On the Inverse Syzygy Problem

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Abstract

We present a constructive solution of the inverse syzygy problem over arbitrary coherent rings. By relating the existence of a kernel representation to torsionlessness instead of the more common torsionfreeness, we do not need to assume the existence of a quotient field. As a by-product, we obtain an algorithm to compute the extension groups of finitely presented modules.

Keywords: Syzygies; Coherent rings; Torsionless modules; Torsionfree modules; Extension groups

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1 Introduction

While the effective computation of syzygies is a very classical problem as a part of the determination of free resolutions, the inverse problem has received much less attention, in particular not from a constructive point of view. Roughly speaking, computing syzygies means to represent a kernel as an image and we will study the converse question when an image is equal to a kernel.

From a purely theoretical point of view, there is not much to add to the treatment given by Auslander and Bridger (1969). However, they use a fairly heavy machinery by embedding the problem into the theory of satellites and stable functors. As already pointed out by Bruns and Vetter (1988, Sec. 16E) for the case of a commutative ring, one can give much simpler direct proofs. None of these references is concerned with constructive aspects of the problem, although it turns out that the obstructions to the existence of a solution are given by an extension group and thus one obtains as a by-product an approach to the effective determination

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of these groups. In mathematical systems theory, a constructive solution of the inverse syzygy problem is of considerable interest (here one usually speaks about the existence of a parametrisation), as it is related to the controllability of linear systems; see (Pommaret 2001, Shankar 2002, Zerz 2001) for a discussion of various notions of controllability and their relation to extension groups.

Oberst (1990, Sec. 7, Thm. 24) was the first to present such a constructive solution for linear differential systems with constant coefficients, i.e., over the usual commutative polynomial ring. He designed an algorithm reducing the inverse syzygy problem via dualisation to two direct syzygy computations. Pommaret (1994, Ch. 7AB) provided later an extension to linear differential systems with variable coefficients using formally adjoint operators instead of dualisation, but otherwise following the same algorithm as Oberst. Chyzak, Quadrat and Robertz (2005) showed that the algorithm can be applied over arbitrary Ore domains. Parts of these results have been rediscovered in (Damiano, Struppa, Vajiac and Vajiac 2009, Moreno-Frías 2003); the computation of extension groups in PBW algebras, including rings of linear differential operators, was also the topic of (Bueso, Gómez-Torrecillas and Lobillo 2001).

Common to all these references is that the existence of a parametrisation is related to the question whether a certain module is *torsionfree*. Furthermore, the given proofs require the introduction of a quotient field so that the underlying ring must be a domain satisfying an Ore condition. One of the main points of this article is to show that, following Auslander and Bridger (1969), it is more natural to use the notion of *torsionlessness* than torsionfreeness.[1] This approach does not only make the proof simpler, one can also dispense with the use of a quotient field and thus work over considerably more general rings. Our only assumption will be that the given ring is both left and right coherent, in order to ensure that the kernels of homomorphisms of finitely generated free modules are always finitely generated.

We will study the relation between the two approaches in Section 4, where we will show that over certain rings, the notions torsionfree and torsionless coincide for finitely presented modules. Our proof of Theorem 3 implicitly contains the classical method for deciding the existence of a parametrisation.

In more mathematical terms, there are two ways to formulate the inverse syzygy problem over a ring $\mathcal{D}$. Auslander and Bridger (1969) and Bruns and Vetter (1988, Sec. 16E) call a finitely generated left $\mathcal{D}$-module C an n-th syzygy if there exists an exact sequence

$$0 \to C \to F_1 \to \cdots \to F_n$$

with finitely generated free $\mathcal{D}$-modules F_i. Then the inverse syzygy problem consists in deciding whether or not a given module C is a first syzygy (by iteration

[1]Note that Auslander and Bridger speak of k-torsionfree modules, but their notion of 1-torsionfree is equivalent to what we call torsionless.

one can then also answer whether it is an n-th syzygy for some $n > 1$).

In the theory of linear systems it is more common to consider instead a homomorphism $\beta : M \to N$ between two finitely generated free left $\mathcal{D}$-modules M, N. Under fairly modest assumptions on the ring $\mathcal{D}$, it is always possible to find a finitely generated free left $\mathcal{D}$-module Q and a left $\mathcal{D}$-module homomorphism $\gamma : Q \to M$ such that $\operatorname{im}(\gamma) = \ker(\beta)$; this is the direct syzygy problem equivalent to the existence of free resolutions. In the inverse problem, one asks whether there exist a finitely generated free left $\mathcal{D}$-module P and a left $\mathcal{D}$-module homomorphism $\alpha : N \to P$ such that $\operatorname{im}(\beta) = \ker(\alpha)$. Such an α is then called a parametrisation of β.

The relation between the two formulations is easy. We set $C := \operatorname{coker}(\beta)$ and if a parametrisation α exists for β, then we obtain an exact sequence $0 \to C \xrightarrow{\tilde{\alpha}} P$ by setting $\tilde{\alpha}\big([n]\big) := \alpha(n)$ for any element $[n] \in C$, so that the module C is a first syzygy in this case. Conversely, given a finitely presented first syzygy C with an exact sequence as above, then we can write C as the cokernel of a homomorphism $\beta : M \to N$ and obtain a parametrisation by defining $\alpha : N \to P$ through $\alpha(n) := \tilde{\alpha}\big([n]\big)$. Thus the two formulations are equivalent.

The existence of a parametrisation plays a major role in fields like mathematical physics or control theory, where $\mathcal{D}$ is usually a ring of differential or difference operators. Given a left $\mathcal{D}$-module $\mathcal{A}$, the problem consists in finding out whether a system of equations $Bw = 0$ can be parametrised in the sense that

$$Bw = 0, w \in \mathcal{A}^q \quad \Longleftrightarrow \quad \exists \ell \in \mathcal{A}^l : w = A\ell \,, \tag{1}$$

where $B \in \mathcal{D}^{g \times q}$, $A \in \mathcal{D}^{q \times l}$. Thus $\mathcal{A}$ should be considered as a function set in which we want to solve the system $Bw = 0$. A prominent example for this situation is given by $\mathcal{D} = \mathbb{R}[\partial_1, \ldots, \partial_n]$ and $\mathcal{A} = \mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R})$. The variable ℓ is interpreted as a potential in physics, and in control theory one calls the representation on the right hand side of (1) an image representation of the system

$$\mathcal{B} = \{w \in \mathcal{A}^q \mid Bw = 0\} = \{w \in \mathcal{A}^q \mid \exists \ell \in \mathcal{A}^l : w = A\ell\} \,.$$

The existence of image representations is closely related to the concept of controllability; see (Pommaret 2001, Shankar 2002, Zerz 2001) for an overview. To decide whether an equivalence as in (1) exists, one sets $M = \mathcal{D}^{1 \times g}$, $N = \mathcal{D}^{1 \times q}$ and $\beta(m) = mB$ for $m \in M$. Then the inverse syzygy problem is solvable if and only if there exists a $\mathcal{D}$-matrix A such that for all $n \in N$, we have

$$\exists m \in M : n = mB \quad \Longleftrightarrow \quad nA = 0 \,, \tag{2}$$

that is, $\operatorname{im}(\beta) = \ker(\alpha)$ for α defined by $\alpha(n) = nA$. If we assume that the $\mathcal{D}$-module $\mathcal{A}$ is an injective cogenerator (Lam 1999, §3A, §19A), then (2) is in turn equivalent to (1). For instance, this holds for $\mathcal{D} = \mathbb{R}[\partial_1, \ldots, \partial_n]$ and $\mathcal{A} = \mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R})$; see (Oberst 1990, Sec. 4, Thm. 69).

2 Solving the inverse syzygy problem

We recall the precise formulation of the inverse syzygy problem: Let $\mathcal{D}$ be a ring (with 1, not necessarily commutative). Let M, N be finite (i.e., finitely generated) free left $\mathcal{D}$-modules. Let $\beta : M \to N$ be a left $\mathcal{D}$-module homomorphism, that is, $\beta(dm) = d\beta(m)$ for all $m \in M$ and $d \in \mathcal{D}$. When does there exist a finite free left $\mathcal{D}$-module P and a left $\mathcal{D}$-module homomorphism $\alpha : N \to P$ such that $\mathrm{im}\,(\beta) = \ker\,(\alpha)$?

For solving the inverse syzygy problem, we use two tools: *dualisation* and *syzygy computation*. Let M be a left $\mathcal{D}$-module. We define the dual module

$$M^* = \mathrm{Hom}_{\mathcal{D}}(M, \mathcal{D})$$

as the set of all left $\mathcal{D}$-linear maps from M to $\mathcal{D}$. Then M^* is a right $\mathcal{D}$-module and there is a natural left $\mathcal{D}$-module homomorphism $\eta_M : M \to M^{**}$ mapping $m \in M$ to $\phi \mapsto \phi(m)$. Given $\beta : M \to N$, we have the dual map (pull-back)

$$\beta^* : N^* \to M^* , \quad \varphi \mapsto \varphi \circ \beta ,$$

which is a right $\mathcal{D}$-module homomorphism. If M is a finite free module, then the dual M^* is finite free, too. Furthermore, in this case η_M is an isomorphism so that we may identify M and M^{**} (in general, η_M is neither injective nor surjective). We remark that $\beta^* = 0$ implies $\beta = 0$ if N is finite free, indeed, the map $\mathrm{Hom}_{\mathcal{D}}(M, N) \cong \mathrm{Hom}_{\mathcal{D}}(N^*, M^*)$, $\beta \mapsto \beta^*$ is an isomorphism of Abelian groups.

Let $\gamma : N \to Q$ be a left module homomorphism of finite free left $\mathcal{D}$-modules N, Q. If $\ker\,(\gamma)$ is finitely generated, then it is the image of a finite free left $\mathcal{D}$-module, say $\hat{M}$, under a left module homomorphism, say $\hat{\beta}$. In other words, $\mathrm{im}\,(\hat{\beta}) = \ker\,(\gamma)$, that is,

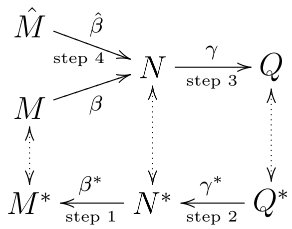
$$\hat{M} \xrightarrow{\hat{\beta}} N \xrightarrow{\gamma} Q$$

is exact. In the sequel we assume that, given a homomorphism γ with a finitely generated kernel, we are able to determine effectively such a homomorphism $\hat{\beta}$ and we call this a syzygy computation.

We now give a simple constructive solution to the inverse syzygy problem: Let a homomorphism $\beta : M \to N$ be given, where M, N are finite free $\mathcal{D}$-modules. We perform the following steps:

1. Dualisation: Consider $\beta^* : N^* \to M^*$ and $\ker\,(\beta^*) \subseteq N^*$.
2. Syzygy computation: Let $\gamma^* : Q^* \to N^*$ be such that $\mathrm{im}\,(\gamma^*) = \ker\,(\beta^*)$.
3. Dualisation: Consider $\gamma : N \to Q$ and $\ker\,(\gamma) \subseteq N$.
4. Syzygy computation: Let $\hat{\beta} : \hat{M} \to N$ be such that $\mathrm{im}\,(\hat{\beta}) = \ker\,(\gamma)$.
5. Check whether $\mathrm{im}\,(\hat{\beta}) = \mathrm{im}\,(\beta)$.

The algorithm is illustrated by the following diagram, where the vertical arrows symbolise dualisation; see (Pommaret 2001, p. 614–615) for an analogous construction for linear differential equations with variable coefficients where instead of the dual map the formal adjoint is used:



(3)

Note that in (3), the sequences $\hat{M} \to N \to Q$ and $M^* \leftarrow N^* \leftarrow Q^*$ are exact, by construction, whereas $M \to N \to Q$ is not necessarily exact (in fact, the exactness of this sequence is precisely what is to be determined).

Obviously, this algorithm will only work if we can guarantee the finiteness of $\ker(\beta^*)$ and $\ker(\gamma)$ in Steps 2 and 4, respectively. For this reason, we must assume that the ring $\mathcal{D}$ is coherent (one could drop the assumption of left coherence by arguing that if $\ker(\gamma)$ is not finitely generated, then the inverse syzygy problem is not solvable). Recall that a ring $\mathcal{D}$ is called *left coherent* if every finitely generated left ideal in $\mathcal{D}$ is finitely presented. This means that for any $d_1, \dots, d_n \in \mathcal{D}$, the left $\mathcal{D}$-module

$$\operatorname{Syz}(d_1, \dots, d_n) = \left\{ (c_1, \dots, c_n) \in \mathcal{D}^{1 \times n} \mid \sum_{i=1}^n c_i d_i = 0 \right\} ,$$

which is the module of relations (syzygy module) of the left ideal $\mathcal{I} = \sum_{i=1}^n \mathcal{D} d_i$, is finitely generated (Lam 1999, §4FG). Equivalently, $\mathcal{D}$ is left coherent if and only if every left $\mathcal{D}$-homomorphism between finite free left $\mathcal{D}$-modules has a finitely generated kernel. Right coherence is defined analogously, and by coherence, we mean both right and left coherence.

It turns out that the notion of a *torsionless* module is decisive for characterising the solvability of the inverse syzygy problem. Recall that a module M is torsionless if the above introduced natural map $\eta_M : M \to M^{**}$ is injective – in other words for any $0 \neq m \in M$ there exists a homomorphism $\psi \in M^*$ with $\psi(m) \neq 0$; see e.g. (Lam 1999, Rem. 4.65(a)). The two for us relevant properties of a torsionless module are collected in the following lemma.

Lemma 1. *Let $\phi : M \to N$ be a homomorphism of left $\mathcal{D}$-modules such that its dual map $\phi^* : N^* \to M^*$ is surjective.*

1. *If M is torsionless, then ϕ is injective.*
2. *If N is torsionless, then* $\ker(\eta_M) = \ker(\phi)$.

Proof. Consider the following commutative diagram:

$$\begin{array}{ccc} M & \xrightarrow{\ \phi\ } & N \\ {\scriptstyle \eta_M}\downarrow & & \downarrow{\scriptstyle \eta_N} \\ M^{**} & \xrightarrow[\ \phi^{**}\ ]{} & N^{**} \end{array}$$

where the bidual map ϕ^{**} is trivially injective, since we assume here that the dual map ϕ^* is surjective.

In the case of the first assertion, η_M is injective, too, as M is assumed to be torsionless. Hence the composed map $\phi^{**} \circ \eta_M = \eta_N \circ \phi$ is injective, which is only possible if ϕ is injective.

For the second assertion, we note that the injectivity of ϕ^{**} implies that

$$\ker(\eta_M) = \ker(\phi^{**} \circ \eta_M) = \ker(\eta_N \circ \phi) = \ker(\phi)\ ,$$

where the last equality follows from the assumption that this time N is torsionless and hence η_N injective. □

Theorem 2. *Let $\mathcal{D}$ be an arbitrary coherent ring and $\beta : M \to N$ a homomorphism of finite free left $\mathcal{D}$-modules. If the map $\hat{\beta}$ is constructed as outlined above, then the following three statements are equivalent.*

1. *There exists a finite free left $\mathcal{D}$-module P and a left $\mathcal{D}$-module homomorphism $\alpha : N \to P$ such that $\operatorname{im}(\beta) = \ker(\alpha)$; in other words, the inverse syzygy problem is solvable.*
2. *The left $\mathcal{D}$-module $C := \operatorname{coker}(\beta) = N/\operatorname{im}(\beta)$ is torsionless.*
3. *We have $\operatorname{im}(\beta) = \operatorname{im}(\hat{\beta})$.*

Proof. The implication "3 ⇒ 1" is obvious, because $\operatorname{im}(\beta) = \operatorname{im}(\hat{\beta}) = \ker(\gamma)$ by construction, and thus we may put $P := Q$ and $\alpha := \gamma$.

The implication "1 ⇒ 2" follows immediately from the homomorphism theorem. As $C = N/\ker(\alpha) \cong \operatorname{im}(\alpha)$, it is isomorphic to a submodule of a free module, and hence clearly torsionless.

Thus, it suffices to show "2 ⇒ 3". Since $\gamma \circ \beta = 0$ and $C = \operatorname{coker}(\beta)$, the homomorphism γ can be decomposed as $\gamma = \phi \circ \pi$ with $\pi : N \to C$ the canonical projection and $\phi : C \to Q$ the induced map. Dually, we obtain $\gamma^* = \pi^* \circ \phi^*$. By construction, we have on one side $\ker(\beta^*) = \operatorname{im}(\gamma^*)$ and on the other side $\ker(\beta^*) = \operatorname{im}(\pi^*)$. Hence $\operatorname{im}(\gamma^*) = \operatorname{im}(\pi^*)$ and since π^* is trivially injective, ϕ^* must be surjective. Now it follows from the first assertion in Lemma 1 that ϕ is injective and hence $\operatorname{im}(\beta) = \ker(\pi) = \ker(\phi \circ \pi) = \ker(\gamma) = \operatorname{im}(\hat{\beta})$. □

The final step of the proof leads to the following observation: As a free module, Q is clearly torsionless. Using again the above decomposition $\gamma = \phi \circ \pi$ and the fact that the dual map ϕ^* is surjective, $\ker(\eta_C) = \ker(\phi)$ by the second assertion in Lemma 1. Furthermore, we have the trivial isomorphism $\ker(\phi) \cong \ker(\gamma)/\ker(\pi) = \operatorname{im}(\hat{\beta})/\operatorname{im}(\beta)$. Hence the procedure outlined above allows us to determine explicitly $\ker(\eta_C)$ and so provides us with an effective test for torsionlessness of a finitely presented module $C = \operatorname{coker}(\beta)$.

For a concrete computational realisation of the procedure outlined above, we assume that $M = \mathcal{D}^{1\times g}$ and $N = \mathcal{D}^{1\times q}$ and that the map $\beta : \mathcal{D}^{1\times g} \to \mathcal{D}^{1\times q}$ is given by $\beta(m) = mB$ for some matrix $B \in \mathcal{D}^{g\times q}$ (in order to obtain a left $\mathcal{D}$-module homomorphism, we must put the matrix to the right and therefore use rows). Using the natural isomorphism $(\mathcal{D}^{1\times q})^* \cong \mathcal{D}^q$ identifying each element of the dual module with its image on the standard basis, it is easy to see that we may then consider β^* as the map $\mathcal{D}^q \to \mathcal{D}^g$ given by $\beta^*(x) = Bx$.[2] Thus the first and the third step of our procedure are computationally trivial.

For notational simplicity, we write in the sequel $\beta = \cdot B$ and $\beta^* = B\cdot$, where the dot indicates the position of the argument. Thus the second step requires to compute the solution set $\ker(B\cdot)$ of a linear system of equations over $\mathcal{D}$. If $\{c_1, \dots, c_s\} \subseteq \mathcal{D}^q$ is a generating set of it, then we may set $C = (c_1, \dots, c_s) \in \mathcal{D}^{q\times s}$, the matrix of the map γ, so that $\ker(B\cdot) = \operatorname{im}(C\cdot)$. In the fourth step we must similarly determine a generating set $\{\hat{b}_1, \dots, \hat{b}_{\hat{g}}\} \subseteq \mathcal{D}^{1\times q}$ of the solution module $\ker(\cdot C)$. Finally, one tests in the fifth step whether the left $\mathcal{D}$-module generated by the rows of the matrix B equals $\langle \hat{b}_1, \dots, \hat{b}_{\hat{g}} \rangle$.

3 Ore rings and self-injective total quotient rings

An element of $\mathcal{D}$ is called *regular* if it is neither a left nor a right zero-divisor. Let S denote the set of all regular elements in $\mathcal{D}$. One calls $\mathcal{D}$ a *left Ore ring* if $Sd \cap \mathcal{D}s \neq \emptyset$ holds for any $d \in \mathcal{D}$ and $s \in S$. This condition is necessary and sufficient (Lam 1999, §10B) for $\mathcal{D}$ to have a classical total ring of left fractions $\mathcal{Q} := S^{-1}\mathcal{D}$. Right Ore rings are defined analogously. By an Ore ring, we mean a right and left Ore ring. We will also refer to $\mathcal{Q}$ as the total quotient ring of $\mathcal{D}$.

Suppose that $\mathcal{D}$ is a left Ore ring. Then the total ring of left fractions $\mathcal{Q}$ of $\mathcal{D}$ is a flat extension of $\mathcal{D}$. First, we may identify $\mathcal{D}$ with a subring of $\mathcal{Q}$ via $d = 1^{-1}d$. Second, we note that $\mathcal{Q}$ is a $\mathcal{D}$-bimodule. Flatness (Bourbaki 1972, Ch. I) amounts to saying that whenever

$$M \to N \to P$$

[2] In the case of a commutative ring $\mathcal{D}$, we may of course use the more common realisation $\beta(m) = Bm$ and obtain then $\beta^*(x) = B^t x$, where B^t denotes the transposed matrix.

is an exact sequence of left $\mathcal{D}$-modules, then

$$\mathcal{Q} \otimes_{\mathcal{D}} M \to \mathcal{Q} \otimes_{\mathcal{D}} N \to \mathcal{Q} \otimes_{\mathcal{D}} P$$

is an exact sequence of left $\mathcal{Q}$-modules (we omit the subscript $\mathcal{D}$ from now on since all tensor products will be taken over $\mathcal{D}$). Here, a left $\mathcal{D}$-module M induces a left $\mathcal{Q}$-module $\mathcal{Q} \otimes M \cong S^{-1}M$, where the kernel of the natural map $M \to S^{-1}M$ is given by

$$t(M) = \{m \in M \mid \exists s \in S : sm = 0\} .$$

For a left $\mathcal{D}$-module homomorphism $\beta : M \to N$, we may identify $\mathcal{Q} \otimes \beta := \mathrm{id}_{\mathcal{Q}} \otimes \beta$ with the mapping

$$S^{-1}\beta : S^{-1}M \to S^{-1}N , \quad s^{-1}m \mapsto s^{-1}\beta(m) .$$

Thus $\mathcal{Q} \otimes \beta$ is injective provided that β is, a condition which is equivalent to $\mathcal{Q}$ being flat. Also, the flatness of $\mathcal{Q}$ implies that $\ker(\mathcal{Q} \otimes \beta) = \mathcal{Q} \otimes \ker(\beta)$ and $\mathrm{im}(\mathcal{Q} \otimes \beta) = \mathcal{Q} \otimes \mathrm{im}(\beta)$ for any left $\mathcal{D}$-homomorphism β (Bourbaki 1972, Ch. I).

We will be particularly interested in the situation where $\mathcal{Q}$ is self-injective. A ring $\mathcal{Q}$ is *left self-injective* if $\mathcal{Q}$, as a left $\mathcal{Q}$-module, is injective. This means that the contravariant functor $\mathrm{Hom}_{\mathcal{Q}}(\cdot, \mathcal{Q})$ from the category of left $\mathcal{Q}$-modules to the category of right $\mathcal{Q}$-modules is exact, i.e., if a sequence

$$U \to V \to W ,$$

where U, V, W are left $\mathcal{Q}$-modules, is exact, then the associated sequence

$$\mathrm{Hom}_{\mathcal{Q}}(U, \mathcal{Q}) \leftarrow \mathrm{Hom}_{\mathcal{Q}}(V, \mathcal{Q}) \leftarrow \mathrm{Hom}_{\mathcal{Q}}(W, \mathcal{Q})$$

of right $\mathcal{Q}$-modules is exact. Right self-injectivity is defined analogously.

The ring $\mathcal{D}$ is a *domain* if $\mathcal{D} \neq \{0\}$ and $\mathcal{D}$ contains no zero-divisors, that is, for all $d_1, d_2 \in \mathcal{D}$, we have $d_1 d_2 = 0 \Rightarrow d_1 = 0$ or $d_2 = 0$.

Finally, $\mathcal{D}$ is a *left Ore domain* if is both a domain and a left Ore ring. Then we have $S = \mathcal{D} \setminus \{0\}$, and $\mathcal{Q}$ is the (skew) field of left fractions of $\mathcal{D}$ (Bourbaki 1974, Ch. I, §9, Ex. 15), which is clearly (right and left) self-injective. This follows from Baer's criterion (Lam 1999, §3A), or directly from the equality of row rank and column rank, which holds over any (not necessarily commutative) field (Lam 2000).

We note that any left Ore domain possesses a self-injective ring of left fractions. The following implications hold for any ring $\mathcal{D}$:

- $\mathcal{D}$ is commutative $\Rightarrow$ $\mathcal{D}$ is a (left and right) Ore ring;
- $\mathcal{D}$ is (left/right) Noetherian $\Rightarrow$ $\mathcal{D}$ is (left/right) coherent;
- $\mathcal{D}$ is a (left/right) Noetherian domain $\Rightarrow$ $\mathcal{D}$ is a (left/right) Ore domain.

Thus the class of coherent left Ore domains comprises, in particular, all Noetherian domains, and thus most of the rings of differential or difference operators considered in systems theory, such as:

- the polynomial ring $\mathbb{R}[\partial_1, \ldots, \partial_n]$, which is the ring of linear differential operators with constant coefficients (as mentioned in the Introduction);
- the Laurent polynomial ring $F[s_1, \ldots, s_n, s_1^{-1}, \ldots, s_n^{-1}]$, which is the ring of linear difference operators with constant coefficients (here, F is a field and s_i denotes the i-th shift operator on $F^{\mathbb{Z}^n}$);
- the Weyl algebra $\mathbb{R}[t][\partial; \mathrm{id}, \frac{d}{dt}]$, where $\partial p = p\partial + \frac{dp}{dt}$ for all $p \in \mathbb{R}[t]$, corresponding to linear differential equations with polynomial coefficients;
- its discrete analogon $F[t][s; \sigma, 0]$, where $sp = \sigma(p)s$ for all $p \in F[t]$, and σ denotes the automorphism on $F[t]$ given by $(\sigma(p))(t) = p(t+1)$.

The last two examples and their multivariate generalisations can be treated within the comprehensive framework of skew polynomial rings (Goodearl and Warfield, Jr. 2004, p. 34). Coherent (but not Noetherian) rings also play a major role in control theory; see for instance (Bourlès and Oberst 2009, Quadrat 2003). On the other hand, the ring $\mathbb{Z}_r[s_1, \ldots, s_n]$, where $\mathbb{Z}_r := \mathbb{Z}/r\mathbb{Z}$ for some $r > 1$, provides an example for a coherent (in fact, even Noetherian) commutative ring with zero-divisors whose total quotient ring is self-injective (Zerz 2007). The same holds, for instance, for the ring $F[s]/\langle s^n - 1\rangle[\partial]$, which is studied in (Brockett and Willems 1974) in connection with ordinary differential equations arising from the spatial discretisation of certain partial differential equations. The ring $F[s]/\langle s^n - 1\rangle$ itself, which turns up naturally with periodic systems and cyclic codes, coincides with its total ring of fractions and is self-injective (Lam 1999, §3B).

4 Connection with torsionfreeness

In many special situations that arise, e.g., in control theory, the solvability of the inverse syzygy problem has been characterised in terms of the torsionfreeness (rather than torsionlessness) of $C = \operatorname{coker}(\beta) = N/\operatorname{im}(\beta)$; see (Oberst 1990, Sec. 7, Thm. 21), (Pommaret 2001, p. 614), (Shankar 2002). This means that $t(C) = 0$, that is, we have $s[n] = 0 \Rightarrow [n] = 0$ for any $s \in S$, $[n] \in C$. It is clear that any torsionless module is torsionfree. The converse is not true in general (for instance, $\mathbb{Q}$ as a $\mathbb{Z}$-module is torsionfree but not torsionless), but holds for finitely generated modules under certain additional assumptions on the ring $\mathcal{D}$. These requirements typically concern the existence of a total quotient ring with desirable properties, for instance, semi-simplicity. A nice presentation of the work of Goldie, Gentile, Levy, and other authors on this subject can be found in (Goodearl and Warfield, Jr. 2004, Ch. 7). Our assumptions have a similar flavour. A different approach to the same problem, in the commutative case, can be found in (Bruns and Vetter 1988, Sec. 16E). Related problems are also studied in (Auslander and Bridger 1969), but note that the authors call torsionless modules (in the sense of the present paper) “1-torsionfree”, and that they write $t(C)$ to denote the kernel of the natural map $C \to C^{**}$ rather than the torsion submodule of C.

Theorem 3. *Let $\mathcal{D}$ be a coherent Ore ring whose total quotient ring is right self-injective. Let C be a finitely presented left $\mathcal{D}$-module. Then C is torsionfree if and only if it is torsionless.*

Proof. Let $C = \operatorname{coker}(\beta)$ be torsionfree. Applying the procedure outlined above, let $\ker(\beta^*) = \operatorname{im}(\gamma^*)$ and $\ker(\gamma) = \operatorname{im}(\hat{\beta})$. Since $\mathcal{Q}$ is flat, we have

$$\ker(\mathcal{Q} \otimes \gamma) = \mathcal{Q} \otimes \ker(\gamma) = \mathcal{Q} \otimes \operatorname{im}(\hat{\beta}) = \operatorname{im}(\mathcal{Q} \otimes \hat{\beta})$$

and similarly

$$\ker\big((\mathcal{Q} \otimes \beta)^*\big) = \ker(\beta^* \otimes \mathcal{Q}) = \operatorname{im}(\gamma^* \otimes \mathcal{Q}) = \operatorname{im}\big((\mathcal{Q} \otimes \gamma)^*\big) \ .$$

Since $\mathcal{Q}$ is right self-injective, this implies

$$\operatorname{im}(\mathcal{Q} \otimes \beta) = \ker(\mathcal{Q} \otimes \gamma) \ .$$

Thus we may conclude that

$$\operatorname{im}(\mathcal{Q} \otimes \beta) = \operatorname{im}(\mathcal{Q} \otimes \hat{\beta}) \ .$$

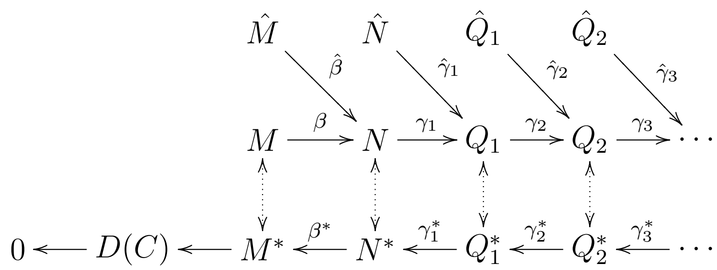
Therefore, any $n \in \operatorname{im}(\hat{\beta})$ can be written in the form $n = s^{-1}\tilde{n}$ for some $s \in S$ and $\tilde{n} \in \operatorname{im}(\beta)$. Then $s[n] = 0$ in $C = N/\operatorname{im}(\beta)$. Since C is torsionfree and s is regular, we have $[n] = 0$, that is, $n \in \operatorname{im}(\beta)$. Thus we have shown $\operatorname{im}(\hat{\beta}) \subseteq \operatorname{im}(\beta)$, and the converse direction holds anyhow. By Theorem 2, we may conclude that C is torsionless. □

5 Computation of extension groups

We noted already above that the procedure outlined in Section 2 allows us to compute the kernel of the natural map $\eta_C : C \to C^{**}$. However, this kernel is actually an extension group and therefore our procedure also provides us with a method to determine certain extension groups. In order to see this, we need the notion of the *Auslander-Bridger dual* $D(C)$ of a finitely presented module C (Auslander and Bridger 1969): if $C = \operatorname{coker}(\beta)$, then we set $D(C) = \operatorname{coker}(\beta^*)$, i.e., the cokernel of the dual map. Since the module C can be presented in many different ways, the dual $D(C)$ is defined only up to projective direct summands. One has then an exact sequence

$$0 \longrightarrow \operatorname{Ext}^1_{\mathcal{D}}\big(D(C), \mathcal{D}\big) \longrightarrow C \xrightarrow{\eta_C} C^{**} \longrightarrow \operatorname{Ext}^2_{\mathcal{D}}\big(D(C), \mathcal{D}\big) \longrightarrow 0 \ . \qquad (4)$$

Thus we may indeed identify $\ker(\eta_C)$ and $\operatorname{Ext}^1_{\mathcal{D}}\big(D(C),\mathcal{D}\big)$. An iteration allows us to compute the higher extension groups, too; each further group requires essentially two additional syzygy computations. We obtain then the following extension of the diagram (3):



Obviously, the bottom row defines a free resolution of $D(C)$ and our results imply therefore the isomorphisms

$$\operatorname{Ext}^1_{\mathcal{D}}\big(D(C),\mathcal{D}\big) \cong \operatorname{im}(\hat{\beta})/\operatorname{im}(\beta)\ , \quad \operatorname{Ext}^{i+1}_{\mathcal{D}}\big(D(C),\mathcal{D}\big) \cong \operatorname{im}(\hat{\gamma}_i)/\operatorname{im}(\gamma_i)\ .$$

Note that the definition of the Auslander-Bridger dual $D(C)$ trivially implies that $D\big(D(C)\big) = C$. Hence, by reverting the roles of β and β^*, we can use our procedure also for computing the extension groups $\operatorname{Ext}^i_{\mathcal{D}}(C,\mathcal{D})$.

Finally, we remark that it is well-known that the extension groups $\operatorname{Ext}^i_{\mathcal{D}}\big(D(C),\mathcal{D}\big)$ depend only on C and not on the chosen presentation β. For $i = 1, 2$ this fact follows immediately from (4). For the higher extension groups we have an isomorphism $\operatorname{Ext}^{i+2}_{\mathcal{D}}\big(D(C),\mathcal{D}\big) \cong \operatorname{Ext}^i_{\mathcal{D}}(C,\mathcal{D})$, since the merging of the defining sequence of $D(C)$ with a free resolution of the dual C^* yields a free resolution of $D(C)$.

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