Minimal Staggered Linear Bases and Their Applications

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Abstract. A staggered linear basis (SLB) provides a particular linear basis for the ideal it generates and contains a Gröbner basis for this ideal. These properties are enhanced through the additional structure of sets of allowed and forbidden terms assigned to each polynomial within the SLB. In the first part of this paper, we report the first implementation of SLB's in the `CoCoALib`. We compare its efficiency with that of the built-in function for Gröbner basis computations, showing that for some classes of examples and coefficient fields, our algorithm outperforms the built-in one. In the second part of this paper, we define and study *minimal* staggered linear bases and present an algorithm for their computation. The third part of the paper explores several applications of SLB's, including the computation of Hilbert functions, Hilbert polynomials, and Hilbert series for polynomial ideals. Furthermore, by leveraging the combinatorial structure of SLB's, we introduce an algorithm for constructing irreducible complementary decompositions for a given monomial ideal. Finally, we present several algorithms that address various aspects of complementary decompositions of ideals.

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1. Introduction

Monomial ideals lie at the heart of commutative algebra and algebraic geometry, forming an essential tool for studying more complex algebraic structures. The combinatorial nature of these ideals allows for the application of discrete methods and algorithms, making them particularly amenable to computational techniques. A thorough understanding of their structure and properties not only deepens the theoretical aspects in algebraic geometry, but also improves practical computational strategies across a range of applications, see [21] for more details.

An important tool for translating questions about an arbitrary polynomial ideal into ones about a monomial ideal is a *Gröbner basis* leading to many applications in various areas in Mathematics and engineering [7, 9, 10, 11]. The basic algorithm for the construction of these bases was introduced by Buchberger in his Ph.D. thesis [5, 8] in 1965. Buchberger later in [6] enhanced his algorithm by incorporating two criteria that we refer to as Buchberger's criteria to reduce the number of unnecessary reductions during the construction of Gröbner bases. In 1988, Gebauer and Möller [15] efficiently incorporated these criteria into Buchberger's algorithm (for a detailed discussion on these criteria and their implementation in Buchberger's algorithm, we refer to [2, page 222]). In 2015, Berkesch and Schreyer in [3] described a simplified variant of this algorithm which will be further discussed later in this paper.

Another related tool for obtaining a canonical representation for an ideal is a *staggered linear basis* (SLB). An SLB of an ideal is essentially a linear basis of the ideal considered as a linear space that includes a Gröbner basis. The combinatorial structure of such a basis induces a representation of the ideal it generates as a direct sum of linear spaces and facilitates the analysis of the ideal. These bases were first introduced by Gebauer and Möller in [14], who provided several illustrative examples. However, the authors did not offer a proof for the correctness of the proposed algorithm. Since then, significant advancements have been made in the construction of these bases, as noted in works such as [24, 25]. Notably, [23] presented the first correct approach for computing staggered linear bases using intermediate syzygies. Following this, [18] developed an incremental, signature-based algorithm for computing these bases, drawing on the structure of the GVW algorithm [13]. Recently, Hashemi and Möller [19] presented a simple and efficient algorithm to compute staggered linear bases by applying Buchberger's criteria within the Berkesch-Schreyer framework.

This paper is structured into three parts. After a brief review of the fundamental definitions and notations related to the theory of SLB's in the next section, we report in the first part (Section 3) on an implementation of the algorithm from [19] in the `CoCoALib` [1]. We compare its efficiency with that of the built-in function for Gröbner basis computations, showing that for some classes of examples and coefficient fields, our algorithm outperforms the built-in one. In a second part (Section 4), we introduce the concept of *minimal* SLB's corresponding to minimal Gröbner bases and provide an algorithm for their computation. The final part of the paper (the last section) explores several applications of SLB's, particularly in the calculation of various invariants of ideals, such as Hilbert functions, Hilbert polynomials, and Hilbert series of polynomial ideals. We also apply the specific structure of staggered linear bases in the study of complementary decompositions of ideals. Additionally, we describe new algorithms for computing an irreducible complementary decomposition of a monomial ideal, identifying the set of all standard pairs for a monomial ideal, testing whether a given set of generalized cones forms a complementary decomposition for an ideal, and determining the closure of a set of cones that may obstruct being an order ideal. We provide illustrative examples to clarify the proposed algorithms.

2. Preliminaries

In this section, we provide an overview of fundamental definitions and notations which will be referenced throughout the rest of the article. We work in the polynomial ring $\mathcal{P} = \mathcal{K}[x_1, \dots, x_n]$, where $\mathcal{K}$ represents a field. Within this context, we consider the polynomials $f_1, \dots, f_s \in \mathcal{P}$, as well as the ideal $\mathcal{I} = \langle f_1, \dots, f_s \rangle$ generated by these polynomials. Furthermore, we write $\mathbb{T} = \{x_1^{\alpha_1} \cdots x_n^{\alpha_n} \mid \alpha_i \geq 0, 1 \leq i \leq n\}$ for the set of all terms (for us, a term is a power product of the variables) in $\mathcal{P}$. Additionally, let us fix a term ordering $\prec$ on $\mathbb{T}$. For a given polynomial $f = \sum_{i=1}^{m} c_i \cdot t_i \neq 0$, where $c_i \in \mathcal{K}$ and $t_i \in \mathbb{T}$, the support of f is defined as $\operatorname{supp}(f) = \{t_i \mid c_i \neq 0\}$. The *leading term* of f with respect to $\prec$, denoted by $\operatorname{lt}(f)$, is the maximum term in $\operatorname{supp}(f)$. The coefficient corresponding to $\operatorname{lt}(f)$ is referred to as the *leading coefficient* of f and is denoted by $\operatorname{lc}(f)$. Furthermore, the *leading monomial* of f is defined as $\operatorname{lm}(f) = \operatorname{lc}(f)\operatorname{lt}(f)$.

If $F \subset \mathcal{P}$ is a finite set of polynomials, we use $\operatorname{lt}(F)$ to represent the set $\{\operatorname{lt}(f) \mid f \in F\}$. The *leading term ideal* of $\mathcal{I}$ is defined as $\operatorname{lt}(\mathcal{I}) = \langle \operatorname{lt}(f) \mid 0 \neq f \in \mathcal{I} \rangle$. A finite set $G \subset \mathcal{I}$ is a *Gröbner basis* for $\mathcal{I}$ with respect to $\prec$ if $\operatorname{lt}(\mathcal{I}) = \langle \operatorname{lt}(G) \rangle$. If every polynomial in the Gröbner basis G has a leading term which is not divisible by the leading term of any other polynomial in G and in addition for each $g \in G$, $\operatorname{lc}(g) = 1$ then G is called a *minimal Gröbner basis* for $\mathcal{I}$. Following [25], in the rest of the paper and for a given sequence $f_1, \dots, f_s \in \mathcal{P}$, we denote the leading term of f_i by $\mathrm{T}(i)$ and the least common multiple of $\mathrm{T}(i)$ and $\mathrm{T}(j)$ for each i, j by $\mathrm{T}(i, j)$. Furthermore, the *S-polynomial* of two polynomials f_i and f_j is defined to be

$$\mathrm{Spoly}(f_i, f_j) = \frac{\mathrm{T}(i,j)}{\mathrm{lm}(f_i)} \cdot f_i - \frac{\mathrm{T}(i,j)}{\mathrm{lm}(f_j)} \cdot f_j.$$

Another key ingredient in the computation of Gröbner bases is the reduction process. For a polynomial $f \in \mathcal{P}$ and a finite set $F \subset \mathcal{P}$, we write $f \longrightarrow_F^* h$ if h is a remainder of the division of f by F. Now, Buchberger's criterion states that a finite set $G \subset \mathcal{P}$ is a Gröbner basis for $\mathcal{I}$ if and only if G generates $\mathcal{I}$ and for every pair of distinct polynomials $g_i, g_j \in G$, we have $\mathrm{S}(g_i, g_j) \longrightarrow_G^* 0$. For more detailed information on the theory of Gröbner bases, we recommend referring to the book [11].

Let us now revisit the concept of staggered linear bases. For a polynomial $f \in \mathcal{P}$ and an arbitrary set $\mathbb{T}' \subseteq \mathbb{T}$ of terms, we define $\mathbb{T}' \cdot f = \{t \cdot f \mid t \in \mathbb{T}'\}$. Given an ideal $\mathcal{I} \subset \mathcal{P}$, we can consider it as a $\mathcal{K}$-linear subspace of $\mathcal{P}$. A staggered linear basis is a basis for this vector space. More precisely, we have:

Definition 2.1. Let $\mathcal{I} = \langle f_1, \ldots, f_s \rangle$ be an ideal and $\mathbf{B} = \bigcup_{i=1}^s \mathbb{A}_i \cdot f_i$ with $\mathbb{A}_1, \ldots, \mathbb{A}_s \subseteq \mathbb{T}$ is called a *staggered linear basis (SLB)* for $\mathcal{I}$, if the following conditions hold:

(1) for each $t_j \in \mathbb{T} \setminus \mathbb{A}_j$ there exists $t_i \in \mathbb{A}_i$ with $i < j$ such that $t_i \cdot \mathrm{lt}(f_i) = t_j \cdot \mathrm{lt}(f_j)$,
(2) for each $t_j \in \mathbb{T} \setminus \mathbb{A}_j$ and each $t \in \mathbb{T}$ it holds $t \cdot t_j \in \mathbb{T} \setminus \mathbb{A}_j$,
(3) for each $t_i \in \mathbb{A}_i, t_j \in \mathbb{A}_j$ with $i \neq j$ we have $t_i \cdot \mathrm{lt}(f_i) \neq t_j \, \mathrm{lt}(f_j)$.

Using the previously defined notations, the set $\mathbb{A}_i$ associated with each polynomial f_i is referred to as the set of *allowed terms* for f_i, whereas the complement of $\mathbb{A}_i$; i.e. $\mathbb{T} \setminus \mathbb{A}_i$, is known as the set of *forbidden terms* for f_i. By applying the second condition in this definition and Dickson's lemma, we can conclude that there exist finitely many terms $t_1, \ldots, t_\ell \in \mathbb{T} \setminus \mathbb{A}_i$ such that $\langle t_1, \ldots, t_\ell \rangle = \mathbb{T} \setminus \mathbb{A}_i$. For convenience, we denote $\{t_1, \ldots, t_\ell\}$ by $\mathbb{F}_i$. Consequently, we can express $\mathbb{A}_i$ as $\mathbb{A}_i = \mathbb{T} \setminus \langle \mathbb{F}_i \rangle$. In the remainder of the paper, instead of denoting a staggered linear basis for the ideal $\mathcal{I}$ as $\mathbb{A}_1 \cdot f_1 \cup \cdots \cup \mathbb{A}_s \cdot f_s$, where each $\mathbb{A}_i \cdot f_i$ is possibly infinite, we adopt the notation $\{(f_1, \mathbb{F}_1), \ldots, (f_s, \mathbb{F}_s)\}$, where $\mathbb{F}_i \subset \mathbb{T}$ for each i is a finite set. For each i, the pair $(f_i, \mathbb{F}_i)$ denotes the set $\mathbb{A}_i \cdot f_i$ and is called a *generalized cone* with the *vertex* f_i. Moreover, we assume that $\mathbb{F}_i$ is the minimal generating set of the ideal $\langle \mathbb{F}_i \rangle$ for each i. The following lemma is useful in the rest of the paper.

Lemma 2.2. *Let $\mathcal{I} = \langle f_1, \ldots, f_s \rangle$ be an ideal and $\{f_1, \ldots, f_s\}$ a Gröbner basis for $\mathcal{I}$. Then, the set $\mathbf{B} = \{(f_1, \mathbb{F}_1), \ldots, (f_s, \mathbb{F}_s)\}$ forms an SLB for $\mathcal{I}$, iff for each i, $\mathbb{F}_i$ is the minimal generating set of $\langle \mathrm{T}(1,i)/\mathrm{T}(i), \ldots, \mathrm{T}(i-1,i)/\mathrm{T}(i) \rangle$.*

Proof. If $\mathbf{B}$ is an SLB, we must show that the equality $\langle \mathbb{F}_i \rangle = \langle \mathrm{T}(1,i)/\mathrm{T}(i), \ldots, \mathrm{T}(i-1,i)/\mathrm{T}(i) \rangle$ for each i holds. Let $u \in \mathbb{F}_i$. Then, by Definition 2.1, there exists $\ell < i$ such that $u \cdot \mathrm{T}(i) = v \cdot \mathrm{T}(\ell)$ where $u \in \mathbb{A}_\ell$. This implies that $\mathrm{T}(\ell,i)/\mathrm{T}(i)$ divides u and consequently u lies in the right-hand side of the desired equality. Now, to prove the other inclusion, we must show that any $\mathrm{T}(\ell,i)/\mathrm{T}(i)$ with $\ell < i$ belongs to $\langle \mathbb{F}_i \rangle$. Let us prove this property using induction on i. For $i = 1, 2$ this property holds true. Assume that for each $j < i$ it holds. We know that $\mathrm{T}(\ell) \mid \mathrm{T}(\ell,i)/\mathrm{T}(i) \cdot \mathrm{T}(i)$ and $\ell < i$. Assume that ℓ is the smallest integer that satisfies this property. Thus, there exists a term u such that $\mathrm{T}(\ell,i)/\mathrm{T}(i) \cdot \mathrm{T}(i) = u \cdot \mathrm{T}(\ell)$. We claim $u \in \mathbb{A}_\ell$. If we prove this claim, then from the definition of SLB's, $\mathrm{T}(\ell,i)/\mathrm{T}(i)$ is forbidden for f_ℓ and lies in $\langle \mathbb{F}_\ell \rangle$. If the claim is not true then $u \in \langle \mathbb{F}_\ell \rangle$ because $\mathbf{B}$ is an SLB. It follows that there exists $v \in \mathbb{F}_\ell$ such that $v \mid u$. From the induction hypothesis, we conclude that $v = \mathrm{T}(k,\ell)/\mathrm{T}(\ell)$ for some integer $k < \ell$. This implies that $\mathrm{T}(k) \mid \mathrm{T}(\ell,i)/\mathrm{T}(i) \cdot \mathrm{T}(i)$; leading to a contradiction with the minimality of ℓ.

Conversely, we know that $\{f_1, \ldots, f_s\}$ is a Gröbner basis. Thus, if we provide this set as the input to the basic algorithm for constructing SLB's as described in [19], then no new polynomial is produced. Additionally, for each i, the set $\mathbb{F}_i$ as the minimal generating set of $\langle \mathrm{T}(1,i)/\mathrm{T}(i), \ldots, \mathrm{T}(i-1,i)/\mathrm{T}(i)\rangle$ is constructed; yielding the desired result. □

In Section 5, we will employ irreducible SLB's. We conclude this section by defining this concept and presenting a straightforward method to compute such an SLB from a given SLB. A generalized cone $(f, \mathbb{F})$ is called *irreducible*, if the monomial ideal generated by $\mathbb{F}$ is irreducible, i.e., generated by pure powers of variables. An SLB is said to be *irreducible*, if each of its elements is an irreducible generalized cone. We remark that Janet-like bases [16, 17] and Pommaret-like bases [20, Section 6] are special types of irreducible SLB's. The next lemma offers a simple technique to resolve an obstruction to being an irreducible SLB.

Lemma 2.3. *Let* $\mathbf{B} = \{(f_1, \mathbb{F}_1), \ldots, (f_s, \mathbb{F}_s)\}$ *be an SLB for the ideal* $\mathcal{I} \subset \mathcal{P}$. *Suppose that* $u = x_i^\ell \cdot v \in \mathbb{F}_1$ *where* $x_i \nmid v \neq 1$. *Then, if we replace* $(f_1, \mathbb{F}_1)$ *in* $\mathbf{B}$ *by* $\{(f_1, \mathrm{Gen}(\mathbb{F}_1 \cup \{x_i^\ell\})), (f_1 \cdot x_i^\ell, \mathrm{Gen}(\langle \mathbb{F}_1\rangle : x_i^\ell))\}$ *we get a new SLB for* $\mathcal{I}$ *where for any given set* A *of terms,* $\mathrm{Gen}(A)$ *stands for the minimal generating set of* $\langle A\rangle$.

Proof. It suffices to demonstrate that $(f_1, \mathbb{F}_1) = (f_1, \mathrm{Gen}(\mathbb{F}_1 \cup \{x_i^\ell\})) \cup (f_1 \cdot x_i^\ell, \mathrm{Gen}(\langle \mathbb{F}_1\rangle : x_i^\ell))$. Let us analyze the elements of both sides. Let $m = f_1 \cdot v$ belong to $(f_1, \mathbb{F}_1)$ where $v \notin \langle \mathbb{F}_1\rangle$. Then, two cases happen: If $x_i^\ell \nmid v$, then m is clearly in the first cone on the right-hand side. Otherwise, we can express m as $f_1 \cdot x_i^\ell \cdot u$ for some term u. From $v \notin \langle \mathbb{F}_1\rangle$, it follows that $u \notin \langle \mathbb{F}_1\rangle : x_i^\ell$ which shows that m belongs to the second cone on the right-hand side. Conversely, since $\mathbb{F}_1$ is a subset of $\mathbb{F}_1 \cup \{x_i^\ell\}$ and $\langle \mathbb{F}_1\rangle : x_i^\ell$, any element in the right-hand side must also belong to the left-hand side, completing the proof. □

This lemma translates immediately into an algorithmic approach to derive an irreducible SLB from an arbitrary SLB. Because of its simplicity, we omit an explicit presentation of the obtained algorithm, but instead demonstrate it in the following example for a concrete ideal.

Example 2.4. In the polynomial ring $\mathcal{K}[x_1, x_2, x_3]$ let us consider the ideal $\mathcal{I}$ generated by the terms $x_3^4, x_2x_3^3, x_2^3x_3^2, x_1x_2x_3, x_1^3x_3, x_1^3x_2^3$. By applying Lemma 2.2 and using the specified order of the generators of $\mathcal{I}$, we obtain the SLB

$$\{(x_3^4, \varnothing), (x_2x_3^3, \{x_3\}), (x_2^3x_3^2, \{x_3\}), (x_1x_2x_3, \{x_2^2x_3, x_3^2\}), (x_1^3x_3, \{x_2, x_3^3\}), (x_1^3x_2^3, \{x_3\})\}$$

for $\mathcal{I}$. In total, only one mixed term appears, namely $x_2^2x_3$ in $\mathbb{F}_4$. Using Lemma 2.3, we derive the following SLB for $\mathcal{I}$

$$\{(x_3^4, \varnothing), (x_2x_3^3, \{x_3\}), (x_2^3x_3^2, \{x_3\}), (x_1x_2x_3^2, \{x_2^2, x_3\}), (x_1x_2x_3, \{x_3\}),$$
$$(x_1^3x_3, \{x_2, x_3^3\}), (x_1^3x_2^3, \{x_3\})\}.$$

Now, no mixed terms appear in any $\mathbb{F}_i$ and hence this SLB is irreducible.

3. Implementation of Staggered Linear Bases

We have implemented a preliminary version of the improved algorithm described in [19], which we refer to as *SLB*. It consists of several `CoCoALib` (v. 0.99818) functions, data types, and classes. `CoCoALib` is a C++ library within `CoCoA`, a system for computations in polynomial rings[1]. We have tested our algorithm on some examples documented in the PHCpack database of polynomial systems; see `https://homepages.math.uic.edu/~jan/demo.html`. We have always used the graded reverse lexicographic order. The calculations have been done on an Intel(R)

[1]For more information, see `http://cocoa.dima.unige.it`.

Core(TM) i5-10400F processor, 2.90 GHz, 2904 MHz, 6 cores, 12 logical processors, 16 GB RAM and 64 bits (Windows 10).

In Table 1, our implementation of *SLB* is compared to `CoCoA`'s built-in function *GBasis* (*GB* for short). All times are given in seconds. The fifth and sixth columns show the number of performed polynomial reductions and, in parentheses, how many of those reductions led to the zero polynomial.

SLB vs GBasis over $\mathbb{Q}$						
Poly set	**time SLB**	**time GB**	**time ratio**	**red SLB**	**red GB**	**red ratio**
Huneke	0.081	0.095	0.85	449 (349)	383 (279)	1.17
Cohn2	0.286	0.094	3.04	111 (76)	88 (46)	1.26
Chemkin	6.154	0.453	13.58	488 (412)	488 (403)	1.00
Eco6	0.005	0.006	0.83	76 (56)	66 (43)	1.15
Eco7	0.045	0.041	1.10	168 (131)	159 (117)	1.06
Eco8	0.304	0.197	1.54	362 (295)	358 (281)	1.01
Noon4	0.005	0.008	0.63	77 (53)	75 (47)	1.03
Noon5	0.048	0.063	0.76	280 (213)	267 (195)	1.05
Cyclic5	0.015	0.014	1.07	127 (86)	113 (75)	1.12
Cyclic6	10.714	0.174	61.57	758 (579)	344 (245)	2.20
Katsura6	0.307	0.161	1.91	164 (128)	169 (128)	0.97
Katsura7	3.417	1.269	2.69	378 (310)	381 (307)	0.99
Katsura8	41.962	12.242	3.43	882 (746)	886 (743)	1.00

TABLE 1. Comparison of *SLB* and *GBasis* over $\mathbb{Q}$.

Although *GBasis* has an advantage over *SLB*, the number of performed reductions does not differ much in most tests. This is a promising sign and an argument in favour of the staggered linear basis approach. In fact, much of the runtime advantage of *GBasis* can be attributed to its use of specialized procedures for dealing with large rational coefficients.

The next set of tests has been done over the finite field $\mathbb{Z}/32003\mathbb{Z}$ instead of $\mathbb{Q}$ – see Table 2. It is remarkable that *SLB* outperforms *GBasis* in some classes of examples. Exceptions are once again the *Cyclic* tests, but the difference is less significant than over $\mathbb{Q}$. Note that some of these runtimes are very short and hence more susceptible to variance.

The most apparent area in which our implementation of *SLB* can be improved is its handling of computations with large rational coefficients during the reduction process. For this *SLB* already makes use of an enriched type of polynomial called *GPoly* that allows the use of `CoCoA`'s built-in function *myReduce*. However, *SLB* cannot utilize *myReduce* to its full effect yet, as this function is deeply connected with the Gröbner basis computation within `CoCoA`. In order to save additional time and storage space, the pairs that still have to be checked during the algorithm are stored as pairs of indices. Lastly, in *SLB* we make use of a special type of vector called *PPVector*. This is a vector for terms in `CoCoA` that accelerates divisibility tests and other operations on monomial ideals.

4. Minimal Staggered Linear Bases

In this section, similar to the concept of a minimal Gröbner basis, we introduce the concept of a minimal SLB. Then, after presenting some auxiliary results, we outline an algorithm that produces a minimal SLB for a given ideal.

SLB vs GBasis over $\mathbb{Z}$ mod 32003			
Poly set	**time SLB**	**time GB**	**time ratio**
Huneke	0.042	0.036	1.17
Cohn2	0.015	0.014	1.07
Chemkin	0.095	0.089	1.07
Eco7	0.014	0.024	0.58
Eco8	0.081	0.095	0.85
Noon4	0.003	0.005	0.60
Noon5	0.019	0.025	0.76
Cyclic5	0.011	0.002	5.50
Cyclic6	0.111	0.042	2.64
Cyclic7	11.449	2.359	4.85
Katsura6	0.037	0.046	0.80
Katsura7	0.264	0.273	0.97
Katsura8	1.873	1.964	0.95

TABLE 2. Comparison of *SLB* and *GBasis* over $\mathbb{Z}/32003\mathbb{Z}$.

Definition 4.1. Let $\mathcal{I} = \langle f_1, \dots, f_s \rangle$ be an ideal and $\mathbf{B} = \{(f_1, \mathbb{F}_1), \dots, (f_s, \mathbb{F}_s)\}$ an SLB for $\mathcal{I}$. Then, $\mathbf{B}$ is called minimal if

- for each i, j with $i \neq j$ the leading term of f_i does not divide that of f_j,
- for each i, we have $\mathrm{lc}(f_i) = 1$.

Similar to this definition, one can also give the definition of a reduced SLB. Now, a natural question that may arise is whether it is possible to transform an SLB into a minimal one. The next proposition shows that it is always possible to obtain a minimal SLB through a finite number of operations applied to a given SLB. Let us state an auxiliary lemma that utilizes the notation introduced in the previous section.

Lemma 4.2. *Let $\mathbf{B} = \{(f_1, \mathbb{F}_1), \dots, (f_s, \mathbb{F}_s)\}$ be an SLB for the ideal $\mathcal{I}$ and $t \in \mathbb{F}_i$ for some i. Then, there exists j such that $t \cdot \mathrm{T}(i) = \mathrm{T}(i, j)$.*

Proof. From Definition 2.1, it follows that there exist j and a term $u \in \mathbb{A}_j$ such that $t \cdot \mathrm{T}(i) = u \cdot \mathrm{T}(j)$. If this is equal to $\mathrm{T}(i, j)$ we are done. Otherwise, it means that $t \cdot \mathrm{T}(i)$ is divisible by $\mathrm{T}(i, j)$. In this case, we can conclude that there exists a term t_1 (different from t) that divides t, and a corresponding term u_1 (different from u) that divides u, satisfying $t_1 \cdot \mathrm{T}(i) = u_1 \cdot \mathrm{T}(j) = \mathrm{T}(i, j)$. Now, two cases for u may occur.

- If $t_1 \in \langle \mathbb{F}_i \rangle$, then this contradicts the minimality of the generating set $\mathbb{F}_i$ for the ideal $\langle \mathbb{F}_i \rangle$.
- Alternatively, if $t_1 \in \mathbb{A}_i$, it implies that $u_1 \in \langle \mathbb{F}_j \rangle$ according to Definition 2.1. Since u_1 divides u based on the same definition, we can conclude that $u \in \langle \mathbb{F}_j \rangle$; which leads to another contradiction.

This finishes the proof. □

Proposition 4.3. *Let $\mathbf{B} = \{(f_1, \mathbb{F}_1), \dots, (f_s, \mathbb{F}_s)\}$ be an SLB for the ideal $\mathcal{I}$ which is not minimal. Suppose that there exist two indices i, j such that $j > i$ and $\mathrm{lt}(f_j) \mid \mathrm{lt}(f_i)$. Then, by removing $(f_i, \mathbb{F}_i)$ and updating the pairs in $\mathbf{B}$ as follows, we get a new SLB for $\mathcal{I}$. For each $\ell = i + 1, \dots, j$,* *if* $\mathrm{T}(i,\ell)/\mathrm{T}(\ell) \in \mathbb{F}_\ell$ *then set* $\mathbb{F}'_\ell$ *to be the minimal generating set of the ideal generated by* $\mathbb{F}_\ell \setminus \{\mathrm{T}(i,\ell)/\mathrm{T}(\ell)\} \cup_{k=1}^{\ell-1} \{\mathrm{T}(k,\ell)/\mathrm{T}(\ell) \mid k \neq i \wedge \mathrm{T}(i) \mid \mathrm{T}(k,\ell)\}$*; otherwise, keep it unchanged.*

Proof. Since we know that $\{f_1,\dots,f_s\}$ forms already a Gröbner basis for $\mathcal{I}$, we can apply Lemma 2.2. From this lemma, it is clear that only $\mathbb{F}_{i+1},\dots,\mathbb{F}_j$ might change and the other sets of forbidden terms remain unchanged. Moreover, for $\ell = i+1,\dots,j$, $\mathbb{F}'_\ell$ will be the minimal generating set of the ideal generated by $\{\mathrm{T}(1,\ell)/\mathrm{T}(\ell),\dots,\mathrm{T}(\ell-1,\ell)/\mathrm{T}(\ell)\} \setminus \{\mathrm{T}(i,\ell)/\mathrm{T}(\ell)\}$. We distinguish the following three cases.

- If $\ell < i$, then the set of forbidden terms does not change and thus $\mathbb{F}'_\ell = \mathbb{F}_\ell$.
- If $i < \ell \leq j$, then we must update $\mathbb{F}_\ell$ to the minimal generating set of the ideal generated by $\{\mathrm{T}(1,\ell)/\mathrm{T}(\ell),\dots,\mathrm{T}(\ell-1,\ell)/\mathrm{T}(\ell)\} \setminus \{\mathrm{T}(i,\ell)/\mathrm{T}(\ell)\}$. If $\mathrm{T}(i,\ell)/\mathrm{T}(\ell) \notin \mathbb{F}_\ell$, then there exists an integer k such that $\mathrm{T}(k,\ell)/\mathrm{T}(\ell)$ divides $\mathrm{T}(i,\ell)/\mathrm{T}(\ell)$ and consequently removing $\mathrm{T}(i,\ell)/\mathrm{T}(\ell)$ does not change anything. Otherwise, $\mathrm{T}(i,\ell)/\mathrm{T}(\ell) \in \mathbb{F}_\ell$ and if we remove $\{\mathrm{T}(i,\ell)/\mathrm{T}(\ell)\}$ from $\{\mathrm{T}(1,\ell)/\mathrm{T}(\ell),\dots,\mathrm{T}(\ell-1,\ell)/\mathrm{T}(\ell)\}$ then we must add $\mathrm{T}(k,\ell)/\mathrm{T}(\ell)$ to $\mathbb{F}_\ell$ for each $k \neq i$ with $k < \ell$ and $\mathrm{T}(i,\ell)/\mathrm{T}(\ell) \mid \mathrm{T}(k,\ell)/\mathrm{T}(\ell)$. However, the last condition is equivalent to $\mathrm{T}(i) \mid \mathrm{T}(k,\ell)$.
- If $j < \ell$, then first note that $\mathrm{T}(j) \mid \mathrm{T}(i)$ and we have $(\mathrm{T}(j,\ell)/\mathrm{T}(\ell)) \mid (\mathrm{T}(i,\ell)/\mathrm{T}(\ell))$. Thus, $\langle \mathrm{T}(k,\ell)/\mathrm{T}(\ell) \mid k < \ell \wedge k \neq i\rangle = \langle \mathrm{T}(k,\ell)/\mathrm{T}(\ell) \mid k < \ell\rangle$ and in turn $\mathbb{F}'_\ell = \mathbb{F}_\ell$. □

Example 4.4. In this simple example, we show how this proposition is applied to minimize an SLB. Let $\mathcal{P} = \mathbb{Q}[x_1,x_2]$ and $\mathcal{I} = \langle x_1^3, x_1x_2, x_2^2\rangle$. From Lemma 2.2, we know that in this ring the set $\mathbf{B} = \{(x_1^3,\varnothing),(x_1x_2^2,\{x_1^2\}),(x_2^2,\{x_1\}),(x_1x_2,\{x_1^2,x_2\})\}$ forms an SLB for the ideal $\mathcal{I}$. Now, by removing the second element using the last one, $\{(x_1^3,\varnothing),(x_2^2,\{x_1^3\}),(x_1x_2,\{x_1^2,x_2\})\}$ is the minimal SLB for $\mathcal{I}$.

As a corollary of the proof of this proposition, let us state a simpler version of the proposition.

Corollary 4.5. *Let* $\mathbf{B} = \{(f_1,\mathbb{F}_1),\dots,(f_s,\mathbb{F}_s)\}$ *be an SLB for the ideal* $\mathcal{I}$ *which is not minimal. Suppose that there exist two indices* i,j *such that* $j > i$ *and* $\mathrm{lt}(f_j) \mid \mathrm{lt}(f_i)$*. Then, by removing* $(f_i,\mathbb{F}_i)$ *and updating the pairs in* $\mathbf{B}$ *as follows, we get a new SLB for* $\mathcal{I}$*. For each* $\ell = i+1,\dots,j$*, set* $\mathbb{F}'_\ell$ *the minimal generating set of* $\langle \mathrm{T}(k,\ell)/\mathrm{T}(\ell) \mid k < \ell \wedge k \neq i\rangle$*; otherwise, keep the sets of minimal forbidden terms unchanged.*

Remark 4.6. We shall note that a simple way also to update $\mathbb{F}_i$'s is as follows. Instead of keeping each $\mathbb{F}_i$ as the minimal generating set of all forbidden terms for f_i, we can keep it as the set $\{\mathrm{T}(1,i)/\mathrm{T}(i),\dots,\mathrm{T}(i-1,i)/\mathrm{T}(i)\}$. Now, with the notations of Proposition 4.3, we can simply update $\mathbb{F}_\ell$ for each $\ell = i+1,\dots,j$ to $\{\mathrm{T}(1,\ell)/\mathrm{T}(\ell),\dots,\mathrm{T}(\ell-1,\ell)/\mathrm{T}(\ell)\} \setminus \{\mathrm{T}(i,\ell)/\mathrm{T}(\ell)\}$. However, it is worth noting from the computational point of view that it is better to keep $\mathbb{F}_i$ for each i as the minimal generating set of all forbidden terms.

In Algorithm 1, $\mathtt{Division}(f,F)$ computes a remainder of the division of f by F.

Theorem 4.7. *Algorithm 1 terminates in finitely many steps and outputs a minimal SLB for the ideal* $\langle f_1,\dots,f_k\rangle$ *with respect to* $\prec$.

Proof. The finite termination of the algorithm is due to the fact that $\mathcal{P}$ is Nœtherian, and it is similar to that of Buchberger's algorithm. To prove the correctness, let us first highlight some features of this algorithm showing the differences between it and the existing algorithms for the computation of SLB's and Gröbner bases. In contrast to Algorithms 1 and 2 in [19], for each i, we initialize $\mathbb{F}_i$ to an empty set in lines 3 and 20 instead of setting it to $\{\mathrm{T}(1),\dots,\mathrm{T}(i-1)\}$. This adjustment allows us to keep $\mathrm{T}(i,j)/\mathrm{T}(j)$ in A in cases where $\mathrm{T}(i)$ and $\mathrm{T}(j)$ are co-prime, thus enabling the removal of more redundant critical pairs in lines 6 and 22. Moreover, if the critical pair (f_i,f_j) satisfies the conditions of Buchberger's first criterion, then we remove it in lines 11 and 27. Another distinction is that when $\mathrm{T}(j)$ divides $\mathrm{T}(i)$, we can not only eliminate the polynomial f_i from F, but also remove

Algorithm 1: MinSLB

Input: A finite set of polynomials $\{f_1, \dots, f_k\}$ and a term ordering $\prec$

Output: A minimal SLB for $\langle f_1, \dots, f_k \rangle$

1 **begin**

2 $F \longleftarrow \{f_1, \dots, f_k\}$ and assume that $\mathrm{lt}(f_1) \succ \cdots \succ \mathrm{lt}(f_k)$

3 $\mathbb{F}_i \longleftarrow \varnothing$ for any $i = 1, \dots, k$

4 $s \longleftarrow k$ and $P \longleftarrow \varnothing$

5 **for** j **from** 2 **to** s **do**

6 $A \longleftarrow \left\{ \mathrm{T}(i,j)/\mathrm{T}(j) \mid i = 1, \dots, j-1, \mathrm{T}(i,j)/\mathrm{T}(i) \notin \langle \mathbb{F}_i \rangle \right\}$

7 **for** $\mathrm{T}(i,j)/\mathrm{T}(j) \in \mathrm{Gen}(A)$ **do**

8 **if** $\mathrm{T}(j) \mid \mathrm{T}(i)$ **then**

9 $P \longleftarrow (P \setminus \{(f_\ell, f_m) \mid \ell, m \in \{1, \dots, s\}, i \in \{\ell, m\}\}) \cup \{(f_i, f_j)\}$

10 remove f_i from F, update $s, \mathbb{F}_{i+1}, \dots, \mathbb{F}_j$ and renumber $f_{i+1}, \dots, f_s$

11 **else if** $\mathrm{T}(i,j) = \mathrm{T}(i) \cdot \mathrm{T}(j)$ **then**

12 $\mathbb{F}_j \longleftarrow \mathbb{F}_j \cup \{\mathrm{T}(i,j)/\mathrm{T}(j)\}$

13 **else**

14 $\mathbb{F}_j \longleftarrow \mathbb{F}_j \cup \{\mathrm{T}(i,j)/\mathrm{T}(j)\}$

15 $P \longleftarrow P \cup \{(f_i, f_j)\}$

16 **while** $P \neq \varnothing$ **do**

17 Select and remove a pair (f_i, f_j) from P with minimal $\mathrm{T}(i,j)$

18 $r \longleftarrow \mathtt{Division}(\mathrm{Spoly}(f_i, f_j), F)$

19 **if** $r \neq 0$ **then**

20 $s \longleftarrow s + 1$ and $f_s \longleftarrow r$ and $\mathbb{F}_s \longleftarrow \varnothing$

21 $F \longleftarrow F \cup \{f_s\}$

22 $A \longleftarrow \left\{ \mathrm{T}(i,s)/\mathrm{T}(s) \mid i = 1, \dots, s-1, \mathrm{T}(i,s)/\mathrm{T}(i) \notin \langle \mathbb{F}_i \rangle \right\}$

23 **for** $\mathrm{T}(i,s)/\mathrm{T}(s) \in \mathrm{Gen}(A)$ **do**

24 **if** $\mathrm{T}(s) \mid \mathrm{T}(i)$ **then**

25 $P \longleftarrow (P \setminus \{(f_\ell, f_m) \mid \ell, m \in \{1, \dots, s\}, i \in \{\ell, m\}\}) \cup \{(f_i, f_s)\}$

26 remove f_i from F, update $s, \mathbb{F}_{i+1}, \dots, \mathbb{F}_s$ and renumber $f_{i+1}, \dots, f_s$

27 **else if** $\mathrm{T}(i,s) = \mathrm{T}(i) \cdot \mathrm{T}(s)$ **then**

28 $\mathbb{F}_s \longleftarrow \mathbb{F}_s \cup \{\mathrm{T}(i,s)/\mathrm{T}(s)\}$

29 **else**

30 $\mathbb{F}_s \longleftarrow \mathbb{F}_s \cup \{\mathrm{T}(i,s)/\mathrm{T}(s)\}$

31 $P \longleftarrow P \cup \{(f_i, f_s)\}$

32 **return** $\{(f_1/\mathrm{lc}(f_1), \mathbb{F}_1), \dots, (f_s/\mathrm{lc}(f_s), \mathbb{F}_s)\}$

all critical pairs from P where one component is f_i, except for (f_i, f_j), as described in lines 10 and 26. This simple improvement can also be implemented in the Update algorithm in [2, page 230].

The correctness of this algorithm is primarily derived from that of Algorithm 2 in [19]. However, we must prove that the mentioned modifications do not compromise the correctness of the algorithm. Suppose that in the first **for**-loop, we select a critical pair (f_i, f_j) associated with the element $\mathrm{T}(i,j)/\mathrm{T}(j) \in \mathrm{Gen}(A)$. Following Algorithm 2 in [19], we should add $\mathrm{T}(i,j)/\mathrm{T}(j)$ into $\mathbb{F}_j$. Note that in the case $\mathrm{T}(j) \mid \mathrm{T}(i)$, according to Proposition 4.3, we do not need to include it into $\mathbb{F}_j$. Furthermore, in this case, we remove f_i from F and all critical pairs involving f_i, except for (f_i, f_j). As per [11, Lemma 3, page 92], f_i is redundant in the final Gröbner basis and it can be removed. Since f_i is no longer in F then we can remove every critical pair containing f_i. To ensure the output basis generates the ideal formed by $f_1, \ldots, f_k$, we must add (f_i, f_j) into P. Given the assumption, we can deduce that $\mathrm{Spoly}(f_i, f_j) = f_i - \frac{\mathrm{T}(i,j)}{\mathrm{lm}(f_j)} \cdot f_j$. Since the algorithm terminates, this S-polynomial is reduced to zero. Consequently, f_i can be written as a combination of other elements in F, indicating that eliminating f_i from F does not cause any problem for F remaining a generating set of the initial ideal. This completes the proof. □

The described algorithm to compute a minimal staggered linear basis has not been implemented successfully in `CoCoA` yet. Attempts to rewrite the existing code of the implementation of *SLB* in Section 3 showed that many changes must be made to accurately reflect the presented pseudocode. This is largely due to the fact that in this new version, polynomials can also be removed from the set *F* which introduces an additional element of bookkeeping. At this stage, no accurate prediction can be made whether the algorithm to compute a minimal staggered linear basis will outperform *SLB*.

5. Applications

In this section, we explore various applications of SLB's related to a complement of an ideal.

5.1. Computing Hilbert Functions Using SLB's

In this subsection, we assume that we are dealing with a *homogeneous* ideal $\mathcal{I}$ and a *homogeneous* SLB $\mathbf{B} = \{(f_1, \mathbb{F}_1), \ldots, (f_s, \mathbb{F}_s)\}$ of it. We demonstrate how to utilize $\mathbf{B}$ to compute the Hilbert function, the Hilbert polynomial, and the Hilbert series of $\mathcal{I}$. We recall that the *volume* and the *Hilbert function*, respectively, of $\mathcal{I}$ are the numerical functions $\mathbb{N}_0 \to \mathbb{N}_0$ defined by

$$V_{\mathcal{I}}(q) = \dim_{\mathcal{K}}(\mathcal{I}_q), \qquad \mathrm{HF}_{\mathcal{I}}(q) = \dim_{\mathcal{K}}(\mathcal{P}_q/\mathcal{I}_q), \tag{1}$$

where $\mathcal{I}_q$ denotes the homogeneous component of degree q of $\mathcal{I}$ and correspondingly for the polynomial ring $\mathcal{P}$. Moreover, if X is a $\mathcal{K}$-vector space, $\dim_{\mathcal{K}}(X)$ refers to the dimension of X as a $\mathcal{K}$-vector space. Obviously,

$$V_{\mathcal{I}}(q) + \mathrm{HF}_{\mathcal{I}}(q) = \dim_{\mathcal{K}}(\mathcal{P}_q) = \binom{n+q-1}{q} \tag{2}$$

for any q and consequently, it suffices to know one of these functions. Since any SLB induces a combinatorial decomposition of the ideal $\mathcal{I}$ as a direct sum of $\mathcal{K}$-linear spaces, it is trivial to read off the volume function of $\mathcal{I}$ from an SLB.

Proposition 5.1. *Let $\{(f_1, \mathbb{F}_1), \ldots, (f_s, \mathbb{F}_s)\}$ be a homogeneous SLB of the homogeneous ideal $\mathcal{I}$. Then the volume function of $\mathcal{I}$ is given by*

$$V_{\mathcal{I}}(q) = \sum_{i=1}^{s} [q \geq \deg(f_i)] \mathrm{HF}_{\langle \mathbb{F}_i \rangle}(q - \deg(f_i)) . \tag{3}$$

Here $[A]$ denotes the Iverson bracket and equals 1, if the statement A is true, and 0 otherwise.

Proof. By the definition of an SLB, we have $V_{\mathcal{I}}(q) = \sum_{i=1}^{s} [q \geq \deg(f_i)] V_{\mathbb{A}_i}(q - \deg f_i)$. Since the order ideal $\mathbb{A}_i$ of allowed terms is the complement of $\langle \mathbb{F}_i \rangle$, its volume function coincides with the Hilbert function of the ideal $\langle \mathbb{F}_i \rangle$ of forbidden terms. □

From a certain degree on, $\mathrm{HF}_{\mathcal{I}}(q)$ is equal to a (unique) polynomial in q, called *Hilbert polynomial*, and denoted by $\mathrm{HP}_{\mathcal{I}}$. The *Hilbert regularity* of $\mathcal{I}$ is

$$\mathrm{hilb}(\mathcal{I}) = \min\{m \mid \forall q \geq m,\ \mathrm{HF}_{\mathcal{I}}(q) = \mathrm{HP}_{\mathcal{I}}(q)\} .$$

We have the identity $\dim(\mathcal{I}) = \deg(\mathrm{HP}_{\mathcal{I}}) + 1$, see [11, Theorem 12, page 494] and in addition from Macaulay's theorem we know that $\mathrm{HF}_{\mathcal{I}} = \mathrm{HF}_{\mathrm{lt}(\mathcal{I})}$. The *Hilbert series* of $\mathcal{I}$ is the power series $\mathrm{HS}_{\mathcal{I}}(t) = \sum_{s=0}^{\infty} \mathrm{HF}_{\mathcal{I}}(s)t^s$. This series can be expressed as the quotient $\mathrm{HS}_{\mathcal{I}}(t) = N(t)/(1-t)^D$ with a polynomial $N \in \mathbb{Q}[t]$ satisfying $N(1) \neq 0$ (see [12, Theorem 7, page 130] or [28]). It is well-known that $\mathrm{hilb}(\mathcal{I}) = \max\{0, \deg(N(t)) - D + 1\}$, see e. g. [4, Proposition 4.1.12]. Based on these definitions and applying the equalities (2) and (3), we can derive the following corollary.

Corollary 5.2. *Let* $\{(f_1, \mathbb{F}_1), \dots, (f_s, \mathbb{F}_s)\}$ *be a homogeneous SLB of the homogeneous ideal* $\mathcal{I}$. *Then the Hilbert polynomial of* $\mathcal{I}$ *can be expressed as*

$$\mathrm{HP}_{\mathcal{I}}(q) = \frac{(q+1)\cdots(q+n-1)}{(n-1)!} - \sum_{i=1}^{s} \mathrm{HP}_{\langle \mathbb{F}_i \rangle}(q - \deg(f_i)) \,. \tag{4}$$

Furthermore, $\max\{\mathrm{hilb}(\langle \mathbb{F}_1 \rangle) + \deg(f_1), \dots, \mathrm{hilb}(\langle \mathbb{F}_s \rangle) + \deg(f_s)\}$ *is an upper bound for* $\mathrm{hilb}(\mathcal{I})$.

Proof. The first claim follows from Proposition 5.1. To prove the other claim, we observe that for each $q \geq \max\{\mathrm{hilb}(\langle \mathbb{F}_i \rangle) + \deg(f_i)\}$, it holds that $\mathrm{HF}_{\langle \mathbb{F}_i \rangle}(q - \deg(f_i)) = \mathrm{HP}_{\langle \mathbb{F}_i \rangle}(q - \deg(f_i))$, which concludes the proof of the claim. □

Corollary 5.3. *Let* $\{(f_1, \mathbb{F}_1), \dots, (f_s, \mathbb{F}_s)\}$ *be a homogeneous SLB of the homogeneous ideal* $\mathcal{I}$. *Then the Hilbert series of* $\mathcal{I}$ *can be expressed as*

$$\mathrm{HS}_{\mathcal{I}}(t) = \frac{1}{(1-t)^n} - \sum_{i=1}^{s} t^{\deg(f_i)} \mathrm{HS}_{\langle \mathbb{F}_i \rangle}(t) \,. \tag{5}$$

Proof. The claim is deduced from Proposition 5.1 and the definition of a Hilbert series. Note that the Hilbert series of the ideal $\langle 0 \rangle$ is given by $1/(1-t)^n$. □

In the case of an irreducible SLB, we can easily calculate $\mathrm{HF}_{\mathcal{I}}, \mathrm{HP}_{\mathcal{I}}$ and $\mathrm{HS}_{\mathcal{I}}$ using these results. To accomplish this, we need to explain how to compute the Hilbert function, the Hilbert polynomial and the Hilbert series of an irreducible ideal. Specifically, without loss of generality, assume that $\mathbb{F}_i = \{x_1^{a_1}, \dots, x_\ell^{a_\ell}\} \subset \mathcal{P}$. We know that $x_1^{a_1}, \dots, x_\ell^{a_\ell}$ form a regular sequence, and consequently, by [12, Lemma 5, page 129], the Hilbert series of $\langle \mathbb{F}_i \rangle$ is given by

$$\mathrm{HS}_{\langle \mathbb{F}_i \rangle}(t) = \prod_{j=1}^{\ell} (1 - t^{a_i})/(1-t)^n \,. \tag{6}$$

Note that for each q the coefficient of t^q in this series gives $\mathrm{HF}_{\langle \mathbb{F}_i \rangle}(q)$. Moreover, $\mathrm{HP}_{\langle \mathbb{F}_i \rangle}$ is a polynomial of degree $m - 1$. Applying an ansatz method, we can compute the coefficients of this polynomial. Finally, we know that $\mathrm{hilb}(\langle \mathbb{F}_i \rangle) = \sum_{i=1}^{\ell}(a_i - 1) + 1$. Therefore, Corollary 5.2 gives an explicit upper bound for $\mathrm{hilb}(\mathcal{I})$.

Example 5.4. Let us consider the irreducible decomposition

$$\mathbf{B} = \{(x_3^4, \varnothing), (x_2 x_3^3, \{x_3\}), (x_2^3 x_3^2, \{x_3\}), (x_1 x_2 x_3^2, \{x_2^2, x_3\}), (x_1 x_2 x_3, \{x_3\}),$$
$$(x_1^3 x_3, \{x_2, x_3^3\}), (x_1^3 x_2^3, \{x_3\})\}$$

computed in Example 2.4. We will use the above results to compute $\mathrm{HP}_{\mathcal{I}}, \mathrm{HS}_{\mathcal{I}}$ and an upper bound for $\mathrm{hilb}(\mathcal{I})$. The Hilbert polynomials of $\langle \mathbb{F}_i \rangle$'s in the given order are respectively $(1/2)q^2 + (3/2)q + 1, q+1, q+1, 2, q+1, 3, q+1$. On the other hand, the Hilbert polynomial of the whole ring is $(1/2)q^2 + (3/2)q + 1$. Using Corollary 5.2, we get

$$\mathrm{HP}_{\mathcal{I}}(q) = (1/2)q^2 + (3/2)q + 1 - \big((1/2)(q-4)^2 + (3/2)q - 5 + q - 3 + q - 4 + 2 + q - 2 + 3 + q - 5\big)$$

which is equal to 7. Moreover, the values $\mathrm{hilb}(\langle \mathbb{F}_i \rangle)$ for $i = 1, \dots, 7$ are respectively $0, 0, 0, 2, 0, 3, 0$. Therefore $\mathrm{hilb}(\mathcal{I})$ is at most 7 and we conclude that the dimension of $\mathcal{I}$ is one. Finally, from (5), we get

$$\frac{1}{(1-t)^3} - \frac{t^4}{(1-t)^3} - \frac{t^4}{(1-t)^2} - \frac{t^5}{(1-t)^2} - \frac{t^4(1-t^2)}{(1-t)^2} - \frac{t^3}{(1-t)^2} - \frac{t^4(1-t^3)}{(1-t)^2} - \frac{t^6}{(1-t)^2}$$

which is equal to $(-t^6 - t^5 + 3t^3 + 3t^2 + 2t + 1)/(1-t)$ and consequently $\mathrm{hilb}(\mathcal{I}) = 6$.

We conclude this subsection by noting that we can present an alternative and effective method to compute the Hilbert function and the Hilbert polynomial of an irreducible monomial ideal. These computations are useful for determining the corresponding notions of an arbitrary monomial ideal (recall that by applying Lemmata 2.2 and 2.3 we are always able to compute an irreducible SLB for a monomial ideal). Let us briefly recall some basic definitions from the theory of involutive bases. Define the class of a term t, denoted by $\mathrm{cls}(t)$, as the smallest integer i such that $x_i \mid t$. The Pommaret division is defined as follows: Any variable x_i with $i \leq \mathrm{cls}(t)$ is Pommaret multiplicative for $t \neq 1$. For 1, all variables are multiplicative. Now, we say that a term t is a Pommaret divisor of another term u if $t \mid u$ and in addition u/t contains only Pommaret multiplicative variables for t. Now, similar to the definition of Gröbner bases, we can define Pommaret bases for polynomial ideals. For simplicity, we give the definition for monomial ideals: a finite subset P of terms is called a *Pommaret basis* for a monomial ideal $\mathcal{I}$ if for each term $u \in \mathcal{I}$ there exists $t \in P$ such that t is a Pommaret divisor of u. In general, a monomial ideal does not possess a finite Pommaret basis. Indeed, a monomial ideal $\mathcal{I}$ has a finite Pommaret basis if and only if it is quasi-stable. A monomial ideal $\mathcal{I} \subseteq \mathcal{P}$ is called *quasi-stable* if for any term $u \in \mathcal{I}$ and for any index $k := \mathrm{cls}(u) < i \leq n$, there exists $s > 0$ such that $u \cdot x_i^s / x_k \in \mathcal{I}$. It is easy to see that any zero-dimensional monomial ideal is quasi-stable. Moreover, if P is a generating set for a monomial ideal $\mathcal{I}$, we can complete P in a finite number of steps into a Pommaret basis for $\mathcal{I}$. For this, we shall multiply any element $u \in P$ by a non-multiplicative variable x_i, and add $x_i \cdot u$ to P. We shall repeat this process until every non-multiplicative prolongation of an element in P has a Pommaret divisor in P. Finally, if P is a Pommaret basis for $\mathcal{I}$, then the set $\{(u, \{x_{\mathrm{cls}(u)+1}, \dots, x_n\}) \mid u \in P\}$ is an SLB for $\mathcal{I}$. For more details on Pommaret bases and their construction, we refer to [26].

Now let $\mathcal{J}$ be an irreducible monomial ideal. For computing the Hilbert function and Hilbert polynomial of $\mathcal{J}$, we can renumber the variables so that $\mathcal{J} = \langle x_k^{a_k}, \dots, x_n^{a_n} \rangle \subset \mathcal{P}$ for some $1 \leq k \leq n$ where $a_i \geq 1$ for each $k \leq i \leq n$. With this assumption, $\mathcal{J}$ is quasi-stable and therefore possesses a finite Pommaret basis given by $\left\{ x_i^{a_i} x_{i+1}^{j_{i+1}} \cdots x_n^{j_n} \mid i = k, \dots, n,\ 0 \leq j_i < a_i \right\}$, see [26, Remark 3.1.17] for more details on this matter. Note that the set of Pommaret multiplicative variables for an element of class i is given by $\{x_1, \dots, x_i\}$. Thus, we can write

$$\mathrm{HF}_{\mathcal{J}}(q) = \binom{n+q-1}{q} - \sum_{i=k}^{n} \sum_{j_{i+1}=0}^{a_{i+1}-1} \cdots \sum_{j_n=0}^{a_n-1} [q \geq a_i + j_{i+1} + \cdots + j_n] \binom{i+q-(a_i+j_{i+1}+\cdots+j_n)-1}{i-1}.$$

If in this formula, we omit the Iverson bracket in the right-hand side, then we get the Hilbert polynomial of $\mathcal{J}$.

5.2. Irreducible Complementary Decompositions

Our aim in this subsection is to use the idea of SLB's to present a simple way to compute an irreducible complementary decomposition for a given ideal. We begin by defining a similar concept of SLB for the complementary decomposition of a monomial ideal.

Definition 5.5. A (generalized) *complementary decomposition* of a monomial ideal $\mathcal{I} \subset \mathcal{P}$ is a finite set of disjoint generalized cones $\mathbf{C} := \{(t_1, \mathbb{F}_1), \dots, (t_r, \mathbb{F}_r)\}$ such that $\mathbb{T} \setminus \mathcal{I} = \bigcup_{i=1}^{r} t_i \cdot (\mathbb{T} \setminus \langle \mathbb{F}_i \rangle)$.

In the case that $\mathcal{I}$ is not monomial, we can compute a Gröbner basis for $\mathcal{I}$, and a complementary decomposition for $\mathrm{lt}(\mathcal{I})$ provides a complementary decomposition for $\mathcal{I}$, see [11, Proposition 1, page 248].

We first note that there always trivially exists a complementary decomposition for any monomial ideal $\mathcal{I} = \langle t_1, \ldots, t_s \rangle \subset \mathcal{P}$, viz., $\{(1, \{t_1, \ldots, t_s\})\}$, i.e., the decomposition having only one cone with vertex 1 and $\mathcal{I}$ as forbidden terms. However, in practice we are interested in an irreducible complementary decomposition, see the next definition.

Definition 5.6. A complementary decomposition is said to be irreducible if each of its elements is an irreducible generalized cone.

Similar to Lemma 2.3, we are able to provide an irreducible complementary decomposition from the trivial complementary decomposition $\{(1, \{t_1, \ldots, t_s\})\}$. Based on this observation, we give the following algorithm for computing an irreducible complementary decomposition for a given monomial ideal.

Algorithm 2: Irreducible Complementary Decomposition

Input: Finite set of terms $t_1, \ldots, t_s$

Output: An irreducible complementary decomposition for the ideal generated by the t_i's

begin

  $\mathbf{C} \leftarrow \{(1, \{t_1, \ldots, t_s\})\}$

  while $\exists (t, \mathbb{F}) \in \mathbf{C}$ `s.t.` $\mathbb{F}$ `contains a term` $x_i^\ell \cdot v$ `with` $x_i \nmid v \neq 1$ **do**

    $\mathbf{C} \leftarrow \mathbf{C} \setminus \{(t, \mathbb{F})\}$

    $\mathbf{C} \leftarrow \mathbf{C} \cup \{(t, \mathrm{Gen}(\mathbb{F} \cup \{x_i^\ell\})), (t \cdot x_i^\ell, \mathrm{Gen}(\langle \mathbb{F} \rangle : x_i^\ell))\}$

  return C

Proposition 5.7. *Algorithm 2 is correct and terminates in finitely many steps.*

Proof. The finite termination of the algorithm is guaranteed by the fact that the number of mixed terms in $t_1, \ldots, t_s$ is finite, and at each step, we reduce at least one of the obstructions. Regarding the correctness of the algorithm, we note that once the **while**-loop in the algorithm terminates, there are no obstructions in $\mathbf{C}$, and consequently, it constitutes an irreducible decomposition. Furthermore, by Lemma 2.3, at each step of the algorithm, $\mathbf{C}$ remains a complementary decomposition for $\langle t_1, \ldots, t_s \rangle$, and this finishes the proof. □

Example 5.8. Consider the ideal $\mathcal{I}$ generated by $t_1 = x_3^4, t_2 = x_2 x_3^3, t_3 = x_2^3 x_3^2, t_4 = x_1 x_2 x_3, t_5 = x_1^3 x_3, t_6 = x_1^3 x_2^3$ in $\mathcal{K}[x_1, x_2, x_3]$, introduced in Example 2.4. By applying Algorithm 2, we compute an irreducible complementary decomposition for $\mathcal{I}$. At the beginning of the algorithm, we set $\mathbf{C} = \{(1, \{t_1, \ldots, t_6\})\}$. We select first t_5, and split $\mathbf{C}$ into two cones

$$(1, \{x_1^3 x_2^3, x_3\}), (x_3, \{x_3^3, x_2 x_3^2, x_2^3 x_3, x_1 x_2, x_1^3\}). \tag{7}$$

Next, choosing the term $x_1^3 x_2^3$ divides the first cone into two cones

$$(1, \{x_2^3, x_3\}), (x_2^3, \{x_1^3, x_3\}).$$

Now, in the second cone of (7), we consider the term $x_1 x_2$ and split it into the following cones:

$$(x_3, \{x_1^3, x_2, x_3^3\}), (x_2 x_3, \{x_1, x_2^2 x_3, x_3^2\}).$$

Finally, in the second cone of this last division, we take the term $x_2^2 x_3$, and further split it into the cones

$$(x_2 x_3, \{x_1, x_3\}), (x_2 x_3^2, \{x_1, x_2^2, x_3\}).$$

Thus, in total, we obtain the following irreducible complementary decomposition for $\mathcal{I}$:

$$\{(1,\{x_2^3,x_3\}),(x_2^3,\{x_1^3,x_3\}),(x_3,\{x_1^3,x_2,x_3^3\}),(x_2x_3,\{x_1,x_3\}),(x_2x_3^2,\{x_1,x_2^2,x_3\})\}.$$

Remark 5.9. Concerning the efficiency of Algorithm 2, we note that at each step, it is advantageous to select a term of the form $x_i^\ell \cdot v$ from a cone in $\mathbf{C}$ where ℓ is the lowest degree of x_i among all terms in that cone. Furthermore, it is beneficial to choose such a term in a way that adding x_i^ℓ to the cone removes multiple terms. This strategy helps streamline the decomposition process and enhances the overall efficiency of the algorithm.

Remark 5.10. It is possible to apply Lemma 2.3 to compute a complementary decomposition $\mathbf{C}$ so that for each $(t,\mathbb{F}) \in \mathbf{C}$, $\mathbb{F}$ contains only variables. To achieve this, if $u = x_i \cdot v \in \mathbb{F}_1$ then we can replace $(t_1,\mathbb{F}_1)$ in $\mathbf{C}$ by $\{(t_1,\mathrm{Gen}(\mathbb{F}_1 \cup \{x_i\})),(t_1 \cdot x_i,\mathrm{Gen}(\langle \mathbb{F}_1\rangle : x_i))\}$. This modification results in a *linear* complementary decomposition for $\mathcal{I}$. In a future paper, we will study the properties of this kind of decomposition.

Remark 5.11. We note that, given an irreducible complementary decomposition $\mathbf{C}$ for an ideal $\mathcal{I}$, we can easily compute certain homological invariants of the ideal; see, for example, subsection 5.1. Specifically, we observe that $\dim(\mathcal{I})$ is equal to $\max\{\dim(\mathbb{F}) \mid \exists t,(t,\mathbb{F}) \in \mathbf{C}\}$. Since each cone $(t,\mathbb{F})$ in $\mathbf{C}$ is irreducible, we can straightforwardly compute $\dim(\mathbb{F})$. For instance, in the ideal presented in Example 5.8, we find that the maximum dimension of the cones in the computed irreducible complementary decomposition of $\mathcal{I}$ is one, which leads us to conclude that $\dim(\mathcal{I}) = 1$.

5.3. Standard Pairs

Another kind of decomposition for the complement of a monomial ideal, albeit not a disjoint one, can be obtained via *standard pairs*. They can be used for deriving bounds for the arithmetic degree and the geometric degree of a homogeneous ideal [27, Section 3]. We will show how they are related to irreducible complementary decompositions. Consider pairs $(t,\mathbb{V}(t))$, where $t \in \mathbb{T}$ is a term and $\mathbb{V}(t) \subseteq \{x_1,\dots,x_n\}$ is a set of variables. Such a pair is called *admissible*, if $\deg_x t = 0$ for all $x \in \mathbb{V}(t)$. On the set of admissible pairs one defines a partial order: $(t,\mathbb{V}(t)) \le (u,\mathbb{V}(u))$ if and only if the restricted cone $\mathbb{T} \cap (u \cdot \mathcal{K}[\mathbb{V}(u)])$ is contained in $\mathbb{T} \cap (t \cdot \mathcal{K}[\mathbb{V}(t)])$. Obviously, this containment is equivalent to $t \mid u$ and any variable x such that either $\deg_x(u) > \deg_x(t)$ or $x \in \mathbb{V}(u)$ also lies in $\mathbb{V}(t)$.

Definition 5.12. Let $\mathcal{I}$ be an arbitrary monomial ideal. An admissible pair $(t,\mathbb{V}(t))$ is called *standard* for $\mathcal{I}$, if $t \cdot \mathcal{K}[\mathbb{V}(t)] \cap \mathcal{I} = \varnothing$ and $(t,\mathbb{V}(t))$ is minimal with respect to $\le$ among all admissible pairs with this property. We denote the set of all standard pairs of the ideal $\mathcal{I}$ by $\mathcal{S}_\mathcal{I}$.

From the set of standard pairs $\mathcal{S}_\mathcal{I}$, an irreducible primary decomposition of $\mathcal{I}$ can be obtained as follows.

Proposition 5.13 ([27, Lemmas 3.3 and 3.5])**.** *Let $\mathcal{I}$ be an arbitrary monomial ideal. Then the complementary set $\bar{\mathcal{I}} = \mathbb{T} \setminus \mathcal{I}$ can be written in the form*

$$\bar{\mathcal{I}} = \bigcup_{(t,\mathbb{V}(t)) \in \mathcal{S}_\mathcal{I}} (\mathbb{T} \cap (t \cdot \mathcal{K}[\mathbb{V}(t)])) \tag{8}$$

and $\mathcal{I}$ can be decomposed as

$$\mathcal{I} = \bigcap_{(t,\mathbb{V}(t)) \in \mathcal{S}_\mathcal{I}} \left\langle x^{\deg_x(t)+1} \mid x \notin \mathbb{V}(t) \right\rangle . \tag{9}$$

Moreover, the arithmetic degree of $\mathcal{I}$ equals the cardinality $|\mathcal{S}_\mathcal{I}|$, and the geometric degree is the number of standard pairs $(t,\mathbb{V}(t))$ such that $(1,\mathbb{V}(t))$ is also standard.

Remark 5.14 ([27, Theorem 2.3 and Proposition 4.1]). Let $\mathcal{I}$ be any homogeneous ideal and $\mathrm{lt}(\mathcal{I})$ its leading ideal with respect to any term order. Then the arithmetic and geometric degrees of $\mathrm{lt}(\mathcal{I})$ are upper and lower bounds, respectively, for the arithmetic and geometric degrees of $\mathcal{I}$.

In general, the primary decomposition (9) is highly redundant. Let Y be an arbitrary subset of variables and consider all standard pairs $(t, \mathbb{V}(t))$ with $\mathbb{V}(t) = Y$. Obviously, among these only the ones with terms t which are maximal with respect to divisibility are relevant for the decomposition (9) and in fact restricting to the corresponding ideals yields the irredundant irreducible decomposition of $\mathcal{I}$.

We now show that $\mathcal{S}_\mathcal{I}$ may be extracted from any irreducible complementary decomposition using the simple Algorithm 3. For a term $t \in \mathbb{T}$, write $\operatorname{var}(t)$ for the set of variables dividing it. For a generalized cone $(s, \mathbb{F}(s))$ of an irreducible complementary decomposition, write $\operatorname{var}(\mathbb{F}(s))$ for the set of allowed variables of the cone, i. e. the variables of which no pure power appears in the set of forbidden terms $\mathbb{F}(s)$.

Algorithm 3: Standard Pairs

Input: Finite irreducible complementary decomposition $\mathbf{C}$ of monomial ideal $\mathcal{I}$
Output: Set $\mathcal{S}_\mathcal{I}$ of standard pairs of $\mathcal{I}$

begin
- $\bar{\mathcal{S}}_\mathcal{I} \leftarrow \varnothing$
- **foreach** $(s, \mathbb{F}(s)) \in \mathbf{C}$ **do**
  - $\bar{\mathbb{V}}(s) \leftarrow \operatorname{var}(s) \cap \operatorname{var}(\mathbb{F}(s))$
  - **if** $\bar{\mathbb{V}}(s) = \varnothing$ **then**
    - $\bar{\mathcal{S}}_\mathcal{I} \leftarrow \bar{\mathcal{S}}_\mathcal{I} \cup \{(s \cdot u, \operatorname{var}(\mathbb{F}(s)) \mid \operatorname{var}(u) \cap \operatorname{var}(\mathbb{F}(s)) = \varnothing \wedge \nexists p \in \mathbb{F}(s) : p \mid u\}$
  - **else**
    - $\bar{s} \leftarrow s / \left(\prod_{x \in \bar{\mathbb{V}}(s)} x^{\deg_x(s)}\right)$
    - $\bar{\mathcal{S}}_\mathcal{I} \leftarrow \bar{\mathcal{S}}_\mathcal{I} \cup \{(\bar{s} \cdot u, \operatorname{var}(\mathbb{F}(s)) \mid \operatorname{var}(u) \cap \operatorname{var}(\mathbb{F}(s)) = \varnothing \wedge \nexists p \in \mathbb{F}(s) : p \mid u\}$
- **return** Set of minimal elements of $\bar{\mathcal{S}}_\mathcal{I}$ with respect to $\leq$

Proposition 5.15. *Let* $\mathbf{C}$ *be a finite irreducible complementary decomposition of the monomial ideal* $\mathcal{I}$*. Then Algorithm 3 terminates in a finite number of steps and computes with* $\mathbf{C}$ *as input the set* $\mathcal{S}_\mathcal{I}$ *of standard pairs of* $\mathcal{I}$*.*

Proof. The algorithm terminates because $\mathbf{C}$ is finite. It is easy to see that the set $\bar{\mathcal{S}}_\mathcal{I}$ computed by Algorithm 3 contains only admissible pairs whose cones are disjoint from $\mathcal{I}$ and cover $\mathbb{T} \setminus \mathcal{I}$, although they are not disjoint any more in general. Thus, it only remains to show that $\bar{\mathcal{S}}_\mathcal{I}$ contains $\mathcal{S}_\mathcal{I}$ as a subset.

Let $(t, \mathbb{V}(t))$ be an admissible pair such that $t \cdot \mathcal{K}[\mathbb{V}(t)] \cap \mathcal{I} = \varnothing$. Since the union of the cones associated to the elements of $\bar{\mathcal{S}}_\mathcal{I}$ by construction still covers $\mathbb{T} \setminus \mathcal{I}$, the finiteness of $\bar{\mathcal{S}}_\mathcal{I}$ implies the existence of a term $\hat{t} \in t \cdot \mathcal{K}[\mathbb{V}(t)]$ and a pair $(u, \mathbb{V}(u)) \in \bar{\mathcal{S}}_\mathcal{I}$ such that $\hat{t} \cdot \mathcal{K}[\mathbb{V}(t)] \subseteq u \cdot \mathcal{K}[\mathbb{V}(u)]$ (obviously, it is not possible to cover $t \cdot \mathcal{K}[\mathbb{V}(t)]$ with a finite number of lower-dimensional cones). As both $(\hat{t}, \mathbb{V}(t))$ and $(u, \mathbb{V}(u))$ are admissible pairs, this entails that in fact $(u, \mathbb{V}(u)) \leq (\hat{t}, \mathbb{V}(t))$. Hence, either $(\hat{t}, \mathbb{V}(t)) \in \bar{\mathcal{S}}_\mathcal{I}$ or it is not a standard pair. $\square$

Example 5.16. We continue Example 5.8. Recall the irreducible complementary decomposition $\mathbf{C}$ of $\mathbb{T} \setminus \mathcal{I}$ given by

$$\begin{array}{ll} (s_1, \mathbb{F}(s_1)) = (x_2x_3^2, \{x_1, x_2^2, x_3\}), & (s_2, \mathbb{F}(s_2)) = (x_2x_3, \{x_1, x_3\}), \\ (s_3, \mathbb{F}(s_3)) = (x_3, \{x_1^3, x_2, x_3^3\}), & (s_4, \mathbb{F}(s_4)) = (x_2^3, \{x_1^3, x_3\}), \\ (s_5, \mathbb{F}(s_5)) = (1, \{x_2^3, x_3\}). & \end{array}$$

We discuss in detail the steps which Algorithm 3 performs for $(s_1, \mathbb{F}(s_1))$ during the loop. We have $\operatorname{var}(s_1) = \{x_2, x_3\}$ and $\operatorname{var}(\mathbb{F}(s_1)) = \varnothing$. Thus $\bar{\mathbb{V}}(s_1) = \varnothing$, and we add all pairs $(s_1 \cdot u, \varnothing)$ to $\bar{\mathcal{S}}_\mathcal{I}$

where u is an allowed term for s_1 on the variables $\{x_1, x_2, x_3\} \smallsetminus \mathrm{var}(\mathbb{F}(s_1)) = \{x_1, x_2, x_3\}$, i. e., $u \in \{1, x_2\}$. Thus, the pairs $(x_2 x_3^2, \varnothing)$ and $(x_2^2 x_3^2, \varnothing)$ are added.

As regards the other pairs, Algorithm 3 adds during the loop:

- for $(s_2, \mathbb{F}(s_2))$ the pair $(x_3, \{x_2\})$;
- for $(s_3, \mathbb{F}(s_3))$ the nine pairs $(x_1^{\ell} x_3^{m+1}, \varnothing)$, where $1 \le \ell, m \le 2$;
- for $(s_4, \mathbb{F}(s_4))$ the three pairs $(1, \{x_2\})$, $(x_1, \{x_2\})$, and $(x_1^2, \{x_2\})$;
- for $(s_5, \mathbb{F}(s_5))$ the three pairs $(1, \{x_1\})$, $(x_2, \{x_1\})$, and $(x_2^2, \{x_1\})$.

Of the 18 generated admissible pairs, only $(x_3, \varnothing)$ is non-standard and removed.

Note that for each of the five input pairs, among the set of admissible pairs induced by them there is a pair with maximal vertex; these five maximal pairs are $(x_2^2 x_3^2, \varnothing)$, $(x_3, \{x_2\})$, $(x_1^2 x_3^3, \varnothing)$, $(x_1^2, \{x_2\})$, and $(x_2^2, \{x_1\})$. These pairs induce the irredundant irreducible decomposition

$$\mathcal{I} = \langle x_1, x_2^3, x_3^3 \rangle \cap \langle x_1, x_3^2 \rangle \cap \langle x_1^3, x_2, x_3^4 \rangle \cap \langle x_1^3, x_3 \rangle \cap \langle x_2^3, x_3 \rangle .$$

5.4. A Complementary Decomposition Test

Let $\mathbf{C} = \{(t_1, \mathbb{F}_1), \dots, (t_r, \mathbb{F}_r)\}$ be a finite set of pairwise disjoint generalized cones. In this subsection, we introduce the novel Algorithm 4 for testing whether $\mathbf{C}$ constitutes a complementary decomposition for a monomial ideal, in other words, whether the union $\mathcal{O} := \bigcup_{i=1}^{r} t_i \cdot \mathbb{A}_i$ of the generalized cones defines an order ideal. Moreover, in the case that $\mathcal{O}$ does not form an order ideal, in the next subsection, we describe Algorithm 5 to return a so-called closure of $\mathbf{C}$.

Recall that a subset $\mathcal{O} \subseteq \mathbb{T}$ is called an order ideal, if for each $t \in \mathcal{O}$ and each $s \mid t$, we have $s \in \mathcal{O}$, or equivalently if its complement $\overline{\mathcal{O}}$ is a monomial ideal. If $\mathcal{O}$ is an order ideal, then Algorithm 4 will return a generating set for the ideal $\overline{\mathcal{O}}$. Within this algorithm, we need the following observations. Assume that $(t_i, \mathbb{F}_i)$ defines a generalized cone and let s be a term in it. Obviously, the intersection $s \cdot \mathbb{T} \cap t_i \cdot \mathbb{A}_i$, i. e. the subset of the generalized cone consisting entirely of multiples of s, is again a generalized cone and one easily verifies that it can be represented by the pair $\big(s, \mathrm{Gen}(\langle \mathbb{F}_i \rangle : s/t_i)\big)$. In addition, to test whether a given generalized cone $(t_i, \mathbb{F}_i) \in \mathbf{C}$ and a term u satisfy $u \cdot \mathbb{T} \cap t_i \cdot \mathbb{A}_i \neq \varnothing$, we can check the membership condition $\mathrm{lcm}(t_i, u)/t_i \notin \langle \mathbb{F}_i \rangle$.

Proposition 5.17. *Algorithm 4 is correct and terminates in finitely many steps.*

Proof. We first consider the termination of the algorithm. The vertices of all the generalized cones ever contained in the set S define a directed graph. Its roots are the vertices in the original set $\mathbf{C}$; and the children of a generalized cone $(t, \mathbb{F})$ chosen in line 6 are the generalized cones added in line 9. Hence, there are at most $|\mathbb{F}|$ children, i. e. finitely many for each vertex in the graph. Furthermore, by construction, no path through the graph can visit the same generalized cone in the original set $\mathbf{C}$ twice. As all added generalized cones are subsets of those in $\mathbf{C}$, the length of any path in the graph is bounded by the cardinality of $\mathbf{C}$. By König's lemma [22, §49, Lemma 10], the graph and hence the set of all generalized cones ever contained in S must be finite. But this observation immediately implies the termination of the algorithm.

Concerning the correctness, we first note that if $1 \notin \mathcal{O}$, then $\mathcal{O}$ trivially cannot be an order ideal. The only way, we can have $1 \in \mathcal{O}$, is that 1 is the vertex of some generalized cone in $\mathbf{C}$. The **if**-clause in line 2 takes care of this condition. Hence we assume from now on that 1 is a vertex.

As second step of the proof of the correctness, we show that if $\mathcal{O}$ is not an order ideal, then the algorithm returns `false`. From above, we know that $1 \in \mathcal{O}$. Assuming that $\mathcal{O}$ is not an order ideal, there exist a term $t \in \mathcal{O}$ and a term u dividing t such that $u \notin \mathcal{O}$. Since $t \in \mathcal{O}$, there exists $(t_i, \mathbb{F}_i) \in \mathbf{C}$ such that $t \in t_i \cdot \mathbb{A}_i$. From $u \notin \mathcal{O}$ and the fact that 1 is a vertex, we conclude that there exists $(t_j, \mathbb{F}_j) \in \mathbf{C}$ with $t_j \mid u$. Let $u = v \cdot t_j$ where $v \in \langle \mathbb{F}_j \rangle$. By the structure of the algorithm, at some stage we choose $(t_j, \mathbb{F}_j)$ from S and study $v_1 \cdot t_j$ with $v_1 \in \mathbb{F}_j$ and $v_1 \mid v$. If there does not exist $(t_\ell, \mathbb{F}_\ell)$ such that $v_1 \cdot t_j \in t_\ell \cdot \mathbb{A}_\ell$, then the term $v_1 \cdot t_j$ is added to B and the test in line 12 yields the output `false` when applied to it since $v_1 \cdot t_j \mid t$. Otherwise, assume that $v_1 \cdot t_j \in t_\ell \cdot \mathbb{A}_\ell$ so that the

Algorithm 4: Test for order ideals

Input: Finite set of disjoint generalized cones $\mathbf{C} = \{(t_1, \mathbb{F}_1), \dots, (t_r, \mathbb{F}_r)\}$
Output: `false`, if $\mathcal{O} := \bigcup_{i=1}^{r} t_i \cdot \mathbb{A}_i$ is not an order ideal, and otherwise a generating set of $\overline{\mathcal{O}}$

1 **begin**
2   **if** $\forall i \colon t_i \neq 1$ **then**
3   **return** `false`
4   $B \leftarrow \varnothing$; $\quad S \leftarrow \mathbf{C}$
5   **while** $S \neq \varnothing$ **do**
6   choose $(t, \mathbb{F}) \in S$; $\quad S \leftarrow S \setminus \{(t, \mathbb{F})\}$
7   **foreach** $u \in \mathbb{F}$ **do**
8   **if** $\exists (t_i, \mathbb{F}_i) \in \mathbf{C}$ `such that` $u \cdot t \in t_i \cdot \mathbb{A}_i$ **then**
9         $S \leftarrow S \cup \{(u \cdot t, \mathrm{Gen}(\langle \mathbb{F}_i \rangle : u \cdot t / t_i))\}$
10   **else**
11         $B \leftarrow B \cup \{u \cdot t\}$
12   **if** $\exists (t_i, \mathbb{F}_i) \in \mathbf{C}$ `such that` $u \cdot t \cdot \mathbb{T} \cap t_i \cdot \mathbb{A}_i \neq \varnothing$ **then**
13   **return** `false`
14   **return** B

algorithm will add the generalized cone $\big(v_1 \cdot t_j, \mathrm{Gen}(\langle \mathbb{F}_\ell \rangle : v_1 \cdot t_j / t_\ell)\big)$ to S. Note that u cannot lie in this generalized cone and hence $v_1 \neq 1$ and consequently $\deg\big(u/(v_1 \cdot t_j)\big) < \deg\big(u/t_j\big)$. Therefore, we have found a vertex closer to u than t_j and we can iterate the argument. Since there are only finitely many such terms, we will encounter after finitely many steps a situation where the test in line 12 returns `false`.

If the algorithm does not return `false`, then we are sure that $\mathcal{O}$ is an order ideal. As the last step of the proof of the correctness of the algorithm, we prove that the returned set B is indeed a generating set of $\overline{\mathcal{O}}$. Consider an arbitrary term $t \in \overline{\mathcal{O}}$. As long as there exists a variable $x \in \mathrm{var}(t)$ (the set of variables appearing in t) such that also $t/x \in \overline{\mathcal{O}}$, we can discard t while searching for minimal generators of $\overline{\mathcal{O}}$. Therefore, it suffices to consider only terms $t \in \overline{\mathcal{O}}$ such that $t/x \in \mathcal{O}$ for all $x \in \mathrm{var}(t)$. We will now show that any such term t will be discovered by our algorithm and put into the set B.

We associate with t its "backwards neighbours", i. e. the terms t/y for some $y \in \mathrm{var}(t)$. If one of them is the vertex of a generalized cone in $\mathbf{C}$, then t will sooner or later be added to B in line 11 and we are done. Otherwise, consider an arbitrary backwards neighbour $t' = t/y$. It must be contained in some generalized cone $(t_i, \mathbb{F}_i) \in \mathbf{C}$. By construction, $t_i \mid t'$, but $t_i \neq t'$, and $t/t_i \in \langle \mathbb{F}_i \rangle$ (otherwise the generalized cone would also contain t). Hence, there exists a generator $u \in \mathbb{F}_i$ such that $s_1 := u \cdot t_i \mid t$.

Assume that $s_1 \notin \mathcal{O}$. Then, from the minimality assumption on t, we conclude that $s_1 = t$. By the structure of the algorithm, the cone $\big(s_1, \mathrm{Gen}(\langle \mathbb{F}_i \rangle : s_1/t_i)\big)$ is added into S. When we study this generalized cone, we have $s_1 \notin t_\ell \cdot \mathbb{A}_\ell$ for each ℓ. Thus, s_1 is added into B and we are done in this case. If $s_1 \in \mathcal{O}$, it lies in some generalized cone $(t_j, \mathbb{F}_j)$ for some j. It follows that $s_1 \notin t_j \cdot \langle \mathbb{F}_j \rangle$ and in turn $s_1/t_j \notin \langle \mathbb{F}_j \rangle$. On the other hand, in line 9, when we select $(t_i, \mathbb{F}_i) \in S$ and study $u \cdot t_i = s_1$ we add $\big(s_1, \mathrm{Gen}(\langle \mathbb{F}_j \rangle : s_1/t_j)\big)$ to S. We know that $s_2 = w \cdot s_1 \mid t$ for some $1 \neq w \in \mathrm{Gen}(\langle \mathbb{F}_j \rangle : s_1/t_j)$ and $\deg(s_2) > \deg(s_1)$. Since the algorithm terminates in finitely many steps, we find $s_\ell \notin \mathcal{O}$ for some ℓ, and the proof finishes. $\square$

We will illustrate the steps of this algorithm using the following simple example.

Example 5.18. Let us consider the complementary decomposition

$$\mathbf{C} = \left\{(1,\{x_2^3,x_3\}),(x_2^3,\{x_1^3,x_3\}),(x_3,\{x_1^3,x_2,x_3^3\}),(x_2x_3,\{x_1,x_3\}),(x_2x_3^2,\{x_1,x_2^2,x_3\})\right\}$$

presented in Example 5.8. Following the steps of the algorithm we will find the generators of an ideal $\mathcal{I}$ such $\mathbf{C}$ forms a decomposition for $\mathcal{P}/\mathcal{I}$. We first let $B := \{\}$ and $S := \mathbf{C}$. Now, we select and remove the element $(1,\{x_2^3,x_3\})$ from S. Now, we check the non-multiplicative prolongations of 1. We observe that x_2^3 lies in the cone $(x_2^3,\{x_1^3,x_3\})$ and the element $(x_2^3,\langle x_1^3,x_3\rangle : 1) = (x_2^3,\{x_1^3,x_3\})$ is already an element of S, so we do not add it into S. Now, let us consider the non-multiplicative prolongation x_3 and again no element is added into S. The next element $(x_2^3,\{x_1^3,x_3\})$ is selected and removed from S. We consider the non-multiplicative multiplication $x_1^3\cdot x_2^3$ has no multiplicative divisor in $\mathbf{C}$, so it is added to B and we have $B = \{x_1^3x_2^3\}$. Now, we consider $x_2^3\cdot x_3$. The element $(x_2x_3,\{x_1,x_3\}) \in \mathbf{C}$ divides it. So, the element $(x_2^3x_3,\langle x_1,x_3\rangle : x_2^2) = (x_2^3x_3,\{x_1,x_3\})$ is added to S. In the next step, we choose and omit $(x_3,\{x_1^3,x_2,x_3^3\})$ from S. We consider the prolongation $x_1^3{\cdot}x_3$, and it is added to B, so $B = \{x_1^3x_2^3, x_1^3x_3\}$. The next prolongation to treat is $x_2{\cdot}x_3$. It lies in the cone $(x_2x_3,\{x_1,x_3\}) \in \mathbf{C}$, and nothing is changed. The last prolongation to study is $x_3^3\cdot x_3$ which is added into B. Hence we have $B = \{x_1^3x_2^3, x_1^3x_3, x_3^4\}$. Now, the element $(x_2x_3,\{x_1,x_3\})$ is selected and removed from S. The prolongation $x_1\cdot x_2x_3$ is added to B and $B = \{x_1^3x_2^3, x_1^3x_3, x_3^4, x_1x_2x_3\}$. On the other hand, the prolongation $x_3{\cdot}x_2x_3$ belongs to the cone $(x_2x_3^2,\{x_1,x_2^2,x_3\}) \in \mathbf{C}$, however, nothing is added to S. Now, we select and eliminate the element $(x_2x_3^2,\{x_1,x_2^2,x_3\})$ from S. All the non-multiplicative prolongations are added to B, and we have

$$B = \{x_1^3x_2^3, x_1^3x_3, x_3^4, x_1x_2x_3^2, x_2^3x_3^2, x_2x_3^3\}.$$

The last element in S is $(x_3x_2^3,\{x_1,x_3\})$. Both non-multiplicative prolongations are added into B, and the algorithm terminates. If we remove redundant elements from B, then we have $B = \{x_3^4, x_2x_3^3, x_2^3x_3^2, x_1x_2x_3, x_1^3x_3, x_1^3x_2^3\}$.

5.5. Order Ideal Closure

In the presented form, Algorithm 4 simply returns `false`, if $\mathcal{O}$ is not an order ideal. In this subsection, we will extend Algorithm 4 so that it outputs the smallest order ideal containing $\mathcal{O}$ (one may call this the *order ideal closure*).

Proposition 5.19. *Algorithm 5 is correct and terminates in finitely many steps.*

Proof. The proof of the correctness of this algorithm is similar to that of Algorithm 4 given in Proposition 5.17. We note that in each branch, we do not meet twice the generalized cones in $\mathbf{C}$. As we showed in the proof of Proposition 5.17, if we add into S the cone of the non-multiplicative prolongation $u\cdot t$ for some cone $(t,\mathbb{F})$ in a branch, $(t,\mathbb{F})$ will no longer appear in that branch. Now, we must only show that in line 14, we do not create infinitely many new cones. Note that in line 12, when we add a term v into U, then $t\cdot v\cdot\mathbb{T}$ does not meet $t_i\cdot\mathbb{A}_i$ for each $(t_i,\mathbb{F}_i)\in\mathcal{O}$. This shows that the number of new cones added into $\mathcal{O}$ in line 14 is finite, thereby completing the proof of finiteness.

To address correctness, we utilize the fact that the algorithm terminates in finitely many steps and that the number of non-multiplicative prolongations we consider is finite. Furthermore, if we encounter any issues in lines 3 or 11, we resolve them by adding new cones.

Finally, we demonstrate that once a new cone is added, it resolves the existing problem without introducing any further issues. According to the structure of the algorithm in line 12, for any new cone $(t,\mathbb{F})$ added to $\mathcal{O}$ and for any other cone $(t_i,\mathbb{F}_i)\in\mathcal{O}$, we have $u\cdot t\cdot\mathbb{T}\cap t_i\cdot\mathbb{A}_i = \varnothing$ for any $u\in\mathbb{F}$. It remains only to show the converse, i.e. $u\cdot t_i\cdot\mathbb{T}\cap t\cdot\mathbb{A} = \varnothing$ for any $u\in\mathbb{F}_i$ where $\mathbb{A}$ represents the set of multiplicative terms of $(t,\mathbb{F})$. However, this claim is true by line 13, thus concluding the proof. □

We summarize the steps of the above algorithm using the following simple example.

Algorithm 5: Order ideal closure

Input: Finite set of disjoint generalized cones $\mathbf{C} = \{(t_1, \mathbb{F}_1), \dots, (t_r, \mathbb{F}_r)\}$

Output: Closure $\mathcal{O}$ of $\mathbf{C}$ and a generating set for the ideal $\overline{\mathcal{O}}$ corresponding to $\mathcal{O}$

1 **begin**
2   $B \leftarrow \emptyset$; $S \leftarrow \mathbf{C}$; $\mathcal{O} \leftarrow \mathbf{C}$
3   **if** $\forall i : t_i \neq 1$ **then**
4     $S \leftarrow S \cup \{(1, \mathrm{Gen}(\langle t_1, \dots, t_r \rangle))\}$
5     $\mathcal{O} \leftarrow \mathcal{O} \cup \{(1, \mathrm{Gen}(\langle t_1, \dots, t_r \rangle))\}$
6   **while** $S \neq \emptyset$ **do**
7   choose $(t, \mathbb{F}) \in S$; $S \leftarrow S \setminus \{(t, \mathbb{F})\}$
8   **foreach** $u \in \mathbb{F}$ **do**
9   **if** $\exists (t_i, \mathbb{F}_i) \in \mathcal{O}$ `such that` $u \cdot t \in t_i \cdot \mathbb{A}_i$ **then**
10         $S \leftarrow S \cup \{(u \cdot t, \mathrm{Gen}(\langle \mathbb{F}_i \rangle : u \cdot t / t_i))\}$
11   **else if** $\exists (t_i, \mathbb{F}_i) \in \mathcal{O}$ `such that` $u \cdot t \cdot \mathbb{T} \cap t_i \cdot \mathbb{A}_i \neq \emptyset$ **then**
12         $U \leftarrow \{\mathrm{lcm}(u \cdot t, t_j)/(u \cdot t) \mid (t_j, \mathbb{F}_j) \in \mathcal{O}, \mathrm{lcm}(u \cdot t, t_j) \in t_j \cdot \mathbb{A}_j\}$
13         $U \leftarrow U \cup \{\mathrm{lcm}(u \cdot t, u_j \cdot t_j)/(u \cdot t) \mid (t_j, \mathbb{F}_j) \in \mathcal{O}, u_j \in \mathbb{F}_j\}$
14         $\mathcal{O} \leftarrow \mathcal{O} \cup \{(u \cdot t, \mathrm{Gen}(\langle U \rangle))\}$
15         $S \leftarrow S \cup \{(u \cdot t, \mathrm{Gen}(\langle U \rangle))\}$
16   **else**
17         $B \leftarrow B \cup \{u \cdot t\}$
18   **return** $(\mathcal{O}, B)$

Example 5.20. Let us consider the decomposition $\mathbf{C}$ given in Example 5.8, from which we will remove the generalized cone $(x_2x_3, \{x_1, x_3\})$. At one step of the algorithm, we select and remove the element $(x_3, \{x_1^3, x_2, x_3^3\})$ from S. Now, we study the non-multiplicative prolongations of x_2 and observe that x_2x_3 does not lie in the other cones. In lines 12 and 13, the set U is constructed whose minimal generating set is $\{x_3, x_2^2, x_1^3\}$. Consequently, only the new cone $(x_2x_3, \{x_3, x_2^2, x_1^3\})$ is added into S and $\mathcal{O}$. Finally, the algorithm then terminates with the minimal generating set $\{x_3^4, x_2x_3^3, x_3x_2^3, x_1x_2x_3^2, x_1^3x_3, x_1^3x_2^3\}$ for $\overline{\mathcal{O}}$. It is important to note that the newly found cone is distinct from the removed one, and consequently, the generating set obtained for $\overline{\mathcal{O}}$ by this algorithm differs from that in Example 5.18.

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