using CoCoA [1], we compute the ideal $\mathscr{R}'$ defining the relative marked scheme $\mathbf{Mf}_{\tilde{I}_{\geq t},\tilde{J}_{\geq t}}$ for $t = \rho_{\tilde{J}} - 1 = 2$. In this case we have $\tilde{J}_{\geq t} = \tilde{J}$ and $\tilde{I}_{\geq t} = \tilde{I}$. The set $\mathscr{H}$ is made of the following polynomials in the ring $\mathbb{Q}[c_1, \dots, c_{20}][x_0, \dots, x_3]$:

$$h_1 = c_1x_0^2 + c_2x_0x_1 + c_3x_1^2 + c_4x_0x_2 + c_5x_1x_2 + c_6x_2^2 + c_7x_0x_3 + c_8x_1x_3 + x_2x_3,$$
$$h_2 = c_9x_0^3 + c_{10}x_0^2x_1 + c_{11}x_0x_1^2 + c_{12}x_1^3 + c_{13}x_0^2x_2 + c_{14}x_0x_1x_2 + c_{15}x_1^2x_2 + c_{16}x_0x_2^2 + c_{17}x_1x_2^2 + c_{18}x_2^3 + c_{19}x_0^2x_3 + c_{20}x_0x_1x_3 + x_1^2x_3.$$

By $\longrightarrow_{\mathscr{H}^*}$ we reduce the polynomials x_3h_1, x_3h_2, x_2h_2, $x_3x_2^5$ and then apply the reduction modulo $\tilde{I}$, obtaining the ideal $\mathscr{R}' \subseteq \mathbb{Q}[c_1,\dots,c_{20}]$. The ring $\mathbb{Q}[c_1,\dots,c_{20}]/\mathscr{R}'$ has Krull dimension 2, so $\mathbf{Mf}_{\tilde{I}_{\geq t},\tilde{J}_{\geq t}}$ has dimension 2. Moreover, the Zariski tangent space to $\mathbf{Mf}_{\tilde{I}_{\geq t},\tilde{J}_{\geq t}}$ at Y has dimension 7 and the point Y defined by $\tilde{J}$ is singular in $\mathbf{Hilb}_X^{p(z)}$, where $p(z) = 5z-3$ is the Hilbert polynomial of $R/\tilde{J}$ and $X = \mathrm{Proj}(R/\tilde{I})$.

Using Macaulay 2 [12] we compute the irreducible components of $\mathscr{R}'$, obtaining that the associated primes have both dimension 2 and are

$$\mathscr{P}_0 = (c_{18},\, c_{17},\, c_{16},\, c_{15},\, c_{14},\, c_{13},\, c_{12},\, c_{11},\, c_{10},\, c_9,\, c_8,\, c_7,\, c_6,\, c_5,\, c_4,\, c_3,\, c_2,\, c_1),$$
$$\mathscr{P}_1 = (c_{18},\, c_{17},\, c_{16},\, c_{15},\, c_{14},\, c_{13},\, c_{12},\, c_{11},\, c_{10},\, c_9,\, c_6,\, c_5,\, c_4,\, c_3,\, c_2,\, c_1\, c_{20}^2 - 4c_{19},\, c_8c_{20} - 2\,c_7,\, 2\,c_8c_{19} - c_7c_{20}).$$

The defining ideals of both the two irreducible components are not prime and contain the point Y as a singular point, because the Zariski tangent spaces at Y to these components have dimensions 7, too. Some ancillary material related to this example is available at http://www.dma.unina.it/~cioffi/RedirectRelative.html.

If we compute the generators for the ideal $\mathscr{U} + \mathscr{V}_Z$ of Theorem 2.9, then we have to deal with 50 parameters instead of 20.

Example 4.7. This is an example of scheme $\mathbf{Mf}_{\tilde{I}_{\geq t},\tilde{J}_{\geq t}}$ which is irreducible but not reduced. Assume $\mathrm{char}(\mathbb{K}) \neq 2$. Given an integer $p > 2$, take the quasi-stable ideals $\tilde{I} = (x_n^2, x_{n-1}^p) \subset \tilde{J} = (x_n, x_{n-1}^p) \subseteq R := \mathbb{K}[x_0,\dots,x_n]$, with $\mathcal{P}_{\tilde{J}} = \{x_n, x_{n-1}^p\}$ and $\mathcal{P}_{\tilde{I}} = \{x_n^2, x_nx_{n-1}^p, x_{n-1}^p\}$. The ideal $\tilde{J}$ defines the only point Y of the Hilbert scheme $\mathbf{Hilb}_X^{p(z)}$, where $p(z)$ is the Hilbert polynomial of $R/\tilde{J}$ and $X = \mathrm{Proj}(R/\tilde{I})$ (see [11], for instance). Following Algorithm 1, by hand we compute the ideal $\mathscr{R}'$ defining $\mathbf{Mf}_{\tilde{I}_{\geq t},\tilde{J}_{\geq t}}$ for $t = \rho_{\tilde{J}} - 1 = 0$, hence $\tilde{J}_{\geq t} = \tilde{J}$ and $\tilde{I}_{\geq t} = \tilde{I}$. The set $\mathscr{H}$ is made only of the polynomial $h = c_1x_0 + c_2x_1 + c_3x_2 + \cdots + c_nx_{n-1} + x_n$. By $\longrightarrow_{\mathscr{H}^*}$ we reduce the terms $x_nx_{n-1}^p$ and x_n^2 and then apply the reduction modulo $\tilde{I}$, obtaining the ideal $\mathscr{R}'$:

$$\mathscr{R}' = (c_n,\dots,c_2,c_1)^2.$$

The affine scheme $\mathrm{Spec}(\mathbb{K}[c_1,c_2,\dots,c_n]/\mathscr{R}')$ is a zero-dimensional scheme supported over the origin and with Zariski tangent space of dimension n at Y. By definition, the multiplicity in $\mathbf{Hilb}_X^{p(z)}$ of the fat point Y is $n+1$.

If we compute the generators for the ideal $\mathscr{U} + \mathscr{V}_Z$ of Theorem 2.9, then we have to deal with $\binom{n-1+p}{p} + n - 1$ parameters instead of n.

5. An open cover of a Hilbert scheme over a Cohen-Macaulay quotient ring on a quasi-stable ideal

Let $\tilde{I}$ be a saturated quasi-stable ideal, $S := R/\tilde{I}$, and consider the projective scheme $X = \mathrm{Proj}(S)$. Hence we can consider the Hilbert scheme $\mathbf{Hilb}_X^{p(z)}$, for an admissible Hilbert polynomial $p(z)$.

When we consider the image in S of an element f of R we mean its image $[f]$ by the projection $\pi : R \to R/\tilde{I}$, that is its residue class modulo $\tilde{I}$.

Following [21], a term τ of R is said *$\tilde{I}$-free* if its residue class $[\tau]$ is non-null, i.e. τ does not belong to $\tilde{I}$. A term of S is the image in S of an $\tilde{I}$-free term of R. If W is any set of terms in S, by abuse of notation we will use the symbol W to also denote the set of terms in $\pi^{-1}(W)$ and vice versa.

The ring S inherits from R a grading for which a term $[t] \in S$ has degree $q \geq 0$ if and only if t has the degree q in R.

An ideal $U \subseteq S$ is *monomial* if it is the image in S of a monomial ideal of R. Every monomial ideal U of S has a unique minimal generating set $\mathcal{B}_U$ made of terms of S. A monomial ideal U of S will be said *quasi-stable in S* if the ideal $(\mathcal{B}_U \cup \mathcal{B}_{\tilde{I}})$ is quasi-stable in R, that is U is the image of a quasi-stable ideal of R.

Definition 5.1. Let U be a quasi-stable ideal in S and let $\tilde{J} = (\mathcal{B}_U \cup \mathcal{B}_{\tilde{I}})$. The image in S of a $\mathcal{P}_{\tilde{J}}$-marked set relative to $\tilde{I}$ is called a *U-marked set*. The image of a $\mathcal{P}_{\tilde{J}}$-marked basis relative to $\tilde{I}$ is called a *U-marked basis*.

From now, we also assume that the ring $S := R/\tilde{I}$ is Cohen-Macaulay.

Proposition 5.2 ([26, Theorem 3.20], [27, Theorem 5.2.9])**.** *Let $\tilde{I}$ be a quasi-stable ideal in R and let $\mathcal{P}_{\tilde{I}}$ be its Pommaret basis. Then $S = R/\tilde{I}$ is Cohen-Macaulay if and only if, for the integer $m = \min\{\min(x^\alpha) \mid x^\alpha \in \mathcal{P}_{\tilde{I}}\}$, there is a pure variable power $x_m^{a_m}$ in $\tilde{I}$.*

Remark 5.3. Note that the criterion given in Proposition 5.2 can be applied to a quasi-stable ideal $\tilde{I}$ by looking at the minimal value of $\min(x^\alpha)$ for x^α in $\mathcal{B}_{\tilde{I}}$.

Let $x_{k+1}, \dots, x_n$ be the variables that divide some minimal generators of $\tilde{I}$. In the following discussion, we will write $\mathbb{T}'$ for the set of all terms in the polynomial ring $\mathbb{K}[x_{k+1}, \dots, x_n]$ and $\mathbb{T}''$ for the set of all terms in the polynomial ring $\mathbb{K}[x_0, \dots, x_k]$.

We need to adapt [3, Proposition 7.2 and Corollary 7.4] to our current setting. We make the following observation as a first step.

Lemma 5.4. *The Cohen-Macaulay quotient ring $S = R/\tilde{I}$ is a finitely generated graded free $\mathbb{K}[x_0, \dots, x_k]$-module*

$$S = \bigoplus_{e=0}^{d} \left(\mathbb{K}[x_0, \dots, x_k](-e)\right)^{m_e},$$

where $d = \max\{\deg(t) \mid t \in \mathbb{T}' \setminus \tilde{I}\}$ and, for each $0 \leq e \leq d$, $m_e = |\{t \in \mathbb{T}' \setminus \tilde{I} \mid \deg(t) = e\}|$.

Proof. First, note that $\tilde{I}$ contains pure powers of all variables x_j with $j > k$; hence, the set $\mathcal{T}(e) := \{t \in \mathbb{T}' \setminus \tilde{I} \mid \deg(t) = e\}$ is finite. Since the generators of $\tilde{I}$ are terms in $\mathbb{T}'$, we have for each $t \in \mathcal{T}(e)$ an injection

$$\iota : \mathbb{T}'' \to S, u \mapsto [u \cdot t].$$

Now, we turn to the graded decomposition. Recall that the ring S inherits a grading from R. It is easy to see that the set of terms of degree $q \geq 0$ in S is disjointly decomposed as follows:

$$S_q \cap \{[u] \mid u \in \mathbb{T}\} = \bigsqcup_{e=0}^{d} \bigsqcup_{t \in \mathcal{T}(e)} t \cdot \left(S_{q-e} \cap \{[u] \mid u \in \mathbb{T}''\}\right). \tag{5.1}$$

It is important to note that all elements in the sets of the right hand side of (5.1) are non-zero; this is guaranteed by the Cohen-Macaulay property of $\tilde{I}$. The claim follows. □

We now highlight that the definition of quasi-stable ideal U of S as the image in S of a quasi-stable ideal $\tilde{J}$ in R containing $\tilde{I}$ is equivalent to the definition of quasi-stable submodule of a free module given in [3, Definition 3.2 item (i)].

Proposition 5.5. *Let $U \subseteq S$ be a monomial ideal with minimal generating set $\mathcal{B}_U$. Then U is quasi-stable if and only if $\mathcal{B}_U$, interpreted as a monomial subset of the $\mathbb{K}[x_0, \dots, x_k]$-module S, generates a quasi-stable submodule.*

Proof. The proof is by routine verification of quasi-stability conditions (as given in [3, Definition 2.2]) for the minimal generators of the ideals and submodules that are considered. □

Thanks to Proposition 5.5, a quasi-stable ideal U of S can be even considered as a quasi-stable $\mathbb{K}[x_0, \dots, x_k]$-submodule of S, and vice versa.

However, the notions of marked set over a quasi-stable ideal U introduced in Definition 5.1 and of marked set over a submodule given in [3, Definition 4.3] are different, as we will see in Example 5.6.

Example 5.6. Take $R = \mathbb{K}[x_0, x_1, x_2]$, $\tilde{I} = (x_2^7)$ and $\tilde{J} = (x_1^2, x_2^7)$. Let U be the ideal that is the image of $\tilde{J}$ in S. Observe that $\mathcal{P}_{\tilde{J}} = \{x_1^2, x_1^2x_2, \dots, x_1^2x_2^6, x_2^7\}$. Thus, according to Definition 5.1, the set $\hat{F} = \{x_0x_1 + x_1^2\}$ is not a U-marked set (for such a set, we would need additionally polynomials marked on each of the terms $x_1^2x_2, \dots, x_1^2x_2^6$). Nevertheless, $\hat{F}$ is marked on the Pommaret basis of the quasi-stable monomial $\mathbb{K}[x_0, x_1]$-submodule of S generated by $\{x_1^2 \cdot [1]\}$, because this singleton set is also the Pommaret basis of the submodule generated by it (see [3, Definition 3.1]).

Nevertheless, for high degrees q, the notions of marked sets over a quasi-stable submodule and of U-marked sets for a quasi-stable ideal $U \subseteq S$ coincide.

Recall that the regularity $\operatorname{reg}(\tilde{I})$ of a quasi-stable ideal $\tilde{I}$ coincides with the maximum degree of a term in its Pommaret basis $\mathcal{P}_{\tilde{I}}$. Hence, for every $q \geq \operatorname{reg}(\tilde{I})$, the Pommaret basis of $U_{\geq q}$ coincides with $\mathcal{B}_{U_{\geq q}}$.

Corollary 5.7. *Let $U \subset S$ be a quasi-stable ideal and $\tilde{J}$ be a quasi-stable ideal whose image in S is U. Let $F \subseteq S_q$ be a finite set of homogeneous elements of S of degree $q \geq \max\{\operatorname{reg}(\tilde{J}), \operatorname{reg}(\tilde{I})\}$. Then, F is a $U_{\geq q}$-marked basis if and only if it is a marked basis over the submodule $U_{\geq q}$.*

Proof. We already observed that in the present setting a quasi-stable monomial ideal U of S can be even considered as a quasi-stable $\mathbb{K}[x_0, \dots, x_k]$-submodule of S, and vice versa. Indeed, by Proposition 5.5, in both the two above interpretations $\mathcal{B}_U$ is a set of generators of U.

The difference between Definition 5.1 and [3, Definition 4.3] appears when $\mathcal{B}_{\tilde{J}}$ does not coincide with $\mathcal{P}_{\tilde{J}}$, so that $\mathcal{B}_U$ does not contain all the terms on which we expect marked polynomials in a U-marked basis.

If $q \geq \max\{\operatorname{reg}(\tilde{J}), \operatorname{reg}(\tilde{I})\}$, then $\mathcal{B}_{\tilde{J}_{\geq q}} = \mathcal{P}_{\tilde{J}_{\geq q}}$ and $\mathcal{B}_{\tilde{I}_{\geq q}} = \mathcal{P}_{\tilde{I}_{\geq q}}$. Hence $\mathcal{B}_{U_{\geq q}} = \mathcal{P}_{\tilde{J}_{\geq q}} \setminus \mathcal{P}_{\tilde{I}_{\geq q}} = \mathcal{B}_{\tilde{J}_{\geq q}} \setminus \mathcal{B}_{\tilde{I}_{\geq q}}$ contains all the expected terms for a $U_{\geq q}$-marked basis. ☐

Example 5.8. Consider $\tilde{I} = (x_2^7) \subseteq R = \mathbb{K}[x_0, x_1, x_2]$, and $\hat{F} = \{x_0x_1^6 + x_1^7\}$, which is marked on $\{x_1^7\}$, and $U = (x_1^7)$ is a quasi-stable ideal in S. $\hat{F}$ is not a U-marked set in the sense of Definition 5.1, because the Pommaret basis $\mathcal{P}_{\tilde{J}}$ of $\tilde{J} = (x_1^7, x_2^7) \subseteq R$ includes also the terms $x_1^7x_2^a$ for $1 \leq a \leq 6$. Note that the degrevlex leading ideal of $J = (\hat{F}, \tilde{I})$ is exactly $\tilde{J}$; this implies that J (and hence also $(F, \tilde{I})$) is 13-regular (13 being the highest degree of an element of $\mathcal{P}_{\tilde{J}}$).

Now consider, as above, $\tilde{I} = (x_2^7)$, but set $F = \{x_1 x_2^6\}$. We have $F \subset S_7$, and $7 = \mathrm{reg}(\tilde{I})$; moreover F is already marked on $\{x_1 x_2^6\}$ which generates a quasi-stable ideal $U = (x_1 x_2^6) \subseteq S$. Since the Pommaret basis $\mathcal{P}_{\tilde{J}}$ of $\tilde{J} = (F, \tilde{I})$ is exactly $F \cup \{x_2^7\}$, F is a U-marked set in the sense of Definition 5.1. Note that the ideal $J = \tilde{J} = (F, \tilde{I}) \subseteq R$ is 7-regular, because it is a quasi-stable monomial ideal whose minimal Pommaret basis has maximal degree 7.

In the remaining part of this section, we assume that the field $\mathbb{K}$ is infinite.

We denote by $\mathrm{PGL}(k+1)$ the subset of $\mathrm{PGL}_{\mathbb{K}}(n+1)$ whose elements define invertible change of coordinates of the following kind:

$$x_i \mapsto x_i \quad \text{for } i = k+1, \dots, n, \quad x_j \mapsto \sum_{t=0}^{k} g_{jt} x_t \quad \text{for } j = 0, \dots, k.$$

For any element $g \in \mathrm{PGL}(k+1)$ we denote by $\tilde{g}$ the automorphism induced by g on S.

Proposition 5.9. *For a given degree $q \geq 0$, let $F \subset S_q$ be a finite set. There exists $g \in \mathrm{PGL}(k+1)$ such that $\tilde{g}(F)$ becomes after an autoreduction a marked set over a quasi-stable monomial submodule of S (according to* [3, Definition 4.3]*).*

Proof. We can directly apply [3, Corollary 7.4], because F is a subset of a single degree component of the finitely generated free graded $\mathbb{K}[x_0, \dots, x_k]$-module S by Lemma 5.4. □

Corollary 5.10. *For every field extension $\mathbb{L}$ of $\mathbb{K}$, let $J \subseteq R_{\mathbb{L}}$ be a saturated ideal containing $\tilde{I}$ and t be an integer such that $t \geq \max\{\mathrm{reg}(J), \mathrm{reg}(\tilde{I})\}$. Then, there exists $g \in \mathrm{PGL}(k+1)$ such that the ideal $\tilde{g}(J_t) \cdot S$ is generated by the image in S of a $\mathcal{P}_{\tilde{J}}$-marked basis H relative to $\tilde{I}$ that belongs to $\underline{\mathbf{Mf}}_{\tilde{I}_{\geq t}, \tilde{J}_{\geq t}}(\mathbb{L})$, for some saturated quasi-stable ideal $\tilde{J}$ containing $\tilde{I}$.*

Proof. By the assumptions on t, we can take a set F of generators of $(J_t)/\tilde{I}$ made only of elements of degree t, so that $F \subseteq S_t$. Recall that we are now assuming that $\mathbb{K}$ is infinite, and hence Zariski dense in any field extension $\mathbb{L}$.

Then by Proposition 5.9, there exists $g \in \mathrm{PGL}(k+1)$ such that $\tilde{g}(F)$ yields after an autoreduction a marked set H over a quasi-stable monomial submodule of S. By [3, Lemma 7.5], we can assume that H is a marked basis over the stable module $U = (\mathrm{Ht}(H))$ (see [3, Definition 3.2]) with regularity $\leq t$. Hence, the ideal $(\mathrm{Ht}(H), \mathcal{P}_{\tilde{I}_{\geq t}})$ has regularity $\leq t$, and its saturation $\tilde{J}$ contains $\tilde{I}$ by Lemma 2.8. Thus, thanks to Corollary 5.7, H is also a U-marked basis and we obtain the thesis by Definition 5.1. □

Recall that if J belongs to $\underline{\mathbf{Mf}}_{\tilde{J}}(\mathbb{K})$ then $J_{\geq t}$ belongs to $\underline{\mathbf{Mf}}_{\tilde{J}_{\geq t}}(\mathbb{K})$, but the converse is not true (e.g. [4, Example 3.8]). However, by Lemma 2.8, if $J_{\geq t}$ belongs to $\underline{\mathbf{Mf}}_{\tilde{J}_{\geq t}}(\mathbb{K})$ and contains $\tilde{I}_{\geq t}$, then J contains $\tilde{I}$.

Given the quasi-stable ideal $\tilde{I}$, let $p(z)$ be any Hilbert polynomial as in Section 2, and consider the sets

$$Q_{p(z)} := \{\tilde{J} \text{ saturated quasi-stable} \mid S/\tilde{J} \text{ has Hilbert polynomial } p(z)\},$$

$$Q_{p(z),\tilde{I}} := \{\tilde{J} \text{ saturated quasi-stable} \mid \tilde{J} \supseteq \tilde{I} \text{ and } S/\tilde{J} \text{ has Hilbert polynomial } p(z)\}.$$

The *Gotzmann number* r of the Hilbert polynomial $p(z)$ is the smallest integer such that $r \geq \operatorname{reg}(J)$ for every saturated ideal J defining a scheme lying on $\mathbf{Hilb}^{p(z)}_{\mathbb{P}^n}$.

For every $g \in \mathrm{PGL}(k+1)$, we consider the functor $\underline{\mathbf{Mf}}^{\tilde{g}}_{\tilde{I}_{\geq t},\tilde{J}_{\geq t}}$ that assigns to every $\mathbb{K}$-algebra A the set $\{(\tilde{g}^{-1}(G)) \subset A[x] \mid (G) \in \underline{\mathbf{Mf}}_{\tilde{I}_{\geq t},\tilde{J}_{\geq t}}(A)\}$ and to every $\mathbb{K}$-algebra morphism $\sigma : A \to A'$, the map

$$\begin{aligned}\underline{\mathbf{Mf}}^{\tilde{g}}_{\tilde{I}_{\geq t},\tilde{J}_{\geq t}}(\sigma) : \underline{\mathbf{Mf}}^{\tilde{g}}_{\tilde{I}_{\geq t},\tilde{J}_{\geq t}}(A) &\to \underline{\mathbf{Mf}}^{\tilde{g}}_{\tilde{I}_{\geq t},\tilde{J}_{\geq t}}(A') \\ \tilde{g}^{-1}(G) &\mapsto \tilde{g}^{-1}(\sigma(G)).\end{aligned}$$

The transformation $\tilde{g}^{-1}$ induces a natural isomorphism of functors between the functors $\underline{\mathbf{Mf}}_{\tilde{I}_{\geq t},\tilde{J}_{\geq t}}$ and $\underline{\mathbf{Mf}}^{\tilde{g}}_{\tilde{I}_{\geq t},\tilde{J}_{\geq t}}$. Hence, $\underline{\mathbf{Mf}}^{\tilde{g}}_{\tilde{I}_{\geq t},\tilde{J}_{\geq t}}$ is an open subfunctor of $\underline{\mathbf{Hilb}}^{p(z)}_X$ for every $g \in \mathrm{PGL}(k+1)$ thanks to Theorem 4.2 item (ii). Analogously, for every $g \in \mathrm{PGL}_{\mathbb{K}}(n+1)$, $\underline{\mathbf{Mf}}^{g}_{\tilde{J}_{\geq t}}$ is the open subfunctor of $\underline{\mathbf{Hilb}}^{p(z)}_{\mathbb{P}^n}$ that we obtain from $\underline{\mathbf{Mf}}_{\tilde{J}_{\geq t}}$ by the natural isomorphism induced by g^{-1}.

Theorem 5.11. *Let $\tilde{I} \subseteq R$ be a saturated quasi-stable ideal such that $S = R/\tilde{I}$ is a Cohen-Macaulay ring and let $X = \mathrm{Proj}(S)$ be the scheme defined by $\tilde{I}$. Let $p(z)$ be a Hilbert polynomial such that $p(t) \leq p_X(t)$ for $t \gg 0$, r be the Gotzmann number of $p(z)$ and $t := \max\{\operatorname{reg}(\tilde{I}), r\}$. Then, there is the open cover*

$$\underline{\mathbf{Hilb}}^{p(z)}_X = \bigcup_{g\in \mathrm{PGL}(k+1)} \left(\bigcup_{\tilde{J}\in Q_{p(z),\tilde{I}}} \underline{\mathbf{Mf}}^{\tilde{g}}_{\tilde{I}_{\geq t},\tilde{J}_{\geq t}} \right).$$

Proof. We can apply [3, Proposition 10.3] to the Hilbert functor $\underline{\mathbf{Hilb}}^{p(z)}_{\mathbb{P}^n}$, obtaining the following open cover

$$\underline{\mathbf{Hilb}}^{p(z)}_{\mathbb{P}^n} = \bigcup_{g\in \mathrm{PGL}_{\mathbb{K}}(n+1)} \left(\bigcup_{\tilde{J}\in Q_{p(z)}} \underline{\mathbf{Mf}}^{g}_{\tilde{J}_{\geq t}} \right). \tag{5.2}$$

Since $\underline{\mathbf{Hilb}}^{p(z)}_X$ is a closed subfunctor of $\underline{\mathbf{Hilb}}^{p(z)}_{\mathbb{P}^n}$, we have an open cover of $\underline{\mathbf{Hilb}}^{p(z)}_X$ intersecting it with the open subfunctors of (5.2). In order to cover $\underline{\mathbf{Hilb}}^{p(z)}_X$ it is enough to consider $J \in Q_{p(z),\tilde{I}}$, thanks to Theorem 3.4.

We now observe that it is even enough to only take $g \in \mathrm{PGL}(k+1)$ thanks to Corollary 5.10. We can now conclude taking into account that $\underline{\mathbf{Mf}}^{g}_{\tilde{J}_{\geq t}} \cap \underline{\mathbf{Hilb}}^{p(z)}_X = \underline{\mathbf{Mf}}^{\tilde{g}}_{\tilde{I}_{\geq t},\tilde{J}_{\geq t}}$ by Theorem 4.2 item (ii), combined with [9, Exercise VI-11]. □

6. Lex-point in Hilbert schemes over Macaulay-Lex quotients on quasi-stable ideals

6.1. **Generalities.** Following [21], the same notation and terminology as in Section 5 are here used also for quotient rings $S := R/M$ when M is any monomial ideal of R.

Definition 6.1. (see [17, 21]) A set W of terms of S is called a *lex-segment of S* if, for all terms $u, v \in S$ of the same degree, if u belongs to W and $v >_{lex} u$ then v belongs to W. A monomial ideal U of S is called a *lex-ideal* if the set of terms in U is a lex-segment of S.

Example 6.2. The image of a lex-ideal of R in S is a lex-ideal of S. However, there are lex-ideals of S that are not the image of a lex-ideal of R. For example, consider $n = 3$ and $\tilde{I} = (x_3^2, x_2^5)$. Then, the image U in S of the ideal $\tilde{J} = (x_3^2, x_3x_2, x_3x_1^2, x_2^5) \subseteq R$ is a lex-ideal in S, but $\tilde{J}$ is not a lex-ideal in R.

The quotient ring S is called a *Macaulay-Lex ring* if, for any homogeneous ideal U of S, there exists a lex-ideal of S having the same Hilbert function as U (e.g. [17]). If the monomial ideal M induces a Macaulay-Lex quotient ring, then we say that M is *Macaulay-Lex* and that M is a *Macaulay-Lex* ideal.

Example 6.3. Various families of Macaulay-Lex monomial ideals $M \subseteq R$ are known. We list some of them explicitly and point to references in other cases.

(1) The most well-known class of Macaulay-Lex ideals are the *Clements-Lindström ideals* [8]. They are ideals generated by a regular sequence $x_n^{d_n}, x_{n-1}^{d_{n-1}}, \dots, x_0^{d_0}$, where $1 \leq d_n \leq d_{n-1} \dots \leq d_0$ are integers or ∞ with $x_i^\infty = 0$. A Clements-Lindström ideal is a quasi-stable ideal $\tilde{I}$ and the quotient ring $S := R/\tilde{I}$, which is called a *Clements-Lindström ring*, is Cohen-Macaulay. If $d_n = 1$, one may as well work in a quotient of $\mathbb{K}[x_0, \dots, x_{n-1}]$ and drop the generator x_n.

(2) Abedelfatah [2, Theorem 4.5] discovered two families of Macaulay-Lex ideals, whose generating sets show some similarities to the generators of Clements-Lindström ideals. In our conventions, they are given as follows, under the conditions $2 \le e_n \le e_{n-1} \le \cdots \le e_0 \le \infty$ and $t_i < e_i$ for all i:
- $I = (x_n^{e_n}, x_n^{t_n} x_{n-1}^{e_{n-1}}, \ldots, x_n^{t_n} x_0^{e_0})$,
- $I = (x_n^{e_n}, x_n^{e_n-1} x_{n-1}^{e_{n-1}}, x_n^{e_n-1} x_{n-1}^{t_{n-1}} x_{n-2}^{e_{n-2}}, \ldots, x_n^{e_n-1} x_{n-1}^{t_{n-1}} \cdots x_1^{t_1} x_0^{e_0})$.

One can show that every such ideal is quasi-stable.

(3) Mermin [19] showed that a monomial regular sequence generates a Macaulay-Lex ideal if and only if it is of the form

$$(x_n^{e_n}, x_{n-1}^{e_{n-1}}, \ldots, x_{r+1}^{e_{r+1}}, x_r^{e_r-1} x_i),$$

where $e_n \le e_{n-1} \le \ldots \le e_r$ and $i \le r$. Note that such an ideal is quasi-stable if and only if $i = r$, i.e., if it is a Clements-Lindström ideal.

(4) A complete characterization of all Macaulay-Lex monomial ideals in $\mathbb{K}[x_0, x_1]$ is known (see e.g. [14]). In particular, there are many quasi-stable Macaulay-Lex ideals in the polynomial ring with two variables.

(5) Given n zero-dimensional Macaulay-Lex monomial ideals M_i, each of them in a polynomial ring with two variables, one can construct [15] a zero-dimensional Macaulay-Lex ideal in R from $M_1, \ldots, M_n$. Being zero-dimensional, this ideal is also quasi-stable. Note that the construction in [15] covers also more general cases.

(6) For each Macaulay-Lex monomial ideal $M_i \subseteq R_i = \mathbb{K}[x_i, \ldots, x_n]$, where $i \in \{1, \ldots, n\}$, also the extension ideal $(M_i) \cdot R \subseteq R$ is Macaulay-Lex. [20, Thm. 4.1]

Remark 6.4. The property of being Macaulay-Lex is not preserved under many common ideal operations like for example saturation.

Consider the monomial ideal $I = (x_3^2, x_3 x_2^7, x_3 x_2 x_1^7, x_3 x_2 x_1^2 x_0^7) \subset R = \mathbb{K}[x_0, \ldots, x_3]$. It is Macaulay-Lex according to Example 6.3 item (2). Its saturation is $I^{sat} = (x_3^2, x_3 x_2 x_1^2, x_3 x_2^7)$. We can apply [15, Proposition 1] to see that I^{sat} is not Macaulay-Lex. Consider the set $J_4 = \{x_3 x_1^3\} \subseteq R/I$. Its lex-segment in R/I^{sat} is $L_4 = \{x_3 x_2^3\}$. Multiplying J_4 with the generators $x_0, \ldots, x_3$ of the homogeneous maximal ideal, we obtain the set $\{x_0, x_1, x_2, x_3\}\{x_3 x_1^3\} = \{x_3 x_1^3 x_0, x_3 x_1^4, x_3 x_2 x_1^3, x_3^2 x_1^3\}$ which, modulo the ideal I^{sat}, is equal to $\{x_3 x_1^3 x_0, x_3 x_1^4\}$, a set with two elements; but multiplying L_4 with $x_0, \ldots, x_3$, we obtain the set $\{x_0, x_1, x_2, x_3\}\{x_3 x_2^3\} = \{x_3 x_2^3 x_0, x_3 x_2^3 x_1, x_3 x_2^4, x_3^2 x_2^3\}$ which, modulo the ideal I^{sat}, is equal to $\{x_3 x_2^4, x_3 x_2^3 x_1, x_3 x_2^3 x_0\}$, with three elements.

Thus, it is not possible to find a lex-ideal in the quotient R/I^{sat} having the same Hilbert function as the ideal $(J_4) + I^{sat}/I^{sat}$.

The Macaulay-Lex property is in general also not preserved when truncating an ideal at a given degree.

Consider the ideal $I = (x_3^2, x_3x_2^3, x_3x_2x_1^3, x_3x_2x_1^2x_0^3) \subset \mathbb{K}[x_0, ..., x_3]$. It is Macaulay-Lex according to Example 6.3 item (2). Consider the same ideal in $\mathbb{K}[w, x_0, ..., x_3]$, where the variable w is ranked lower than the others. The ideal is still Macaulay-Lex, see Example 6.3 item (6). Moreover, it is also saturated in this ring. As mentioned in Example 6.3 item (2), the ideal is quasi-stable. Its regularity is 8, as its minimal Pommaret basis has maximal degree 8 [27, Corollary 5.5.18] (one element of degree 8 being $x_3x_2^2x_1^2x_0^3$). The truncation $I_{\geq 8}$ is a stable ideal [27, Proposition 5.5.19]. A minimal generator of $I_{\geq 8}$ is $a = x_3x_2x_1^2x_0^4$. Now consider the term $b = x_3x_2^2x_1x_0^3$. b is also of degree 8, is lexicographically larger than a, and $\min(a) = \min(b)$. But b is not in the ideal $I_{\geq 8.}$ This shows that $I_{\geq 8}$ is not piecewise lexsegment [2, Definition 3.1] although it is stable. It follows [2, Theorem 3.6] that $I_{\geq 8}$ is not Macaulay-Lex.

From now we assume that $M := \tilde{I}$ is a quasi-stable ideal in R and that $S = R/\tilde{I}$ is a Macaulay-Lex ring. Recall that a monomial ideal U of S is quasi-stable in S if the ideal $(\mathcal{B}_U \cup \mathcal{B}_{\tilde{I}})$ is quasi-stable in R.

Lemma 6.5. *With the notation above,*

(i) If W is a lex-segment in S then $\{x_0, \dots, x_n\} \cdot W$ is a lex-segment in S.
(ii) A lex-ideal U in S is quasi-stable.
(iii) If U is a lex-ideal of S, then $(\mathcal{B}_U \cup \mathcal{B}_{\tilde{I}})^{sat}/\tilde{I}$ is a lex-ideal.

Proof. For item (i) see [20, Proposition 2.5]. For item (ii), let τ be a term of U with minimal variable x_i and let $x_j > x_i$. Since $x_j \frac{\tau}{x_i} >_{lex} \tau$, we must have that $x_j \frac{\tau}{x_i}$ belongs to U unless it belongs to $\tilde{I}$. Then, we conclude because $\tilde{I}$ is quasi-stable. Item (iii) now follows from item (ii) and from the properties of the lexicographic term order, because thanks to the properties of quasi-stable ideals we obtain the saturation replacing x_0 by 1 in every generator. □

If $\tilde{I}$ is a saturated ideal we can consider the projective scheme $X = \mathrm{Proj}(S)$ and the Hilbert scheme $\mathbf{Hilb}_X^{p(z)}$ on the Macaulay-Lex ring S, for an admissible Hilbert polynomial $p(z)$.

Theorem 6.6. *Let $S = R/\tilde{I}$ be a Macaulay-Lex ring and $X = \mathrm{Proj}(S)$. Then $\mathbf{Hilb}_X^{p(z)}$ is non-empty if and only if it contains a (unique) point Y defined by a lex-ideal of S. Moreover, Y has the minimal possible Hilbert function in $\mathbf{Hilb}_X^{p(z)}$.*

Proof. Let r be the maximum between the Gotzmann number of $p(z)$ and the regularity of $\tilde{I}$. If $\mathbf{Hilb}_X^{p(z)}$ is non-empty, then there exists at least a lex-ideal U of S such that S/U has Hilbert polynomial $p(z)$, because S is Macaulay-Lex. In particular, letting $\tilde{p}(z)$ be the Hilbert polynomial of $S = R/\tilde{I}$, the set W made of the $\tilde{p}(r)-p(r)$ lex-largest terms of U of degree r is a lex-segment. Thanks to Lemma 6.5 item (iii), the ideal $(W \cup \mathcal{B}_{\tilde{I}})^{sat}/\tilde{I}$ is a lex-ideal too and by construction defines the desired point Y in $\mathbf{Hilb}_X^{p(z)}$. Indeed, by definition of saturation we obtain the same point Y starting from any other lex-ideal of S with Hilbert polynomial $p(z)$, so that the last assertion also follows. □

Definition 6.7. Let S be a Macaulay-Lex ring over a saturated quasi-stable ideal $\tilde{I}$ and $X = \mathrm{Proj}(S)$. If $\mathbf{Hilb}_X^{p(z)}$ is non-empty, then its unique point defined by a lex-ideal of S is called the *lex-point* of $\mathbf{Hilb}_X^{p(z)}$.

6.2. **Examples of smooth and singular lex-points.** We now give some examples both of smooth and singular lex-points in Hilbert schemes over Macaulay-Lex rings.

When it is necessary, we apply Algorithm 1 and, like in Section 4, take into account the fact that a scheme $\mathbf{Mf}_{\tilde{I}_{\geq t}, \tilde{J}_{\geq t}}$ is an open subscheme of $\mathbf{Hilb}_X^{p(z)}$. Hence, the Zariski tangent space to $\mathbf{Mf}_{\tilde{I}_{\geq t}, \tilde{J}_{\geq t}}$ at one of its points is equal to the Zariski tangent space to $\mathbf{Hilb}_X^{p(z)}$ at the same point (see also [5, Corollary 1.9]).

Example 6.8. Here is a trivial case of smooth lex-points. When $\tilde{I} = (x_3, \dots, x_n)$, X is the projective plane and so every point of the Hilbert scheme $\mathbf{Hilb}_X^{p(z)}$ is smooth, for a constant polynomial $p(z)$. Hence, in this case every point in $\mathbf{Hilb}_{\mathbb{P}^n_{\mathbb{K}}}^{p(z)}$, even singular, corresponds to a smooth point in $\mathbf{Hilb}_X^{p(z)}$. This is the extremal case of the trivial situation in which $\tilde{I}$ is generated by variables. In fact, in such case the image in S of a lex-point of R is still smooth because it is simply the lex-point in a Hilbert scheme over a lower dimensional projective space.

Example 6.9. The following saturated quasi-stable ideals $\tilde{I} \subset \tilde{J}$ in the ring $R = \mathbb{K}[x_0, \dots, x_n]$ give *smooth* lex-points Y in the Hilbert scheme $\mathbf{Hilb}_X^{p(z)}$, where $X = \mathrm{Proj}(R/\tilde{I})$, $p(z)$ is the Hilbert polynomial of $R/\tilde{J}$ and Y is defined by $\tilde{J}/\tilde{I}$:

(i) $\tilde{I} = (x_n^k, x_n^{k-1}x_{n-1}) \subset \tilde{J} = (x_n^k, x_n^{k-1}x_{n-1}, x_n^{k-1}x_{n-2})$, for every $n \geq 3$ and $k \geq 2$

(ii) $\tilde{I} = (x_n, x_{n-1})^2 \subset \tilde{J} = (x_n^2, x_n x_{n-1}, x_n x_{n-2}, x_{n-1}^2)$, for every $n \geq 3$.

In case (i), the ideal $\tilde{I}$ is Macaulay-Lex thanks to [14, Theorem 4] and [20, Theorem 4.1] and the ideal $\tilde{J}$ already defines a lex-point in R. By a pencil-and-paper work, we apply Algorithm 1 and obtain the following results. The set $\mathscr{H}$ is made of the polynomial $h = x_n^{k-1}x_{n-2} + c_1 x_0^k + \dots c_s x_n^{k-1}x_{n-3}$ in the ring $\mathbb{Q}[c_1,\dots,c_s][x_0,\dots,x_n]$, with $s = \binom{n+k}{n} - 3$. By $\longrightarrow_{\mathscr{H}^*}$ we need to reduce the three polynomials $x_n h$, $x_{n-1}h$, $x_n \cdot x_n^{k-1}x_{n-1}$ and then apply the reduction modulo $\tilde{I}$. The last polynomial $x_n \cdot x_n^{k-1}x_{n-1}$ is already reduced with respect to $\longrightarrow_{\mathscr{H}^*}$ and moreover belongs to $\tilde{I}$.

The terms in the polynomial $x_{n-1}h$ that do not belong to $\tilde{I}$ are already reduced modulo $\longrightarrow_{\mathscr{H}^*}$. Hence, their coefficient must be null. On the other hand, the coefficients of the terms belonging to $\tilde{I}$ are free. Such terms are of type $x_{n-1}\cdot x_n^{k-1}x_j$, where $0 \le j \le n-3$, hence they are $n-2$.

The terms in the polynomial $x_n h$ that belong to $\tilde{I}$ are of type $x_n \cdot x_n^{k-1}x_j$, where $0 \le j \le n-3$, hence they have the same coefficients of the analogous terms in $x_{n-1}h$. All the other terms must have a null coefficient because of the polynomial $x_{n-1}h$, even those that are not reduced with respect to $\longrightarrow_{\mathscr{H}^*}$.

In conclusion, the ideal $\mathscr{R}'$ is generated by $s-(n-2)$ parameters c_i, and hence $\mathbf{Mf}_{\tilde{I}_{\ge t},\tilde{J}_{\ge t}}$ is a linear variety of dimension $n-2$. Hence, it is smooth and, in particular, the point Y is smooth in $\mathbf{Hilb}_X^{p(z)}$.

In case (ii), the ideal $\tilde{I}$ is Macaulay-Lex thanks to [18, Theorem 2.1] and the ideal $\tilde{J}$ does not define a lex-point in R, but $\tilde{J}/\tilde{I}$ defines a lex point in $S = R/\tilde{I}$. As for case (i), we apply Algorithm 1. We proceed in an analogous way as for case (i), obtaining that $\mathbf{Mf}_{\tilde{I}_{\ge t},\tilde{J}_{\ge t}}$ is a linear variety of dimension $2n-3$.

Example 6.10. The saturated quasi-stable ideals $\tilde{I} = (x_3^3, x_3^2x_2) \subset \tilde{J} = (x_3^2, x_3x_2, x_3x_1) \subset R = \mathbb{K}[x_0,\dots,x_3]$ give a *singular* lex-point Y in the Hilbert scheme $\mathbf{Hilb}_X^{p(z)}$, where $X = \mathrm{Proj}(R/\tilde{I})$, $p(z)$ is the Hilbert polynomial of $R/\tilde{J}$ and Y is defined by $\tilde{J}/\tilde{I}$. Like in Example 6.9, the ideal $\tilde{I}$ is Macaulay-Lex thanks to [14, Theorem 4] and [20, Theorem 4.1]. For $t \ge 1$, the dimension of $\mathbf{Mf}_{\tilde{I}_{\ge t},\tilde{J}_{\ge t}}$ is 2 and the dimension of its Zariski tangent space at Y is 6. In this case we have $\mathcal{P}_{\tilde{J}} \setminus \mathcal{P}_{\tilde{I}} = \mathcal{P}_{\tilde{J}}$ and so we apply the computation described at the end of Section 2.

Remark 6.11. The points Y of Examples 4.6 and 4.7 are singular lex-points in Hilbert schemes over Clements-Lindström rings (see also Example 6.2). Actually, it seems not obvious to find a non-trivial example of smooth lex-point in a Clements-Lindström ring. However, there are other points that are smooth. For example, letting $\tilde{I} = (x_3^2) \subset \tilde{J} = (x_3^2, x_2^2) \subset R = \mathbb{K}[x_0,\dots,x_3]$, both the dimensions of $\mathbf{Mf}_{\tilde{I}_{\ge 1},\tilde{J}_{\ge 1}}$ and of the Zariski tangent space at the point defined by $\tilde{J}/\tilde{I}$, which is not a lex-point, are 8 in the Hilbert scheme $\mathbf{Hilb}_X^{p(z)}$ with $X = \mathrm{Proj}(R/\tilde{I})$ and $p(z) = 4z$.

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