Involution Analysis of the Partial Differential Equations Characterising Hamiltonian Vector Fields

Werner M. Seiler
Lehrstuhl für Mathematik I, Universität Mannheim
68131 Mannheim, Germany
Email: `werner.seiler@math.uni-mannheim.de`
Web: `www.math.uni-mannheim.de/~wms`
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Abstract

In a recent article, certain underdetermined linear systems of partial differential equations connected with Lie-Poisson structures have been studied. They were constructed via power series solutions of the evolution equation for a given Hamiltonian. We extend the results to arbitrary Poisson manifolds, correct an error in the case of degenerate Poisson structures, and show that these linear systems simply characterise Hamiltonian vector fields. Our basic tool is the formal theory of differential equations with its central concept of an involutive system.

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I. INTRODUCTION

In a recent article, Bender et al.[1] studied certain underdetermined linear systems of partial differential equations connected with Lie-Poisson structures. These systems arose from considering power series solutions of the corresponding evolution equations. The purpose of the present work is to extend their results to arbitrary Poisson manifolds and to clarify the relation to Poisson geometry. In particular, it turns out that these systems are simply the equations characterising the Hamiltonian vector fields on the manifold.

Furthermore, we correct an error in the cited work. For degenerate Poisson structures, i. e. if the Poisson matrix J is singular, the claimed expression is *not* the general solution of the underdetermined system; further solutions exist. Let us take for example the system related to the Lie-Poisson structure induced by the Lie algebra E_2:

$$\begin{aligned} y\frac{\partial G}{\partial z} + x\frac{\partial F}{\partial z} &= 0\,, \\ y\frac{\partial H}{\partial z} + y\frac{\partial F}{\partial x} - x\frac{\partial F}{\partial y} - G &= 0\,, \\ x\frac{\partial H}{\partial z} - y\frac{\partial G}{\partial x} + x\frac{\partial G}{\partial y} - F &= 0\,. \end{aligned} \tag{1}$$

Bender et al.[1] claimed that its general solution is

$$F = -y\frac{\partial K}{\partial z}\,, \quad G = x\frac{\partial K}{\partial z}\,, \quad H = y\frac{\partial K}{\partial x} - x\frac{\partial K}{\partial y} \tag{2}$$

with an arbitrary function $K(x, y, z)$. However, one easily checks that $F = c_1/x$, $G = c_2/y$ and $H = (c_1/x^2 + c_2/y^2)z$ solves (1) for arbitrary values of the constants c_1, c_2 and only for $c_1 = -c_2 = c$ these expressions are contained in the solution family (2), namely for the choice $K = -cz/xy$. We will show that (2) represents the general solution only, if we augment (1) by the algebraic constraint $xF + yG = 0$.

Our basic tool will be the formal theory of differential equations[2–4]. Some elements of this theory are about a century old, but it still seems to be fairly unknown. In fact, Bender et al.[1] rose a number of questions concerning the arbitrariness of the general solution of a system of partial differential equations many of which can be answered with the help of the formal theory (see e. g. Ref. 5).

A key concept within the formal theory is *involution*. For lack of space, we cannot give a detailed introduction but must refer to the above cited works. Any serious analysis of a system of differential equations not in Cauchy-Kovalevskaya form requires usually that the system is involutive. Involution comprises in particular the absence of integrability conditions. Any system can be completed to an involutive one (under some mild regularity assumptions). Fortunately, we do not have to bother with a completion, as it turns out that due to the properties of Poisson manifolds our system is already involutive.

The next section reviews those basic properties of Poisson manifolds that are needed later. Section III derives an underdetermined system of partial differential equations characterising Hamiltonian vector fields and analyses the conditions under which it is involutive. The relation to the work of Bender et al.[1] is found in Section IV studying the fundamental power series solutions of the Hamiltonian evolution equation. Finally, after considering explicitly the case of the Lie-Poisson structure of E_2, some conclusions are given in Section VI.

II. POISSON MANIFOLDS

For easier comparison with the results presented by Bender et al.[1], we use throughout local coordinates; the intrinsic theory of Poisson manifolds can be found in the book of Vaisman[6] (with emphasis on the geometry, see also Ref. 7) or in the books of Marsden and Ratiu[8] resp. Libermann and Marle[9] (with emphasis on applications in mechanics).

A *Poisson manifold* is a (smooth) manifold M equipped with a bracket structure on the ring $\mathcal{F}(M)$ of smooth functions on M. This bracket $\{\cdot,\cdot\}:\mathcal{F}(M)\times\mathcal{F}(M)\to\mathcal{F}(M)$ must satisfy four axioms for arbitrary functions $F,G,H\in\mathcal{F}(M)$ and real constants λ,μ:

(i) $\{F,G\}=-\{G,F\}$ (skew-symmetry),

(ii) $\{\lambda F+\mu G,H\}=\lambda\{F,H\}+\mu\{G,H\}$ (linearity),

(iii) $\{FG,H\}=F\{G,H\}+\{F,H\}G$ (Leibniz rule),

(iv) $\Big\{F,\{G,H\}\Big\}+\Big\{G,\{H,F\}\Big\}+\Big\{H,\{F,G\}\Big\}=0$ (Jacobi identity).

Thus the ring $\mathcal{F}(M)$ acquires the structure of an infinite-dimensional Lie algebra.

In terms of local coordinates $(z^1,\dots,z^n)$ on the manifold M, the bracket is uniquely determined by the *Poisson matrix* $J^{kl}=\{z^k,z^l\}$. For two arbitrary functions $F,G\in\mathcal{F}(M)$ we find $\{F,G\}=(\nabla F)^t J\nabla G$. The matrix J is obviously skew-symmetric and the Jacobi identity induces the following differential equations for its components:

$$J^{kl}\frac{\partial J^{ij}}{\partial z^l}+J^{il}\frac{\partial J^{jk}}{\partial z^l}+J^{jl}\frac{\partial J^{ki}}{\partial z^l}=0\,,\qquad 1\leq i,j,k\leq n\,. \tag{3}$$

In an intrinsic language, the Poisson bracket is defined in terms of the Poisson bivector $w\in\Lambda^2(TM)$ by $\{F,G\}=w(dF,dG)$. The matrix J is a coordinate representation of this bivector, $w=J^{kl}\partial_{z^k}\wedge\partial_{z^l}$, and the Jacobi identity (3) expresses the vanishing of the Schouten-Nijenhuis bracket of w with itself.

If the matrix J is regular (which obviously can only happen on even-dimensional manifolds), then M is in fact a symplectic manifold with the symplectic structure locally defined by the inverse of J. If $\mathcal{A}$ is an n-dimensional Lie algebra, then its dual space $\mathcal{A}^*$ carries a canonical *Lie-Poisson structure*. Since $T\mathcal{A}^*\cong\mathcal{A}^*\times\mathcal{A}^*$, we may identify for any smooth function $\phi\in\mathcal{F}(\mathcal{A}^*)$ at the point $\lambda\in\mathcal{A}^*$ the tangent map $T_\lambda\phi:\mathcal{A}^*\to I\!R$ with an element of $\mathcal{A}^{**}=\mathcal{A}$ and define for $\phi,\psi\in\mathcal{F}(\mathcal{A}^*)$

$$\{\phi,\psi\}(\lambda)=\lambda\Big([T_\lambda\phi,T_\lambda\psi]\Big)\,.$$

If C^i_{kl} denotes the structure constants of $\mathcal{A}$ for some basis $\{A_1,\dots,A_n\}$, i. e. $[A_k,A_l]=C^i_{kl}A_i$, then in coordinates $(z_1,\dots,z_n)$ with respect to the dual basis $\{A^1,\dots,A^n\}$ of $\mathcal{A}^*$ the Poisson matrix is given by $J_{kl}=C^i_{kl}z_i$. It is trivial to check that it satisfies (3). Bender et al.[1] considered exclusively Poisson structures of this form.

A Poisson structure defines a homomorphism $\sharp:T^*M\to TM$ by $\omega_1(\omega_2^\sharp)=w(\omega_1,\omega_2)$ for arbitrary one-forms $\omega_1,\omega_2\in T^*M$. The structure w is *non-degenerate*, if $\sharp$ is an isomorphism or, equivalently, the Poisson matrix J is regular. More generally, the rank of a Poisson structure is defined as the dimension of $\operatorname{im}\sharp$ (we restrict here and in the sequel to the case that $\mathrm{im}\,\sharp$ defines a regular distribution on M). If $\mathrm{rank}\,w = n - r$, then r linearly independent one-forms $\chi^{(a)} = \chi^{(a)}_k dz^k$ exist on M such that $\ker \sharp = \langle \chi^{(1)}, \dots, \chi^{(r)} \rangle$. If a function $C \in \mathcal{F}(M)$ is such that $dC \in \ker \sharp$, then $\{C, F\} = 0$ for all $F \in \mathcal{F}(M)$ and C is called a *Casimir function.*

We associate with any function $H \in \mathcal{F}(M)$ its *Hamiltonian vector field*

$$X_H = (dH)^\sharp = \{\cdot, H\} = J^{kl} \frac{\partial H}{\partial z^l} \frac{\partial}{\partial z^k} \,.$$

For a Casimir function C, X_C is obviously the zero vector field. For later use, we introduce as special case the short hand

$$X^{(k)} = X_{-z^k} = \{z^k, \cdot\} = J^{kl} \frac{\partial}{\partial z^l} \,.$$

The Poisson bracket may now conveniently be expressed with these fields:

$$\{F, G\} = \frac{\partial F}{\partial z^k} X^{(k)} G = -\frac{\partial G}{\partial z^k} X^{(k)} F \,.$$

The Hamiltonian vector fields form a Lie algebra under the Lie bracket. Indeed, one easily obtains for arbitrary functions $F, G, H \in \mathcal{F}(M)$ with the help of the Jacobi identity

$$X_{\{F,G\}}(H) = \Big\{H, \{F, G\}\Big\} = \Big\{\{H, F\}, G\Big\} - \Big\{\{H, G\}, F\Big\} = [X_G, X_F](H) \,. \tag{4}$$

Thus the mapping $F \mapsto X_F$ defines a Lie algebra *anti*homomorphism between $\mathcal{F}(M)$ and the Hamiltonian vector fields (if we had defined $X_F = \{F, \cdot\}$, we would have obtained a homomorphism[6]). It follows from (4) that the distribution $\mathrm{im}\,\sharp$ is involutive and thus defines a foliation of the manifold M. By a classical result in Poisson geometry, the leaves are $(n-r)$-dimensional symplectic manifolds, i. e. they carry a non-degenerate Poisson structure. Obviously, the codistribution $\ker \sharp$ is just the annihilator of $\mathrm{im}\,\sharp$.

A Hamiltonian vector field X_H may be written either in the form $X_H = (X^{(k)} H) \partial_{z^k}$ or as $X_H = -H_{z^k} X^{(k)}$. Thus the fields $X^{(k)}$ span the distribution $\mathrm{im}\,\sharp$ and (4) takes the form

$$[X^{(k)}, X^{(l)}] = \frac{\partial J^{kl}}{\partial z^j} X^{(j)} \,. \tag{5}$$

Note that in the special case of a Lie-Poisson structure the vector fields $X^{(k)}$ provide us with a representation of the underlying Lie algebra, as then $\partial J_{kl} / \partial z_j = C^j_{kl}$.

The vector field X_H defines a dynamical system on the Poisson manifold M. We denote its flow by ϕ, i. e. ϕ is a map $\mathbb{R} \times M \to M$ (as we are only interested in local properties, we do not bother about the precise domain of definition of ϕ) and $\phi_z(t) = \phi(t, z)$ yields the integral curve of X_H passing through the point $z \in M$. A central property of Hamiltonian flows is that they preserve the Poisson structure. Indeed, it follows again from the Jacobi identity that for arbitrary functions $F, G, H \in \mathcal{F}(M)$

$$\begin{aligned} (L_{X_H} w)(dF, dG) &= X_H\Big(w(dF, dG)\Big) - w(L_{X_H} dF, dG) - w(dF, L_{X_H} dG) \\ &= X_H\Big(\{F, G\}\Big) - \{X_H F, G\} - \{F, X_H G\} = 0 \end{aligned}$$

where L_X denotes the Lie derivative with respect to the vector field X. Thus $L_{X_H}w = 0$ for every Hamiltonian vector field X_H.

If we consider how functions $F \in \mathcal{F}(M)$ vary along the integral curves of the Hamiltonian vector field X_H, then we find for arbitrary points $z \in M$

$$\frac{d}{dt}(\phi_z^* F)(t) = \phi_z^*\Big(dF(X_H)\Big)(t) = \phi_z^*\Big(\{F, H\}\Big)(t)\,. \tag{6}$$

Often this equation is briefly written as $\dot{F} = \{F, H\}$.

Let us expand the function $\phi^* F$ in a formal power series in t

$$(\phi^* F)(t, z) = \sum_{\alpha=0}^{\infty} F_\alpha(z) t^\alpha \tag{7}$$

with coefficients $F_\alpha \in \mathcal{F}(M)$. A trivial computation shows then that

$$\Big(\{F, G\}\Big)_\alpha = \sum_{\beta=0}^{\alpha} \Big\{F_{\alpha-\beta}, G_\beta\Big\}\,. \tag{8}$$

Since, by definition of a flow, $(\phi^* F)(0, z) = F(z)$, we may express the coefficients F_α in a closed form: entering the expansion (7) into the differential equation (6) yields the relation $\alpha F_\alpha = \{F_{\alpha-1}, H\}$ and thus

$$\begin{gathered} F_0 = F\,, \quad F_1 = \{F, H\}\,, \quad F_2 = \frac{1}{2}\Big\{\{F, H\}, H\Big\}\,, \\ F_\alpha = \frac{1}{\alpha!}\Big\{\cdots\Big\{\{F, \underbrace{H\}, H\Big\}\cdots, H\Big\}}_{\alpha \text{ brackets}}\,. \end{gathered} \tag{9}$$

III. A PARTIAL DIFFERENTIAL SYSTEM AND ITS INVOLUTION ANALYSIS

Assume we are given an arbitrary vector field X on the Poisson manifold M. A natural question is whether or not a function $H \in \mathcal{F}(M)$ exists such that $X = X_H = (dH)^\sharp$. By the results of Section II, this corresponds in local coordinates where $X = \xi^k \partial_{z^k}$ to studying the solvability of the following overdetermined inhomogeneous linear first-order system for H:

$$X^{(k)} H = J^{kl} \frac{\partial H}{\partial z^l} = \xi^k\,. \tag{10}$$

It should be noted that we consider in the sequel only the *formal* solvability, i. e. the existence of formal power series solutions H.

In the language of the formal theory, conditions on the right hand side $\vec{\xi}$ for the existence of solutions are called *compatibility conditions*. They are determined by rendering (10) involutive. For linear first-order systems with one unknown function this is a classical problem much studied in the 19th century (see e. g. the references in the textbooks[10,11]); in modern geometric language it leads to the Frobenius theorem. The system (10) is involutive, if and only if the vector fields $X^{(k)}$ span an involutive distribution, i. e. if the distribution is closed under the Lie bracket.

But we have already determined in (5) that this is indeed the case. Applying these commutator relations to the function H and using (10) immediately yields the following compatibility conditions for the right hand side $\vec{\xi}$:

$$X^{(k)}\xi^l - X^{(l)}\xi^k = \frac{\partial J^{kl}}{\partial z^j}\xi^j\,, \qquad 1 \leq k < l \leq n\,. \tag{11}$$

Restricted to the special case of a Lie-Poisson manifold, this is the underdetermined system studied by Bender et al.[1] In general, (11) does *not* contain all compatibility conditions of (10). If the Poisson structure is degenerate, the vector fields $X^{(k)}$ are not linearly independent, as trivially $\chi^{(a)}_k X^{(k)} = 0$ for the one-forms $\chi^{(a)}$ spanning $\ker \sharp$. Thus we find in addition

$$\chi^{(a)}_k \xi^k = 0\,, \qquad 1 \leq a \leq r\,. \tag{12}$$

Bender et al.[1] ignored these equations and they are the reason for the problem mentioned in the introduction.

In an intrinsic language, (11) is (up to a constant factor) the local coordinate form of $L_X w = 0$ and (12) of $\chi^{(a)}(X) = 0$. This observation is not too surprising: by definition of the one-forms $\chi^{(a)}$, the second equation simply says that the vector field X must lie in $\operatorname{im} \sharp$ and we showed above that for every Hamiltonian vector field the Lie derivative of the Poisson bivector w vanishes. Thus both conditions are obviously *necessary* for X being Hamiltonian. Because of the involution of the linear system (10), they are also *sufficient* for the vector field X being (formally) Hamiltonian.

It follows from these considerations that the general formal solution of (11,12) is

$$\xi^k = X^{(k)}H = J^{kl}\frac{\partial H}{\partial z^l}$$

where the function $H \in \mathcal{F}(M)$ is arbitrary. Hence (11,12) indeed always represents an underdetermined system, although it comprises more equations than unknown functions.

We proceed with an involution analysis of the combined system (11,12). We first analyse whether it is possible to generate integrability conditions via cross-differentiations within the differential equations (11). We introduce the short hand $A^{kl} = X^{(k)}\xi^l - X^{(l)}\xi^k - (\partial J^{kl}/\partial z^j)\xi^j$ (the antisymmetric tensor corresponding to the bivector $L_X w$) and thus (11) may be written as $A^{kl} = 0$. If we take the cyclic combination

$$X^{(m)}A^{kl} + X^{(k)}A^{lm} + X^{(l)}A^{mk} = 0\,, \tag{13}$$

all second-order derivatives vanish and thus an integrability condition might be hidden here.

(13) contains the difference $(X^{(m)}X^{(k)} - X^{(k)}X^{(m)})\xi^l$ (and cyclic permutations of it). Here we enter the commutation relation (5). The arising equation contains expressions of the form $(\partial J^{mk}/\partial z^j)\big(X^{(j)}\xi^l - X^{(l)}\xi^j\big)$ which are simplified using (11). These operations finally yield the algebraic integrability conditions

$$\left(\frac{\partial J^{mk}}{\partial z^i}\frac{\partial J^{il}}{\partial z^n} + \frac{\partial J^{lm}}{\partial z^i}\frac{\partial J^{ik}}{\partial z^n} + \frac{\partial J^{kl}}{\partial z^i}\frac{\partial J^{im}}{\partial z^n} + J^{il}\frac{\partial^2 J^{mk}}{\partial z^n \partial z^i} + J^{ik}\frac{\partial^2 J^{lm}}{\partial z^n \partial z^i} + J^{im}\frac{\partial^2 J^{kl}}{\partial z^n \partial z^i}\right)\xi^n = 0\,.$$

As the coefficient of ξ^n is just the z^n-derivative of the Jacobi identity (3), this condition is always satisfied. If our Poisson structure is non-degenerate (so that the equations (12) do not arise), this result suffices to conclude that (11) is involutive, as (13) is the only linear combination of differentiated equations where all second-order derivatives vanish.

In the case of a degenerate Poisson structure we must also analyse prolongations of the algebraic equations (12). We first consider the application of one of the vector fields $X^{(l)}$. Upon entering the differential equations (11), a trivial computation yields

$$X^{(l)}(\chi_k^{(a)}\xi^k) = \left(J^{lm}\frac{\partial\chi_k^{(a)}}{\partial z^m} + \frac{\partial J^{lm}}{\partial z^k}\chi_m^{(a)}\right)\xi^k\,.$$

Exploiting the relation $J^{lm}\chi_m^{(a)} = 0$ we have thus found the integrability conditions

$$J^{lm}\left(\frac{\partial\chi_k^{(a)}}{\partial z^m} - \frac{\partial\chi_m^{(a)}}{\partial z^k}\right)\xi^k = 0\,, \qquad 1 \leq l \leq n\,. \tag{14}$$

A necessary condition for involution is that these conditions are linearly dependent of the algebraic equations (12) or, equivalently, that contracting the coefficients with J^{nk} yields zero. Relabelling the indices m and k in the second term, we arrive at the condition

$$\left(J^{nk}J^{lm} - J^{lk}J^{nm}\right)\frac{\partial\chi_k^{(a)}}{\partial z^m} = 0\,.$$

Exploiting again the relation $J^{lm}\chi_m^{(a)} = 0$ and using the Jacobi identity (3) we find

$$\left(J^{nk}J^{lm} - J^{lk}J^{nm}\right)\frac{\partial\chi_k^{(a)}}{\partial z^m} = \left(J^{lm}\frac{\partial J^{kn}}{\partial z^m} + J^{nm}\frac{\partial J^{lk}}{\partial z^m}\right)\chi_k^{(a)} = J^{km}\frac{\partial J^{ln}}{\partial z^m}\chi_k^{(a)} = 0$$

and hence this condition is always satisfied by the definition of the one-forms $\chi^{(a)}$.

As the vector fields $X^{(l)}$ are not linearly independent, the analysis above does not cover all possible prolongations of the constraints (12). Let $Y^{(b)}$ be r further vector fields such that together with the fields $X^{(l)}$ they span the full tangent space TM. This implies the existence of functions A_k^{lb} and B_c^{lb} such that

$$[X^{(l)}, Y^{(b)}] = A_k^{lb}X^{(k)} + B_c^{lb}Y^{(c)}\,. \tag{15}$$

Applying the vector fields $Y^{(b)}$ to (12) yields differential equations

$$Y^{(b)}(\chi_k^{(a)}\xi^k) = 0 \tag{16}$$

which are obviously algebraically independent of the system (11). In a strict geometric sense, they represent integrability conditions, albeit of a rather trivial nature. The more interesting question is whether it is possible to derive further integrability conditions via cross-differentiations between equations in (16) and (11), respectively. But one fairly easily sees that this cannot be the case.

A further integrability condition could only be generated by applying one of the vector fields $X^{(l)}$ to (16) and eliminating all arising second-order derivatives using equations obtained by applying some of the fields $Y^{(b)}$ to (11). But according to (15) we may write

$$X^{(l)}Y^{(b)}(\chi_k^{(a)}\xi^k) = \Bigl(Y^{(b)}X^{(l)} + A_k^{lb}X^{(k)} + B_c^{lb}Y^{(c)}\Bigr)(\chi_k^{(a)}\xi^k)\,.$$

The second and the third term on the right hand side consist of linear combinations of equations already present in the augmented system (11,12,16). In the first term we find the expression $X^{(l)}(\chi_k^{(a)}\xi^k)$ of which we have shown above that it may also be written as a linear combination of equations in (11,12). Thus the right hand side is algebraically dependent of (11,12,16) plus the equations obtained by applying $Y^{(b)}$ to (11) and there are no further integrability conditions hidden.

Hence we have proven that for non-degenerate Poisson structures the system (11) is directly involutive, whereas in the degenerate case we must only add the trivial integrability conditions (16) to the system (11,12) in order to achieve involution.

The Schouten-Nijenhuis bracket allows us to reformulate these lengthy coordinate calculations in an intrinsic language. As already mentioned, the system (11) is equivalent to $L_X w = -[X, w] = 0$. Let Y be an arbitrary vector field, then the prolongation of (11) in the direction Y is simply given by $L_Y L_X w = [Y, [X, w]] = 0$. The graded Jacobi identity of the Schouten-Nijenhuis bracket yields $L_Y L_X w = L_X L_Y w - L_{[X,Y]} w$. On the right hand side, all second-order derivatives of X are located in the second term. It vanishes, if Y is a Hamiltonian vector field (i. e. a linear combination of the fields $X^{(k)}$), as $[X, Y]$ is then Hamiltonian, too. However, in this case we also have $L_Y w = 0$, so that no integrability conditions appear.

In a similar manner, we can analyse the prolongation of the equations $\chi^{(a)}(X) = 0$. If Y is a Hamiltonian vector field, then $L_Y(\chi^{(a)}(X)) = (L_Y\chi^{(a)})(X) - \chi^{(a)}([X, Y]) = (L_Y\chi^{(a)})(X)$. By Cartan's formula, $L_Y\chi^{(a)} = \iota_Y d\chi^{(a)} + d(\iota_Y\chi^{(a)})$ where ι_Y denotes the interior derivative with respect to Y. The second term vanishes, as Y is Hamiltonian and thus $\iota_Y\chi^{(a)} = 0$. For the first term, we exploit that the codistribution $\ker\sharp$ spanned by the forms $\chi^{(a)}$ is involutive, as it annihilates the involutive distribution $\operatorname{im}\sharp$. This implies by the Frobenius theorem that $d\chi^{(a)} = \omega_b^{(a)} \wedge \chi^{(b)}$ for some one-forms $\omega_b^{(a)}$ and hence $\iota_X\iota_Y d\chi^{(a)} = 0$. Thus no integrability conditions appear in the prolongation with respect to a Hamiltonian vector fields and the trivial ones (16) in any other prolongation.

IV. THE FUNDAMENTAL POWER SERIES SOLUTIONS

Bender et al.[1] derived the underdetermined system (11) differently. They considered in some local coordinate chart the n special functions $F^{(k)} \in \mathcal{F}(M)$ given by $F^{(k)}(z) = z^k$. We may call the functions $\phi^* F^{(k)}$ the *fundamental solutions* of the differential equation (6), as it obeys a kind of non-linear superposition principle: for any function $F \in \mathcal{F}(M)$ we have

$$(\phi^* F)(t, z) = F\Bigl((\phi^* F^{(1)})(t, z), \ldots, (\phi^* F^{(n)})(t, z)\Bigr)\,.$$

Indeed, the functions $\phi^* F^{(k)}$ are obviously nothing but the components of the flow in the local coordinates $(z^1, \ldots, z^n)$.

Bender et al.[1] expanded the fundamental solutions into power series of the form (7) and determined partial differential equations for the coefficients $F_\alpha^{(k)}$. Using our results above it is straightforward to find these equations. We first apply the relation (8) for $\alpha = 1$:

$$\Bigl(\{F^{(k)}, F^{(l)}\}\Bigr)_1 = \{F_1^{(k)}, F_0^{(l)}\} + \{F_0^{(k)}, F_1^{(l)}\}\,.$$

By the definition of the functions $F^{(k)}$, the right hand side may obviously be written in the form $X^{(k)}F_1^{(l)} - X^{(l)}F_1^{(k)}$. Because of $(\{F^{(k)}, F^{(l)}\})_0 = \{F_0^{(k)}, F_0^{(l)}\} = J^{kl}$, the left hand side evaluates to

$$\Bigl(\{F^{(k)}, F^{(l)}\}\Bigr)_1 = \{J^{kl}, H\} = \frac{\partial J^{kl}}{\partial z^j}\{z^j, H\} = \frac{\partial J^{kl}}{\partial z^j}\{F_0^{(j)}, H\} = \frac{\partial J^{kl}}{\partial z^j}F_1^{(j)}\,.$$

Putting the pieces together, we find that the first-order coefficients $F_1^{(k)}$ satisfy the compatibility conditions (11). This is of course not surprising: by definition of the flow ϕ and the functions $F^{(k)}$, the coefficients $F_1^{(k)}$ are nothing but the components ξ^k of the Hamiltonian vector field X_H in the given local coordinates.

We may derive similarly systems of partial differential equations for the higher-order coefficients $F_k^{(i)}$. In second order, the starting point is the relation

$$\Bigl(\{F^{(k)}, F^{(l)}\}\Bigr)_2 = \{F_2^{(k)}, F_0^{(l)}\} + \{F_1^{(k)}, F_1^{(l)}\} + \{F_0^{(k)}, F_2^{(l)}\}\,.$$

The right hand side equals $X^{(k)}F_2^{(l)} - X^{(l)}F_2^{(k)} + \{F_1^{(k)}, F_1^{(l)}\}$. Using the results above we evaluate the left hand side to

$$\Bigl(\{F^{(k)}, F^{(l)}\}\Bigr)_2 = \Bigl\{\Bigl(\{F^{(k)}, F^{(l)}\}\Bigr)_1, H\Bigr\} = \Bigl\{\frac{\partial J^{kl}}{\partial z^j}F_1^{(j)}, H\Bigr\} = \frac{\partial J^{kl}}{\partial z^j}F_2^{(j)} + \Bigl\{\frac{\partial J^{kl}}{\partial z^j}, H\Bigr\}F_1^{(j)}\,.$$

Thus we obtain the following system for the second-order coefficients:

$$X^{(k)}F_2^{(l)} - X^{(l)}F_2^{(k)} - \frac{\partial J^{kl}}{\partial z^j}F_2^{(j)} = \Bigl\{\frac{\partial J^{kl}}{\partial z^j}, H\Bigr\}F_1^{(j)} - \{F_1^{(k)}, F_1^{(l)}\}\,. \tag{17}$$

Note that the homogeneous part of this linear system is identical with our compatibility conditions (11). It is easy to see that if we continue in this manner, the homogeneous part always remains unchanged. Only the right hand side becomes a more and more complicated expression in the coefficients of lower order. More precisely, we find

$$X^{(k)}F_\alpha^{(l)} - X^{(l)}F_\alpha^{(k)} - \frac{\partial J^{kl}}{\partial z^j}F_\alpha^{(j)} = R_\alpha^{kl} - \sum_{\beta=1}^{\alpha-1}\{F_{\alpha-\beta}^{(k)}, F_\beta^{(l)}\} \tag{18}$$

where the term R_α^{kl} is determined by the recurrence relation

$$R_{\alpha+1}^{kl} = \Bigl\{\frac{\partial J^{kl}}{\partial z^j}, H\Bigr\}F_\alpha^{(j)} + \{R_\alpha^{kl}, H\}\,, \qquad R_1^{kl} = 0\,.$$

It should be noted that so far we have only shown that the functions $F_\alpha^{(k)}$ satisfy certain differential equations. We have *not* shown that they form their respective general solution. Indeed, we already know from the last section that this is the case only for non-degenerate Poisson structures. It is trivial to see that

$$\chi_k^{(a)}F_1^{(k)} = \chi_k^{(a)}\{F_0^{(k)}, H\} = \chi_k^{(a)}X^{(k)}H = 0\,, \qquad 1 \le a \le r\,.$$

Thus we recover (12). However, using the power series approach we still have no rigorous proof that we have found all equations satisfied by the first-order coefficients $F_1^{(\alpha)}$ whereas this followed trivially from our compatibility analysis in the last section.

The higher-order coefficients $F^{(k)}_\alpha$ also satisfy some algebraic constraints which may be determined recursively. If we set $G^{(a)}_\alpha = \chi^{(a)}_k F^{(k)}_\alpha$, then a trivial computation yields

$$G^{(a)}_{\alpha+1} = \{G^{(a)}_\alpha, H\} - \{\chi^{(a)}_k, H\} F^{(k)}_\alpha \,. \tag{19}$$

Of course, such a relation holds for arbitrary one-forms χ. The special property of the one-forms $\chi^{(a)} \in \ker\sharp$ is that for them $G^{(a)}_1 = 0$.

Bender et al.[1] explicitly determined and solved for several concrete instances of a Lie-Poisson manifold the inhomogeneous system (17) for the second-order coefficients. They used the traditional approach of finding a particular solution of the inhomogeneous system and adding the general solution of the homogenous system. But obviously, (9) provides us with simpler closed form expressions for the general solution of the combined system (18,19) for any order α.

V. AN EXPLICIT EXAMPLE

We detail the calculations presented so far for the Lie-Poisson structure associated with the three-dimensional Lie algebra E_2 generating the Euclidean motions in $\mathbb{R}^2$. This structure is characterised by the Poisson matrix

$$J = \begin{pmatrix} 0 & 0 & -y \\ 0 & 0 & x \\ y & -x & 0 \end{pmatrix} .$$

If we exclude the origin and confine ourselves to $M = \mathbb{R}^2 \setminus \{0\}$, its constant rank is 2 and $\ker\sharp$ is spanned by the single one-form $\chi = x dx + y dy$. The vector fields $X^{(k)}$ have the form

$$X^{(1)} = -y\partial_z\,, \qquad X^{(2)} = x\partial_z\,, \qquad X^{(3)} = y\partial_x - x\partial_y\,.$$

They yield a representation of E_2 on $\mathbb{R}^3$, as the only non-vanishing Lie brackets are

$$[X^{(1)}, X^{(3)}] = -X^{(2)}\,, \qquad [X^{(2)}, X^{(3)}] = X^{(1)}\,.$$

If we write $\xi^1 = F$, $\xi^2 = G$, $\xi^3 = H$, then the evaluation of (11) yields the system (1).

Obviously, $X^{(1)}$ and $X^{(2)}$ are linearly dependent: $\chi_k X^{(k)} = xX^{(1)} + yX^{(2)} = 0$. Hence (12) takes the form $xF + yG = 0$ and all solutions given by (2) satisfy this equation. However, the system (1) implies only the relations $\partial/\partial z(xF + yG) = 0$ and $(y\partial/\partial x - x\partial/\partial y)(xF + yG) = 0$. Thus it possesses further solutions for which $xF + yG \neq 0$; one example has been given in the introduction.

The Hamiltonian vector field X_K associated to an arbitrary function K is of the form $X_K = -yK_z\partial_x + xK_z\partial_y + (yK_x - xK_y)\partial_z$. As expected, the components are just the components of the general solution (2). A direct compatibility analysis yields that a general vector field $X = F\partial_x + G\partial_y + H\partial_z$ is Hamiltonian, if and only if its components satisfy

$$xF + yG = 0\,, \qquad F_x + G_y + H_z = 0\,.$$

This system is equivalent to (1) *plus* the algebraic constraint $xF + yG = 0$.

Let us perform an involution analysis of the system consisting of (1) and the constraint $xF + yG = 0$. Cross-differentiations of the equations within (1) do not lead to any new equations. There is not short-cut for showing this; one must perform some lengthy computations which are equivalent to our analysis of (13).

The z-prolongation of the constraint is already contained in (1); thus applying neither $X^{(1)}$ nor $X^{(2)}$ to the constraint yields an integrability condition. If we apply $X^{(3)}$ to it, we obtain an equation which is a linear combination of the second and the third equation in the system (1). Indeed, it is easy to see that in our special case the one-form χ is such that in (14) the expression in the parentheses vanishes.

However, the vector fields $X^{(k)}$ do not form a basis of TM. Thus we must also study either the x- or the y-prolongation of the constraint. Either one leads to a trivial integrability condition, say, $xF_x + yG_x + F = 0$, if we choose $Y = \partial_x$. Now, as a last step, we must check whether a cross-differentiation of this equation with some equation contained in (1) yields something non-trivial. But it is easy to see that this is not the case and we have arrived at an involutive system.

VI. CONCLUSIONS

We rederived the results of Bender et al.[1] on Lie-Poisson structures in a more geometric fashion using the formal theory of differential equations and generalised them to arbitrary Poisson manifolds. It turned out that their system simply describes necessary and sufficient conditions on the components of a vector field for the field to be Hamiltonian with respect to the given Poisson structure. We also corrected an error of Bender et al.[1] in the case of degenerate Poisson structures where the Poisson matrix admits non-trivial null vectors.

We studied only the formal solvability of our differential systems. In the analytic category we could use the Cartan-Kähler theorem for proving the existence of analytic solutions (because of the linearity of the systems, the Holmgren theorem ensures that these solutions are unique even in much larger function spaces). The assumption that the Poisson matrix is analytic is not so restrictive, as it is for example trivially true for any Lie-Poisson structure. In contrast, the restriction to analytic vector field is surely too severe for applications.

An existence theory in more general function spaces would require a deeper analysis of the systems. In the case of a non-degenerate Poisson structure it is not difficult to prove that both (10) and (11,12) form elliptic systems in the sense that their symbol maps are injective. Thus one could apply the results outlined in the encyclopaedia article by Dudnikov and Samborski.[12] But as for our purposes the formal analysis was sufficient, we refrain from going further in this direction.

It is very instructive to see how deeply the involution analysis of our partial differential system (11,12) is related to Poisson geometry. The two characteristic properties of the Poisson matrix J, skew-symmetry and the Jacobi identity (3), must be applied repeatedly for showing the vanishing of all integrability conditions. It is tempting to conjecture that conversely the system has the wanted properties, if and only if the matrix J defines a Poisson structure, but we have not been able to prove this. Such a proof would require to solve the *inverse* problem of compatibility for the system (11,12). Fairly recently, an algorithmic solution of this inverse problem has been developed.[13,14] However, the involved computations are rather tedious and it seems hopeless to perform them for our system.

We studied in this article only classical Poisson systems. Bender et al.[1] also considered quantum systems where the coordinates z^j become non-commutative. We developed recently a theory of involutive bases in a fairly large class of algebras, the polynomial algebras of solvable type.[4] This class contains in particular the universal enveloping algebras of finite-dimensional Lie algebras. The latter should provide the right setting for an analogous analysis of the quantum version of the evolution equation (6).

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