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On the minimal free resolution of the Rees algebra of some monomial ideals

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Abstract We give a complete description of the defining ideal $\mathcal{R}(I)$ of the Rees algebra of any monomial ideal I minimally generated by three monomials in two variables. We give a Gröbner basis and a complete explicit description of the minimal free resolution of $\mathcal{R}(I)$ based on arithmetical properties of the generators of the ideal I. We also present algorithms for the computation of $\mathcal{R}(I)$ and its minimal free resolution. These results extend and generalize previous work by Cox and by Cortadellas and D'Andrea on parametrizations of monomial plane curves.

Keywords Rees algebra, binomial ideals, free resolution, Gröbner bases

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1 Introduction

Let $I \subseteq \mathbb{K}[t_1, \ldots, t_n]$ be an ideal in a polynomial ring on n indeterminates over a field $\mathbb{K}$. The Rees algebra of I encodes important algebraic information about I and geometrical information about the variety defined by I. In particular, the Rees algebra is used to compute integral closure of powers of ideals and the asymptotic behaviour of these powers and products of ideals [3]; the homological properties of the Rees algebra characterize for instance ideals having linear powers, or linear products [3,4,19]. In the geometric setting, the Rees algebra is a fundamental tool in the resolution of singularities of algebraic varieties (in this context, the Rees algebra is usually referred to as the blow-up algebra), cf. [2,12,23]. A recent application of Rees algebras is to the method of moving curves for the implicitization of a rational parametrization, which was developed by Sederberg and others in [20,21]. In [9], Cox established a connection between this problem and the computation of the defining ideal of the Rees algebra of the ideal associated to the parametrization. After this connection was established, several cases have been analyzed. In particular, in [8], the authors study the case of monomial plane curves. They give a complete

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description of the minimal free resolution of the Rees algebra of the parametrization by means of an arithmetical procedure on the data defining the curve.

In general, little is known about the defining ideals and the structure of free resolutions of Rees algebras, therefore the work in [8] and others like [5,10,11,15,16,24,25] are valuable contributions to the understanding of this important object. This paper is a contribution to this area that extends some of the previous work on Rees algebras. The main result of the first part of the paper is Algorithm 1, see Theorem 1. It computes the minimal generating set of the defining ideal of the Rees algebra of any monomial ideal I minimally generated by three monomials in two variables. Which we later prove to be unique up to a possible sign change, Theorem 2. As particular case, we apply our methods to the problem of monomial (projective) plane curve parametrizations, as studied in [8]. Using the intermediate computations of this algorithm, we describe a graph that encodes all the data in the minimal free resolution of the Rees algebra of the ideal I. This resolution is described in the main result of the paper, Theorem 7. In the process of proving this result, we find a minimal Gröbner basis for the defining ideal of the Rees algebra of I for a suitable term order.

The outline of the paper is the following: In Section 2 we give the necessary preliminaries and basic notations on Rees algebras. The main algorithm is described in Section 3, and we apply it to the particular case of parametrizations of monomial plane curves in Section 4. Finally, Section 5 is devoted to the description of the minimal free resolution of the Rees algebra. This section starts with the description of the graph that encodes the minimal free resolution of the Rees algebra of the ideal (Section 5.1), then we obtain a Gröbner basis of the ideal (Section 5.2), a non-minimal free resolution (Section 5.3) and finally the minimal free resolution (Section 5.4).

2 Preliminaries and problem setting

Let $I \subset R = \mathbb{K}[T_1, \ldots, T_n] = \mathbb{K}[\mathbf{T}]$ be a monomial ideal, let $G(I)$ be the unique minimal set of monomial generators of I, and let z be an extra variable. The *Rees algebra* of I, denoted by $\mathcal{R}_I \subset R[z]$ is the R-subalgebra

$$\mathcal{R}_I = R \oplus Iz \oplus I^2 z^2 \oplus \cdots$$

We can describe the Rees algebra in the following way. Let S denote the polynomial ring $S = \mathbb{K}[T_1, \ldots, T_n][X_w \mid w \in G(I)] = \mathbb{K}[\mathbf{T}][\mathbf{X}]$. The *Rees map* is the R-algebra map $\phi : S \to R[z]$ given by $\phi(X_w) = wz$. The kernel of ϕ is called the *Rees ideal* of I (or the *defining ideal of the Rees algebra of I*), denoted by $\mathcal{R}(I)$. The Rees algebra of I is given by the quotient $\mathcal{R}_I \simeq S/\mathcal{R}(I)$, which is isomorphic to the image of ϕ. In addition to the Rees map, we will make use of the *toric map* $\psi : S \to R$ given by $\psi(X_w) = w$. The generators of $\mathcal{R}(I) = \ker(\phi)$ are binomials of the form $\mathbf{T}^\alpha \mathbf{X}^\beta - \mathbf{T}^{\alpha'} \mathbf{X}^{\beta'}$, where $\phi(\mathbf{T}^\alpha \mathbf{X}^\beta) = \phi(\mathbf{T}^{\alpha'} \mathbf{X}^{\beta'})$. These correspond, by dropping the powers of z, to binomials such that $\psi(\mathbf{T}^\alpha \mathbf{X}^\beta) = \psi(\mathbf{T}^{\alpha'} \mathbf{X}^{\beta'})$ where $|\beta| = |\beta'|$.

Example 1 In the case of a monomial plane curve, we start with a parametrization $\varphi : \mathbb{P}^1_{\mathbb{K}} \to \mathbb{P}^2_{\mathbb{K}}$ given by $(t_0 : t_1) \mapsto (t_0^d : t_0^u t_1^{d-u} : t_1^d)$. We set now $R = \mathbb{K}[T_0, T_1]$ and $S = \mathbb{K}[T_0, T_1, X_0, X_1, X_2]$. The ideal of the parametrization is the monomial ideal $I \subseteq R$ given by $I = \langle T_0^d, T_0^u T_1^{d-u}, T_1^d \rangle$ and the Rees map is given by $\phi(X_0) = T_0^d z$, $\phi(X_1) = T_0^u T_1^{d-u} z$, $\phi(X_2) = T_1^d z$. The Rees algebra of I is fully studied in [8] for this case.

The main approach for the study of $\mathcal{R}_I$ is to obtain a minimal set of generators of the Rees ideal $\mathcal{R}(I)$, which we denote $G(\mathcal{R}(I))$. In order to do so we will make use of the following term:

Definition 1 We say a binomial is an *essential binomial* for a binomial ideal, if (up to a non-zero constant) it belongs to some minimal generating set of the ideal.

Recall that binomial ideals may not have a unique minimal set of generators, cf. [6] for more on toric ideals with a unique one; thus, these binomials may be replaced by equivalent binomials in different minimal generating sets. Nonetheless, since they belong to a minimal set of generators, these binomials cannot be expressed as combinations of the remaining essential binomials in the set. Obtaining them is difficult in general, both for toric ideals as $\ker(\psi)$ and for Rees ideals, cf. [7,8,18]. Such a minimal set of generators $G(\mathcal{R}(I))$ is given by relations among powers of the minimal generators of I. These are essentially minimal first syzygies of I^t for each t, but also some divisibility relations among powers of minimal generators of I might give rise to essential binomials.

Let $I \subseteq R = \mathbb{K}[T_0, T_1] = \mathbb{K}[\mathbf{T}]$ be a monomial ideal minimally generated by the set $G(I) = \{\mathbf{T}^\alpha, \mathbf{T}^\beta, \mathbf{T}^\gamma\}$. Let $h = T_0^{\mu_0} T_1^{\mu_1}$ be the greatest common divisor of the elements of $G(I)$ and $I' = \langle \frac{g}{h} | g \in G(I)\rangle$. We have that $s \cdot \mathbf{T}^i - s' \cdot \mathbf{T}^j = 0$ if and only if $s \cdot \frac{\mathbf{T}^i}{h} - s' \cdot \frac{\mathbf{T}^j}{h} = 0$, therefore we only need to study the Rees algebra of ideals like I', which are of the form $I' = \langle T_0^{d_1}, T_0^{u_1} T_1^{u_2}, T_1^{d_2}\rangle$, where d_1, u_1, d_2 and u_2 are four integers. Thus, we can restrict our analysis to zero-dimensional ideals.

3 Algorithm for obtaining a minimal generating set of the Rees algebra of some monomial ideals

Let $I = \langle T_0^{d_1}, T_0^{u_1} T_1^{u_2}, T_1^{d_2}\rangle \subseteq \mathbb{K}[T_0, T_1]$. In order to compute a generating set of the Rees ideal of I we need to find the essential binomials, which are of the form $\mathbf{T}^\alpha \mathbf{X}^\beta - \mathbf{T}^{\alpha'} \mathbf{X}^{\beta'}$ with $|\beta| = |\beta'| = t$, for every t, such that $\phi(\mathbf{T}^\alpha \mathbf{X}^\beta - \mathbf{T}^{\alpha'} \mathbf{X}^{\beta'}) = \psi(\mathbf{T}^\alpha \mathbf{X}^\beta - \mathbf{T}^{\alpha'} \mathbf{X}^{\beta'}) = 0$. Lemma 1 and Propositions 1 and 2 below give necessary conditions for essential binomials, while Proposition 3 gives a sufficient condition. These propositions are constructive in the sense that we explicitly enumerate all the essential binomials, i.e. a minimal set of generators of $\mathcal{R}(I)$. This enumeration gives rise to Algorithm 1, which is the main result of this section, see Theorem 1. Furthermore, Theorem 2 below establishes that this class of Rees ideals belongs to the family of binomial ideals with a unique minimal generating set.

First observe that since $T_1 \nmid \psi(X_0)$ and $T_0 \nmid \psi(X_2)$ only powers of $\psi(X_1)$ can be expressed as a product of powers of $\psi(X_0)$ and $\psi(X_2)$, i.e. there is some tuple $(k, k_1, k_2) \in \mathbb{N}_0^3$ such that $\psi(X_1^k) = \psi(X_0^{k_1} X_2^{k_2})$. For the degrees in T_0 and T_1, such a relation implies $k \cdot u_i = k_i \cdot d_i$ for $i = 1, 2$, and then k must be a multiple of $d_i / \gcd(d_i, u_i)$. Hence, the smallest integers satisfying that relation are

$$k = \operatorname{lcm}\left(\frac{d_1}{\gcd(d_1, u_1)}, \frac{d_2}{\gcd(d_2, u_2)}\right)\ ,\ k_1 = \frac{k u_1}{d_1} \text{ and } k_2 = \frac{k u_2}{d_2}.$$

Let $G^{(t)}(I) = \{g_1^{(t)}, \ldots, g_{k_t}^{(t)}\}$ be the set of products of t elements of $G(I)$ ordered lexicographically. We denote by $\psi_t'(g_i^{(t)})$ the monomial $X_0^a X_1^b X_2^c \in S$ such that $\psi(X_0^a X_1^b X_2^c) = g_i^{(t)}$ and $a+b+c = t$. Observe that this is unique for $t < k$, and thus by a slight abuse of notation, we can consider ψ_t' and ψ as mutually inverse for $t < k$. In the following sections, we will say that a binomial $\mathbf{T}^\alpha \mathbf{X}^\beta - \mathbf{T}^{\alpha'} \mathbf{X}^{\beta'}$ in $S = \mathbb{K}[\mathbf{T}][\mathbf{X}]$ corresponds to the relation $s \cdot g_i^{(t)} - s' \cdot g_j^{(t)} = 0$ in $\mathbb{K}[\mathbf{T}]$ if $\mathbf{T}^\alpha = s$, $\mathbf{T}^{\alpha'} = s'$, $\mathbf{X}^\beta = \psi_t'(g_i^{(t)})$ and $\mathbf{X}^{\beta'} = \psi_t'(g_j^{(t)})$ and vice-versa.

Lemma 1 *Given $G(\mathcal{R}(I))$, a minimal set of generators of the Rees ideal $\mathcal{R}(I)$ where $I = \langle T_0^{d_1}, T_0^{u_1} T_1^{u_2}, T_1^{d_2}\rangle \subseteq \mathbb{K}[T_0, T_1]$. Any essential binomial of $G(\mathcal{R}(I))$ must correspond to some relations of the form*

$$s_{i,i+1}^{(t)} \cdot g_i^{(t)} - s_{i,i+1}^{\prime(t)} \cdot g_{i+1}^{(t)} = 0, \text{ for } i = 1, 2, \ldots, k_t - 1.$$

where

$$s_{i,j}^{(t)} = \frac{\operatorname{lcm}(g_i^{(t)}, g_j^{(t)})}{g_i^{(t)}} \text{ , and } s_{i,j}'^{(t)} = \frac{\operatorname{lcm}(g_i^{(t)}, g_j^{(t)})}{g_j^{(t)}}, \text{ for } i < j.$$

Proof Consider a binomial corresponding to the relation formed by two monomials that are not lexicographically consecutive in $G^{(t)}(I)$, i.e.

$$s_{i,i+j}^{(t)} \cdot g_i^{(t)} - s_{i,i+j}'^{(t)} \cdot g_{i+j}^{(t)} = 0, \text{ for some } j > 1.$$

Then,

$$\begin{aligned} s_{i,i+j}^{(t)} \cdot \psi_t'(g_i^{(t)}) - s_{i,i+j}'^{(t)} \cdot \psi_t'(g_{i+j}^{(t)}) =& c_{i,i+1}\big(s_{i,i+1}^{(t)} \cdot \psi_t'(g_i^{(t)}) - s_{i,i+1}'^{(t)} \cdot \psi_t'(g_{i+1}^{(t)})\big) \\ &+ c_{i+1,i+j}'\big(s_{i+1,i+j}^{(t)} \cdot \psi_t'(g_{i+1}^{(t)}) - s_{i+1,i+j}'^{(t)} \cdot \psi_t'(g_{i+j}^{(t)})\big), \end{aligned}$$

where $c_{i,i+1} = \frac{s_{i,i+j}^{(t)}}{s_{i,i+1}^{(t)}} = \frac{\operatorname{lcm}(g_i^{(t)}, g_{i+j}^{(t)})}{\operatorname{lcm}(g_i^{(t)}, g_{i+1}^{(t)})}$ and $c_{i+1,i+j}' = \frac{s_{i,i+j}'^{(t)}}{s_{i+1,i+j}'^{(t)}} = \frac{\operatorname{lcm}(g_i^{(t)}, g_{i+j}^{(t)})}{\operatorname{lcm}(g_{i+1}^{(t)}, g_{i+j}^{(t)})}$. Hence, any binomial $s_{i,i+j}^{(t)} \cdot \psi_t'(g_i^{(t)}) - s_{i,i+j}'^{(t)} \cdot \psi_t'(g_{i+j}^{(t)})$ with $j > 1$ is not essential since it can be expressed as some combination of other binomials.

Observe that for $t = 1$ we have two essential binomials, namely those corresponding to the only two minimal syzygies in $\operatorname{Syz}(I)$:

$$s_{1,2}^{(1)} \cdot g_1^{(1)} - s_{1,2}'^{(1)} \cdot g_2^{(1)} = 0 \text{ corresponds to } T_1^{u_2} X_0 - T_0^{d_1 - u_1} X_1 \in G(\mathcal{R}(I)) \tag{1}$$

$$s_{2,3}^{(1)} \cdot g_2^{(1)} - s_{2,3}'^{(1)} \cdot g_3^{(1)} = 0 \text{ corresponds to } T_1^{d_2 - u_2} X_1 - T_0^{u_1} X_2 \in G(\mathcal{R}(I)). \tag{2}$$

As mentioned above, a minimal generating set of $\mathcal{R}(I)$ is composed of binomials of the form $\mathbf{T}^\alpha \mathbf{X}^\beta - \mathbf{T}^{\alpha'} \mathbf{X}^{\beta'}$ with $|\beta| = |\beta'| = t$, for all t, such that $\psi(\mathbf{T}^\alpha \mathbf{X}^\beta - \mathbf{T}^{\alpha'} \mathbf{X}^{\beta'}) = 0$. This implies that both terms in each essential binomial must not have any common factor, which can only be achieved if $\mathbf{X}^\beta$ or $\mathbf{X}^{\beta'}$ is actually a pure power of one of the X_i. Note that for every $t > 0$, $\psi(X_0^t)$ is the largest lexicographically ordered element in $G^{(t)}(I)$ and $\psi(X_2^t)$ is the smallest. By Lemma 1 we know that essential binomials come uniquely from the relations between two consecutive elements in $G^{(t)}(I)$; however, the only consecutive elements of $\psi(X_0^t)$ and $\psi(X_2^t)$ are $\psi(X_0^{t-1} X_1)$ and $\psi(X_1 X_2^{t-1})$ respectively. In both instances we have a common factor (X_0^{t-1} and X_2^{t-1} respectively); therefore, for $t > 1$ there can not be any essential binomial involving the generators $X_0^t = \psi_t'(g_1^{(t)})$ and $X_2^t = \psi_t'(g_{k_t}^{(t)})$. Hence, we must only look for consecutive pairs in $G^{(t)}(I)$ involving the generator $g_i^{(t)} = \psi(X_1^t)$, because any potential essential binomial can only come from those.

Notation Using some arithmetical properties on the initial parameters u_1, u_2, t_1, and t_2, we introduce the following notation, which will be of great importance all throughout this paper:

- $q = \min\left\{\frac{d_1}{\gcd(d_1,u_1)}, \frac{d_2}{\gcd(d_2,u_2)}\right\}$ and $[q] = \{1, \ldots, q\}$.
- $a_i^{(k)} = u_k \cdot i \pmod{d_k}$ with $0 \le a_i^{(k)} < d_k$ and $b_i^{(k)} = \lfloor \frac{u_k \cdot i}{d_k} \rfloor$ for $i = 1, 2, \ldots, q$ and $k = 1, 2$.
- $\Delta_{\max}(d_k, u_k) = \{i \in [q] \mid a_i^{(k)} = \max\{a_1^{(k)}, \ldots, a_i^{(k)}\}\}$
- $\Delta_{\min}(d_k, u_k) = \{i \in [q] \mid a_i^{(k)} = \min\{a_1^{(k)}, \ldots, a_i^{(k)}\}\}$

Lemma 2 $\Delta_{\max}(d_k, u_k) \cap \Delta_{\min}(d_k, u_k) = \{1\}$. *Furthermore, for any* $\delta \in \Delta_{\min}(d_k, u_k)$ *and* $\epsilon \in \Delta_{\max}(d_k, u_k)$, *we have* $a_\delta^{(k)} \le a_\epsilon^{(k)}$ *and this inequality is strict if* $\delta \ne \epsilon$.

Proof This is clear from the definitions.

Lemma 3 *We have the following relations between $\Delta_{\min}(d_k, u_k)$ and $\Delta_{\max}(d_k, u_k)$:*

1. *Let $\delta \in \Delta_{\min}(d_k, u_k)$, and let $\epsilon = \min\{\gamma \in \Delta_{\min}(d_k, u_k) \mid \gamma > \delta\}$. Then,*
   (a) $\epsilon - \delta \in \Delta_{\max}(d_k, u_k)$,
   (b) There is no element of $\Delta_{\max}(d_k, u_k)$ between $\epsilon - \delta$ and ϵ.
2. *Let $\delta \in \Delta_{\max}(d_k, u_k)$, and let $\epsilon = \min\{\gamma \in \Delta_{\max}(d_k, u_k) \mid \gamma > \delta\}$. Then,*
   (a) $\epsilon - \delta \in \Delta_{\min}(d_k, u_k)$,
   (b) There is no element of $\Delta_{\min}(d_k, u_k)$ between $\epsilon - \delta$ and ϵ.
3. *In the situation of item 1, let $\zeta_1 < \cdots < \zeta_r$ be the integers in $\Delta_{\max}(d_k, u_k)$ between δ and ϵ. Then, for each n with $1 \leq n \leq r$, we have:*
   (a) $\zeta_n - \delta \in \Delta_{\max}(d_k, u_k)$,
   (b) There is no element of $\Delta_{\max}(d_k, u_k)$ between $\zeta_n - \delta$ and ζ_n.
4. *In the situation of item 2, let $\zeta_1 < \cdots < \zeta_r$ be the integers in $\Delta_{\min}(d_k, u_k)$ between δ and ϵ. Then, for each n with $1 \leq n \leq r$, we have:*
   (a) $\zeta_n - \delta \in \Delta_{\min}(d_k, u_k)$,
   (b) There is no element of $\Delta_{\min}(d_k, u_k)$ between $\zeta_n - \delta$ and ζ_n.

Proof We first prove items 1a and 2a. For this it suffices to prove item 1a, because item 2a is dual to it.

Regarding item 1a, let $\delta \in \Delta_{\min}(d_k, u_k)$ and $\epsilon = \min\{\gamma \in \Delta_{\min}(d_k, u_k) \mid \gamma > \delta\}$. We know that $a_\delta^{(k)} = \min\{a_i^{(k)} \mid 1 \leq i \leq \delta\}$ and $a_\epsilon^{(k)} = \min\{a_i^{(k)} \mid 1 \leq i \leq \epsilon\}$. This implies in particular that $a_\delta^{(k)} > a_\epsilon^{(k)} \geq 0$. Moreover, there is no element of $\Delta_{\min}(d_k, u_k)$ between δ and ϵ. Now we argue by *reductio ad absurdum.* Assume that $\epsilon - \delta \notin \Delta_{\max}(d_k, u_k)$. Then there is an integer $1 \leq j < \epsilon - \delta$ such that $a_j^{(k)} > a_{\epsilon-\delta}^{(k)}$, and we have $a_\epsilon^{(k)} = a_\delta^{(k)} + a_{\epsilon-\delta}^{(k)} - d_k < a_\delta^{(k)} + a_j^{(k)} - d_k < a_\delta^{(k)}$. This implies

$$a_\epsilon^{(k)} < a_{\delta+j}^{(k)} < a_\delta^{(k)},$$

but then $\Delta_{\min}(d_k, u_k)$ would contain an element between δ and ϵ, a contradiction. Thus the assumption was false and $\epsilon - \delta \in \Delta_{\max}(d_k, u_k)$ as claimed.

Now we prove items 1b and 2b. For this it suffices to prove item 1b, because item 2b is dual to it.

Regarding item 1b, we argue by *reductio ad absurdum.* Assume there were an integer $\zeta \in \Delta_{\max}(d_k, u_k)$ with $\epsilon - \delta < \zeta < \epsilon$, and take the minimal such ζ. Then $\zeta = \min\{\gamma \in \Delta_{\max}(d_k, u_k) \mid \gamma > \epsilon - \delta\}$, and, using item 2a, we obtain $\eta := \zeta - (\epsilon - \delta) \in \Delta_{\min}(d_k, u_k)$. Note that $\eta < \epsilon - (\epsilon - \delta) = \delta$. Moreover, $a_\zeta^{(k)} > a_{\epsilon-\delta}^{(k)}$; but this implies

$$a_\epsilon^{(k)} < a_{\delta+\zeta}^{(k)} < a_\delta^{(k)},$$

yielding $a_{\delta+\zeta-\epsilon}^{(k)} = a_\eta^{(k)} < a_\delta^{(k)}$, in contradiction to $\delta \in \Delta_{\min}(d_k, u_k)$. Hence the assumption was false and 1b holds.

Now we prove items 3a and 4a. For this it suffices to prove item 3a, because item 4a is dual to it.

Regarding item 3a, by hypothesis, we know that $a_\delta^{(k)} = \min\{a_i^{(k)} \mid 1 \leq i \leq \delta\}$ and $a_{\zeta_n}^{(k)} = \max\{a_i^{(k)} \mid 1 \leq i \leq \zeta_n\}$. This implies in particular that $0 < a_\delta^{(k)} < a_{\zeta_n}^{(k)} < d$. Moreover, there is no element of $\Delta_{\min}(d_k, u_k)$ between δ and ζ_n. Now we argue by *reductio ad absurdum.* Assume that $\zeta_n - \delta \notin \Delta_{\max}(d_k, u_k)$. Then there is an integer $1 \leq j < \zeta_n - \delta$ such that $a_j^{(k)} > a_{\zeta_n-\delta}^{(k)}$. This implies one of the following: Either,

$$a_\delta^{(k)} < a_{\zeta_n}^{(k)} < a_{\delta+j}^{(k)} < d,$$

in contradiction to $a^{(k)}_{\zeta_n} = \max\{a^{(k)}_i \mid 1 \leq i \leq \zeta_n\}$; or,

$$0 \leq a^{(k)}_{\delta+j} < a^{(k)}_{\delta} < a^{(k)}_{\zeta_n},$$

in which case $\Delta_{\min}(d_k, u_k)$ would contain an element between δ and ζ_n, again a contradiction. Thus, the assumption was false and $\zeta_n - \delta \in \Delta_{\max}(d_k, u_k)$ as claimed.

Now we prove items 3b and 4b. For this it suffices to prove item 3b, because item 4b is dual to it.

Regarding item 3b, we argue by *reductio ad absurdum*. Assume that for some n with $1 \leq n \leq r$ there were an element $\eta_n \in \Delta_{\max}(d_k, u_k)$ with $\zeta_n - \delta < \eta_n < \zeta_n$, and take the minimal such η_n. Then $\eta_n = \min\{\gamma \in \Delta_{\max}(d_k, u_k) \mid \gamma > \zeta_n - \delta\}$, and, using item 2a, we obtain $\rho_n := \eta_n - (\zeta_n - \delta) \in \Delta_{\min}(d_k, u_k)$. Note that $\rho_n < \zeta_n - (\zeta_n - \delta) = \delta$. Using item 2b, we get the additional information that there is no element of $\Delta_{\min}(d_k, u_k)$ between ρ_n and η_n. But by hypothesis $\delta \in \Delta_{\min}(d_k, u_k)$, and $\rho_n < \delta$. Thus necessarily $\zeta_n - \delta < \eta_n < \delta$. As a consequence, $\zeta_1 - \delta < \eta_n < \zeta_1$. Thus, from now on we may assume that $n = 1$. We also write $\eta := \eta_1$ and $\rho := \rho_1$.

We aim to show that, under the made assumption, $r = 1$. To this end, if $r > 1$, consider $\tilde{\eta} := \zeta_2 - \delta > \zeta_1 - \delta$. By item 3a, $\tilde{\eta} \in \Delta_{\max}(d_k, u_k)$. There cannot be any element $\hat{\eta} \in \Delta_{\max}(d_k, u_k)$ between $\zeta_1 - \delta$ and $\tilde{\eta}$, as this would induce an element of $\Delta_{\max}(d_k, u_k)$ between ζ_1 and ζ_2. Hence, $\tilde{\eta} = \eta$, and $\rho = (\zeta_2 - \delta) - (\zeta_1 - \delta) = \zeta_2 - \zeta_1$. However, $\rho < \delta < \zeta_2$, and this contradicts item 2b applied to ζ_1 and ζ_2. Thus indeed we must have $r = 1$.

Thus, under the made assumption, $\zeta := \zeta_1$ is the only element in $\Delta_{\max}(d_k, u_k)$ between δ and ϵ. By item 1a, $\epsilon - \delta \in \Delta_{\max}(d_k, u_k)$. By item 1b, there is no element of $\Delta_{\max}(d_k, u_k)$ between $\epsilon - \delta$ and ϵ. This implies $\epsilon \geq \zeta + \delta$. Consider the integer $\sigma := \zeta + \rho$. Since $\zeta < \sigma < \epsilon$, $\sigma \notin \Delta_{\min}(d_k, u_k) \cup \Delta_{\max}(d_k, u_k)$. Hence, $a^{(k)}_{\delta} < a^{(k)}_{\sigma} < a^{(k)}_{\zeta}$; otherwise, $a^{(k)}_{\sigma}$ would be a maximal or minimal value. Moreover, also $a^{(k)}_{\rho}$ lies between $a^{(k)}_{\delta}$ and $a^{(k)}_{\zeta}$: $a^{(k)}_{\rho} > a^{(k)}_{\delta}$, because $\{\rho, \delta\} \subseteq \Delta_{\min}(d_k, u_k)$ and $\rho < \delta$; $a^{(k)}_{\rho} < a^{(k)}_{\zeta}$, because $\rho \neq \zeta$ and $\zeta \in \Delta_{\max}(d_k, u_k)$. Furthermore, $a^{(k)}_{\sigma} < a^{(k)}_{\rho}$; otherwise, $a^{(k)}_{\rho} < a^{(k)}_{\zeta+\rho} < a^{(k)}_{\zeta}$, implying $a^{(k)}_{\zeta} < a^{(k)}_{\zeta}$, which is absurd. Thus, we have: $a^{(k)}_{\delta} < a^{(k)}_{\zeta+\rho} < a^{(k)}_{\rho} < a^{(k)}_{\zeta}$; and, as $\rho = \eta - \zeta + \delta$, we get $a^{(k)}_{\delta} < a^{(k)}_{\delta+\eta} < a^{(k)}_{\rho}$. This implies, finally, that $a^{(k)}_{\eta} < a^{(k)}_{\rho}$, the desired contradiction (note that $\eta \in \Delta_{\max}(d_k, u_k)$ and $\rho \in \Delta_{\min}(d_k, u_k)$). Hence the assumption was false and 3b holds.

From parts (1) and (2) of this last lemma one can deduce the following:

Corollary 1 *If $\delta \in \Delta_{\min}(d_k, u_k)$ where $\theta = \max\{\gamma \in \Delta_{\max}(d_k, u_k) \mid \gamma \leq \delta\}$, then we have that $\delta + \theta$ is either the index of the immediate next minimum after δ in $\Delta_{\min}(d_k, u_k)$ or the index of the immediate next maximum after θ in $\Delta_{\max}(d_k, u_k)$. Furthermore, there does not exist $\lambda \in \Delta_{\max}(d_k, u_k) \cup \Delta_{\min}(d_k, u_k)$ such that $\delta < \lambda < \delta + \theta$.*
The same holds true for any $\delta \in \Delta_{\max}(d_k, u_k)$ where $\theta = \max\{\gamma \in \Delta_{\min}(d_k, u_k) \mid \gamma \leq \delta\}$.

Lemma 3 tells us that, at any given point i in the sequence, if $a^{(k)}_{\delta} = \max\{a^{(k)}_1, \ldots, a^{(k)}_i\}$ and $a^{(k)}_{\theta} = \min\{a^{(k)}_1, \ldots, a^{(k)}_i\}$, then either $\delta + \theta \in \Delta_{\max}(d_k, u_k)$ or $\delta + \theta \in \Delta_{\min}(d_k, u_k)$ and there are no elements between i and $\delta + \theta$ in either $\Delta_{\max}(d_k, u_k)$ or $\Delta_{\min}(d_k, u_k)$. To put it simply, by adding the indices of the last maximum and the last minimum at a given point in the sequence we get either the subindex of the next minimum or maximum in the sequence.

Lemma 4 *Let $i > 1$ be the smallest integer which satisfies $b^{(1)}_i - b^{(1)}_{i-1} = b^{(2)}_i - b^{(2)}_{i-1}$. Then $\Delta_{\min}(d_1, u_1) \cap [i-1] = \Delta_{\max}(d_2, u_2) \cap [i-1]$, and $\Delta_{\max}(d_1, u_1) \cap [i-1] = \Delta_{\min}(d_2, u_2) \cap [i-1]$. In other words, for $j < i$, one of $a^{(1)}_j$ and $a^{(2)}_j$ is a maximum if and only if the other is a minimum.*

Proof Let us first assume $i > 1$ is the smallest integer which satisfies $b_i^{(1)} - b_{i-1}^{(1)} = b_i^{(2)} - b_{i-1}^{(2)}$. By Lemma 2 note that 1 is in all Deltas. Hence, by Corollary 1, at $j = 2$ we must have either a maximum or a minimum in both sequences. If both $a_2^{(1)}$ and $a_2^{(2)}$ were maxima on their respective sequence, then we are done because $b_2^{(1)} - b_1^{(1)} = 0$ and $b_2^{(2)} - b_1^{(2)} = 0$; so, $i = 2$. This is clear since by assumption $u_k < d_k$. Similarly, if both $a_2^{(1)}$ and $a_2^{(2)}$ were minima of their sequence, then $b_2^{(1)} - b_1^{(1)} = b_2^{(2)} - b_1^{(2)} = 1$; so, $i = 2$ and we are done. Thus, the only case left is if one is a maximum and the other a minimum, in which case we repeat the process by adding the positions of the "updated" last maxima and minima. Note that we get the same outcome from these sums in both sequences since up until this point the sequences are complementary (in terms of maxima and minima) of each other, leading to the same scenario as described at $j = 2$.

Furthermore, note that in a similar way one can easily conclude the following fact, which will be later used to show Propositions 2, 3 and 4, and Lemma 5.

Remark 1 Let $i > 1$ be the smallest integer such that $b_i^{(1)} - b_{i-1}^{(1)} = b_i^{(2)} - b_{i-1}^{(2)}$. For each $j < i$ we have that either $b_j^{(1)} = b_{j-1}^{(1)} + 1$ while $b_j^{(2)} = b_{j-1}^{(2)}$ or $b_j^{(2)} = b_{j-1}^{(2)} + 1$ while $b_j^{(1)} = b_{j-1}^{(1)}$, and since $b_1^{(1)} = b_1^{(2)} = 0$ then $b_j^{(1)} + b_j^{(2)} = j - 1$.

Proposition 1 *There are no essential binomials involving elements of $G^{(t)}(I)$ for $t > q$.*

Proof Recall that $q = \min\{d_k / \gcd(d_k, u_k) \mid k = 1, 2\}$. Without loss of generality assume that $q = d_1 / \gcd(d_1, u_1)$. At $t = q$ we have that $a_q^{(1)} = 0$. Thus,

$$\deg_{T_0} \psi(X_1^q) = u_1 \cdot q = \deg_{T_0} \psi(X_0^{b_q^{(1)}} X_2^{q - b_q^{(1)}}) = d_1 \cdot b_q^{(1)} .$$

This fact implies that:

$$T_1^{|r|} X_1^q - X_0^{b_q^{(1)}} X_2^{q-b_q^{(1)}} \in \mathcal{R}(I), \text{ or} \tag{3a}$$

$$T_1^{|r|} X_0^{b_q^{(1)}} X_2^{q-b_q^{(1)}} - X_1^q \in \mathcal{R}(I), \tag{3b}$$

for $r := d_2(q - b_q^{(1)}) - u_2 q$. If $r \geq 0$ we are in the case (3a), otherwise we are in the case (3b).

Let now $t > q$ i.e. $t = q + j$ for some $j \in \mathbb{Z}^+$. From Lemma 1 we know that any essential binomial whose monomials correspond to elements of $G^{(t)}(I)$ must come from some lexicographically consecutive pair of elements of $G^{(t)}(I)$, we also know that one of them must correspond to the pure power t of X_1 and the other one to some combination of only X_0 and X_2. Since $d_1 \cdot (b_{q+j-1}^{(1)} + 1) \geq u_1 \cdot (q + j) \geq d_1 \cdot b_{q+j-1}^{(1)}$ we know that the generator lexicographically closest to $\psi(X_1^{q+j})$ satisfying the previous condition are $\psi\big(X_0^{b_{q+j-1}^{(1)}+1} X_2^{q+j-(b_{q+j-1}^{(1)}+1)}\big)$ and $\psi\big(X_0^{b_{q+j-1}^{(1)}} X_2^{q+j-(b_{q+j-1}^{(1)})}\big)$ yielding the following potentially essential binomials:

$$T_1^{|r_1|} X_0^{b_{q+j-1}^{(1)}+1} X_2^{q+j-(b_{q+j-1}^{(1)}+1)} - T_0^{|r_2|} \cdot X_1^{q+j}, \text{ and}$$
$$T_1^{|r_1'|} \cdot X_1^{q+j} - T_0^{|r_2'|} X_0^{b_{q+j-1}^{(1)}} X_2^{q+j-b_{q+j-1}^{(1)}},$$

where $r_1 = u_2 \cdot (q + j) - d_2 \cdot \big(q + j - (b_{q+j-1}^{(1)} + 1)\big)$, $r_2 = d_1 \cdot (b_{q+j-1}^{(1)} + 1) - u_1 \cdot (q + j)$, $r_1' = d_2 \cdot \big(q + j - (b_{q+j-1}^{(1)})\big) - u_2 \cdot (q + j)$, and $r_2' = u_1 \cdot (q + j) - d_1 \cdot (b_{q+j-1}^{(1)})$.

Note that if for $t = q$ we have that the binomial (3b), i.e. $T_1^{|r|} X_0^{b_q^{(1)}} X_2^{q-b_q^{(1)}} - X_1^q$, is in $\mathcal{R}(I)$, then one can express the term containing the pure power X_1^{q+j} as a combination of all X_0, X_1

and X_2 in both of the potentially essential binomials, henceforth guaranteeing a common factor between the two terms in both binomials. On the other hand, if we have that the element (3a), i.e $\left(T_1^r \cdot X_1^q - X_0^{b_q^{(1)}} X_2^{q-b_q^{(1)}}\right)$ is in $\mathcal{R}(I)$, then we need to see that $X_0^{b_q^{(1)}} X_2^{q-b_q^{(1)}}$ divides both $X_0^{b_{q+j-1}^{(1)}+1} X_2^{q+j-(b_{q+j-1}^{(1)}+1)}$ and $X_0^{b_{q+j-1}^{(1)}} X_2^{q+j-b_{q+j-1}^{(1)}}$, in order to demonstrate that there exists a common factor X_1 between the terms in the binomials. In other words, we need to show that $b_q^{(1)} \leq b_{q+j-1}^{(1)}$ and $q - b_q^{(1)} \leq q + j - (b_{q+j-1}^{(1)} + 1)$. The first inequality is pretty straightforward since $b_\alpha^{(1)} \geq b_\beta^{(1)}$ for any $\alpha \geq \beta$ by definition, hence, $b_q^{(1)} \leq b_{q+j-1}^{(1)}$ since we assumed $j \in \mathbb{Z}^+$. For the second inequality, note that one can easily deduce from the definitions that $\alpha - \beta \geq b_\alpha^{(1)} - b_\beta^{(1)}$ for any two $\alpha, \beta \in \mathbb{Z}_0^+$ such that $\alpha \geq \beta$. Since $q + j - 1 \geq q$ and, as shown above, $b_{q+j-1}^{(1)} \geq b_q^{(1)}$, then we can write the following inequality:

$$
\begin{aligned}
& & (q+j-1) - q \geq b_{q+j-1}^{(1)} - b_q^{(1)} \\
\Leftrightarrow & & b_q^{(1)} - q \geq b_{q+j-1}^{(1)} - (q+j) + 1 \\
\Leftrightarrow & & q - b_q^{(1)} \leq q + j - (b_{q+j-1}^{(1)} + 1)
\end{aligned}
$$

This concludes the proof, since it shows that the only two possible binomials whose monomials correspond to elements in $G^{(t)}(I)$ for $t > q$ that could potentially be essential are indeed non-essential.

Proposition 2 *If $b_i^{(1)} - b_{i-1}^{(1)} = b_i^{(2)} - b_{i-1}^{(2)}$ for some integer i, then there do not exist any essential binomials involving elements of $G^{(t)}(I)$ for any $t > i$.*

Proof Let $i > 1$ be the smallest integer such that $b_i^{(1)} - b_{i-1}^{(1)} = b_i^{(2)} - b_{i-1}^{(2)}$, then $b_j^{(1)} + b_j^{(2)} = j - 1$ for every $j < i$ by Remark 1. Hence it is straightforward to see that $\psi\big(X_0^{b_j^{(1)}+1} X_2^{b_j^{(2)}}\big)$ and $\psi\big(X_0^{b_j^{(1)}} X_2^{b_j^{(2)}+1}\big)$ are the two elements of $G^{(j)}(I)$ that can be written as a product of powers of $\psi(X_0)$ and $\psi(X_2)$, which are lexicographically closest to $\psi\big(X_1^j\big)$, for $1 < j < i$. This points to the only 2 pairs of monomials in $G^{(j)}(I)$ that can form a candidate for the essential binomials.

Now, for $G^{(i)}$, consider the two possible cases:

i) Let $b_i^{(1)} - b_{i-1}^{(1)} = b_i^{(2)} - b_{i-1}^{(2)} = 0$, then we have that $\psi\big(X_1^i\big)$ is the next lexicographically ordered element after $\psi\big(X_0^{b_i^{(1)}+1} X_2^{b_i^{(2)}+1}\big)$, where:

$$\deg_{T_0}\left(\psi\big(X_0^{b_i^{(1)}+1} X_2^{b_i^{(2)}+1}\big)\right) > \deg_{T_0}\left(\psi\big(X_1^i\big)\right)$$

and

$$\deg_{T_1}\left(\psi\big(X_0^{b_i^{(1)}+1} X_2^{b_i^{(2)}+1}\big)\right) > \deg_{T_1}\left(\psi\big(X_1^i\big)\right).$$

Hence, we have that the following binomial is in $\mathcal{R}(I)$:

$$T_0^{d_1(b_i^{(1)}+1)-u_1 i} T_1^{d_2(b_i^{(2)}+1)-u_2 i} X_1^i - X_0^{b_i^{(1)}+1} X_2^{b_i^{(2)}+1}. \tag{4}$$

ii) Let $b_i^{(1)} - b_{i-1}^{(1)} = b_i^{(2)} - b_{i-1}^{(2)} = 1$, then we have that $\psi\big(X_0^{b_i^{(1)}+1} X_2^{b_i^{(2)}+1}\big)$ is the next lexicographically ordered element after $\psi\big(X_1^i\big)$, where:

$$\deg_{T_0}\left(\psi\big(X_0^{b_i^{(1)}} X_2^{b_i^{(2)}}\big)\right) \leq \deg_{T_0}\left(\psi\big(X_1^i\big)\right)$$

and

$$\deg_{T_1}\left(\psi\left(X_0^{b_i^{(1)}}X_2^{b_i^{(2)}}\right)\right) \leq \deg_{T_1}\left(\psi\left(X_1^i\right)\right).$$

Hence, the following binomial is in $\mathcal{R}(I)$:

$$T_0^{u_1 i - d_1 b_i^{(1)}} T_1^{u_2 i - d_2 b_i^{(2)}} X_0^{b_i^{(1)}} X_2^{b_i^{(2)}} - X_1^i. \tag{5}$$

The rest of the proof follows the lines of the proof of Proposition 1. There are no essential binomials induced by $G^{(t)}(I)$, $t > i$ since any relation between elements in $G^{(t)}(I)$ for $t > i$ can be expressed in terms of either (4) or (5).

Proposition 3 *Let i be the smallest integer which satisfies $b_i^{(1)} - b_{i-1}^{(1)} = b_i^{(2)} - b_{i-1}^{(2)}$ and let $j < \min\{i, q\}$.*

There exists an essential binomial, induced by a syzygy of $G^{(j)}(I)$, of the form:

$$T_1^{a_j^{(2)}} X_0^{b_j^{(1)}+1} X_2^{b_j^{(2)}} - T_0^{d_1 - a_j^{(1)}} X_1^j$$

if and only if $a_j^{(1)} = \max\{a_k^{(1)} | k = 1, \ldots, j\}$ and $a_j^{(2)} = \min\{a_k^{(2)} | k = 1, \ldots, j\}$.
Similarly, there exists an essential binomial, induced by a syzygy of $G^{(j)}(I)$, of the form:

$$T_1^{d_2 - a_j^{(2)}} X_1^j - T_0^{a_j^{(1)}} X_0^{b_j^{(1)}} X_2^{b_j^{(2)}+1}$$

if and only if $a_j^{(1)} = \min\{a_k^{(1)} | k = 1, \ldots, j\}$ and $a_j^{(2)} = \max\{a_k^{(2)} | k = 1, \ldots, j\}$.

Proof Note that due to Lemma 4 we know that for all $j < i$, one of $a_j^{(1)}$ and $a_j^{(2)}$ is a maximum if and only if the other is a minimum. Since we assumed $j < \min\{q, i\} \leq i$, we only need show the statements hold for one of the sequences.

Let us consider the case where $a_j^{(1)}$ is a maximum for some $j < \min\{i, q\}$, the case that $a_j^{(1)}$ is a minimum is analogous.

As we have already mentioned, an essential binomial comes from a relation of a consecutive pair of monomials in $G^{(j)}(I)$ where one of the two is the monomial $h = \psi(X_1^j) = T_0^{u_1 \cdot j} \cdot T_1^{u_2 \cdot j}$. Since $a_j^{(1)} = u_1 \cdot i \pmod{d_1}$, we have that $u_1 \cdot j = d_1 \cdot k + a_j^{(1)}$ for some k. Hence $k = b_j^{(1)}$, and $\deg_{T_0}(h) = d_1 \cdot b_j^{(1)} + a_j^{(1)}$. Correspondingly, $\deg_{T_1}(h) = d_2 \cdot b_j^{(2)} + a_j^{(2)}$. Since we know that $b_j^{(1)} + b_j^{(2)} = j - 1$ by Remark 1, then $b_j^{(2)} = j - (b_j^{(1)} + 1)$, and $\deg_{T_1}(h) = d_2 \cdot (j - (b_j^{(1)} + 1)) + a_j^{(2)}$.

Let h' be the immediate previous element in $G^{(j)}(I)$ i.e. $h' \prec_{lex} h$, and there does not exist $\hat{h} \in G^{(j)}(I)$ such that $h' \prec_{lex} \hat{h} \prec_{lex} h$. Recall that $\psi'_t(h') = X_0^a X_1^b X_2^c$ where (a, b, c) is the unique 3-tuple such that $a + b + c = j$. Observe that the essential binomial coming from the pair (h', h) is relevant if $b = 0$. Thus, we have that $\deg_{T_0}(h') = d_1 \cdot a$ and so we get:

$$d_1 \cdot b_j^{(1)} + a_j^{(1)} < d_1 \cdot a$$

Note that $b_j^{(1)} \leq \frac{u_1}{d_1} \cdot j$, and by assumption $u_1 < d_1$; hence, we have that $b_j^{(1)} < j$. Thus, there exists an element $g \in G^{(j)}(I)$ corresponding to the 3-tuple $\left(b_j^{(1)} + 1, 0, j - (b_j^{(1)} + 1)\right) = \left(b_j^{(1)} + 1, 0, b_j^{(2)}\right)$ with $\deg_{T_0}(g) = d_1 \cdot b_j^{(1)} + d_1$, which implies $g \prec_{lex} h$. We must show that $g = h'$, i.e. there does not exist $\hat{h} \in G^{(j)}(I)$ such that $\deg_{T_0}(g) > \deg_{T_0}(\hat{h}) > \deg_{T_0}(h)$ if and only if $a_j^{(1)} = \max\{a_k^{(1)} | k = 1, \ldots, j\}$. In order to get a contradiction on the first direction let us assume that there exists such an element $\hat{h} \in G^{(j)}(I)$, with $\psi'_t(\hat{h}) = X_0^{\hat{a}} X_1^{\hat{b}} X_2^{\hat{c}}$. Then, we have that:

$$d_1 \cdot b_j^{(1)} + a_j^{(1)} < d_1 \cdot \hat{a} + u_1 \cdot \hat{b} < d_1 \cdot b_j^{(1)} + d_1$$

for some $\hat{a}, \hat{b} \geq 0$ such that $\hat{a} + \hat{b} \leq j$. However, if we subtract $d_1 \cdot b_j^{(1)}$ from every side of the inequality we get that:

$$a_j^{(1)} < d_1 \cdot (\hat{a} - b_j^{(1)}) + u_1 \cdot \hat{b} < d_1.$$

Since the difference between $a_j^{(1)}$ and d_1 is by definition smaller that d_1 this implies $\hat{a} = b_j^{(1)}$ and $u_1 \cdot \hat{b} \pmod{d_1} > a_j^{(1)}$ where $\hat{b} < j$ (because $\hat{b} = j$ implies that $h = \hat{h}$). This contradicts our assumption that $a_j^{(1)} = \max\{a_k^{(1)} | k = 1, \ldots, j\}$.

For the other direction, let's assume there is some $p < j$ with $d_1 > a_p^{(1)} > a_j^{(1)}$. By adding $d_1 \cdot b_j^{(1)}$ to the inequality we get

$$d_1 \cdot b_j^{(1)} + a_j^{(1)} < d_1 \cdot b_j^{(1)} + a_p^{(1)} < d_1 \cdot b_j^{(1)} + d_1$$

which can be rewritten, using that $a_p^{(1)} = u_1 \cdot p - d_1 \cdot b_p^{(1)}$, as:

$$d_1 \cdot b_j^{(1)} + a_j^{(1)} < d_1 \cdot (b_j^{(1)} - b_p^{(1)}) + u_1 \cdot p < d_1 \cdot b_j^{(1)} + d_1$$

which implies that exists an element $\hat{h} \in G^{(j)}(I)$ such that

$$\psi'_j(\hat{h}) = X_0^{j-p-(b_j^{(1)} - b_p^{(1)})} X_1^p X_2^{(b_j^{(1)} - b_p^{(1)})}.$$

This element sits lexicographically between h and h' and thus, by Lemma 1, there does not exist such an essential binomial in $\mathcal{R}(I)$ corresponding to the pair (h', h).

Thus there does not exist any element in $G^{(j)}(I)$ between $\psi\big(X_0^{b_j^{(1)}+1} X_2^{b_j^{(2)}}\big)$ and $\psi\big(X_1^j\big)$ in the lexicographic order, if and only if $a_j^{(1)} = \max\{a_k^{(1)} | k = 1, \ldots, j\}$ and $a_j^{(2)} = \min\{a_k^{(2)} | k = 1, \ldots, j\}$. In other words we have the lexicographically consecutive pair

$$\Big(\psi\big(X_0^{b_j^{(1)}+1} X_2^{b_j^{(2)}}\big), \psi\big(X_1^j\big)\Big)$$

which yields the essential binomial corresponding to the syzygy

$$T_1^{a_j^{(2)}} \psi(X_0^{b_j^{(1)}+1} X_2^{b_j^{(2)}}) - T_0^{d_1 - a_j^{(1)}} \psi(X_1^j) = 0,$$

which is in $\mathrm{Syz}(I^j)$ for every $j < \min\{i, q\}$ if and only if $a_j^{(1)} = \max\{a_k^{(1)} | k = 1, \ldots, j\}$ and $a_j^{(2)} = \min\{a_k^{(2)} | k = 1, \ldots, j\}$.

The previous conditions are used in Algorithm 1 to compute a minimal generating set of the Rees ideal of any monomial ideal in two variables with three minimal generators.

Theorem 1 *Let $I = \langle T_0^{d_1}, T_0^{u_1} T_1^{u_2}, T_1^{d_2} \rangle \subseteq R = \mathbf{k}[T_0, T_1]$ a monomial ideal in two variables generated by three monomials, and let $q = \min\{d_1/\gcd(d_1, u_1), d_2/\gcd(d_2, u_2)\}$. Algorithm 1 terminates in at most q steps and returns a minimal generating set of the Rees ideal of I.*

Proof As we saw above, a minimal generating set of $\mathcal{R}(I)$ is made of essential binomials, remember that these correspond to relations between pairs of products of powers of the same degree of $g_0 = T_0^{d_1}, g_1 = T_0^{u_1} T_1^{u_2}$, and $g_2 = T_1^{d_2}$ that cannot be expressed as any combination of other such relations. Let us consider the lexicographically ordered set of all these products of powers of degree t, then Lemma 1 tells us that all the relations between any pair that is not made of consecutive elements in the set does not correspond to a minimal generator for all $t \in \mathbb{N}^+$, since any such relation can be expressed in terms of two other relations. Excluding all those and given

Algorithm 1 Minimal generating set of $\mathcal{R}(I)$ where I is a zero-dimensional monomial ideal in the polynomial ring in two variables generated by any three minimal generators

Input: The minimal generators $f_1 = T_0^{d_1}$, $f_2 = T_0^{u_1}T_1^{u_2}$ and $f_3 = T_1^{d_2}$ of the monomial ideal $I \subset \mathbb{K}[T_0, T_1]$.
Output: A minimal generating set of $\mathcal{R}(I)$

get_param(f_1, f_2, f_3) ▷ subalgorithm retrieving u_1, u_2, d_1, and d_2 from $\{f_1, f_2, f_3\}$
$a^{(1)} \leftarrow u_1, a^{(2)} \leftarrow u_2, b^{(1)} \leftarrow 0, b^{(2)} \leftarrow 0, b_2^{(1)} \leftarrow 0, b_2^{(2)} \leftarrow 0, j \leftarrow 1;$
$gens \leftarrow \{\}$
$min^{(1)} \leftarrow a^{(1)}, min^{(2)} \leftarrow a^{(2)}$
$max^{(1)} \leftarrow a^{(1)}, max^{(2)} \leftarrow a^{(2)}$
$gens \leftarrow gens \cup \{T_1^{u_2}X_0 - T_0^{d_1-u_1}X_1\}$
$gens \leftarrow gens \cup \{T_1^{d_2-u_2}X_1 - T_0^{u_1}X_2\}$
while $a^{(1)} \neq 0$ and $a^{(2)} \neq 0$ **do**
  $j \leftarrow j + 1$
  $a^{(1)} \leftarrow (u_1 * j) \mod d_1$
  $a^{(2)} \leftarrow (u_2 * j) \mod d_2$
  $b^{(1)} \leftarrow u_1 * j/d_1$
  $b^{(2)} \leftarrow u_2 * j/d_2$
  if $b^{(1)} - b_2^{(1)} == b^{(1)} - b_2^{(1)}$ **then**
    if $b^{(1)} - b_2^{(1)} == 0$ **then**
      $gens \leftarrow gens \cup \{T_0^{d_1-a^{(1)}}T_1^{d_2-a^{(2)}}X_1^j - X_0^{b^{(1)}+1}X_2^{b^{(2)}+1}\}$
      break
    end if
    if $b^{(1)} - b_2^{(1)} == 1$ **then**
      $gens \leftarrow gens \cup \{T_0^{a^{(1)}}T_1^{a^{(2)}}X_0^{b^{(1)}}X_2^{b^{(2)}} - X_1^j\}$
      break
    end if
  else
    if $(a^{(1)} > max^{(1)}$ **and** $a^{(2)} < min^{(2)})$ **then**
      $gens \leftarrow gens \cup \{T_1^{a^{(2)}}X_0^{b^{(1)}+1}X_2^{b^{(2)}} - T_0^{d_1-a^{(1)}}X_1^j\}$
      $max^{(1)} \leftarrow a^{(1)}, min^{(2)} \leftarrow a^{(2)}$
    end if
    if $(a^{(1)} < min^{(1)}$ **and** $a^{(2)} > max^{(2)})$ **then**
      $gens \leftarrow gens \cup \{T_1^{d_2-a^{(2)}}X_1^j - T_0^{a^{(1)}}X_0^{b^{(1)}}X_2^{b^{(2)}+1}\}$
      $min^{(1)} \leftarrow a^{(1)}, max^{(2)} \leftarrow a^{(2)}$
    end if
    $b_2^{(1)} \leftarrow b^{(1)}$
    $b_2^{(2)} \leftarrow b^{(2)}$
  end if
end while
return $gens$

the fact that one of the two elements in the pair must be a pure power g_1^t as claimed at the beginning of the section; that yields the conclusion that the only minimal generators are the binomials that correspond to the relation of this pure power and a product of powers of g_0 and g_2 that is located immediately in front or prior to g_1^t in the ordered set.

Let i be the smallest integer which satisfies $b_i^{(1)} - b_{i-1}^{(1)} = b_i^{(2)} - b_{i-1}^{(2)}$. For $t < \min\{q, i\}$ Proposition 3 finds all such binomials. Then, by Propositions 1 and 2 at $t = q$ and at $t = i$, we expect to find some relation of divisibility between g_1^t and $g_0^j g_2^{t-j}$, for some integer $j < t$. Whichever of these comes from the smaller degree is the one corresponding to a binomial that is a minimal generator. In fact, this is the last element in this minimal generating set since due to these last two propositions we know that all other relations after this one can be expressed as combinations of the ones we have already found. This shows the correctness and termination of Algorithm 1.

In general, it is possible for a binomial ideal to have more than one minimal set of generators. A recent problem arising from Algebraic Statistics, see [22], is when a toric ideal possesses a unique minimal set of binomial generators. To study this problem, Ohsugi and Hibi introduced in [17] the notion of indispensable binomials. A binomial is called *indispensable* for a binomial ideal, if (up to a non-zero constant) it belongs to every binomial generating set of the ideal. Hence, a binomial ideal I possesses a unique minimal generating set, if and only if all binomials in a minimal generating set of I are indispensable. Next we show that the Rees ideals treated in this paper belong to this class of binomial ideals.

Theorem 2 *The Rees ideal $\mathcal{R}(I)$ of any monomial ideal $I = \langle T_0^{d_1}, T_0^{u_1}T_1^{u_2}, T_1^{d_2}\rangle \subset \mathbb{K}[T_0, T_1]$ possesses a unique minimal binomial generating set (up to a sign change).*

Proof Let $\mathcal{H}$ be the minimal generating set of the Rees ideal $\mathcal{R}(I) \subset \mathbb{K}[T_0, T_1, X_0, X_1, X_2]$ obtained as computed by Algorithm 1. Note that, as we saw at Section 2, all binomials in $\mathcal{R}(I)$ must be of the form $\mathbf{T}^{\alpha}\mathbf{X}^{\beta} - \mathbf{T}^{\alpha'}\mathbf{X}^{\beta'}$ with $|\beta| = |\beta'|$, such that $\phi(\mathbf{T}^{\alpha}\mathbf{X}^{\beta}) = \phi(\mathbf{T}^{\alpha'}\mathbf{X}^{\beta'})$; hence, we can easily see that by assigning the following "natural" weighting of the variables

$$w_{X_0} = \deg(g_0) = d_1,\;\; w_{X_1} = \deg(g_1) = u_1 + u_2,\;\; w_{X_2} = \deg(g_2) = d_2,\;\; w_{T_0} = 1,\;\; w_{T_1} = 1$$

then all binomials in the ideal can be considered homogeneous, and thus $\mathcal{R}(I)$ can also be regarded as homogeneous. It is well-known that although individual generators in a minimal generating set can be replaced by linear combinations of others, the total number of independent generators in a homogeneous polynomial ideal I remains unchanged because they span the same module. Therefore, in order to show that $\mathcal{H}$ is the unique minimal generating set of some Rees ideal $\mathcal{R}(I)$, let us assume that there exists an essential binomial $g \in \mathcal{H}$ which is not indispensable. This means that then there exists another minimal generating set $\mathcal{H}' = \mathcal{H} \setminus \{g\} \cup \{g'\}$ such that $\mathcal{R}(I) = \langle \mathcal{H}' \rangle$. Again, the binomial substitute must be of the form $g' = \mathbf{T}^{\alpha}\mathbf{X}^{\beta} - \mathbf{T}^{\alpha'}\mathbf{X}^{\beta'}$ with $|\beta| = |\beta'|$, and moreover $\gcd(\mathbf{T}^{\alpha}\mathbf{X}^{\beta}, \mathbf{T}^{\alpha'}\mathbf{X}^{\beta'}) = 1$ since $\mathcal{H}'$ is a minimal generating set of $\mathcal{R}(I)$. Remember that $g \in \mathcal{R}(I)$; hence, $g = c' \cdot g' + \sum_{h \in \mathcal{H}' \setminus \{g'\}} c_h \cdot h$, where $c' \neq 0$ since the binomial g is essential and cannot be expressed as a combination of the rest of the elements in $\mathcal{H}$. Recall that one of the monomials, let us call it τ, composing g was divisible by T_1, moreover $\deg_{X_1}(\tau) = \deg_{\mathbf{X}}(\tau)$. Note now how every essential binomial obtained from Algorithm 1 satisfies this, and also how as the X_1 degree gets larger on every step both T_0 and T_1 degrees get smaller. This implies that there does not exist any monomial in the essential binomials of $\mathcal{H} \setminus \{g\}$ which divides τ. Therefore, one of the monomials in g' has to be of the form $\sigma = \frac{\tau}{c'}$, while the other monomial, ρ, has to be divisible by X_0 or X_2. Note that all essential binomials of this form are already in the set $\mathcal{H}$ up to a sign change. Thus, we are left with no other option than $c' = -1$, or equivalently $g = -g'$, which then shows that actually the minimal generating set of $\mathcal{R}(I)$, $\mathcal{H}$, is unique up to a sign change.

Note that since both terms in every binomial generator of the Rees ideal of I have constant coefficients ± 1, we will refer to this minimal set of generators as the unique normalized set of binomials up to a possible sign change of the elements. Here normalized means that the leading coefficient of these binomials is 1, see Section 5.2.

j	$a^{(1)}$	$b^{(1)}$	$a^{(2)}$	$b^{(2)}$	$max^{(1)}$	$max^{(2)}$	$min^{(1)}$	$min^{(2)}$	$gens$
1	9	0	6	0	9	6	9	6	$T_1^6X_0 - T_0^6X_1, T_1^7X_1 - T_0^9X_2$
2	3	1	12	0		12	3		$T_1X_1^2 - T_0^3X_0X_2$
3	12	1	5	1	12			5	$T_1^5X_0^2X_2 - T_0^3X_1^3$
4	6	2	11	1					
5	0	3	4	2			0	4	$T_1^4X_0^3X_2^2 - X_1^5$

Table 1 Trace of Algorithm 1 for $I = \langle T_0^{15}, T_0^9T_1^6, T_1^{13}\rangle$.

Example 2 Let $I = \langle T_0^{15}, T_0^9T_1^6, T_1^{13}\rangle$. Both $q = 5$ and $i = 5$, so Algorithm 1 finishes in 5 steps and returns the unique minimal generating set of $\mathcal{R}(I)$. First, the first two elements are added to the set of minimal generators of $\mathcal{R}(I)$ as seen in (1) and (2). They correspond to each of the two minimal first syzygies of I^1, namely:

$$g_1 = T_1^6X_0 - T_0^6X_1 \text{ and } g_2 = T_1^7X_1 - T_0^9X_2$$

The algorithm proceeds as follows: at $j = 2$ we have a min from $a_2^{(1)}$ and (as expected by Lemma 4) a max from $a_2^{(2)}$; hence, the essential binomial $g_3 = T_1X_1^2 - T_0^3X_0X_2$ is added. At $j = 3$ we get a max from $a_3^{(1)}$ and a min from $a_3^{(2)}$, so the essential binomial $g_4 = T_1^5X_0^2X_2 - T_0^3X_1^3$ is added. At $j = 4$ nothing relevant is found as expected from Corollary 1. Finally, at $j = 5 = \min\{i, q\}$ the Algorithm detects the divisibility relation, the corresponding binomial $g_5 = T_1^4X_0^3X_2^2 - X_1^5$ is added, and the Algorithm terminates.

Table 1 shows the trace of the variables across the steps of the algorithm for this example. Columns max and min only show numbers when they are actually updated and therefore new elements are added to the set of minimal generators of $\mathcal{R}(I)$.

Remark 2 Corollary 1 tells us that if one adds the indices of the last maximum and minimum at any given point in one of the sequences then one gets the index of the immediately next relevant element (a maximum or a minimum) in the sequence. By repeating this process with the upgraded indices of the maximum and minimum then one can compute all the indices where minimal generators of the Rees ideal are added, without actually having to go through all indices. This can save some computations in Algorithm 1 but we have opted not to include this optimization in order to ease the description of the algorithm. To make use of such optimization, one would need to add two more variables initialized to 1, for example: $c_1 \leftarrow 1$ and $c_2 \leftarrow 1$. Then, line 9 in Algorithm 1 would need to be changed to $j \leftarrow c_1 + c_2$, and last: the new indices need to be updated so lines 26 and 30 should update also the variables c_1 and c_2 respectively, i.e. by setting $c_1 \leftarrow j$ and $c_2 \leftarrow j$ respectively.

4 Rees algebras of monomial plane curve parametrizations

A particularly interesting case of application of Theorem 1 is to the ideal of a parametrization of a plane monomial curve, see [8]. In this case, the ideal under consideration is of the form $I = \langle T_0^d, T_0^uT_1^{d-u}, T_1^d\rangle \subset R = \mathbb{K}[T_0, T_1]$, i.e. $d_1 = d_2 = d$ and $u_2 = d - u_1$. These ideals correspond to the equigenerated case of the ideals studied in the previous section. Because of that we can describe their minimal generating set using just two parameters. As a result, we are able to adapt the propositions in the preceding sections to a simpler form and hence to give an adapted version of Algorithm 1 for this particular case. We provide in this way an alternative approach to the computation of $\mathcal{R}(I)$, which allows us to give a more combinatorial description of its minimal free resolution, see Section 5.

Proposition 4 (Adaptation of a combination of Proposition 1 and Proposition 2) *Let $t = \frac{d}{\gcd(d,u)}$. There are no essential binomials involving elements of $G^{(j)}(I)$ for $j > t$.*

Proof The proof of Proposition 1 can be easily adapted to this case since $\gcd(d,u) = \gcd(d, d-u)$ which yields $t = q$ and as a result we obtain that the binomial

$$X_1^{\frac{d}{\gcd(d,u)}} - X_0^{\frac{u}{\gcd(d,u)}} X_2^{\frac{d-u}{\gcd(d,u)}}$$

is an essential binomial for $\mathcal{R}(I)$ and it comes from the one relation of divisibility between elements in $G^{(t)}(I)$.

To see that $i = t$ is the smallest integer such that $b_i^{(1)} - b_{i-1}^{(1)} = b_i^{(2)} - b_{i-1}^{(2)}$, is slightly less straightforward. Remember that we saw in the previous section (Remark 1) that $b_j^{(1)} - b_{j-1}^{(1)} \neq b_j^{(2)} - b_{j-1}^{(2)}$ is equivalent to saying that $b_j^{(1)} + b_j^{(2)} = j - 1$ for every $j < t$.

Note that $u_1 \cdot i = u \cdot i = d \cdot k + a_j^{(1)}$ for some integer k, where $0 < a_j^{(1)} < d$ for every $1 \leq j < t$ by definition. Then, $u_2 \cdot j = (d-u) \cdot j = d \cdot j - (d \cdot k + a_j^{(1)}) = d(j - k - 1) + d - a_j^{(1)}$. Since $0 < d - a_j^{(1)} < d$, for every $1 \leq j < t$; we then have that $b_j^{(1)} + b_j^{(2)} = k + (j - k - 1) = j - 1$, as desired. Note that also at $j = t$ we have $a_t^{(1)} = 0$ and thus $b_t^{(1)} + b_t^{(2)} = k + (t - k) = t$. But, $b_{t-1}^{(1)} + b_{t-1}^{(2)} = k + ((t-1) - k - 1) = t - 2$, which implies as desired that $i = t$ is the smallest integer such that $b_i^{(1)} - b_{i-1}^{(1)} = b_i^{(2)} - b_{i-1}^{(2)} = 1$.

Therefore, for the particular case of the ideal of the parametrization, it does not make much sense to continue using two sequences since each one contains the same information as the other one and we just saw that the stopping step of the algorithms is always $t = \frac{d}{\gcd(d,u)}$.

For integers below t, the essential binomials are described by the following adaptation of Proposition 3.

Proposition 5 (Adaptation of Proposition 3) *Let $a_i = u \cdot i \pmod d$ and $b_i = \lfloor \frac{u \cdot i}{d} \rfloor$ for $i = 1, 2, \ldots, \frac{d}{\gcd(d,u)}$. Then, there exists an essential binomial formed by elements of $G^{(j)}(I)$ of the form:*

$$T_1^{d-a_j} \cdot X_0^{b_j+1} X_2^{j-(b_j+1)} - T_0^{d-a_j} \cdot X_1^j$$

if and only if $a_j = \max\{a_i | i = 1, \ldots, j\}$ for every $j \leq \frac{d}{\gcd(d,u)}$.

Similarly, there exists an essential binomial formed by elements of $G^{(j)}(I)$ of the form:

$$T_1^{a_j} \cdot X_1^j - T_0^{a_j} X_0^{b_j} X_2^{j-b_j}$$

if and only if $a_j = \min\{a_i | i = 1, \ldots, j\}$ for every $j \leq \frac{d}{\gcd(d,u)}$.

As a result of the last two propositions we are able to adapt Algorithm 1 for the case of parametrizations of monomial plane curves obtaining a quite simpler version in the form of Algorithm 2.

Example 3 Let $I = \langle T_0^{21}, T_0^6 T_1^{15}, T_1^{21} \rangle$, so $d = 21$ and $u = 6$. Algorithm 2 finishes in $\frac{d}{\gcd(d,u)} = 7$ steps. First, we add the first two elements to this set of minimal generators of $\mathcal{R}(I)$, one for each of the two minimal first syzygies of I, corresponding to the minimal generators of $\mathrm{Syz}(I)$, namely:

$$g_1 = T_1^{15} X_0 - T_0^{15} X_1 \text{ and } g_2 = T_1^6 X_1 - T_0^6 X_2.$$

The algorithm proceeds until step 7 in which $g_6 = X_1^7 - X_0^2 X_2^5$ is added. Table 2 shows the trace of the variables across the steps of the algorithm for this example, and the rest of the generators, g_3, g_4, g_5. Columns max and min only show numbers when they are actually updated, and therefore new elements are added to the set of minimal generators of $\mathcal{R}(I)$.

Algorithm 2 Minimal generating set of Rees ideal $\mathcal{R}(I)$ where I is the associated ideal to some monomial plane curve parametrization

Input: Parameters d and u of the monomial ideal defining the monomial plane curve.
Output: A minimal generating set of $\mathcal{R}(I)$

$gens \leftarrow \{\}$
$j \leftarrow 1,\ a \leftarrow u,\ b \leftarrow 0$
$min \leftarrow a$
$max \leftarrow a$
$gens \leftarrow gens \cup \left\{T_1^{d-u}X_0 - T_0^{d-u}X_1\right\}$
$gens \leftarrow gens \cup \left\{T_1^{u}X_1 - T_0^{u}X_2\right\}$
while $a \neq 0$ **do**
  $j \leftarrow j+1$
  $a \leftarrow (u * j) \mod d$
  $b \leftarrow (u * j)/d$
  if $a > max$ **then**
    $gens \leftarrow gens \cup \left\{T_1^{d-a}X_0^{b+1}X_2^{j-(b+1)} - T_0^{d-a}X_1^{j}\right\}$
    $max \leftarrow a$
  end if
  if $a < min$ **then**
    $gens \leftarrow gens \cup \left\{T_1^{a}X_1^{j} - T_0^{a}X_0^{b}X_2^{j-b}\right\}$
    $min \leftarrow a$
  end if
end while
return $gens$

j	a	b	max	min	gens
1	6	0	6	6	$T_1^{15}X_0 - T_0^{15}X_1$, $T_1^{6}X_1 - T_0^{6}X_2$
2	12	0	12		$T_1^{9}X_0X_2 - T_0^{9}X_1^2$
3	18	0	18		$T_1^{3}X_0X_2^2 - T_0^{3}X_1^3$
4	3	1		3	$T_1^{3}X_1^4 - T_0^{3}X_0X_2^3$
5	9	1			
6	15	1			
7	0	2		0	$X_1^7 - X_0^2X_2^5$

Table 2 Trace of Algorithm 2 for $I = \langle T_0^{21}, T_0^{6}T_1^{15}, T_1^{21}\rangle$.

5 Minimal free resolution of the Rees ideal

In this section we give an explicit description of the minimal free resolution of the defining ideal of the Rees algebra associated to tri-generated monomial ideals in two variables. We thus generalize the results given in [8] for the case of Rees algebras associated to plane curve parametrizations using a different approach. In Section 5.1 we describe what we call the *Rees graph* of the ideal I, which is constructed using the data obtained in Algorithm 1. We claim that this graph encodes the minimal free resolution of $\mathcal{R}(I)$ and that all the information about the minimal resolution can be read off from it (Section 5.4). To prove this, we need as preparatory step a Gröbner basis of $\mathcal{R}(I)$ (Section 5.2) and from it we obtain a non-minimal free resolution of $\mathcal{R}(I)$ (Section 5.3). Minimalizing this resolution using two basic algebraic reductions yields finally the minimal free resolution (Section 5.4).

5.1 The Rees graph of I

Consider four given integers d_1, d_2, u_1, u_2, and the monomial ideal

$$I = I_{(d_1,d_2,u_1,u_2)} = \langle T_0^{d_1}, T_0^{u_1} T_1^{u_2}, T_1^{d_2} \rangle \subset \mathbb{K}[T_0, T_1].$$

Let $G(\mathcal{R}(I)) = \{g_1, \ldots, g_r\}$ be the minimal generating set for the Rees ideal $\mathcal{R}(I)$, that is an ideal in $\mathbb{K}[T_0, T_1, X_0, X_1, X_2]$, sorted as obtained via Algorithm 1 (up to a possible multiplication by (-1) of some elements). Using the data in Algorithm 1 we shall construct a graph that encodes the minimal free resolution of $\mathcal{R}(I)$. The nodes of the graph correspond to each of the generators in $G(\mathcal{R}(I))$. To ease the description of the graph, we classify the elements of $G(\mathcal{R}(I))$ into two types: we say that $g_i \in G(\mathcal{R}(I))$ is an *upper generator* if it was included in step 6, 16 or 25 of Algorithm 1, i.e. if it comes from an element of $\Delta_{\max}(d_1, u_1)$, and we say that g_i is a *lower generator* if it comes from an element of $\Delta_{\min}(d_1, u_1)$, i.e. it was included in step 7, 20 or 29 of Algorithm 1. In particular, g_1 is an upper generator, and g_2 is a lower generator.

We describe step by step the construction of the graph corresponding to I. For this, we will accommodate the nodes of the graph in two rows, a top row formed by nodes corresponding to the *upper generators* and a bottom row corresponding to the *lower generators*. This display is of course arbitrary, but will help making the description of the graph more convenient. The graph is built in two main steps:

1. Place the node corresponding to g_1 in the top row, and the one corresponding to g_2 in the bottom row.
2. For each of the generators g_i, $i \in \{3, \ldots, r\}$ proceed in order, doing the following: If g_i is an *upper generator* place it as the rightmost element in the top row and draw an edge from the rightmost node in both the top and the bottom row of the graph. Otherwise, if g_i is a *lower generator* place it as the rightmost element in the bottom row and draw an edge from the rightmost node in both the top and the bottom row of the graph.

We call this graph the *Rees graph of I* and denote it by $\mathrm{Graph}(\mathcal{R}(I))$ or more explicitly by $\mathrm{Graph}(d_1, d_2, u_1, u_2)$. It has r nodes, $2(r-2)$ edges and $r-3$ triangles. Figure 1 shows the three steps in the construction of $\mathrm{Graph}(15, 13, 9, 6)$, corresponding to the ideal in Example 2.

Example 4 Let $(d_1, d_2, u_1, u_2) = (15, 13, 9, 6)$ as in Example 2.

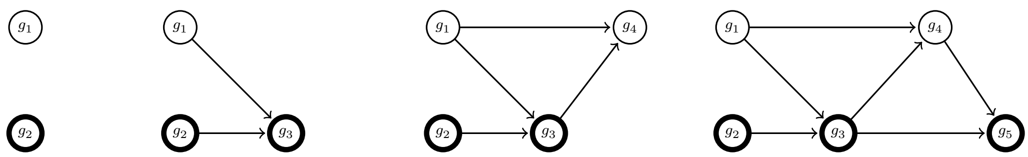

Fig. 1 Step by step of the construction of $\mathrm{Graph}(15, 13, 9, 6)$. The thick nodes correspond to *lower generators*.

Notation We will use the notation $\mathrm{DE}(\Gamma)$ for the set of directed edges in a graph $\Gamma = \mathrm{Graph}(d_1, d_2, u_1, u_2)$. As an abuse of notation sometimes we write $(j \longrightarrow k)$ instead of $(g_j \longrightarrow g_k)$ for the edges in $\mathrm{DE}(\Gamma)$. In every triangle of Γ, there is a unique source j and a unique sink ℓ. Denote the third vertex as k; write (j, k, ℓ) for the whole triangle. This means in particular that the arrows of Γ supported on g_j, g_k, g_ℓ are exactly $(j \longrightarrow k)$, $(j \longrightarrow \ell)$, and $(k \longrightarrow \ell)$. Write $\mathrm{Tri}(\Gamma)$ for the set of triangles in Γ.

The main claim of this section (see Theorem 7) is that the minimal free resolution of $\mathcal{R}(I)$ is of the form

$$0 \longrightarrow S^{r-3} \xrightarrow{\phi_2} S^{2(r-2)} \xrightarrow{\phi_1} S^r \xrightarrow{\phi_0} \mathcal{R}(I) \longrightarrow 0$$

and that Graph($\mathcal{R}(I)$) encodes this resolution in the sense that there is one generator of the first module for each node of the graph, one generator of the second module for each edge in DE (Graph($\mathcal{R}(I)$)) and one generator of the third module for each triangle in Tri (Graph($\mathcal{R}(I)$)). In particular, we have that $\mathrm{pd}(\mathcal{R}(I)) = 2$ if $r > 3$ (if $r = 3$ then $\mathrm{pd}(\mathcal{R}(I)) = 1$). Also, $\beta_0(\mathcal{R}(I)) = r$, $\beta_1(\mathcal{R}(I)) = |\mathrm{DE}\,(\mathrm{Graph}(\mathcal{R}(I)))\,| = 2(r-2)$ and $\beta_2(\mathcal{R}(I)) = |\mathrm{Tri}\,(\mathrm{Graph}(\mathcal{R}(I)))\,| = r-3$ (if $r > 3$). Moreover, the differentials in the resolution can be read off from the data in the graph and in Algorithm 1. These claims are proven in the rest of the section, see Theorems 4 and 5.

5.2 A Gröbner basis for $\mathcal{R}(I)$

Our first step is to build a Gröbner basis for $\mathcal{R}(I)$, which will be obtained by adding just one element to its minimal generating set. For this, we use a block term order, different from the one used in [8]. The reason for this choice is that this term order will allow us to obtain a Gröbner bases of the first and second syzygy modules of $\mathcal{R}(I)$ and construct a free resolution. Furthermore, the choice of this term order and the Gröbner basis associated to it is convenient for the computation of some homological invariants of $\mathcal{R}(I)$ as well as an involutive basis for it, as can be seen in [13].

We recall some terminology related to term orders and Gröbner bases that we will use. Given a term order $\prec$ and an ideal I, we say that a Gröbner basis G of I is *minimal* if it is of minimal cardinality, i.e., there is a bijection from G to the minimal generating set of the leading ideal of I. We say that a set of polynomials F is *normalized* if all its elements have the leading coefficient 1. Finally, we call a term order $\prec$ an *elimination order* for some subset of variables Y if the terms only divisible by variables of Y are larger than all other terms with respect to $\prec$.

Definition 2 Let $\sigma = T_0^{a_0} T_1^{a_1} X_0^{b_0} X_1^{b_1} X_2^{b_2}$ and $\tau = T_0^{c_0} T_1^{c_1} X_0^{d_0} X_1^{d_1} X_2^{d_2}$. Then,

$$\sigma \prec \tau \Leftrightarrow \begin{cases} X_0^{b_0} X_1^{b_1} X_2^{b_2} \prec_{\mathrm{drl}} X_0^{d_0} X_1^{d_1} X_2^{d_2}, \text{ or} \\ X_0^{b_0} X_1^{b_1} X_2^{b_2} = X_0^{d_0} X_1^{d_1} X_2^{d_2} \quad \wedge \quad T_0^{a_0} T_1^{a_1} \prec_{\mathrm{drl}} T_0^{c_0} T_1^{c_1} \end{cases},$$

where $\prec_{\mathrm{drl}}$ indicates the degree reverse lexicographic ordering with $X_2 \prec_{\mathrm{drl}} X_0 \prec_{\mathrm{drl}} X_1$ and $T_0 \prec_{\mathrm{drl}} T_1$.

We will be using the same notation from Section 3, where the integer $i > 1$ is the smallest integer which satisfies $b_i^{(1)} - b_{i-1}^{(1)} = b_i^{(2)} - b_{i-1}^{(2)}$ and $q = \min\{\frac{d_1}{\gcd(d_1,u_1)}, \frac{d_2}{\gcd(d_2,u_2)}\}$.

Lemma 5 *Let j be a positive integer such that $j < \min\{i, q\}$. Then, for each $k \in \{1, 2\}$, there exists exactly one integer $0 \le \ell_k < d_k$ such that $\mathcal{R}(I) \subset \mathbb{K}[T_0, T_1, X_0, X_1, X_2]$ contains a binomial $\alpha - \beta$ with $\alpha = T_{k-1}^{\ell_k} X_1^j$ and $\gcd(\alpha, \beta) = 1$. Moreover, $\ell_k = d_k - a_j^{(k)}$.*

Furthermore, if we consider the normalized minimal generating set $G(\mathcal{R}(I))$ with respect to the term ordering described above, then all its elements except for the last one, g_r, are binomials of the form studied in this lemma. The leading term of the element $g_r = \alpha_r - \beta_r$ is of the form $\alpha_r = T_0^{\ell_1} T_1^{\ell_2} X_1^j$ with $0 \le \ell_1 < d_1$, $0 \le \ell_2 < d_2$, and $j = \min\{i, q\}$. In fact, only if $i < q$ and $b_i^{(1)} - b_{i-1}^{(1)} = b_i^{(2)} - b_{i-1}^{(2)} = 0$ then both $T_0, T_1 \mid \alpha_r$; otherwise, $\alpha_r = T_{k-1}^{\ell_k} X_1^j$ for some $k \in \{1, 2\}$ (just like for the rest of the minimal generators).

Proof In the same way as we did in Section 3 (Propositions 2 and 3), for every power X_1^j for $j < \min\{i,q\}$ we consider the two elements that come from a product of powers of X_0 and X_2 that are lexicographically closest to $\psi(X_1^j)$ in the ordered set $G^{(j)}(I)$, one prior and one after. We must show that the only two binomials described above come from the two relations between the power of X_1 and these two elements.

Note that we have $\deg_{T_0}(\phi(X_1^j)) = u_1 j$. For any integers $t \geq 0$ and $s \geq 0$, we have $\deg_{T_0}(\phi(X_0^t X_2^s)) = d_1 t$. Remember from Section 3 (Remark 1) that $b_j^{(1)} + b_j^{(2)} = j-1$ for $j < i$; hence, $X_0^{b_j^{(1)}+1} X_2^{b_j^{(2)}}$ is the element lexicographically before X_1^j, and $X_0^{b_j^{(1)}} X_2^{b_j^{(2)}+1}$ is the element after it. As desired, we get from the first relation $\alpha_1 = T_0^\ell X_1^j$ with $\ell = d_1 - a_j^{(1)}$; and from the second $\alpha_1 = T_1^\ell X_1^j$ with $\ell = d_2 - a_j^{(2)}$.

The statements about the elements of $G(\mathcal{R}(I))$ are clear from Propositions 1, 2, and 3. This can also be seen in Algorithm 1.

For this minimal generating set to become a Gröbner basis for the order previously defined, we only add one element representing the trivial syzygy between $\psi(X_0) = T_0^{d_1}$ and $\psi(X_2) = T_1^{d_2}$.

Theorem 3 *Let d_1, d_2, u_1, u_2 be four integers and $G(\mathcal{R}(I))$ be the normalized minimal generating set w.r.t. the term order described above for the Rees ideal $\mathcal{R}(I) \subset \mathbb{K}[T_0, T_1, X_0, X_1, X_2]$ of the monomial ideal $I = \langle T_0^{d_1}, T_0^{u_1} T_1^{u_2}, T_1^{d_2} \rangle \subset \mathbb{K}[T_0, T_1]$. Furthermore, let $g_0 := T_1^{d_2} \cdot X_0 - T_0^{d_1} \cdot X_2$.*

Then, $\overline{G}(\mathcal{R}(I)) := \{g_0\} \cup G(\mathcal{R}(I))$ is a minimal Gröbner basis of $\mathcal{R}(I)$ for that elimination order.

Proof For a given monomial $\sigma = T_0^{a_0} T_1^{a_1} X_0^{b_0} X_1^{b_1} X_2^{b_2}$, write σ_X for the specialization $\sigma|_{T_0=T_1=1}$. By construction, it is clear that for any binomial $g = \sigma - \tau \in \overline{G}(\mathcal{R}(I))$, we have $\sigma_X \neq \tau_X$, and thus the leading term of g with respect to our elimination order $\prec$ only depends on σ_X and τ_X. Hence, $\mathrm{lt}(g_0) = T_1^{d_2} X_0$, and by Lemma 5, $\mathrm{lt}(g_r) = T_0^{\ell_1} T_1^{\ell_2} X_1^j$, for some integers ℓ_1, ℓ_2 and j with $0 \leq \ell_1 < d_1$, $0 \leq \ell_2 < d_2$, and $j = \min\{i,q\}$; and for any other $g \in \overline{G}(\mathcal{R}(I))$, we have $\mathrm{lt}(g)$ is either $T_0^{\ell_1} X_1^j$ or $T_1^{\ell_2} X_1^j$ for some $1 \leq j < \min\{i,q\}$ as seen in Lemma 5.

Let now $\sigma = T_i^\ell X_1^m \in \mathrm{lt}(G(\mathcal{R}(I)))$ where $i \in \{0,1\}$. Then $\mathrm{lt}(\mathcal{R}(I))$ contains no strict divisor of σ. Indeed, as the exponent ℓ of T_i is derived in Algorithm 1 from an extreme value, and as such a divisor α would be of the form stated in Lemma 5, the existence of a strict divisor would contradict the correctness of the algorithm (Theorem 1).

Now consider any binomial $b = \alpha - \beta \in \mathcal{R}(I)$, where $\alpha \succ \beta$. We need to show that there is an element $g \in \overline{G}(I)$ such that $\mathrm{lt}(g)$ divides $\mathrm{lt}(b) = \alpha$. If the monomials α and β are not coprime, then we have that $b/\gcd(\alpha,\beta)$ is also in $\ker(\phi)$. So from now on we consider that α and β are coprime. It is clear that $\deg(\alpha_X) = \deg(\beta_X) > 0$. We now analyze several cases:

Case 1: $\alpha_X = X_1^s$ for some positive integer s. If $s \geq \min\{i,q\}$, then α is either divisible by $\mathrm{lt}(g_r)$ as we saw from Propositions 1 and 2; or otherwise we are not looking at a binomial with the smallest coefficients T_i and then $\mathrm{lt}(g_1)$ or $\mathrm{lt}(g_2)$ must divide α. If $s < \min\{i,q\}$, then write $\alpha = T_i^r X_1^s$ for some integer r and for some $i \in \{0,1\}$. If $r \geq d$, it is obvious that α is divisible by either $\mathrm{lt}(g_1)$ or $\mathrm{lt}(g_2)$. If $r < d$, then $\alpha - \beta$ is a binomial of the form stated in Lemma 5. Moreover, there is a maximal integer $1 \leq j \leq s$ such that $\mathrm{lt}(G(I))$ contains a term of the form $\sigma = T_k^{\ell_{k+1}} X_1^j$, where ℓ_{k+1} is chosen via an extreme property in Algorithm 1. It follows that σ divides α.

Case 2: α_X is divisible by X_0 and β_X is divisible by X_2. Then $\deg_{T_1}(\phi(\alpha_X)) \leq \deg_{T_0}(\phi(\beta_X)) - d_2$. But then, since $\phi(\alpha) = \phi(\beta)$, $T_1^{d_2} | \alpha$. Hence, $T_1^{d_2} X_0 = \mathrm{lt}(g_0) | \alpha$.

Case 3: α_X is divisible by X_2 and β_X is not divisible by X_0, This case cannot occur, because then β_X is a pure power of X_1 and then β would be the leading term of the binomial $\alpha - \beta$.

By Lemma 5 it is clear that $X_1 \mid \operatorname{lt}(g_i)$ and $X_0, X_2 \nmid \operatorname{lt}(g_i)$ for all $g_i \in G(\mathcal{R}(I))$ and it is also clear (from Algorithm 1) that there is no divisibility among the leading terms of the minimal generators. Besides, note that $X_1 \nmid \operatorname{lt}(g_0)$ but $X_0 \mid \operatorname{lt}(g_0)$, hence we can guarantee that there is no divisibility either among the leading terms of $\overline{G}(\mathcal{R}(I))$. Moreover, since this set is already a normalized Gröbner basis for that specific term order with the smallest size possible we deduce its minimality.

5.3 A free resolution of $\mathcal{R}(I)$

The goal of this section is to set the ground for the proof that $\operatorname{Graph}(\mathcal{R}(I))$ encodes the minimal free resolution of $\mathcal{R}(I)$. For this, we first augment the graph with the new generator g_0 introduced in the previous paragraphs. As we will see below, the element g_0 behaves in a similar way as all other lower generators so we place it as the leftmost element of the bottom row of the graph. The new augmented graph $\overline{\operatorname{Graph}}(\mathcal{R}(I)) = \overline{\operatorname{Graph}}(d_1, d_2, u_1, u_2)$ has one more vertex, corresponding to g_0, two new edges, namely $g_0 \to g_2$ and $g_1 \to g_2$; and a new triangle, formed by the vertices g_1, g_2, g_3 and the edges that connect them. The augmented graph $\overline{\operatorname{Graph}}(\mathcal{R}(I))$ encodes a free resolution of $\mathcal{R}(I)$. In Section 5.4 this resolution will be reduced to a minimal one.

Example 5 Figure 2 shows the augmented graph $\overline{\operatorname{Graph}}(15, 13, 9, 6)$.

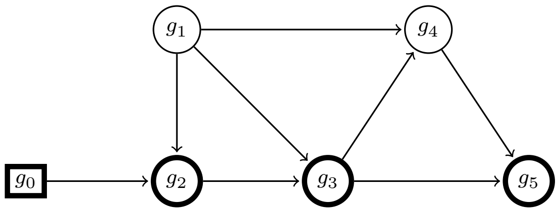

Fig. 2 $\overline{\operatorname{Graph}}(15, 13, 9, 6)$.

For the analysis of the syzygies of the Gröbner basis $\overline{G}(\mathcal{R}(I))$, recall the sets $\Delta_{\max / \min}(d_k, u_k)$ defined in Section 3. Since nothing relevant happens for steps after $t = \min\{i, q\}$, we can remove all values strictly higher than t from every $\Delta_{\max / \min}(d_k, u_k)$. Also, let $\Delta(d_1, d_2, u_1, u_2) := \{\deg_{X_1}(\operatorname{lt}(g)) \mid g \in G(\mathcal{R}(I))\}$. Note that $\Delta(d_1, d_2, u_1, u_2) = \Delta_{\max}(d_1, u_1) \cup \Delta_{\min}(d_1, u_1) = \Delta_{\min}(d_2, u_2) \cup \Delta_{\max}(d_2, u_2)$.

We also need to make use of the following monomial ideals:

Definition 3 Write $\overline{G}(\mathcal{R}(I)) = \{g_0, g_1, \ldots, g_r\}$. For $0 \leq j < r$ consider the quotient monomial ideals

$$M_j = \langle \operatorname{lt}(g_{j+1}), \ldots, \operatorname{lt}(g_r) \rangle : \langle \operatorname{lt}(g_j) \rangle,$$

and let $G(M_j)$ be the minimal monomial generating set of M_j.

By a well-known construction, the set of syzygies of $\overline{G}(\mathcal{R}(I))$ induced by multiplying any $g_j \in \overline{G}(\mathcal{R}(I))$ by any minimal generator $m \in G(M_j)$ and reducing to zero with respect to $\overline{G}(\mathcal{R}(I))$ is a Gröbner basis of $\operatorname{Syz}\left(\overline{G}(\mathcal{R}(I))\right)$ with respect to a suitable module term order (e.g. [1, Cor. 1.11], [14]). We will use this fact.

In order to understand the structure of $\overline{\operatorname{Graph}}(\mathcal{R}(I))$, we will use the colon ideals M_j. For their description, we define an auxiliary map.

Definition 4 For each $\delta \in \Delta(d_1, d_2, u_1, u_2) \setminus \{1\}$, there is a unique $g_j \in \overline{G}(\mathcal{R}(I)) \setminus \{g_0, g_1, g_2\}$ such that $\deg_{X_1}(g_j) = \delta$. Write $\iota(\delta) := j$.

Note that $\iota : \Delta(d_1, d_2, u_1, u_2) \setminus \{1\} \longrightarrow \{3, \dots, r\}$ is bijective.

Proposition 6 *Let $t = \min\{i, q\}$. $\overline{G}(\mathcal{R}(I))$, M_j, and $G(M_j)$ be given as in Definition 3. Then:*

- $G(M_0) = \{X_1\}$,
- $G(M_1) = \{T_1^{d_2-u_2}\} \cup \{\mathrm{lt}(g_{\iota(\zeta_1)})/X_1, \dots, \mathrm{lt}(g_{\iota(\zeta_\ell)})/X_1, \mathrm{lt}(g_{\iota(\epsilon)})/T_0^{\deg_{T_0}(\mathrm{lt}(g_{\iota(\epsilon)}))} X_1\}$, *where:* $\epsilon = \min\{\gamma \in \Delta_{\max}(d_1, u_1) \cup \{t\} \mid \gamma > 1\}$, *and* $\zeta_1 < \cdots < \zeta_\ell$ *are the elements of* $\Delta_{\min}(d_1, u_1)$ *between* 1 *and* ϵ,
- $G(M_2) = \{\mathrm{lt}(g_{\iota(\zeta_1)})/X_1, \dots, \mathrm{lt}(g_{\iota(\zeta_\ell)})/X_1, \mathrm{lt}(g_{\iota(\epsilon)})/T_1^{\deg_{T_1}(\mathrm{lt}(g_{\iota(\epsilon)}))} X_1\}$, *where:* $\epsilon = \min\{\gamma \in \Delta_{\min}(d_1, u_1) \cup \{t\} \mid \gamma > 1\}$, *and* $\zeta_1 < \cdots < \zeta_\ell$ *are the elements of* $\Delta_{\max}(d_1, u_1)$ *between* 1 *and* ϵ,
- *For* $2 < j < t$,
  - *If* $\delta := \iota^{-1}(j) \in \Delta_{\min}(d_1, u_1)$, *then*
  $$G(M_j) = \{\mathrm{lt}(g_{\iota(\zeta_1)})/X_1^\delta, \dots, \mathrm{lt}(g_{\iota(\zeta_\ell)})/X_1^\delta, \mathrm{lt}(g_{\iota(\epsilon)})/T_1^{\deg_{T_1}(\mathrm{lt}(g_{\iota(\epsilon)}))} X_1^\delta\},$$
  where: $\epsilon = \min\{\gamma \in \Delta_{\min}(d_1, u_1) \cup \{t\} \mid \gamma > \delta\}$, *and* $\zeta_1 < \cdots < \zeta_\ell$ *are the elements of* $\Delta_{\max}(d_1, u_1)$ *between* δ *and* ϵ,
  - *If* $\delta := \iota^{-1}(j) \in \Delta_{\max}(d_1, u_1)$, *then*
  $$G(M_j) = \{\mathrm{lt}(g_{\iota(\zeta_1)})/X_1^\delta, \dots, \mathrm{lt}(g_{\iota(\zeta_\ell)})/X_1^\delta, \mathrm{lt}(g_{\iota(\epsilon)})/T_0^{\deg_{T_0}(\mathrm{lt}(g_{\iota(\epsilon)}))} X_1^\delta\},$$
  where: $\epsilon = \min\{\gamma \in \Delta_{\max}(d_1, u_1) \cup \{t\} \mid \gamma > \delta\}$, *and* $\zeta_1 < \cdots < \zeta_\ell$ *are the elements of* $\Delta_{\min}(d_1, u_1)$ *between* δ *and* ϵ.

Proof Recall that $\overline{G}(\mathcal{R}(I)) = \{g_0, g_1, g_2, \dots, g_r\}$, where $g_0 = T_1^{d_2} X_0 - T_0^{d_1} X_2$, $g_1 = T_0^{d_1-u_1} X_1 - T_1^{u_2} X_0$, and $g_2 = T_1^{d_2-u_2} X_1 - T_0^{u_1} X_2$. Moreover, the leading terms of the other elements g_j, where $2 < j < r$ are of the form $\mathrm{lt}(g_j) = P_j \cdot X_1^{q_j}$ with $2 \le q_j < t$ and P_j a pure power of T_0 or of T_1. Moreover, by Algorithm 1, P_j is a power of T_0 with positive exponent if and only if $j < r$ and $\iota^{-1}(j) \in \Delta_{\max}(d_1, u_1) \cap \Delta_{\min}(d_2, u_2)$; it is a power of T_1 with positive exponent if and only if $j < r$ and $\iota^{-1}(j) \in \Delta_{\min}(d_1, u_1) \cap \Delta_{\max}(d_2, u_2)$. It is also immediate from Algorithm 1 that the exponents of T_0 respectively T_1 form decreasing sequences for increasing indices j. Furthermore, $\mathrm{lt}(g_r)$ is either a pure power of X_1 or a power of X_1 with coefficients T_0 and T_1 of lowest degree.

From what has been stated, it is clear that $G(M_0) = \{X_1\}$. For any j with $0 < j < r$, the X_1-degrees of $\mathrm{lcm}(\mathrm{lt}(g_j), \mathrm{lt}(g_{j+1}))/\mathrm{lt}(g_j), \dots, \mathrm{lcm}(\mathrm{lt}(g_j), \mathrm{lt}(g_r))/\mathrm{lt}(g_j)$ form an increasing sequence. Let $\mathrm{lt}(g_j) = T_k^{p_j} X_1^{q_j}$, where $k \in \{0, 1\}$ (i.e. if g_j is an upper generator $k = 0$, and if it is a lower generator $k = 1$). It is clear that, for the first index $\ell > j$ for which $\mathrm{lt}(g_\ell)$ is not divisible by T_{1-k}, we have that $\mathrm{lcm}(\mathrm{lt}(g_j), \mathrm{lt}(g_\ell)/\mathrm{lt}(g_j)$ is a pure power of X_1. This implies

$$G(M_j) = \{\mathrm{lcm}(\mathrm{lt}(g_j), \mathrm{lt}(g_{j+1}))/\mathrm{lt}(g_j), \dots, \mathrm{lcm}(\mathrm{lt}(g_j), \mathrm{lt}(g_\ell))/\mathrm{lt}(g_j)\}.$$

Moreover, $\mathrm{lt}(g_{j+1}), \dots, \mathrm{lt}(g_{\ell-1})$ are all divisible by T_{1-k} and thus, they are all of the same type of generators, while both $\mathrm{lt}(g_j)$ and $\mathrm{lt}(g_\ell)$ are of the opposite type (lower generators vs. upper generators).

In the case that such a index l cannot be found it means that $l = t$ with $\iota(t) = r$, and since $\mathrm{lt}(g_r)$ is either a pure power X_1^t or the highest power of X_1 with lowest T_k coefficient, then either way the same argument that we claimed for the other case still holds. The claimed statements are now obvious.

Proposition 7 *Let the monomial ideals M_j and their minimal generating sets $G(M_j)$ be given as in Definition 3 for the Gröbner basis $\overline{G}\,(\mathcal{R}(I))$. There exists a directed edge $(j \longrightarrow \ell) \in \mathrm{DE}(\overline{\mathrm{Graph}}(\mathcal{R}(I)))$ for every pair of indices (j,ℓ) with $j<\ell$ such that $\mathrm{lcm}(\mathrm{lt}(g_j),\mathrm{lt}(g_\ell))/\,\mathrm{lt}(g_j) \in G(M_j)$.*

Proof The result is obtained immediately from the description of the ideals M_j that was just given in Proposition 6.

Corollary 2 *For $\overline{G}(I)=\{g_0,g_1,\dots,g_r\}$, $\overline{\mathrm{Graph}}(\mathcal{R}(I))$ is acyclic and contains $2r-2$ directed edges and $r-2$ triangles.*

Proof $\overline{\mathrm{Graph}}(\mathcal{R}(I))$ is acyclic because for any directed edge $(j \longrightarrow k)$ in it, we must satisfy, by construction, $j<k$.

There are two directed edges pointing to g_2 in the graph, these correspond to the $X_1 \in G(M_0)$ and $T_1^{d_2-u_2} \in G(M_1)$. All other edges have targets g_k for some $k>2$. Moreover, from each g_j with $j \geq 1$ of a given type ($\Delta_{\min}(d_1,u_1)$ vs. $\Delta_{\max}(d_1,u_1)$), there is an arrow to the next g_k of the same type and to all g_ℓ between g_j and g_k (of the the opposite type). Thus, for any g_k with $k \geq 2$ there are exactly two edges with target g_k: One from the previous g_j of the same type as g_k, and one from the g_ℓ of the opposite type where ℓ is the maximal possible index. Thus, there are $2(r-1)=2r-2$ edges.

Now consider any triangle in the graph. By acyclicity there is a unique sink g_k in the triangle. As g_2 is targeted by two edges coming from g_0 and g_1, but no edge from g_0 to g_1 can ever exist, we have $k>2$. Without loss of generality, assume $\deg_{X_1}(g_k) \in \Delta_{\max}(d_1,u_1)$ i.e. an upper generator. As we saw g_k is targeted by exactly two edges, one coming from the previous upper generator g_j, and the other from the previous lower generator i.e. corresponding to the minimum with largest subindex $\ell<k$, g_ℓ. It is obvious that there is at most one triangle with sink g_k. So now, we show that there does exist a triangle with this sink: If g_ℓ lies between g_j and g_k i.e. $j<\ell<k$, then there is a directed edge $(j \longrightarrow \ell)$, completing the triangle; otherwise, $\ell<j$. But then, as g_ℓ targets the next minimum g_h with $h>k$, and all other maximums in between, so it targets g_j and g_k as desired. Either way, we obtain a triangle.

Notation Observe that we can write every binomial $g_j \in \overline{G}(\mathcal{R}(I))$ as $g_j=\mathrm{lt}(g_j)-\mathrm{tt}(g_j)$, where $\mathrm{tt}(g_j)$ stands for the tail term of the binomial g_j. To ease the description of the formulas of the next two theorems let us now introduce the following notation:

- $v(a,b)=\mathrm{lcm}(\mathrm{lt}(g_a),\mathrm{lt}(g_b))/\,\mathrm{lt}(g_a)$,
- $w(a,b)=\gcd(\mathrm{tt}(g_a),\mathrm{tt}(g_b))$.

Using this notation we can state and prove the first main result of the section.

Theorem 4 *Write $\overline{G}\,(\mathcal{R}(I))=\{g_0,g_1,\dots,g_r\}$, and denote by $\{\mathbf{e}_0,\mathbf{e}_1,\dots,\mathbf{e}_r\}$ the canonical basis of $(\mathbb{K}[T_0,T_1,X_0,X_1,X_2])^{r+1}$. A Gröbner basis $S^{(1)}$ of $\mathrm{Syz}\left(\overline{G}\,(\mathcal{R}(I))\right)$ is given by:*

$$S^{(1)}=\{\mathbf{s}^{(1)}_{(j,k)}=v(j,k)\mathbf{e}_j-v(k,j)\mathbf{e}_k+w(j,k)\mathbf{e}_h \mid \{(j\to k),(h\to k)\}\subset \mathrm{DE}\left(\overline{\mathrm{Graph}}(\mathcal{R}(I))\right)\},$$

Observe that $|S^{(1)}|=|\mathrm{DE}\left(\overline{\mathrm{Graph}}(\mathcal{R}(I))\right)|=2r-2$.

Proof Recall that $\overline{G}\,(\mathcal{R}(I))=\{g_0,g_1,g_2,\dots,g_r\}$, where $g_0=T_1^{d_2}X_0-T_0^{d_1}X_2$, $g_1=T_0^{d_1-u_1}X_1-T_1^{u_2}X_0$, and $g_2=T_1^{d_2-u_2}X_1-T_0^{u_1}X_2$. Thus, the two nodes that target $g_{k=2}$ are g_0 and g_1, which gives us as a result the first two elements of $S^{(1)}$: $s^{(1)}_{(0,2)}=X_1\mathbf{e}_0-T_1^{u_2}\mathbf{e}_2+T_0^{u_1}X_2\mathbf{e}_1$, and $s^{(1)}_{(1,2)}=T_1^{d_2-u_2}\mathbf{e}_1-T_0^{d_1-u_1}\mathbf{e}_2+1\mathbf{e}_0$. One verifies by direct computation that these two elements of $S^{(1)}$ are in Syz $\left(\overline{G}\left(\mathcal{R}(I)\right)\right)$ and arise from reductions to zero with respect to $\overline{G}\left(\mathcal{R}(I)\right)$ of $X_1 g_0$ and $T_1^{d_2-u_2} g_1$, respectively.

It remains to be shown that the other elements of $S^{(1)}$ are in bijection to the remaining elements of $\cup_j G(M_j)$ as listed in Proposition 6 and arise from reductions to zero of mg_j with respect to $\overline{G}\left(\mathcal{R}(I)\right)$, where $m \in G(M_j)$. A bijection is clearly given via the arrow count in DE $\left(\overline{\text{Graph}}(\mathcal{R}(I))\right)$, using the arrow $(j \longrightarrow k)$. Consider the element g_j. Clearly, $m := v(j,k) \in G(M_j)$. It is also obvious that the first step of the reduction of mg_j is subtraction of $v(k,j)g_k =: \tilde{m}g_k$. There remains the S-polynomial $s := -m \cdot \text{tt}(g_j) + \tilde{m} \cdot \text{tt}(g_k)$. Since $j < k$, $\deg_{X_1}(\text{lt}(g_j)) < \deg_{X_1}(\text{lt}(g_k))$; thus, $\deg_{X_1}\left(m \cdot \text{tt}(g_j)\right) = \deg_{X_1}(m) > 0 = \deg_{X_1}(\tilde{m} \cdot \text{tt}(g_k))$. This implies $\text{lt}(s) = m \cdot \text{tt}(g_j)$. Without loss of generality, assume $\deg_{X_1}(\text{lt}(g_j)) \in \Delta_{\min}(d_1, u_1)$. Then by Lemma 3 item 1a or 3a, $\deg_{X_1}(\text{lt}(s)) \in \Delta_{\max}(d_1, u_1)$ and by item 1b or 3b of the same lemma it is the highest such degree below $\deg_{X_1}(\text{lt}(g_k))$. Thus g_h with $h := \iota\left(\deg_{X_1}(\text{lt}(s))\right)$ is the second node in $\overline{\text{Graph}}(\mathcal{R}(I))$ targeting g_k. Finally, the binomial $\tilde{s} := s/w(j,k)$ is made up out of two coprime monomials and $\deg_{X_1}(\text{lt}(\tilde{s})) = \deg_{X_1}(\text{lt}(s))$. Thus by Lemma 5, $\tilde{s} = -g_h$; hence, $s + w(j,k)g_h = 0$, finishing the reduction, as claimed.

A Gröbner basis for the second syzygy module $\text{Syz}^2\left(\overline{G}\left(\mathcal{R}(I)\right)\right)$ can be also obtained from $\overline{\text{Graph}}(\mathcal{R}(I))$. In this case, the triangles in the graph play a fundamental role.

Theorem 5 *Write $\overline{G} = \{g_0, g_1, \ldots, g_r\}$, and denote by $\{\mathbf{f}_{(j,k)} \mid (j \longrightarrow k) \in \text{DE}\left(\overline{\text{Graph}}(\mathcal{R}(I))\right)\}$ the canonical basis of $(\mathbb{K}[T_0, T_1, X_0, X_1, X_2])^{2r-2}$. A Gröbner basis $S^{(2)}$ of $\text{Syz}^2\left(\overline{G}\left(\mathcal{R}(I)\right)\right)$ is given by:*

$$
\begin{aligned}
S^{(2)} = \{\mathbf{s}^{(2)}_{(j,k,\ell)} &= v(h,k)\mathbf{f}_{(j,k)} - v(\ell,k)\mathbf{f}_{(j,\ell)} + v(\ell,j)\mathbf{f}_{(k,\ell)} - w(j,k)\mathbf{f}_{(h,k)} \mid \\
&(j,k,\ell) \in \text{Tri}\left(\overline{\text{Graph}}(\mathcal{R}(I))\right) \wedge (h \longrightarrow k) \in \text{DE}\left(\overline{\text{Graph}}(\mathcal{R}(I))\right) \wedge h \neq j\},
\end{aligned}
$$

Moreover, $\langle S^{(2)} \rangle \subset (\mathbb{K}[T_0, T_1, X_0, X_1, X_2])^{2r-2}$ is a free submodule. Observe that $|S^{(2)}| = |\text{Tri}\left(\overline{\text{Graph}}(\mathcal{R}(I))\right)| = r - 2$.

Proof We first look at the leading term set of $S^{(1)}$. It is in bijection to $\cup_j G(M_j)$, where the monomial ideals M_j are as in Proposition 6. We order $S^{(1)}$ linearly in the way induced by $G(M_0) \cup G(M_1) \cup \ldots \cup G(M_{r-1})$ taken in the order of writing; the sets $G(M_j)$ themselves are ordered as written in Proposition 6. We now take the induced colon ideals. Their analysis can be split into that of the colon ideals of $G(M_j)$, which are associated to the module component supported on $\mathbf{e}_j$ $(0 \leq j \leq r)$. But, as noted in the proof of Proposition 6, $G(M_j) = \{T_i^{a_1} X_1^{b_1}, \ldots, T_i^{a_s} X_1^{b_s}\}$ $(i \in \{0,1\})$ for a decreasing sequence of non-negative integers $a_1 > a_2 > \cdots > a_s \geq 0$ and an ascending sequence of non-negative integers $0 \leq b_1 < \cdots < b_s < c$. (These integers depend on j.) Thus, all colon ideals are generated by a single pure X_1-power. (Note that the singleton $G(M_0)$ does not induce any colon ideal.) The freeness of $\text{Syz}^2\left(\overline{G}(I)\right)$ follows.

Now, for a given j, let $p = X_1^{b_{s+1}-b_s} = \text{lcm}(T_i^{a_{s+1}} X_1^{b_{s+1}}, T_i^{a_s} X_1^{b_s}) / T_i^{a_s} X_1^{b_s}$ be the single generator of one of the colon ideals induced by $G(M_j)$. There are indexes $k < \ell$ (in fact, $\ell = k+1$), such that $T_i^{a_s} X_1^{b_s} = v(j,k)$ and $T_i^{a_{s+1}} X_1^{b_{s+1}} = v(j,\ell)$. The module term $p\mathbf{f}_{j,k}$ is the leading term of one of the generators of $\text{Syz}\left(S^{(1)}\right)$ and every leading term arises in this way. Thus the generating set of $\text{Syz}\left(S^{(1)}\right)$ induced by the colon ideal structure of the leading term set of $S^{(1)}$ is in bijection to $\text{Tri}\left(\overline{\text{Graph}}(\mathcal{R}(I))\right)$ via $p\mathbf{f}_{j,k} \mapsto (j,k,\ell)$.

Now consider $\mathbf{s}^{(2)}_{(j,k,\ell)}$ for a given triangle (j,k,ℓ) with $\ell \geq 3$. As g_k does not correspond to the last element in the enumeration of $G(M_j)$, $\deg_{X_1}(g_j)$ and $\deg_{X_1}(g_k)$ are of opposite types (upper vs. lower generators) by Corollary 2. Let g_h be the second node in $\overline{\text{Graph}}(\mathcal{R}(I))$ targeting g_k. Necessarily, g_h is of the same type as g_k. Thus, by Lemma 3, we have that $\deg\left(v(h,k)\right) = \deg_{X_1}\left(v(h,k)\right) = \deg_{X_1}(g_k) - \deg_{X_1}(g_h)$ is in the same type as $\deg_{X_1}(g_j)$ and it is the maximal such degree below $\deg_{X_1}(g_k)$. But g_j targets g_k. Therefore, $\deg_{X_1}(g_j) = \deg_{X_1}(g_k) - \deg_{X_1}(g_h)$, this can also be deduced from Corollary 1 since $\deg_{X_1}(g_j) + \deg_{X_1}(g_h) = \deg_{X_1}(g_k)$. A similar argument shows that $\deg(p) = \deg_{X_1}(g_\ell) - \deg_{X_1}(g_k) = \deg_{X_1}(g_j)$. Thus, $p = v(h,k)$ as claimed and the leading module term of $\mathbf{s}^{(2)}_{(j,k,\ell)}$ is $v(h,k)\mathbf{f}_{(j,k)}$. Substituting $\mathbf{s}^{(1)}_{(j,k)}$, we obtain an element $\mathbf{s}' \in \text{Syz}\left(\overline{G}\right)$ with leading term $v(h,k)v(j,k)\mathbf{e}_j = \left(X_1^{\deg_{X_1}(g_j)}\right)\left(\text{lt}(g_k)/X_1^{\deg_{X_1}(g_j)}\right)\mathbf{e}_j = \text{lt}(g_k)\mathbf{e}_j$. We can reduce $\mathbf{s}'$ using $\mathbf{s}^{(1)}_{(j,\ell)}$, which has the leading term $v(j,\ell)\mathbf{e}_j$. Again using Proposition 6 and Lemma 3, we see that $\deg_{X_1}\left(v(j,\ell)\right) = \deg_{X_1}(g_k)$. It is possible that $v(j,\ell)$ is additionally divisible by a T_i $(i \in \{0,1\})$; but in any case, $v(j,\ell)v(\ell,k) = \text{lt}(g_k)$. Thus, we reduce $\mathbf{s}'$ by subtracting $v(\ell,k)\mathbf{s}^{(1)}_{(j,\ell)}$, obtaining $\mathbf{s}''$. In its support, we find $v(h,k)w(j,k)\mathbf{e}_h$. The syzygy $\mathbf{s}^{(1)}_{(h,k)} \in S^{(1)}$ has leading term $v(h,k)\mathbf{e}_h$. Hence, we reduce $\mathbf{s}''$ by subtracting $w(j,k)\mathbf{s}^{(1)}_{(h,k)}$, obtaining $\mathbf{s}'''$. There is only one module monomial in its support which is divisible by X_1, namely $-v(h,k)v(k,j)\mathbf{e}_k$. Using Lemma 3, we can simplify this to $-\text{lt}(g_j)\mathbf{e}_k$. We notice that the element $\mathbf{s}^{(1)}_{(k,l)} \in S^{(1)}$ has leading term $v(k,\ell)\mathbf{e}_k$. By Lemma 3, $\deg_{X_1}\left(v(k,\ell)\right) = \deg_{X_1}(g_j)$. It is possible that $v(k,\ell)$ is additionally divisible by a T_i $(i \in \{0,1\})$; but in any case, $v(k,\ell)v(\ell,j) = \text{lt}(g_j)$. Hence, we reduce $\mathbf{s}'''$ by adding $v(\ell,j)\mathbf{s}^{(1)}_{(k,\ell)}$, obtaining $\mathbf{s}''''$. One can check that no term divisible by X_1 remains in $\mathbf{s}''''$. Assume $\mathbf{s}'''' \neq \mathbf{0}$; then its leading term would not be divisible by X_1, in contradiction to Theorem 4. Hence, $\mathbf{s}'''' = \mathbf{0}$.

Note that even though $g_{h=0}$ is technically of neither type, but we have that $G(M_0) = X_1$ and $(0 \to 2) \in \text{DE}\left(\overline{\text{Graph}}(\mathcal{R}(I))\right)$. Hence, g_0 behaves exactly like a lower generator of the type coming from $\Delta_{\min}(d_1,u_1)$. Moreover, it can be verified by direct computation that the claims stated before also hold for the particular triangle $(1,2,3) \in \text{Tri}\left(\overline{\text{Graph}}(\mathcal{R}(I))\right)$ with $h = 0$, more about this particular triangle will follow in the next subsection.

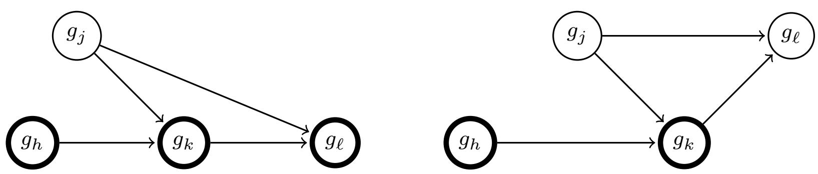

Fig. 3 The nodes g_h, g_j, g_k, g_ℓ appearing in the the construction of the syzygy $\mathbf{s}^{(2)}_{(j,k,\ell)}$ in the proof of Theorem 5 are situated as illustrated here. There are two cases. Note that g_j, g_k must belong to opposite types (upper vs. lower generators); the only difference between the two cases is whether g_ℓ is of the same type as g_k (seen on the left) or of the same the type as g_j (seen on the right). Without loss of generality we assumed both g_h and g_k are lower generators, there exists the other possibility of both being upper generators in which case the images would be flipped upside down.

Since the second syzygy module of $\mathcal{R}(I)$ is free, the resolution stops here. The above results can be summarized in the main theorem of this section:

Theorem 6 *There exists a free resolution of $\mathcal{R}(I)$ of the form*

$$0 \longrightarrow S^{r-2} \xrightarrow{\phi_2} S^{2(r-1)} \xrightarrow{\phi_1} S^{r+1} \xrightarrow{\phi_0} \mathcal{R}(I) \longrightarrow 0$$

$$
\begin{array}{c}
\\ h \\ j \\ k \\ \ell
\end{array}
\begin{array}{c}
\begin{array}{ccccc} (h,k) & (j,k) & (j,\ell) & (k,\ell) \end{array} \\
\begin{bmatrix}
\boldsymbol{v(h,k)} & w(j,k) & 0 & 0 \\
w(h,k) & \boldsymbol{v(j,k)} & \boldsymbol{v(j,\ell)} & w(k,\ell) \\
-v(k,h) & -v(k,j) & w(j,\ell) & \boldsymbol{v(k,\ell)} \\
0 & 0 & -v(\ell,j) & -v(\ell,k)
\end{bmatrix}
\end{array}
\begin{array}{c}
(j,k,\ell) \\
\begin{bmatrix}
-w(j,k) \\
\boldsymbol{v(h,k)} \\
-v(\ell,k) \\
v(\ell,j)
\end{bmatrix}
\end{array}
$$

Fig. 4 Illustration of the construction of the syzygy $\mathbf{s}^{(2)}_{(j,k,\ell)}$ in the proof of Theorem 5. Only the terms in bold and coloured in ***red*** are divisible by X_1, note that these terms can only come from $v(a,b)$ with $a < b$ and $(a,b) \neq (1,2)$. In the proof, it is shown that both $v(h,k)v(j,k) = \mathrm{lt}(g_k)$ and $v(j,\ell)v(\ell,k) = \mathrm{lt}(g_k)$; hence, $v(h,k)v(j,k) - v(j,\ell)v(\ell,k) = 0$. Analogously both $v(h,k)v(k,j) = \mathrm{lt}(g_j)$ and $v(k,\ell)v(\ell,j) = \mathrm{lt}(g_j)$; so, $-v(h,k)v(k,j) + v(k,\ell)v(\ell,j) = 0$. These two identities relate the middle rows of the matrix with columns $\mathbf{s}^{(1)}_{(h,k)}, \ldots, \mathbf{s}^{(1)}_{(k,\ell)}$ to the column representing $\mathbf{s}^{(2)}_{(j,k,\ell)}$. Moreover, it is clearly to be seen that after a multiplication of these matrices, the top and bottom entries vanish. There remain only sums of products of entries that are not divisible by X_1 (which must reduce to zero as explained at the end of proof of Theorem 5).

encoded in the graph $\overline{\mathrm{Graph}}(\mathcal{R}(I))$*, with differential maps* ϕ_1 *and* ϕ_2 *deduced in the usual way from Theorems 4 and 5, respectively.*

5.4 The minimal free resolution of $\mathcal{R}(I)$

As expected, the free resolution obtained in Theorem 6 is not minimal, since we added a redundant element to our minimal generating set in order to obtain a Gröbner basis. Nonetheless, it is very close to being minimal, in fact the minimal resolution can be recovered with a couple of simple algebraic reductions which do not depend on the parameters. These reductions are explicitly described in the present section.

Note that the non-zero constant coefficients in the differentials that keep this resolution from being minimal can only come from the coefficient $w(j,k)$ in $s^{(1)}_{(j,k)}$ and from $-w(j,k)$ in $s^{(2)}_{(j,k,l)}$. This is very straightforward since all other coefficients are of the type $v(a,b)$ and the l.c.m. of any two nonconstant terms can never give you a constant. Remember that $w(j,k) = \gcd(\mathrm{tt}(g_j), \mathrm{tt}(g_k))$, and note that $X_0, X_2 \mid \mathrm{tt}(g_\rho)$ for every $\rho \geq 3$. However, we know that: $\mathrm{tt}(g_0) = T_0^{d_1} X_2$, $\mathrm{tt}(g_1) = T_1^{u_2} X_0$, and $\mathrm{tt}(g_2) = T_0^{u_1} X_2$. Thus, $w(j,k) = \gcd(\mathrm{tt}(g_j), \mathrm{tt}(g_k)) = 1$ only for pairs $(0,1)$ and $(1,2)$. Nevertheless, there does not exist any first nor second syzygy where $j = 0$ and $k = 1$; hence, the only first syzygy with a non-zero constant coefficient is $s^{(1)}_{(1,2)}$ and the only second syzygy where that also happens is $s^{(2)}_{(1,2,3)}$. Since the binomials g_0, g_1 and g_2 are trivially obtained directly from u_1, u_2, d_1 and d_2, let us illustrate the matrices from the differentials as in Figure 4:

On this figure one can clearly see that after two algebraic reductions one obtains the minimal free resolution. Moreover, if we take a closer look, the reduction on the first matrix gets rid of $s^{(1)}_{(1,2)}$ as well as our redundant generator g_0, i.e. erasing the whole first row on the first matrix with label (0), and both the second column on the first matrix as well as the second row on the second matrix with labels $(1,2)$, while only altering the coefficients of one other first syzygy namely $s^{(1)}_{(0,2)}$ (the one represented on the first column). However, note that this same syzygy is removed immediately after due to the one reduction on the second matrix, together with the second syzygy $s^{(2)}_{(1,2,3)}$. Since, this first syzygy is not present in any of all other second syzygies,

$$
\begin{array}{c}
\\ (0) \\ (1) \\ (2) \\ (3) \\ \vdots
\end{array}
\begin{array}{c}
\begin{array}{ccccc} (0,2) & (1,2) & (1,3) & (2,3) & \dots \end{array} \\
\left[\begin{array}{ccccc}
\boldsymbol{X_1} & \mathbf{1} & 0 & 0 & 0\dots 0 \\
T_0^{u_1}X_2 & T_1^{d_2-u_2} & \boldsymbol{v(1,3)} & w(2,3) & \dots \\
-T_1^{u_2X_0} & -T_0^{d_1-u_1} & w(1,3) & \boldsymbol{v(2,3)} & \dots \\
0 & 0 & -v(3,1) & -v(3,2) & \dots \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{array}\right]
\end{array}
\quad
\begin{array}{c}
\\ (0,2) \\ (1,2) \\ (1,3) \\ (2,3) \\ \vdots
\end{array}
\begin{array}{c}
\begin{array}{cc} (1,2,3) & \dots \end{array} \\
\left[\begin{array}{cc}
\mathbf{-1} & 0\dots 0 \\
\boldsymbol{v(2,3)} & \dots \\
-v(3,2) & \dots \\
v(3,1) & \dots \\
\vdots & \ddots
\end{array}\right]
\end{array}
$$

all entries in the matrix remain unchanged. Thus, simply omitting these few columns and rows gives you the differential maps from the minimal free resolution of $\mathcal{R}(I)$.

Theorem 7 *The minimal free resolution of $\mathcal{R}(I)$ is of the form:*

$$0 \longrightarrow S^{r-3} \xrightarrow{\phi_2} S^{2(r-2)} \xrightarrow{\phi_1} S^r \xrightarrow{\phi_0} \mathcal{R}(I) \longrightarrow 0$$

encoded in a graph of the form described in Section 5.1, with differential maps ϕ_1 and ϕ_2 deduced in the usual way from Theorems 4 and 5 respectively, acting on the Rees graph of I instead.

Proof The proof follows naturally from the explanations given above. The reductions on the free resolution from Theorem 6 give us a result a minimal free resolution with one less element in the initial S-module of the sequence, two fewer elements in the generating set of the first module of the syzygies and one less in the second module of the sequences. This is equivalent to removing from $\overline{\text{Graph}}(\mathcal{R}(I))$ the node g_0, the edges $(0 \to 2)$ and $(1 \to 2)$, and the triangle $(1,2,3)$. Thus, we are left with the Rees graph $\text{Graph}(\mathcal{R}(I))$ of the form described in Section 5.1, which supports the minimal free resolution in the same way since as we saw above the rest of the entries in the matrices remain unchanged.

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