Structural Identifiability and Discrete Symmetries

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Abstract

We discuss the use of symmetries for analysing the structural identifiability and observability of control systems. Special emphasis is put on the role of discrete symmetries, in contrast to the more commonly studied continuous or Lie symmetries. We argue that discrete symmetries are the origin of parameters which are structurally locally identifiable, but not globally. We exploit this fact to present a methodology for structural identifiability analysis that detects such parameters and characterizes the symmetries in which they are involved. We demonstrate the use of our methodology by applying it to four case studies.

1 Introduction

We are interested in two closely related properties of dynamic models, namely structural identifiability and observability. A state variable is *observable* if its initial value can be determined from knowledge of the model's subsequent input and output trajectories, and a parameter in a model is structurally *identifiable* if it can be determined in the same way. These properties are *structural*, in the sense that they only depend on the model equations; they are not affected by the quality of the experimental data. Since a parameter can be considered as a constant state variable, structural identifiability will be treated as a particular case of observability [7, 44].

Symmetries [3, 26] are a useful tool for analysing control systems [21, 22, 31, 34, 43, 44, 45, 46, 47, 48]. In particular, the existence of continuous symmetries leaving the output invariant represents an obstruction to local observability. Recently, two of the present authors analysed for a larger number of parametric models the difference between *local* and *global* structural identifiability [29]. If a parameter is locally, but not globally identifiable, then for a given output finitely

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many different values are possible for this parameter. One goal of the current paper is to argue that this is related to the existence of *discrete* symmetries.

Discrete symmetries have been studied much less than their continuous counterpart. The most popular approach is due to Hydon [15] and an alternative approach was proposed by Gaeta and Rodríguez [10]. Both approaches are not completely algorithmic. Furthermore, Hydon's method can only be used, if continuous symmetries exist. In the particularly interesting situation that a control system is structurally locally identifiable, but not globally, this will not be the case.

We will follow an approach proposed by Reid *et al.* [27] and work with the finite determining system instead of the infinitesimal one. This system is generally highly non-linear and thus hard to analyse. Reid *et al.* relied on an algorithm `rifsimp` [28] which uses many heuristics and hence may fail. We will use rigorous techniques from differential algebra (see [40] for a concise introduction), in particular the Thomas decomposition originally developed by Thomas [41, 42] and later revived by Gerdt [11]. It is implemented in Maple as a package called TDDS [12] (for an earlier version see [2]).

For just deciding identifiability, it is not necessary to *compute* symmetries; it suffices to *detect* their existence. This can be done algorithmically with methods from the formal theory of differential equations (see e. g. [38] and references therein for an extensive introduction); some applications of such methods in the context of symmetry theory have e. g. been discussed in [32, 36, 37].

2 Discrete and Continuous Symmetries for Observability

We will discuss finite-dimensional nonlinear control systems

$$\dot{\mathbf{x}} = \mathbf{f}(t, \mathbf{x}, \mathbf{u})\,, \tag{1a}$$

$$\mathbf{y} = \mathbf{h}(t, \mathbf{x}, \mathbf{u})\,. \tag{1b}$$

Here, as usual, $\mathbf{x} \in \mathbb{R}^n$ denotes the *state*, $\mathbf{u} \in \mathbb{R}^m$ the *input*, $\mathbf{y} \in \mathbb{R}^\ell$ the *output* and all these dependent variables are assumed to be smooth functions of a single independent variable, the time t. We will assume that the right hand sides are rational functions of their arguments. This covers most systems appearing in typical applications.

For an observability analysis, one considers *point transformations* of the restricted form[1]

$$\tilde{t} = t\,, \qquad \tilde{\mathbf{x}} = \mathbf{X}(t, \mathbf{x})\,, \qquad \tilde{\mathbf{u}} = \mathbf{u}\,. \tag{2a}$$

Since the input is considered as known, we should indeed admit only transformations that do not change it. And as observability is concerned with reconstructing the state $\mathbf{x}$ from the output $\mathbf{y}$, we cannot allow for transformation of the time t. Finally, the transformation of the output $\mathbf{y}$ is determined by entering (2a) into (1b). A point transformation must be invertible requiring that the Jacobian $\partial\mathbf{X}/\partial\mathbf{x}$ must possess full rank almost everywhere.

[1] One could admit that $\mathbf{X}$ also depends on $\mathbf{u}$. But one will find as one determining equation that $\mathbf{X}_\mathbf{u} = 0$. Hence our restricted ansatz suffices.

The transformation of derivatives of arbitrary order is completely determined by the chain rule (see e. g. [3]); for first-order one obtains the *prolongation*

$$\dot{\tilde{\mathbf{x}}} = \mathbf{X}^{(1)}(t, \mathbf{x}, \dot{\mathbf{x}}) = \mathbf{X}_t + \mathbf{X}_{\mathbf{x}}\dot{\mathbf{x}}\,. \tag{2b}$$

A control system is *not* observable, if it possesses symmetries that leave the output invariant. Hence, we require that both the state equation (1a) and the output equation (1b) remain invariant under the point transformation (2). Entering (2) into (1) yields the conditions

$$\mathbf{X}^{(1)}(t, \mathbf{x}, \dot{\mathbf{x}}) = \mathbf{f}\big(t, \mathbf{X}(t, \mathbf{x}), \mathbf{u}\big)\,, \tag{3a}$$

$$\mathbf{h}(t, \mathbf{x}, \mathbf{u}) = \mathbf{h}\big(t, \mathbf{X}(t, \mathbf{x}), \mathbf{u}\big)\,. \tag{3b}$$

Note that (3b) is a purely algebraic equation, i. e. no derivatives of X appear in it.

We eliminate $\dot{\mathbf{x}}$ in the left hand side of (3a) by substituting $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{u})$, i. e. by using (1a), and then clear denominators. Then each equation in (3) can be considered as a multivariate polynomial in the inputs $\mathbf{u}$ with coefficients Δ_j which are polynomials in the variables t, $\mathbf{x}$ and the functions $\mathbf{X}$ plus their first derivatives $\mathbf{X}_t$, $\mathbf{X}_{\mathbf{x}}$. The *finite determining system* for point symmetries of (1a) is then obtained by requiring that all these coefficients Δ_j vanish, as the invariance must hold for arbitrary inputs $\mathbf{u}$. It represents a complicated system of first-order polynomially nonlinear partial differential equations,

$$\Delta_j(t, \mathbf{x}, \mathbf{X}, \mathbf{X}_t, \mathbf{X}_{\mathbf{x}}) = 0\,, \qquad j = 1, \dots, J\,, \tag{4}$$

for the functions $\mathbf{X}$ which is generally not solved for any derivatives, i. e. fully implicit. The number J of equations in (4) depends not only on the dimensions n, m, ℓ of the vectors $\mathbf{x}$, $\mathbf{u}$, $\mathbf{y}$, but also on the degrees with which the inputs $\mathbf{u}$ appear in (1). If we assume that (1) is affine in the inputs (as it is often the case in applications), then $J = (n + \ell)(m + 1)$.

Most people using symmetry methods do not set up the finite determining system (4). Instead, they use the infinitesimal approach developed by Lie already in the 19th century. Mathematically, it corresponds to determining the Lie algebra of the Lie group of point symmetries. In this approach, one assumes that one has at least a one-parameter group of transformations, i. e. one must augment (2a) by a parameter ϵ:

$$\tilde{t} = t\,, \qquad \tilde{\mathbf{x}} = \mathbf{X}(t, \mathbf{x}, \epsilon)\,, \qquad \tilde{\mathbf{u}} = u\,. \tag{5}$$

Here we always assume that we obtain the identity transformation for $\epsilon = 0$ and that the concatenation of two transformations with the parameter values ϵ_1 and ϵ_2, respectively, yields the transformation for $\epsilon_1 + \epsilon_2$.

For small ϵ, one can linearise (5) around $\epsilon = 0$ and obtains the corresponding *infinitesimal generator*, the vector field

$$V = \left.\frac{\partial \mathbf{X}}{\partial \epsilon}\right|_{\epsilon=0} \cdot \partial_{\mathbf{x}}\,. \tag{6}$$

One can obtain the *prolonged infinitesimal generator* $V^{(1)}$ by linearising (2b). But given an arbitrary vector field

$$V = \boldsymbol{\xi}(t, \mathbf{x}) \cdot \partial_{\mathbf{x}}\,, \tag{7}$$

$V^{(1)}$ can also be determined directly: with the ansatz $V^{(1)} = V + \boldsymbol{\xi}^{(1)}(t, \mathbf{x}, \dot{\mathbf{x}}) \cdot \partial_{\dot{\mathbf{x}}}$ a short calculation yields $\boldsymbol{\xi}^{(1)} = \boldsymbol{\xi}_t + \boldsymbol{\xi}_{\mathbf{x}} \dot{\mathbf{x}}$. This formula can be extended to higher derivatives [3], but the expressions are getting rapidly rather complicated.

The ansatz (7) defines an *infinitesimal symmetry* of the control system (1), if and only if it satisfies

$$V^{(1)}\big(\dot{\mathbf{x}} - \mathbf{f}(t, \mathbf{x}, \mathbf{u})\big)\Big|_{\dot{\mathbf{x}}=\mathbf{f}(t,\mathbf{x},\mathbf{u})} = 0\,, \tag{8a}$$

$$V\mathbf{h}(t, \mathbf{x}, \mathbf{u}) = 0\,. \tag{8b}$$

The left hand side of the first equations means that one applies first the prolonged generator $V^{(1)}$ to the expression $\dot{\mathbf{x}} - \mathbf{f}(t, \mathbf{x}, \mathbf{u})$ and then replaces in the result every occurrence of $\dot{\mathbf{x}}$ by $\mathbf{f}(t, \mathbf{x}, \mathbf{u})$. For a rational $\mathbf{f}$, one has to clear denominators in the result. Then one can again consider each equation in (8) as a multivariate polynomial in the inputs $\mathbf{u}$ with coefficients δ_j which are polynomials in the variables t, $\mathbf{x}$ and the functions $\boldsymbol{\xi}$ plus their partial derivatives $\boldsymbol{\xi}_t$, $\boldsymbol{\xi}_x$ and the *infinitesimal determining system* is obtained by requiring that all coefficients of these polynomials vanish. In contrast to the finite determining system, we obtain now *linear* partial differential equations

$$\delta_j(t, \mathbf{x}, \boldsymbol{\xi}, \boldsymbol{\xi}_t, \boldsymbol{\xi}_{\mathbf{x}}) = 0\,, \qquad j = 1, \dots, J\,, \tag{9}$$

for the coefficients $\boldsymbol{\xi}$ of the vector field V.

If $\hat{\boldsymbol{\xi}}(t, \mathbf{x})$ is a solution of the infinitesimal determining system (9), then integrating the system $\frac{d\mathbf{X}}{d\epsilon} = \hat{\boldsymbol{\xi}}(T, \mathbf{X})$ yields a one-parameter group of transformations of the form (5). For any choice of the parameter ϵ, the obtained point transformation is a solution of the finite determining system (4). But we cannot expect that *all* solutions of (4) arise in this way, i. e. can be embedded in a one-parameter family of transformations.

From a mathematical point of view, the deeper reason is that firstly one assumes in the infinitesimal approach that one has a whole Lie group of symmetries, i. e. *continuous symmetries*. Secondly, even then, this group may consist of several disjoint components and infinitesimal methods see only the connected component of the identity. If a dynamical system has only *discrete symmetries*, then these do not form a Lie group, but typically a finite group.

3 Symmetries and Structural Identifiability

In practise, most control systems also depend on *parameters*, i. e. the right hand sides of (1) are also functions of additional arguments $\boldsymbol{\theta} \in \mathbb{R}^k$ representing the parameters:

$$\dot{\mathbf{x}} = \mathbf{f}(t, \mathbf{x}, \mathbf{u}, \boldsymbol{\theta})\,, \tag{10a}$$

$$\mathbf{y} = \mathbf{h}(t, \mathbf{x}, \mathbf{u}, \boldsymbol{\theta})\,. \tag{10b}$$

We distinguish between local and global versions of structural identifiability. A parameter θ_i of a model of the form (10) is *structurally globally identifiable* (SGI) if, for any admissible inputs and almost all parameter vectors $\boldsymbol{\theta}$, $\boldsymbol{\theta}^*$, the equality $\mathbf{y}(t, \boldsymbol{\theta}) = \mathbf{y}(t, \boldsymbol{\theta}^*)$ implies $\theta_i = \theta_i^*$. A model is SGI, if every parameter θ_i is SGI. Likewise, a parameter is *structurally locally identifiable* (SLI) if, for almost all vectors $\boldsymbol{\theta}$, $\boldsymbol{\theta}^*$ and almost all initial conditions, the equality $\mathbf{y}(t,\boldsymbol{\theta}) = \mathbf{y}(t,\boldsymbol{\theta}^*)$ allows only finitely many different values of θ_i.

Structural identifiability may be considered as a special case of observability, if we treat the parameters as additional state variables with a trivial dynamics, i. e. if we augment (10) by

$$\dot{\boldsymbol{\theta}} = 0\,. \tag{10c}$$

Consequently, we have to extend (2a) to

$$\tilde{t} = t\,, \quad \tilde{\mathbf{x}} = \mathbf{X}(t,\mathbf{x},\boldsymbol{\theta})\,, \quad \tilde{\boldsymbol{\theta}} = \boldsymbol{\Theta}(t,\mathbf{x},\boldsymbol{\theta})\,, \quad \tilde{\mathbf{u}} = \mathbf{u} \tag{11a}$$

and the prolongation (2b) to

$$\dot{\tilde{\mathbf{x}}} = \mathbf{X}^{(1)}(t,\mathbf{x},\boldsymbol{\theta},\dot{\mathbf{x}},\dot{\boldsymbol{\theta}}) = \mathbf{X}_t + \mathbf{X}_{\mathbf{x}}\dot{\mathbf{x}} + \mathbf{X}_{\boldsymbol{\theta}}\dot{\boldsymbol{\theta}}\,, \tag{11b}$$

$$\dot{\tilde{\boldsymbol{\theta}}} = \boldsymbol{\Theta}^{(1)}(t,\mathbf{x},\boldsymbol{\theta},\dot{\mathbf{x}},\dot{\boldsymbol{\theta}}) = \boldsymbol{\Theta}_t + \boldsymbol{\Theta}_{\mathbf{x}}\dot{\mathbf{x}} + \boldsymbol{\Theta}_{\boldsymbol{\theta}}\dot{\boldsymbol{\theta}}\,. \tag{11c}$$

The invariance conditions (3) take now the extended form

$$\mathbf{X}^{(1)}(t,\mathbf{x},\boldsymbol{\theta},\dot{\mathbf{x}},\dot{\boldsymbol{\theta}}) = \mathbf{f}\big(t,\mathbf{X}(t,\mathbf{x},\boldsymbol{\theta}),\mathbf{u},\boldsymbol{\Theta}(t,\mathbf{x},\boldsymbol{\theta})\big)\,, \tag{12a}$$

$$\boldsymbol{\Theta}^{(1)}(t,\mathbf{x},\boldsymbol{\theta},\dot{\mathbf{x}},\dot{\boldsymbol{\theta}}) = 0\,, \tag{12b}$$

$$\mathbf{h}(t,\mathbf{x},\mathbf{u},\boldsymbol{\theta}) = \mathbf{h}\big(t,\mathbf{X}(t,\mathbf{x},\boldsymbol{\theta}),\mathbf{u},\boldsymbol{\Theta}(t,\mathbf{x},\boldsymbol{\theta})\big)\,. \tag{12c}$$

In these equations, we substitute $\dot{\mathbf{x}}$ by $\mathbf{f}(t,\mathbf{x},\mathbf{u},\boldsymbol{\theta})$ and $\dot{\boldsymbol{\theta}}$ by 0. Again, if the control system (1) is even polynomial, then we obtain immediately expressions which can be considered as multivariate polynomials in the inputs $\mathbf{u}$ with coefficients Δ_j which are polynomials in the variables t, $\mathbf{x}$, $\boldsymbol{\theta}$ and the functions $\mathbf{X}$, $\boldsymbol{\Theta}$ plus their derivatives $\mathbf{X}_t$, $\mathbf{X}_{\mathbf{x}}$, $\boldsymbol{\Theta}_t$, $\boldsymbol{\Theta}_{\mathbf{x}}$ (since we have for the parameters the trivial dynamics $\dot{\boldsymbol{\theta}} = 0$, the derivatives $\mathbf{X}_{\boldsymbol{\theta}}$ and $\boldsymbol{\Theta}_{\boldsymbol{\theta}}$ will not show up). For the finite determining system, the only difference to (4) is now that the functions Δ_j depend on more arguments.

We are not aware of a rigorous argument why $\boldsymbol{\Theta}$ should not depend on $\mathbf{x}$, but in our experience this rarely happens. Hence, it appears natural to impose the restriction $\tilde{\boldsymbol{\theta}} = \boldsymbol{\Theta}(t,\boldsymbol{\theta})$ or equivalently to add the determining equations $\boldsymbol{\Theta}_{\mathbf{x}} = 0$. This considerably simplifies concrete computations and in some cases makes them feasible at all.

For the infinitesimal approach, we write the ansatz (7) as

$$V = \boldsymbol{\xi}(t,\mathbf{x},\boldsymbol{\theta}) \cdot \partial_{\mathbf{x}} + \boldsymbol{\zeta}(t,\mathbf{x},\boldsymbol{\theta}) \cdot \partial_{\boldsymbol{\theta}} \tag{13}$$

with its first prolongation given by

$$V^{(1)} = V + \boldsymbol{\xi}^{(1)}(t,\mathbf{x},\boldsymbol{\theta},\dot{\mathbf{x}},\dot{\boldsymbol{\theta}}) \cdot \partial_{\dot{\mathbf{x}}} + \boldsymbol{\zeta}^{(1)}(t,\mathbf{x},\boldsymbol{\theta},\dot{\mathbf{x}},\dot{\boldsymbol{\theta}}) \cdot \partial_{\dot{\boldsymbol{\theta}}} \tag{14}$$

where the coefficients possess the explicit representation

$$\boldsymbol{\xi}^{(1)} = \boldsymbol{\xi}_t + \boldsymbol{\xi}_{\mathbf{x}}\dot{\mathbf{x}} + \boldsymbol{\xi}_{\boldsymbol{\theta}}\dot{\boldsymbol{\theta}}\,, \tag{15a}$$

$$\boldsymbol{\zeta}^{(1)} = \boldsymbol{\zeta}_t + \boldsymbol{\zeta}_{\mathbf{x}}\dot{\mathbf{x}} + \boldsymbol{\zeta}_{\boldsymbol{\theta}}\dot{\boldsymbol{\theta}}\,. \tag{15b}$$

The infinitesimal symmetry condition (8) takes now the form

$$V^{(1)}\big(\dot{\mathbf{x}} - \mathbf{f}(t,\mathbf{x},\mathbf{u},\boldsymbol{\theta})\big)\Big|_{\dot{\mathbf{x}}=\mathbf{f}(t,\mathbf{x},\mathbf{u},\boldsymbol{\theta}),\ \dot{\boldsymbol{\theta}}=0} = 0\,, \tag{16a}$$

$$V^{(1)}\big(\dot{\boldsymbol{\theta}}\big)\Big|_{\dot{\mathbf{x}}=\mathbf{f}(t,\mathbf{x},\mathbf{u},\boldsymbol{\theta}),\ \dot{\boldsymbol{\theta}}=0} = 0\,, \tag{16b}$$

$$V\mathbf{h}(t,\mathbf{x},\mathbf{u},\boldsymbol{\theta}) = 0\,. \tag{16c}$$

The second equation is easy to evaluate: it yields the partial differential equation

$$\boldsymbol{\zeta}^{(1)}\big(t, \mathbf{x}, \mathbf{f}(t, \mathbf{x}, \mathbf{u}, \boldsymbol{\theta}), 0\big) = \boldsymbol{\zeta}_t + \boldsymbol{\zeta}_{\mathbf{x}} \cdot \mathbf{f}(t, \mathbf{x}, \mathbf{u}, \boldsymbol{\theta}) = 0\,. \tag{17}$$

If a control system (10) admits continuous symmetries preserving the output, then it cannot be locally identifiable, as infinitely many solutions yield the same output. These solutions form a continuous family and thus cannot even locally be distinguished. Thus, the existence of such symmetries implies unidentifiability. The absence of such symmetries guarantees structural local identifiability [46, Cor. 3.1]. In the case of a finite group of discrete symmetries, only finitely many solutions lead to the same output. The system is then locally identifiable, as in a sufficiently small neighbourhood only one appropriate solution exists, but not globally. Following [29, 8, 24], we use the acronym SLING for a parameter that is Structurally Locally Identifiable, but Not Globally.

4 Formal Analysis of Determining Systems

Even the linear determining system for *infinitesimal* symmetries is in practise difficult to solve explicitly for systems of ordinary differential equations (the situation is fairly different for partial differential equations where such determining systems are routinely solved by computer algebra packages relying on a combination of heuristics and systematic theory). The *finite* determining systems will be explicitly solvable only in exceptional cases. However, a formal analysis of the determining system is possible much more often. "Formal" has here two meanings: it implies that firstly one only works with the differential equations themselves, trying for example to bring them into a suitable normal form, andy that secondl one considers only formal power series solutions (see [38] for a more extensive discussion).

In our context, one of the most important information is the size of the solution space: if there is more than one solution, then non-trivial symmetries exist and the given system is not observable. There are methods for formally counting solutions (or more precisely freely choosable Taylor coefficients of formal solutions) — see e. g. [17, 18, 35] — and for the infinitesimal determining system such a counting was already proposed by Schwarz [33].

In this letter, we concentrate only on distinguishing between infinitely many and only finitely many solutions corresponding to the distinction between not even locally observable and locally but not globally observable. For this purpose, it suffices to bring the determining system into a kind of normal form. For the linear infinitesimal determining system, *Janet bases* provide such a normal form [38, 39]; in the nonlinear case, one needs the *Thomas decomposition* [30].

As the infinitesimal determining system has always the zero solution and the finite determining system has always the identity map as trivial solutions, determining systems can never be inconsistent. If the infinitesimal determining system has more solutions than only the zero solution, then it has automatically infinitely many solutions implying local non-observability. This is easy to detect from a Janet basis, as the basis becomes trivial, if only the zero solution exists.

The analysis of nonlinear systems is more involved. The first difference is that one cannot simply transform the given system into a normal form, but must perform case distinctions so that the decomposition actually consists of a finite number of so-called simple systems which have to be studied separately. We have infinitely many solutions, if at least one of these simple systems contains an equation which is still a differential equation, i. e. in which still a derivative appears. Indeed, in this case one can choose freely at least one Taylor coefficient of the formal solution.

The simplest case for a finite solution space arises, if each simple system admits exactly one solution (in this case, the simple system contains for each unknown function one linear equation determining it uniquely). Then the total number of solutions is just the number of simple systems. Generally, a simple system may also contain algebraic equations of higher degree. If we define the degree of a simple system as the product of the degrees of all the equations contained in it, then the number of solutions is exactly this degree. The total number of solutions of the determining system is then the sum of the degrees of the arising simple systems, as the theory of the Thomas decomposition asserts that the solutions spaces of the simple systems are disjoint (this is an important difference to an alternative decomposition method due to Boulier *et al.* [4, 5] where solution spaces can intersect).

5 Examples

We present four different examples. In the first one, a simple linear compartmental model from physiology, only discrete symmetries appear (hence all parameters and states are at least locally structural identifiable). As they consist of a simple permutation, they could have been guessed by a direct inspection of the system. The main advantage of our systematic approach is that it guarantees that there are indeed no further symmetries (continuous or discrete). In what follows, if a simple system does not include an equation for a transformed variable, it is understood to admit the trivial identity transformation.

Example 5.1. The 4-compartmental mammillary model from [23, Ex. 7] consists of the linear control system:

$$\begin{cases} \dot{x}_1 = -(k_{21}+k_{31}+k_{41}+k_{01})x_1 + \\ \qquad k_{12}x_2 + k_{13}x_3 + k_{14}x_4 + u\,, \\ \dot{x}_2 = k_{21}x_1 - k_{12}x_2\,, \\ \dot{x}_3 = k_{31}x_1 - k_{13}x_3\,, \\ \dot{x}_4 = k_{41}x_1 - k_{14}x_4\,, \\ y = x_1\,. \end{cases} \tag{18}$$

By direct inspection, one easily identifies an invariance under an action of the finite group S_3 (the symmetric group for sets with three elements) containing six elements: if we form the three triples (k_{12}, k_{13}, k_{14}), (k_{21}, k_{31}, k_{41}), (x_2, x_3, x_4) and perform on them simultaneously the same permutation of the indices $2, 3, 4$, then the system (18) remains unchanged. Setting up the finite determining system and applying the Thomas decomposition reveals that this is indeed the complete solution space. Each of the six arising simple systems admits exactly one solution representing one of the transformations described above. Among these simple systems, one is provided here as a representative example:

$$\tilde{x}_2 = x_3,\ \tilde{x}_3 = x_4,\ \tilde{x}_4 = x_2,\ \tilde{k}_{12} = k_{13},\ \tilde{k}_{13} = k_{14},$$
$$\tilde{k}_{14} = k_{12},\ \tilde{k}_{21} = k_{31},\ \tilde{k}_{31} = k_{41},\ \tilde{k}_{41} = k_{21}.$$

Hence we can conclude that in this example all states are at least locally observable and all parameters are at least locally identifiable. But globally, only x_1 and k_{01} are observable or identifiable, respectively.

Our second example is a well studied model from biology with a more complex finite solution space, demonstrating the rise of multiple solution from a single simple system.

Example 5.2. We consider the Goodwin oscillator [13]

$$\begin{cases} \dot{X} = \frac{1}{1+Z^m} - X\,, \\ \dot{Y} = X - k_2 Y\,, \\ \dot{Z} = k_1 Y - k_3 Z\,, \\ y = X \end{cases} \tag{19}$$

in a non-dimensional form. With a Hill coefficient $m = 4$, the finite determining system decomposes into six simple systems of purely algebraic equations. Two of them involve quadratic equations in k_1 and the remaining systems are linear, thus the discrete symmetries form a group of order 8. It is not difficult to identify it as the commutative group $C_4 \times C_2$ (the product of two cyclic groups). The generator for the subgroup C_2 is given as the single solution of the simple system

$$\tilde{Y} = Y + \frac{Z}{k_1}(k_2 - k_3), \quad \tilde{k}_2 = k_3, \quad \tilde{k}_3 = k_2. \tag{20}$$

It is easy to see that this transformation does not affect the first equation of the system and is a symmetry of the other two equations. The existence of this subgroup implies that the state Y is only locally observable and the parameters k_2, k_3 are SLING. As generator of the subgroup C_4, one can take any of the two solutions of the simple system

$$\tilde{Z} = -\frac{k_1}{\tilde{k}_1} Z, \quad \tilde{k}_1^2 = -k_1^2. \tag{21}$$

These transformations leave both the Hill function in the first equation and the complete third equation invariant. However, they are biologically not relevant, as they lead to negative or even complex values for k_1 and Z. Thus, from a biological point of view Z is globally observable and k_1 globally identifiable. One can verify by direct computation that for an arbitrary Hill coefficient $m \in \mathbb{N}$ one always finds the commutative group $C_m \times C_2$ as a discrete symmetry group and we conjecture that there are no other symmetries.

Our next example is a classical model from the control theory literature. It also possesses an easy to find permutation symmetry, but in addition also a continuous symmetry group.

Example 5.3. The following model stems from [19]:

$$\begin{cases} \dot{x}_1 = -\theta_1 x_1 + \theta_2 u\,, \\ \dot{x}_2 = -\theta_3 x_2 + \theta_4 u\,, \\ \dot{x}_3 = -(\theta_1 + \theta_3) x_3 + (\theta_4 x_1 + \theta_2 x_2) u\,, \\ y = x_3\;. \end{cases} \tag{22}$$

Direct inspection yields again a discrete S_2 symmetry: if we form the three pairs (θ_1, θ_3), (θ_2, θ_4), (x_1, x_2) and simultaneously swap their elements, then the system (22) remains unchanged. However, this time the symmetry group is larger. The Thomas decomposition of the finite determining system consists of two simple systems, each comprising a set of algebraic equations and an identical set of differential equations for $\tilde{\theta}_4$, indicating the presence of infinitely many solutions. Solving for $\tilde{\theta}_4$ reveals that the infinite part of the solution space can be parametrized by a single function of the form

$$\varphi := \varphi\Big(\theta_1, \theta_2, \theta_3, \theta_4, (-x_1 x_2 + x_3)\exp\big[t(\theta_1 + \theta_3)\big]\Big)\,.$$

This result implies that the model is invariant under a Lie group consisting of two connected components. The component containing the identity is given by the transformations

$$\tilde{x}_1 = \frac{\theta_4 x_1}{\varphi}\,,\ \tilde{x}_2 = \frac{x_2\varphi}{\theta_4}\,,$$
$$\tilde{\theta}_1 = \theta_1\,,\ \tilde{\theta}_2 = \frac{\theta_2\theta_4}{\varphi}\,,\ \tilde{\theta}_3 = \theta_3\,,\ \tilde{\theta}_4 = \varphi$$

(the identity is obtained by choosing $\varphi = \theta_4$). The second connected component is given by the transformations

$$\tilde{x}_1 = \frac{\theta_2 x_2}{\varphi}\,,\ \tilde{x}_2 = \frac{x_1\varphi}{\theta_2}\,,$$
$$\tilde{\theta}_1 = \theta_3\,,\ \tilde{\theta}_2 = \frac{\theta_2\theta_4}{\varphi}\,,\ \tilde{\theta}_3 = \theta_1\,,\ \tilde{\theta}_4 = \varphi$$

(the second permutation mentioned above arises by choosing $\varphi = \theta_2$). Thus the symmetry analysis reveals that only the two parameters θ_1 and θ_3 are locally identifiable though not globally, whereas all other states and parameters are not even locally observable or identifiable.

Our last example is an epidemiological model where the appearing discrete symmetry group operates in a highly non-trivial way and where it appears impossible to guess the form of the symmetries by a simple inspection of the control system.

Example 5.4. In [49], the authors augmented a classical SEIR model for an epidemic by a quarantine compartment, which is also the output, leading to the input-free system:

$$\begin{cases} \dot{S} = -\beta S I\,, \\ \dot{E} = \beta S I - \nu E\,, \\ \dot{I} = \nu E - \psi I - (1-\psi)\gamma I\,, \\ \dot{Q} = \psi I - \gamma Q\,, \\ \dot{R} = (1-\psi)\gamma I + \gamma Q\,, \\ y = Q\,. \end{cases} \tag{23}$$

Here, not only direct inspection fails, but even the application of our approach reaches its (computational) limits. Hence, we applied certain simplifications to the finite determining system. The dynamics of the state R in (23) is independent of the rest of the system and does not affect the output. An infinitesimal analysis reveals that only R admits continuous symmetries. Considering only a reduced system, obtained by removing the equation for R, simplifies the determining system. Furthermore, we adopt our heuristic that the transformed parameters are independent of $\mathbf{x}$ and assume that γ remains invariant. The resulting determining system then decomposes into two simple systems: one corresponding to the identity transformation and the other yielding a non-trivial symmetry:

$$\tilde{S} = \frac{\nu\psi S(\gamma-1)}{(\nu-\gamma)(\psi\gamma-\psi-\gamma)},\ \tilde{I} = \frac{I\psi(1-\gamma)}{\nu-\gamma},$$
$$\tilde{E} = \frac{\big(I(\psi-1)\gamma-\psi I+\nu(E+I)\big)(\gamma-1)\psi}{(\nu-\gamma)\big((\psi-1)\gamma-\psi\big)},$$
$$\tilde{\beta} = \frac{\beta(\gamma-\nu)}{(\gamma-1)\psi},\ \tilde{\nu} = (1-\psi)\gamma+\psi,\ \tilde{\psi} = \frac{\gamma-\nu}{\gamma-1}.$$

Due to our simplifications, we cannot exclude the existence of further discrete symmetry transformations. Nevertheless, we can make statements about the identifiability and observability. While the state R is not even locally observable, the states S, E, I are locally but not globally observable. In addition, the parameters β, ν, ψ are SLING. No statement about γ is possible. The explicit form of the non-trivial symmetry allows us furthermore to conclude that the found discrete symmetry is biologically not relevant: biologically meaningful values of γ and ν lie between 0 and 1 and normally γ is larger than ν; applying the above transformation then leads to negative values of $\tilde{\psi}$ and $\tilde{I}$ which are biologically meaningless.

6 Conclusions

In this paper, we have presented an approach to study the structural identifiability and observability (SIO) of nonlinear dynamic models using discrete symmetries. It is known that continuous symmetries are sources of non-identifiability and/or non-observability, i.e., a single input-output trajectory is compatible with an infinite number of parameter or state variable values. This fact has already been exploited by several authors [46, 22, 1, 25, 6], who introduced methods to analyse SIO by searching for continuous symmetries. However, said methods can only distinguish between non-identifiability and (at least) local identifiability (and similarly observability). In contrast, here we have exploited the fact that the existence of purely discrete symmetries does not lead to non-identifiable parameters, but to parameters that are structurally locally identifiable, but not globally (SLING). Thus, by determining the discrete symmetries in a model we obtain a more precise SIO analysis, which can distinguish between SLING and globally identifiable parameters. Here, we have presented a methodology to perform this analysis. The code that implements the method and reproduces the results can be accessed at: `https://doi.org/10.5281/zenodo.16410656`. Key elements in our procedure are obtaining the finite determining system, analysing it using the Thomas decomposition, and finding the number of solutions. Admittedly, our method seems to be computationally less efficient than some differential algebraic algorithms currently available [14, 9]. However, it provides more information, since it can fully characterize the form of the symmetries in which the parameters and state variables are involved. This information is valuable for the purposes of finding all possible solutions and reparameterizing the model [25, 20]. A future line of work is to improve the computational efficiency of our method. Additionally, we envision its extension to systems of partial differential equations, for which comparatively few techniques for structural identifiability analysis exist.

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References

[1] Milena Anguelova, Johan Karlsson, and Mats Jirstrand. Minimal output sets for identifiability. *Mathematical biosciences*, 239(1):139–153, 2012.

[2] T. Bächler, V.P. Gerdt, M. Lange-Hegermann, and D. Robertz. Algorithmic Thomas decomposition of algebraic and differential systems. *J. Symb. Comp.*, 47:1233–1266, 2012.

[3] G.W. Bluman and S. Kumei. *Symmetries and Differential Equations.* Applied Mathematical Sciences 81. Springer-Verlag, New York, 1989.

[4] F. Boulier, D. Lazard, F. Ollivier, and M. Petitot. Representation for the radical of a finitely generated differential ideal. In A.H.M. Levelt, editor, *Proc. ISSAC '95*, pages 158–166. ACM Press, New York, 1995.

[5] F. Boulier, D. Lazard, F. Ollivier, and M. Petitot. Computing representations for radicals of finitely generated differential ideals. *Appl. Alg. Eng. Comm. Comp.*, 20:73–121, 2009.

[6] Mario Castro and Rob J De Boer. Testing structural identifiability by a simple scaling method. *PLOS Computational Biology*, 16(11):e1008248, 2020.

[7] Manolis N Chatzis, Eleni N Chatzi, and Andrew W Smyth. On the observability and identifiability of nonlinear structural and mechanical systems. *Structural Control and Health Monitoring*, 22(3):574–593, 2015.

[8] Katherine Clemens, Jonathan Martinez, Anne Shiu, Michaela Thompson, and Benjamin Warren. Parameter identifiability of linear-compartmental mammillary models. *arXiv preprint arXiv:2506.21889*, 2025.

[9] Ruiwen Dong, Christian Goodbrake, Heather A Harrington, and Gleb Pogudin. Differential elimination for dynamical models via projections with applications to structural identifiability. *SIAM Journal on Applied Algebra and Geometry*, 7(1):194–235, 2023.

[10] G. Gaeta and M.A. Rodríguez. Discrete symmetries of differential equations. *J. Phys. A: Math. Gen.*, 29:859–880, 1996.

[11] V.P. Gerdt. On decomposition of algebraic PDE systems into simple subsystems. *Acta Appl. Math.*, 101:39–51, 2008.

[12] V.P. Gerdt, M. Lange-Hegermann, and D. Robertz. The Maple package TDDS for computing Thomas decompositions of systems of nonlinear PDEs. *Comp. Phys. Comm.*, 234:202–215, 2019.

[13] Brian C. Goodwin. Oscillatory behavior in enzymatic control processes. *Advances in Enzyme Regulation*, 3:425–437, 1965.

[14] Hoon Hong, Alexey Ovchinnikov, Gleb Pogudin, and Chee Yap. Sian: software for structural identifiability analysis of ode models. *Bioinformatics*, 35(16):2873–2874, 2019.

[15] P.E. Hydon. Discrete point symmetries of ordinary differential equations. *Proc. Royal Soc.: Math. Phys. Eng. Sci.*, 454:1961–1972, 1998.

[16] N. Ibragimov, M. Torrisi, and A. Valenti, editors. *Proc. Modern Group Analysis: Advanced Analytical and Computational Methods in Mathematical Physics*. Kluwer, Dordrecht, 1993.

[17] M. Lange-Hegermann. *Counting Solutions of Differential Equations*. PhD thesis, RWTH Aachen, Germany, 2014. Available at `http://darwin.bth.rwth-aachen.de/opus3/frontdoor.php?source_opus=4993`.

[18] M. Lange-Hegermann. The differential counting polynomial. *Found. Comp. Math.*, 18:291–308, 2018.

[19] Y. Lecourtier, F. Lamnabhi-Lagarrigue, and E. Walter. A method to prove that nonlinear models can be unidentifiable. In *Proc. 26th IEEE Conf. Decision Control*, pages 2144–2145, 1987.

[20] Gemma Massonis, Julio R Banga, and Alejandro F Villaverde. Autorepar: a method to obtain identifiable and observable reparameterizations of dynamic models with mechanistic insights. *International Journal of Robust and Nonlinear Control*, 33(9):5039–5057, 2023.

[21] Gemma Massonis and Alejandro F Villaverde. Finding and breaking Lie symmetries: Implications for structural identifiability and observability in biological modelling. *Symmetry*, 12:469, 2020.

[22] Benjamin Merkt, Jens Timmer, and Daniel Kaschek. Higher-order Lie symmetries in identifiability and predictability analysis of dynamic models. *Physical Review E*, 92(1):012920, 2015.

[23] N. Meshkat, C.E. Kuo, and J. DiStefano III. On finding and using identifiable parameter combinations in nonlinear dynamic systems biology models and COMBOS: A novel web implementation. *PLOS One*, 9:e110261, 2014.

[24] Nicolette Meshkat and Anne Shiu. Structural identifiability of compartmental models: Recent progress and future directions. *arXiv preprint arXiv:2507.04496*, 2025.

[25] Nicolette Meshkat and Seth Sullivant. Identifiable reparametrizations of linear compartment models. *Journal of Symbolic Computation*, 63:46–67, 2014.

[26] P.J. Olver. *Applications of Lie Groups to Differential Equations*. Graduate Texts in Mathematics 107. Springer-Verlag, New York, 1986.

[27] G.J. Reid, D.T. Weih, and A.D. Wittkopf. A point symmetry group of a differential equation which cannot be found using infinitesimal methods. In Ibragimov et al. [16], pages 311–316.

[28] G.J. Reid, A.D. Wittkopf, and A. Boulton. Reduction of systems of nonlinear partial differential equations to simplified involutive forms. *Eur. J. Appl. Math.*, 7:635–666, 1996.

[29] Xabier Rey Barreiro and Alejandro F Villaverde. On the origins and rarity of locally but not globally identifiable parameters in biological modeling. *IEEE Access*, 11, 2023.

[30] D. Robertz. *Formal Algorithmic Elimination for PDEs*, volume 2121 of *Lecture Notes in Mathematics*. Springer-Verlag, Cham, 2014.

[31] K. Schlacher, A. Kugi, and K. Zehetleitner. A Lie-group approach for nonlinear dynamic systems described by implicit ordinary differential equations. In *Proc. Mathematical Theory of Network and Systems (MTNS)*, 2002.

[32] J. Schü, W.M. Seiler, and J. Calmet. Algorithmic methods for Lie pseudogroups. In Ibragimov et al. [16], pages 337–344.

[33] F. Schwarz. An algorithm for determining the size of symmetry groups. *Comp.*, 49:95–115, 1992.

[34] Alexandre Sedoglavic. A probabilistic algorithm to test local algebraic observability in polynomial time. *Journal of Symbolic Computation*, 33(5):735–755, 2002.

[35] W.M. Seiler. On the arbitrariness of the general solution of an involutive partial differential equation. *J. Math. Phys.*, 35:486–498, 1994.

[36] W.M. Seiler. Arbitrariness of the general solution and symmetries. *Acta Appl. Math.*, 41:311–322, 1995.

[37] W.M. Seiler. Involution and symmetry reductions. *Math. Comp. Model.*, 25:63–73, 1997.

[38] W.M. Seiler. *Involution — The Formal Theory of Differential Equations and its Applications in Computer Algebra.* Algorithms and Computation in Mathematics 24. Springer-Verlag, Berlin, 2010.

[39] W.M. Seiler and E. Zerz. Algebraic theory of linear systems: A survey. In A. Ilchmann and T. Reis, editors, *Surveys in Differential-Algebraic Equations II*, Differential-Algebraic Equations Forum, pages 287–333. Springer-Verlag, 2015.

[40] W.Y. Sit. The Ritt-Kolchin theory for differential polynomials. In L. Guo, W.F. Keigher, P.J. Cassedy, and W.Y. Sit, editors, *Differential Algebra and Related Topics*, pages 1–70. World Scientific, 2002.

[41] J.M. Thomas. *Differential Systems.* Colloquium Publications XXI. American Mathematical Society, New York, 1937.

[42] J.M. Thomas. *Systems and Roots.* W. Byrd Press, 1962.

[43] Alejandro F Villaverde. Symmetries in dynamic models of biological systems: Mathematical foundations and implications. *Symmetry*, 14(3):467, 2022.

[44] Alejandro F Villaverde et al. Observability and structural identifiability of nonlinear biological systems. *Complexity*, 2019, 2019.

[45] Alejandro F Villaverde and Gemma Massonis. On testing structural identifiability by a simple scaling method: relying on scaling symmetries can be misleading. *PLoS Computational Biology*, 17(10):e1009032, 2021.

[46] James WT Yates, Neil D Evans, and Michael J Chappell. Structural identifiability analysis via symmetries of differential equations. *Automatica*, 45(11):2585–2591, 2009.

[47] K. Zehetleitner. *A Geometric Approach to Equivalence Problems for Implicit Dynamic Systems.* PhD thesis, Johannes Kepler Universität Linz, 2005.

[48] K. Zehetleitner and K. Schlacher. Computer algebra algorithms for the test on accessibility and observability for implicit dynamical systems. In *Proc. 2003 European Control Conference (ECC)*, pages 282–287. IEEE, 2003.

[49] W.-T. Zha, F.-R. Pang, N. Zhou, B. Wu, Y. Liu, Y.-B. Du, X.-Q. Hong, and Y. Lv. Research about the optimal strategies for prevention and control of varicella outbreak in a school in a central city of China: Based on an SEIR dynamic model. *Epidemiol. Infection*, 148:e56, 2020.