Algebraic and Geometric Analysis of Singularities of Implicit Differential Equations (Invited Talk)

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Abstract. We review our recent works on singularities of implicit ordinary or partial differential equations. This includes firstly the development of a general framework combining algebraic and geometric methods for dealing with general systems of ordinary or partial differential equations and for defining the type of singularities considered here. We also present an algorithm for detecting all singularities of an algebraic differential equation over the complex numbers. We then discuss the adaptions required for the analysis over the real numbers. We further outline for a class of singular initial value problems for a second-order ordinary differential equation how geometric methods allow us to determine the local solution behaviour in the neighbourhood of a singularity including the regularity of the solution. Finally, we show for some simple cases of algebraic singularities how there such an analysis can be performed.

Keywords: Implicit differential equations · Algebraic differential equations · Singularities · Vessiot spaces · Regularity decomposition · Singular initial value problems

1 Introduction

Many different forms of "singular" behaviour appear in the context of differential equations and many different views have been developed for them. Most of them are related to singularities of individual solutions of a given differential equation like blow-ups or shocks, i. e. either a solution component or some derivative of it becomes infinite. Other interpretations are concerned with bifurcations, with multivalued solutions or with singular integrals. In dynamical systems theory, many authors call stationary points (or equilibria) singularities.

We will identify a (system of) differential equations with a geometric object and singularities are points on it which are "different" from the generic points. "Different" means e. g. that the dimensions of certain geometric structures jump. Therefore, from all the "singularities" mentioned above, stationary points are closest to our singularities. In fact, in the case of ordinary differential equations, we will analyse the local solution behaviour by constructing a dynamical system for which the singularity is a stationary point.

In this article, we will given an overview over some recent results of ours; for all details and in particular proofs, we must refer to the original works [22,34,35].

In [22], a *general framework* for dealing with singularities of *arbitrary systems* of ordinary or partial differential equations was developed by combining methods from differential algebra, algebraic geometry and differential topology. Conceptually, it follows the classical geometric approach to define singularities that dates back at least to Clebsch and Poincaré (see [27] for a review and historical perspective) and extends it also to partial differential equations and to situations where one has no longer a manifold but a variety. Because of the use of algebraic methods, the theory could be made fully algorithmic and was implemented in Maple. We will present these results in the Sections 2–4.

A central algebraic method used, the Thomas decomposition, assumes that the underlying field is algebraically closed. For our use of the differential Thomas decomposition the base field is largely irrelevant and we can continue to use it also for *real differential equations*. The actual identification of singularities is done via an algebraic Thomas decomposition and – as shown for some concrete examples in [35] – its application to real equations is problematic. As the key step for the detection of singularities is the analysis of a linear system of equations over an algebraic set, we can replace it in the real case by a parametric Gaussian elimination followed by a quantifier elimination. Simultaneously, this allows us to extend to semialgebraic equations, i. e. to systems comprising not only equations and inequations, but also more general inequalities like positivity constraints. These considerations from [35] are the topic of Section 5.

Once we have identified a singularity, we want to analyse the local solution behaviour. This cannot be done at the same level of generality as the detection. In Section 6, we study *geometric singularities* of real *ordinary* differential equations using methods from dynamical systems theory. Then we restrict further to quasilinear equations and show in Section 7 – following [34] – how, for a specific class of scalar second-order singular initial value problems, non-trivial existence, (non)uniqueness and regularity results can be obtained.

While the analysis of geometric singularities is a classical topic, *algebraic singularities* have been essentially ignored in the context of differential equations. One of the few exceptions is the work by Falkensteiner and Sendra [12] where the theory of plane algebraic curves is used to analyse first-order scalar autonomous ordinary differential equations. We will show in Section 8 how our geometric approach allows us to analyse certain simple situations with ad hoc methods.

2 Differential Systems & Algebraic Differential Equations

In this section we introduce most of the algebraic and geometric techniques used in this article. For lack of space, we cannot provide a completely self-contained introduction. For any unexplained terminology on the (differential) algebraic side we refer to [30] and on the geometric side to [32].

We begin with the algebraic point of view. Consider the polynomial ring $\mathcal{P} = \mathbb{C}[x_1, \dots, x_n]$ with the ranking defined by $x_i < x_j$ for $i < j$. The largest variable appearing in a polynomial $p \in \mathcal{P}$ is called its *leader* $\operatorname{ld} p$. Considering p as a univariate polynomial in this variable, the *initial* $\operatorname{init} p$ is defined as leading coefficient and the *separant* $\operatorname{sep} p$ as the derivative $\partial p/\partial \operatorname{ld} p$. An *algebraic system* S is a finite set of polynomial equations and inequations

$$S = \{\ p_1 = 0,\ \dots,\ p_s = 0,\ q_1 \neq 0,\ \dots,\ q_t \neq 0\ \} \tag{A}$$

with polynomials $p_i, q_j \in \mathcal{P}$ and $s, t \in \mathbb{N}_0$. Its *solution set* $\operatorname{Sol} S = \{a \in \mathbb{C}^n \mid p_i(a) = 0,\ q_j(a) \neq 0\ \forall\, i, j\}$ is a locally Zariski closed set, namely the difference of the two varieties $\operatorname{Sol}(\{p_1 = 0, \dots, p_s = 0\})$ and $\operatorname{Sol}(\{q_1 = 0, \dots, q_t = 0\})$. The algebraic system (A) is *simple*, if (i) it is *triangular*, (ii) it has *non-vanishing initials*, i. e. for each $r \in \{p_1, \dots, p_s, q_1, \dots, q_t\}$, the equation $\operatorname{init} r = 0$ has no solution in $\operatorname{Sol} S$ and (iii) it is *square-free*, i. e. for each $r \in \{p_1, \dots, p_s, q_1, \dots, q_t\}$, the equation $\operatorname{sep} r = 0$ has no solution in $\operatorname{Sol} S$. Simple systems behave "better" in many respects than general systems. One can show that for a simple system S the saturated ideal

$$\mathcal{I}_{\mathrm{alg}}(S) := \langle p_1, \dots, p_s\rangle : q^\infty \subset \mathcal{P} \qquad \text{where } q = \operatorname{init} p_1 \cdots \operatorname{init} p_s \tag{1}$$

is the vanishing ideal of the Zariski closure of $\operatorname{Sol} S$ [30, Prop. 2.2.7].

A *Thomas decomposition* of (A) consists of finitely many simple algebraic systems $S_1, \dots, S_k$ such that $\operatorname{Sol} S$ is the *disjoint* union of $\operatorname{Sol} S_1, \dots, \operatorname{Sol} S_k$. Any algebraic system admits a Thomas decomposition (which is not unique). This decomposition was introduced by Thomas [39,40] in the context of differential algebra. It follows the general philosophy of treating algebraic or differential systems via triangular sets (see [16,17] for a survey). A special feature of it is its disjointness. It had been largely forgotten, until it was revived by Gerdt [13]; a modern presentation can also be found in [30]. Concrete implementations have been provided in [2,3,14] and some more theoretical applications in [21,25].

In the differential case, we consider the ring of *differential polynomials* $K\{U\}$ where $K = \mathbb{C}(x_1, \dots, x_n)$, $U = \{u_1, \dots, u_m\}$ are finitely many differential indeterminates and where we take the partial derivatives $\delta_i = \partial/\partial x_i$ as derivations. Given some differential polynomials $p_1, \dots, p_s \in K\{U\}$, we must distinguish between the algebraic ideal $\langle p_1, \dots, p_s\rangle$ and the differential ideal $\langle p_1, \dots, p_s\rangle_\Delta$ generated by them. The latter one contains in addition all differential consequences $\delta^\mu p$ of any element p of it. We also introduce the subring $\mathcal{D} \subset K\{U\}$ of those differential polynomials where also the coefficients are polynomials in the variables x^i. For any $\ell \in \mathbb{N}_0$, we define the finitely generated subalgebra

$$\mathcal{D}_\ell = \mathbb{C}\big[\,x^i,\ u^\alpha_\mu \mid 1 \leq i \leq n,\ 1 \leq \alpha \leq m,\ |\mu| \leq \ell\,\big]$$

which may be considered as the coordinate ring of a jet bundle (see below).

We choose on $K\{U\}$ an orderly Riquier ranking $<$. The notion of leader, initial and separant can be extended straightforwardly. A *differential system* S is a finite set of differential polynomial equations and inequations

$$S = \{\ p_1 = 0,\ \dots,\ p_s = 0,\ q_1 \neq 0,\ \dots,\ q_t \neq 0\ \} \tag{D}$$

with $p_i, q_j \in \mathcal{D}$ and $s, t \in \mathbb{N}_0$. As *solution set* $\operatorname{Sol} S$, we take for simplicity all formal power series solutions. The differential system (D) is *simple*, if (i) it is simple as an algebraic system in the finitely many jet variables u^α_μ which actually occur in S ordered according to $<$, (ii) its equation part forms a passive system in the sense of Janet–Riquier theory for the Janet division and (iii) no leader of an inequation q_j is an (iterated) derivative of the leader of an equation p_k.

A *Thomas decomposition* of the differential system (D) consists of finitely many simple differential systems $S_1, \dots, S_k$ such that $\operatorname{Sol} S$ is the disjoint union of the solution sets $\operatorname{Sol} S_1, \dots, \operatorname{Sol} S_k$. Any differential system admits such a decomposition which can be computed algorithmically by interweaving algebraic Thomas decompositions and the Janet–Riquier theory.

The key tool in the geometric theory of differential equations (see [32] and references therein) are *jet bundles*. For K being either $\mathbb{R}$ or $\mathbb{C}$, we set $\mathcal{X} = K^n$, $\mathcal{U} = K^m$ and consider maps $\phi : \mathcal{X} \to \mathcal{U}$ which in the real case are assumed to be smooth and in the complex case to be holomorphic and which need be defined only on some open subset of $\mathcal{X}$. The coordinates $\mathbf{x} = (x^1, \dots, x^n)$ on $\mathcal{X}$ are the independent variables and the coordinates $\mathbf{u} = (u^1, \dots, u^m)$ on $\mathcal{U}$ represent the dependent variables or unknown functions. The ℓth order jet bundle $J_\ell(\mathcal{X},\mathcal{U})$ consist of all Taylor polynomials of degree ℓ of such maps ϕ. Coordinates on $J_\ell(\mathcal{X},\mathcal{U})$ are therefore $(\mathbf{x}, \mathbf{u}^{(\ell)})$ where $\mathbf{x}$ gives the expansion point and the jet variables $\mathbf{u}^{(\ell)}$ represent the Taylor coefficients up to order ℓ which we may identify with the corresponding derivatives of ϕ at the point $\mathbf{x}$. For the components of $\mathbf{u}^{(\ell)}$ we use the usual multi-index notation $u^\alpha_\mu = \partial^{|\mu|}\phi^\alpha / \partial \mathbf{x}^\mu$ with $1 \leq \alpha \leq m$ and $\mu \in \mathbb{N}_0^n$ satisfying $0 \leq |\mu| \leq \ell$. Hence we find that $J_\ell(\mathcal{X},\mathcal{U}) \cong K^{d_\ell}$ with $d_\ell = n + m\binom{m+\ell}{\ell}$. For our purposes, two topologies on $J_\ell(\mathcal{X},\mathcal{U})$ are relevant: one is induced by the Euclidean metric on K^{d_ℓ} and the other one is the Zariski topology on K^{d_ℓ} with varieties as closed sets. Finally, we introduce the canonical projection maps $\pi^\ell_k : J_\ell(\mathcal{X},\mathcal{U}) \to J_k(\mathcal{X},\mathcal{U})$ with $\pi^\ell_k(\mathbf{x}, \mathbf{u}^{(\ell)}) = (\mathbf{x}, \mathbf{u}^{(k)})$ for $\ell > k \geq 0$ and $\pi^\ell : J_\ell(\mathcal{X},\mathcal{U}) \to \mathcal{X}$ with $\pi^\ell(\mathbf{x}, \mathbf{u}^{(\ell)}) = \mathbf{x}$.

Definition 1. *An* algebraic jet set *of order ℓ is a locally Zariski closed subset $\mathcal{J}_\ell \subseteq J_\ell(\mathcal{X},\mathcal{U})$ (i. e. $\mathcal{J}_\ell$ is the difference of two varieties in $J_\ell(\mathcal{X},\mathcal{U})$). Such a set $\mathcal{J}_\ell$ is an* algebraic differential equation *of order ℓ, if in addition the Euclidean closure of $\pi^\ell(\mathcal{J}_\ell)$ equals $\mathcal{X}$. An algebraic jet set or an algebraic differential equation is called* irreducible*, if it is an irreducible locally Zariski closed subset.*

We define here differential equations as a geometric object and do not distinguish between scalar equations and systems. An algebraic jet set is obtained by considering the solution set of an algebraic system on $J_\ell(\mathcal{X},\mathcal{U})$. The additional condition for an algebraic differential equation ensures that the independent variables are indeed independent. It excludes equations like $x^1 + x^2 = 0$ which obviously is not a differential equation. Admitted is an equation like $xu' = 1$ where $x = 0$ is not contained in the projection but in its closure. We use here the Euclidean closure, as we would like to be able to express each point outside of the set $\pi^\ell(\mathcal{J}_\ell)$ as the limit of a sequence of points inside.

Any map $\phi : \mathcal{X} \to \mathcal{U}$ induces a map $j_0\phi : \mathcal{X} \to J_0(\mathcal{X},\mathcal{U}) = \mathcal{X} \times \mathcal{U}$ defined by $j_0\phi(\mathbf{x}) = \big(\mathbf{x}, \phi(\mathbf{x})\big)$. The graph Γ_ϕ of ϕ is the image of $j_0\phi$. For any order $\ell > 0$, we may consider the *prolongations* $j_\ell\phi : \mathcal{X} \to J_\ell(\mathcal{X},\mathcal{U})$ given by

$j_\ell\phi(\mathbf{x}) = \big(\mathbf{x}, \phi(\mathbf{x}), \partial_{\mathbf{x}}\phi(\mathbf{x}), \dots, \partial_{\mathbf{x}}^{\ell}\phi(\mathbf{x})\big)$ where $\partial_{\mathbf{x}}^{k}\phi(\mathbf{x})$ represents all derivatives of ϕ of order k. The next definition reformulates geometrically the classical one.

Definition 2. *A* (classical) solution *of the algebraic differential equation* $\mathcal{J}_\ell \subseteq J_\ell(\mathcal{X}, \mathcal{U})$ *is a map* $\phi : \mathcal{X} \to \mathcal{U}$ *such that its prolongation satisfies* $\operatorname{im} j_\ell\phi \subseteq \mathcal{J}_\ell$*.*

Example 3. Let us consider the ordinary differential equation $u' = xu^2$ from this geometric view point. It represents a typical differential equation with singular solutions, as an elementary integration yields the general solution $\phi_c(x) = 2/(2c - x^2)$ parametrised by an arbitrary constant $c \in \mathbb{R}$.

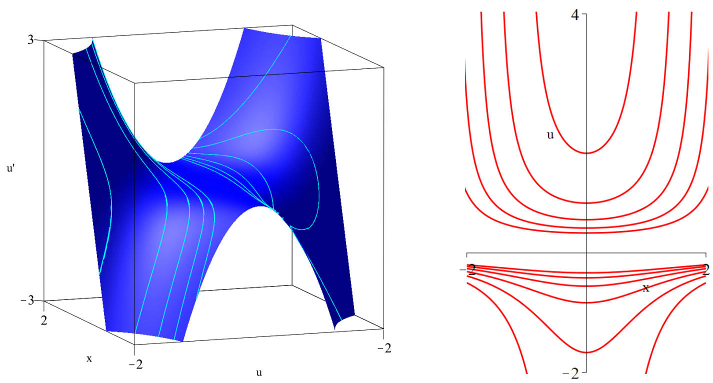

Fig. 1. Ordinary differential equation with singular solutions. Left: prolonged solutions in $J_1(\mathbb{R}, \mathbb{R})$. Right: classical solution graphs in x-u plane.

The left half of Fig. 1 shows in dark blue the corresponding algebraic differential equation $\mathcal{J}_1 \subset J_1(\mathbb{R}, \mathbb{R})$ and in light blue for some values of c the prolonged solutions $j_1\phi_c(x)$. In the right half, traditional graphs of these solutions show clearly the poles for positive values of c. We will see later that this differential equation has *no* singularities in the sense relevant for this article.

If one tries to combine the algebraic and the geometric point of view, one has to note some fundamental differences between the two. A differential ideal automatically contains all differential consequences of its generators. By choosing a jet bundle of a certain order ℓ, we immediately restrict to equations of order at most ℓ. On the other hand, geometric notions like the singularities we will consider in the next section cannot even be formulated in differential algebra. Thus moving from one point of view to the other one requires some care, as otherwise information is lost.

In applications, one usually starts with a differential system S like (D). The following approach appears to be very natural to associate with it for any given order $\ell \in \mathbb{N}_0$ an algebraic jet set in $J_\ell(\mathcal{X}, \mathcal{U})$. We take the *differential* ideal

$$\hat{\mathcal{I}}_{\text{diff}}(S) = \langle p_1, \dots, p_s \rangle_\Delta \subseteq \mathcal{D}$$

generated by the equations in S. It induces the *algebraic* ideal

$$\hat{\mathcal{I}}_\ell(S) = \hat{\mathcal{I}}_{\mathrm{diff}}(S) \cap \mathcal{D}_\ell \subseteq \mathcal{D}_\ell$$

as the corresponding finite-dimensional truncation. It automatically contains all hidden integrability conditions up to order ℓ. The inequations in S are also used to define an *algebraic* ideal: $\mathcal{K}_\ell(S) = \langle \hat{Q}_\ell \rangle_{\mathcal{D}_\ell}$ with $\hat{Q}_\ell = \prod_{\mathrm{ord}\,(q_j) \leq \ell} q_j$. We then define the algebraic jet set

$$\hat{\mathcal{J}}_\ell(S) = \mathrm{Sol}\left(\hat{\mathcal{I}}_\ell(S)\right) \setminus \mathrm{Sol}\left(\mathcal{K}_\ell(S)\right) \subseteq J_\ell(\mathcal{X},\mathcal{U})$$

consisting of all points of $J_\ell(\mathcal{X},\mathcal{U})$ satisfying both the equations and the inequations in S interpreted as algebraic (in)equations in $J_\ell(\mathcal{X},\mathcal{U})$.

However, this procedure leads to many problems. The ideals $\hat{\mathcal{I}}_\ell(S)$ are often too small (not radical) and the algebraic jet sets $\hat{\mathcal{J}}_\ell(S)$ are not necessarily algebraic differential equations. Furthermore, the effective determination of $\hat{\mathcal{I}}_\ell(S)$ is difficult. Finally, the sets $\hat{\mathcal{J}}_\ell(S)$ are possibly too small, as an algebraic interpretation of inequations is much stronger than a differential one. Differentially, the inequation $u' \neq 0$ simply excludes the zero function; algebraically, it excludes all points with a vanishing u'-coordinate and thus e. g. all critical points of solutions. A more extensive discussion of these problems can be found in [22].

The situation improves, if one assumes that S is a *simple* differential system. Taking – following the reasoning behind (1) – the saturated differential ideal

$$\mathcal{I}_{\mathrm{diff}} = \hat{\mathcal{I}}_{\mathrm{diff}}(S) : \Big(\prod_{j=1}^{s} \mathrm{init}\,(p_j)\,\mathrm{sep}\,(p_j)\Big) \tag{2}$$

instead of $\hat{\mathcal{I}}_{\mathrm{diff}}(S)$ and then using the same procedure as above to define algebraic ideals $\mathcal{I}_\ell(S)$ and algebraic jet sets $\mathcal{J}_\ell(S)$, one can show that these ideals are automatically radical and that explicit generators of the algebraic ideals $\mathcal{I}_\ell(S)$ are easily computable (see [22] for details).

The saturation in (2) leads to a Zariski closure. The inequations of a simple differential system exclude all points where an initial or separant vanishes and thus most of the singularities studied later. The saturation restores some of them and we only exclude irreducible components completely consisting of such points.

Example 4. Consider the system $\hat{S}$ consisting of the two partial differential equations $p_1 = uu_x - yu - y^2$ and $p_2 = yu_yu$ which is *not* simple. A differential Thomas decomposition yields only one simple system S obtained by augmenting $\hat{S}$ by the inequation $q = \mathrm{sep}\,p_1 = u$. The algebraic ideal $\hat{\mathcal{I}}_1(S)$ obtained by truncating the differential ideal $\langle p_1, p_2 \rangle_\Delta$ has the prime decomposition $\hat{\mathcal{I}}_1(S) = \langle p_2, p_3 \rangle \cap \langle u, y \rangle$ where $p_3 = u_xu_y - u - y$ implying that also the differential ideal is not prime. Saturating with respect to $Q = yu$ removes the prime component $\langle u, y \rangle$ and we find that $\mathcal{I}_{\mathrm{diff}}(S) = \langle p_2, p_3 \rangle_\Delta$ and $\mathcal{I}_1(S) = \langle p_2, p_3 \rangle \subset \mathcal{D}_1$.

Definition 5. *An algebraic differential equation $\mathcal{J}_\ell \subset J_\ell(\mathcal{X},\mathcal{U})$ is* locally integrable*, if there exists a Zariski open and dense subset $\mathcal{R}_\ell \subseteq \mathcal{J}_\ell$ such that $\mathcal{J}_\ell$ possesses for each point $\rho \in \mathcal{R}_\ell$ at least one solution ϕ with $\rho \in \mathrm{im}\, j_\ell\phi$.*

Local integrability is for many purposes an important concept. If a point $\rho \in \mathcal{J}_\ell$ is "far away" from any (prolonged) solution, then one can argue how relevant such a point is. In fact, the existence of such points is a clear indication that $\mathcal{J}_\ell$ has not been well chosen. A typical problem are overlooked hidden integrability conditions. As any simple differential system is passive, we obtain via the existence theorem of Riquier the following result for our construction.

Proposition 6 ([22, Prop. 3.6, Lemma 3.7]). *Let S be a simple differential system. Then the Zariski closure $\overline{\mathcal{J}_\ell(S)} = \operatorname{Sol}\bigl(\mathcal{I}_\ell(S)\bigr)$ is a locally integrable algebraic differential equation.*

3 Singularities of Algebraic Differential Equations

In the affine space K^{d_ℓ} all coordinates are equal. In the jet bundle $J_\ell(\mathcal{X},\mathcal{U})$ we distinguish different types like independent and dependent variables or derivatives. The *contact distribution* $\mathcal{C}_\ell \subset TJ_\ell(\mathcal{X},\mathcal{U})$ encodes these different roles and is generated by the vector fields

$$C_i^{(\ell)} = \partial_{x^i} + \sum_{\alpha=1}^{m} \sum_{0\leq|\mu|<\ell} u^\alpha_{\mu+1_i}\partial_{u^\alpha_\mu}\,, \qquad 1 \leq i \leq n\,, \tag{3}$$

$$C^\mu_\alpha = \partial_{u^\alpha_\mu}\,, \qquad 1 \leq \alpha \leq m\,,\ |\mu| = \ell \tag{4}$$

where $\mu + 1_i$ is obtained by increasing the ith entry of μ by one.

Definition 7. *Let $\mathcal{J}_\ell \subseteq J_\ell(\mathcal{X},\mathcal{U})$ be an algebraic jet set. The* Vessiot cone *$\mathcal{V}_\rho[\mathcal{J}_\ell]$ at $\rho \in \mathcal{J}_\ell$ is the intersection of the tangent cone $C_\rho\mathcal{J}_\ell$ with the contact space $\mathcal{C}_\ell|_\rho$.*

At a smooth point, the tangent cone $C_\rho\mathcal{J}_\ell$ and the tangent space $T_\rho\mathcal{J}_\ell$ coincide and thus the Vessiot cone becomes a *Vessiot space*, i. e. a K-linear space, which can be computed by linear algebra. Since the Vessiot spaces are contained in the contact distribution, we make for any vector $\mathbf{v} \in \mathcal{V}_\rho[\mathcal{J}_\ell]$ the ansatz

$$\mathbf{v} = \sum_i a^i C_i^{(\ell)}|_\rho + \sum_{|\mu|=\ell}\sum_\alpha b^\alpha_\mu C^\mu_\alpha|_\rho \tag{5}$$

with yet to be determined coefficients a^i, $b^\alpha_\mu \in K$. Let the jet set $\mathcal{J}_\ell$ be given as the solution set of an algebraic system on $J_\ell(\mathcal{X},\mathcal{U})$ with equations $p_\tau = 0$. At a smooth point ρ, $\mathbf{v}$ is tangential to $\mathcal{J}_\ell$, if and only if $dp_\tau|_\rho(\mathbf{v}) = 0$ for all τ leading to a homogeneous linear system for the coefficient vectors $\mathbf{a}$, $\mathbf{b}$,

$$D(\rho)\mathbf{a} + M_\ell(\rho)\mathbf{b} = 0\,, \tag{6}$$

where the entries of the matrices D, M_ℓ are given by $D_{i\tau}(\rho) = C_i^{(\ell)}(p_\tau)(\rho)$ and $(M_\ell)^\mu_{\alpha\tau}(\rho) = C^\mu_\alpha(p_\tau)(\rho)$. The rank of (6) and thus the dimension of $\mathcal{V}_\rho[\mathcal{J}_\ell]$ may vary over $\mathcal{J}_\ell$. Considered as functions of ρ, the solutions of (6) are smooth outside of a Zariski closed set and – by potentially enlarging this set – we may even assume that the dimension remains constant, since dimension is an upper semicontinuous function. Thus on a Zariski open and dense set we obtain a smooth regular distribution.

The projection $\pi^\ell_{\ell-1} : J_\ell(\mathcal{X},\mathcal{U}) \to J_{\ell-1}(\mathcal{X},\mathcal{U})$ induces at any point $\rho \in J_\ell(\mathcal{X},\mathcal{U})$ the vertical space $V_\rho\pi^\ell_{\ell-1} = \ker T_\rho\pi^\ell_{\ell-1}$ spanned by the vectors $C^\mu_\alpha|_\rho$. The vertical part of the Vessiot cone at a point $\rho \in \mathcal{J}_\ell$ is the *symbol cone* $\mathcal{N}_\rho[\mathcal{J}_\ell] = \mathcal{V}_\rho[\mathcal{J}_\ell] \cap V_\rho\pi^\ell_{\ell-1}$. At smooth points, we will speak of the *symbol space*. Again, on a Zariski open subset of $\mathcal{J}_\ell$ the symbol spaces $\mathcal{N}_\rho[\mathcal{J}_\ell]$ define a smooth regular distribution $\mathcal{N}[\mathcal{J}_\ell]$.

At a smooth point $\rho \in \mathcal{J}_\ell$, the symbol space $\mathcal{N}_\rho[\mathcal{J}_\ell]$ consists of those solutions of (6) where all coefficients $\mathbf{a}$ vanish: it is the kernel of the *symbol matrix* $M_\ell(\rho)$. Hence, we can write the Vessiot space as a direct sum $\mathcal{V}_\rho[\mathcal{J}_\ell] = \mathcal{N}_\rho[\mathcal{J}_\ell] \oplus \mathcal{H}_\rho$ with some π^ℓ-transversal complement $\mathcal{H}_\rho$ which is not uniquely determined. $\mathcal{J}_\ell$ is a *differential equation of finite type*, if on a Zariski open and dense subset the symbol cones vanish. For such equations, we expect that generically to every point $\rho \in \mathcal{J}_\ell$ there exists a unique solution ϕ with $\rho \in \operatorname{im} j_q\phi$, i. e. we may consider ρ as initial data for an initial value problem.

Remark 8. Computing the Vessiot space via (6) can be seen as a "projective" version of prolongation. Indeed, the formal derivative with respect to x^i of a differential equation $p_\tau = 0$ of order ℓ is

$$D_i p_\tau = C^{(\ell)}_i(p_\tau) + \sum_{\alpha=1}^{m} \sum_{|\mu|<\ell} C^\mu_\alpha(p_\tau) u^\alpha_{\mu+1_i} \,. \tag{7}$$

For an *ordinary* differential equation of finite type $\mathbf{a} = a$ is scalar and we have one coefficient b^α for each unknown function u^α. If $(a, \mathbf{b})$ is a solution of (6), then the unique solution ϕ with $\rho \in \operatorname{im} j_q\phi$ satisfies $\phi^{(\ell+1)}(x_0) = \mathbf{b}/a$ where $x_0 = \pi^\ell(\rho)$, i. e. the Vessiot space contains information about the derivatives in the next order. Obviously, if $a = 0$, then $\phi^{(\ell+1)}$ blows up, as x approaches x_0.

We will denote the family of Vessiot cones by $\mathcal{V}[\mathcal{J}_q]$ and call it briefly the *Vessiot distribution* of $\mathcal{J}_\ell$, although strictly speaking we obtain a distribution only on a subset of $\mathcal{J}_q$. But the considerations above justify this slight abuse of language. The Vessiot distribution can be interpreted as a kind of "infinitesimal solution space" of $\mathcal{J}_\ell$: if ϕ is any solution of $\mathcal{J}_\ell$ and ρ lies on $\operatorname{im} j_\ell\phi$, then the tangent space $T_\rho \operatorname{im} j_\ell\phi$ lies in the Vessiot cone $\mathcal{V}_\rho[\mathcal{J}_\ell]$.

Definition 9. *A* generalised solution *of the algebraic differential equation $\mathcal{J}_\ell \subseteq J_\ell(\mathcal{X},\mathcal{U})$ with $\dim\mathcal{X} = n$ is an n-dimensional submanifold $\mathcal{N} \subseteq \mathcal{J}_\ell$ such that $T_\rho\mathcal{N} \subseteq \mathcal{V}_\rho[\mathcal{J}_\ell]$ at every point $\rho \in \mathcal{N}$. The projection $\pi^\ell_0(\mathcal{N}) \subset J_0(\mathcal{X},\mathcal{U})$ of a generalised solution is called a* geometric solution.

If ϕ is a classical solution of $\mathcal{J}_\ell$, then $\operatorname{im} j_\ell\phi$ is a generalised solution and the graph $\Gamma_\phi = \operatorname{im} j_0\phi$ of ϕ the corresponding geometric solution. Furthermore, at any point $\rho \in \operatorname{im} j_\ell\phi$ we find that $\mathcal{V}_\rho[\mathcal{J}_\ell] = \mathcal{N}_\ell[\mathcal{J}_\rho] \oplus T_\rho \operatorname{im} j_\ell\phi$. As we will see later, an algebraic differential equation may possess further generalised solutions.

Definition 10 ([22, Def. 4.1]). *Let $\mathcal{J}_\ell \subseteq J_\ell(\mathcal{X},\mathcal{U})$ be a locally integrable algebraic differential equation and* $\dim \mathcal{X} = n$. *A point $\rho \in \mathcal{J}_\ell$ is an* algebraic singularity *of $\mathcal{J}_\ell$, if ρ is a non-smooth point of $\mathcal{J}_\ell$ in the sense of algebraic geometry. A smooth point $\rho \in \mathcal{J}_\ell$ is called*

(i) regular*, if ρ possesses a Euclidean open neighbourhood $\mathcal{U} \subseteq \mathcal{J}_\ell$ such that the Vessiot cones form on $\mathcal{U}$ a regular distribution which is decomposable as $\mathcal{V}[\mathcal{J}_\ell]|_\mathcal{U} = \mathcal{N}[\mathcal{J}_\ell]|_\mathcal{U} \oplus \mathcal{H}$ with an n-dimensional, transversal, involutive, smooth distribution $\mathcal{H} \subseteq T\mathcal{U}$;*

(ii) regular singular*, if ρ possesses a Euclidean open neighbourhood $\mathcal{U} \subseteq \mathcal{J}_\ell$ such that the Vessiot cones form on $\mathcal{U}$ a regular distribution but where* $\dim \mathcal{V}_\rho[\mathcal{J}_\ell] - \dim \mathcal{N}_\rho[\mathcal{J}_\ell] < n$*;*

(iii) irregular singular*, if there does not exist a Euclidean open neighbourhood $\mathcal{U} \subseteq \mathcal{J}_\ell$ such $\mathcal{V}[\mathcal{J}_\ell]|_\mathcal{U}$ is a regular distribution, i. e. any such neighbourhood contains a point $\bar{\rho}$ such that* $\dim \mathcal{V}_\rho[\mathcal{J}_\ell] > \dim \mathcal{V}_{\bar{\rho}}[\mathcal{J}_\ell]$.

An irregular singularity ρ is purely irregular*, if* $\dim \mathcal{V}_\rho[\mathcal{J}_\ell] - \dim \mathcal{N}_\rho[\mathcal{J}_\ell] = n$. *Regular and irregular singular points are also called* geometric singularities.

Algebraic singularities are not considered in the differential topological theory and one finds there much simpler definitions (see e. g. [1] or [27]), as only *ordinary* differential equations are considered where it is not necessary to consider neighbourhoods. One knows in advance the "right" dimension of the Vessiot spaces and can thus compare pointwise with this value. For *partial* differential equations, this is generally no longer the case and Definition 10 represents to our knowledge the first definition of geometric singularities for general systems of partial differential equations (the much simpler intermediate case of partial differential equations of finite type was already considered in [18]).

Remark 11. The three cases distinguished in Definition 10 for smooth points correspond essentially to an analysis of the linear system (6). At an irregular singular point, its rank does not take the maximal possible value attained in the other two cases. At a regular point, the symbol matrix alone is already of this rank. Thus geometric singularities are characterised by a rank drop of the symbol matrix. At non-singular points of an *ordinary* differential equations, the complement $\mathcal{H}$ is always one-dimensional and thus trivially involutive. In this case (or more generally for any locally integrable differential equation of finite type), the taxonomy of Definition 10 is complete. For *partial* differential equations, it is still an open question whether points can exist on $\mathcal{J}_\ell$ which satisfy all conditions for a regular point except the involutivity of $\mathcal{H}$ (see [22] for a more extensive discussion of this topic).

Example 12. We consider the algebraic differential equation $\mathcal{J}_2 \subset J_2(\mathbb{C}^2, \mathbb{C})$ for one unknown function u in two independent variables x, y defined by:

$$x^2 u_{xx} + x u_x + (x-1)^2 u = 0\,, \qquad (1-y^2) u_{yy} + 2y u_y + 2u = 0\,.$$

Seven cases arise in the analysis of the linear system (6) for the Vessiot spaces:

1. *Regular points* on $\mathcal{J}_2$ are characterised by the conditions $x \neq 0$ and $y^2 - 1 \neq 0$. They have a three-dimensional Vessiot space.

2. Points where $x = 0$, $y^2 - 1 \neq 0$ and either $u_x \neq 0$ or $u_y \neq 0$ are *regular singular*. They also possess a three-dimensional Vessiot space. As the coefficients a_1 and a_2 in (5) must satisfy the equation $2u_x a_1 + u_y a_2 = 0$, only a one-dimensional transversal complement exists.
3. Basically the same holds for points where $y^2 - 1 = 0$, $x \neq 0$ and either $yu_x + u_{xy} \neq 0$ or $u \neq 0$: they are *regular singular* and have a three-dimensional Vessiot space with a one-dimensional transversal complement defined by the equation $(yu_x + u_{xy})a_1 - 2ua_2 = 0$.
4. Points where $x = 0$, $y^2 - 1 = 0$ and either $u_x \neq 0$ or $yu_{xy} + u_x \neq 0$ are *irregular singularities* which are not purely irregular: the Vessiot space is four-dimensional with a one-dimensional transversal complement defined by the condition $a_1 = 0$.
5. Points where $x = 0$, $u_x = 0$, $u_y = 0$ and $y^2 - 1 \neq 0$ are *purely irregular singular* and possess a four-dimensional Vessiot space defined by the equation $(y^2 - 1)b_{02} - 2yu_{xy}a_1 = 0$.
6. The same behaviour is shown by points with $y^2 - 1 = 0$, $u = 0$, $u_y = 0$, $x \neq 0$, but with the Vessiot space defined by the equation $x^2 b_{20} + (x^2 - xy - 2x - 1)u_x a_1 = 0$.
7. Finally, the points where $x = 0$, $y^2 - 1 = 0$, $u_x y = 0$ and $u = 0$ are also *purely irregular singular* but now with a five-dimensional Vessiot space.

Note that the cases 2, 3 and 4 do not correspond to an algebraic jet set but the union of two such sets, because of the disjunctions in their defining conditions. Hence, if one applies the algorithm we will present in the next section to this example, then one obtains actually $10 = 7 + 3$ cases.

Any definition of a "singularity" is only meaningful, if generic points are regular. For equations of finite type, this is obvious and not even discussed in the literature. However, for general partial differential equations, such a statement becomes highly non-trivial and its proof requires major results from the geometric theory of differential equations. The key issue is to prove the involutivity of the complement $\mathcal{H}$ over a Zariski open and dense subset.

Theorem 13 ([22, Thm. 4.7]). *Let S be a simple differential system which contains no equation of an order greater than $\ell \in \mathbb{N}$ and $\mathcal{J}_\ell(S)$ the associated algebraic differential equation. Then the regular points in the Zariski closure $\overline{\mathcal{J}_\ell(S)}$ contain a Zariski open and dense subset.*

4 Regularity Decompositions

(Geometric) singularities are points where the dimensions of some geometric structures like symbol or Vessiot spaces jump. An algebraic jet set $\mathcal{J}_\ell \subseteq J_\ell(\mathcal{X}, \mathcal{U})$ is *regular*, if it consists only of smooth points and both its Vessiot distribution $\mathcal{V}[\mathcal{J}_\ell]$ and its symbol $\mathcal{N}[\mathcal{J}_\ell]$ define smooth vector bundles over $\mathcal{J}_\ell$. The solution space of the linear system (6) behaves uniformly over a regular algebraic jet set and thus all points on such a set are classified identically by Definition 10.

Definition 14. *Let $S \subset \mathcal{D}$ be a simple differential system and $\overline{\mathcal{J}_\ell(S)} \subset J_\ell(\mathcal{X},\mathcal{U})$ the associated algebraic jet set in a sufficiently high order ℓ. Let furthermore $\overline{\mathcal{J}_\ell(S)} = \mathcal{J}_{\ell,1} \cup \cdots \cup \mathcal{J}_{\ell,t}$ be its decomposition into irreducible varieties. A* regularity decomposition *of the variety $\mathcal{J}_{\ell,k}$ represents it as a disjoint union of finitely many regular algebraic jet sets $\mathcal{J}^{(1)}_{\ell,k}, \dots, \mathcal{J}^{(r)}_{\ell,k}$, the* regularity components *of $\mathcal{J}_{\ell,k}$, and of the set* $\operatorname{ASing}\bigl(\overline{\mathcal{J}_\ell(S)}\bigr)$ *of algebraic singularities.*

A constructive proof of the existence of regularity decompositions for any simple differential system is provided by Algorithm 1 below. Regularity decompositions are not unique and thus this algorithm simply returns one possible decomposition. The first two lines represent an algebraic preprocessing. In Line 2, the algebraic ideal $\mathcal{I}_\ell(S)$ is constructed explicitly via Janet–Riquier and Gröbner theory – for details see [22, Rem. 3.8]. The determination of a prime decomposition in Line 3 is a standard task in commutative algebra. Then the algorithm loops over each prime component. In Line 5, two simultaneous linear systems are set up over each prime component, i. e. we consider the combined system

$$\left\{\begin{aligned} \mathbf{J}(p_{k,j}) &= 0\,, \\ \mathbf{v}(p_{k,j}) &= 0\,, \\ p_{k,j} &= 0\,, \end{aligned}\right\} \qquad j = 1, \dots, s_k\,. \tag{8}$$

Here the polynomials $p_{k,j}$ form a basis of the kth prime ideal $\mathcal{I}_{\ell,k}(S)$. The two linear systems are obtained by applying two vector fields to these generators. The first one, $\mathbf{J} = \sum_\mu \sum_\alpha c^\alpha_\mu \partial_{u^\alpha_\mu} + \sum_i d^i \partial_{x^i}$ represents a general tangent vector in the jet bundle with yet undetermined coefficients $\mathbf{c}$ and $\mathbf{d}$. The first linear system in (8) encodes the condition that $\mathbf{J}$ is tangential to the kth prime component. By the Jacobian criterion, a jump in its rank characterises algebraic singularities. The second linear system is constructed with a general contact vector (5) and thus represents (6) for determining the Vessiot spaces. Changes in its behaviour indicate geometric singularities.

The undetermined coefficients $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, $\mathbf{d}$ represent the unknowns of the linear systems and we consider the left hand sides of the equations as elements of $\mathcal{D}^{\mathrm{ex}}_\ell = \mathcal{D}_\ell[\mathbf{a},\mathbf{b},\mathbf{c},\mathbf{d}]$. As changes in the behaviour of the linear systems indicate singularities, these can be detected by an algebraic Thomas decomposition of (8) for a suitably chosen ordering. More precisely, we must have: (i) $\mathbf{d} > \mathbf{c} > \mathbf{b} > \mathbf{a} > \mathbf{u} > \mathbf{x}$, (ii) restricted to the jet variables $\mathbf{u}$ it must correspond to an orderly ranking and (iii) the variables c^α_μ and b^α_μ are ordered among themselves in the same way as the derivatives u^α_μ.

Given a simple algebraic system $S^{\mathrm{ex}}_{k,\ell}$ in the obtained decomposition, the subsystem $S_{k,\ell}$ obtained by eliminating all equations and inequations containing some of the auxiliary variables $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, $\mathbf{d}$ describes a regular jet set. In practise, one is nevertheless strongly interested in getting the extended systems $S^{\mathrm{ex}}_{k,\ell}$, as the appearing leaders allow us to deduce the dimensions of the Vessiot and symbol spaces and thus to classify automatically the points on the jet set [22, Prop. 5.10]. The proof of the correctness of Algorithm 1 requires a number of rather technical issues and cannot be discussed here – see [22, Thm. 5.13].

Algorithm 1: Regularity Decomposition for a Simple Differential System

Input: a simple differential system S over the ring $K\{U\}$ of differential polynomials and a sufficiently high order $\ell \in \mathbb{N}$

Output: a regularity decomposition for each prime component $\mathcal{I}_{\ell,k}(S)$ of the algebraic ideal $\mathcal{I}_\ell(S) \subset \mathcal{D}_\ell$

begin

  compute a generating set $\{p_1, \dots, p_s\}$ of the radical ideal $\mathcal{I}_\ell(S)$

  compute a prime decomposition $\mathcal{I}_\ell(S) = \mathcal{I}_{\ell,1}(S) \cap \ldots \cap \mathcal{I}_{\ell,t}(S)$ of $\mathcal{I}_\ell(S)$ and a generating set $\{p_{k,1}, \dots, p_{k,s_k}\}$ for each prime component $\mathcal{I}_{\ell,k}(S)$

  for $k = 1$ **to** t **do**

    compute an algebraic Thomas decomposition $S^{\mathrm{ex}}_{k,1}, \dots, S^{\mathrm{ex}}_{k,r_k}$ of the algebraic system defined over $\mathcal{D}^{\mathrm{ex}}_\ell$ for an ordering as described above

  return *the systems* $S_{k,i}$ *consisting of those equations* $p = 0$ *and inequations* $q \neq 0$ *in* $S^{\mathrm{ex}}_{k,i}$ *with* $p \in \mathcal{D}_\ell$ *and* $q \in \mathcal{D}_\ell$

Example 15. The *hyperbolic gather* is one of the elementary catastrophes. Interpreted as a first-order ordinary differential equation, it is given by $\mathcal{J}_1 = \{(u')^3 + uu' - x = 0\}$. A real picture of it is contained in Figure 2 presented in Example 19 below. Despite its simplicity, it well illustrates some of the problems appearing in the practical use of Algorithm 1. Using the implementation of the Thomas decomposition described in [2], our algorithm returns a regularity decomposition with seven components all consisting of smooth points. One component contains the two irregular singularities, namely the points $(2, -3, -1)$ and $(-2, -3, 1)$ shown in Figure 2 in red. The regular singularities fill three components. Two of them correspond to the fold line shown in Figure 2 in white which arises as the common zero set of our equation and its separant. The “tip” of the fold line is put in a separate component. The third component contains only complex points and is thus not visible in Figure 2. Finally, there are three components with regular points. A closer analysis of the Vessiot spaces at these points (presented in [22, Ex. 7.2]) reveals that they can be combined into a single regularity component; the splitting into three separate components is solely an artefact of the Thomas decomposition due to its internal use of projections along each coordinate axis.

It represents a general problem of Algorithm 1 that it performs implicitly a Thomas decomposition of the considered irreducible varieties. Some singularities indeed arise from the geometry of these varieties: it was no coincidence that in Example 15 all singularities lie on the fold line. But the Thomas decomposition automatically also puts all points on the differential equation lying under or over the fold line in separate components, although this is generally unnecessary for the singularity analysis. In systems with several unknown functions $\mathbf{u}$ (and corresponding derivatives), this effect can be much more pronounced and its size depends generally on the ordering of the entries of $\mathbf{u}$ (and the induced effect on the ordering of their derivatives $\mathbf{u}^{(\ell)}$), although this ordering is irrelevant for the analysis of the differential equation.

5 Semialgebraic Differential Equations

So far, we have exclusively considered the case of complex differential equations, as the Thomas decomposition assumes that the underlying field is algebraically closed. As in applications real equations dominate, we discuss now following [35] an extension of the ideas of the last sections to differential equations over the real numbers. We will not provide a complete solution to this problem. The approach presented above consists essentially of three phases. In the first phase, we perform a differential Thomas decomposition. Here the underlying field plays only a minor role, as a crucial point is the completion to a passive system.

In the second phase, we construct from an obtained simple differential system a suitable algebraic differential equation. This step involves some problematic operations. Firstly, we perform a saturation which provides us with a radical ideal. Over the real numbers, we should actually strive for the real radical according to the real nullstellensatz (see e. g. [6, Sect. 4.1] for a discussion). An algorithm for determining the real radical was proposed by Becker and Neuhaus [5,24]; an implementation over the rational numbers exists in Singular [37]. Secondly, we need irreducible varieties and hence a prime decomposition. Since computing such a decomposition is related to factorisation, it strongly depends on the underlying field. Again, effective methods exist only over the rational numbers. Thus we conclude that for arbitrary simple differential systems the second phase cannot be done completely algorithmically.

It is unclear whether all steps of the second phase are really necessary. For example, for determining the tangent space at a smooth point, one does not need the real radical. Many problems in practise automatically lead to prime ideals. We will in the sequel assume that we are able to perform all required computations by saying that we are dealing with a *well-prepared system* and concentrate on a real variant of the third phase.

In the third phase, we solve two linear systems over a locally Zariski closed set. In the previous section, we simply threw all (in)equations together and computed an algebraic Thomas decomposition. Now we will present an alternative approach making stronger use of the "staggered" structure of the problem and the partial linearity. Furthermore, we also extend the class of differential equations considered. An algebraic differential equation was essentially defined as a locally Zariski closed sets, i. e. it was described by equations $p_i = 0$ and inequations $q_j \neq 0$. Over the real numbers, it is desirable to include also inequalities $q_j \diamond 0$ where $\diamond$ stands for some relation in $\{<, >, \leq, \geq, \neq\}$. Thus we replace the condition "locally Zariski closed" by "semialgebraic" (see e. g. [6, Chap. 2]).

Definition 16. *A* semialgebraic jet set *of order ℓ is a semialgebraic subset $\mathcal{J}_\ell \subseteq J_\ell(\mathcal{X},\mathcal{U})$. Such a set $\mathcal{J}_\ell$ is a* semialgebraic differential equation *of order ℓ, if in addition the Euclidean closure of $\pi^\ell(\mathcal{J}_\ell)$ equals $\mathcal{X}$.*

We call a semialgebraic jet set $\mathcal{J}_\ell \subseteq J_\ell\pi$ *basic*, if it can be described by a finite set of equations $p_i = 0$ and a finite set of inequalities $q_j > 0$. We call such a pair of sets a *basic semialgebraic system* on $J_\ell\pi$. It follows from an elementary result in real algebraic geometry [6, Prop. 2.1.8] that any semialgebraic jet set can be expressed as a union of finitely many basic semialgebraic jet sets. We will always assume that our sets are given in this form and study each basic semialgebraic system separately, as for some steps in our analysis it is crucial that at least the equation part of the system is a pure conjunction.

The basic idea underlying the approach of [35] is to treat the system (8) in stages. So far, we computed an algebraic Thomas decomposition of the full system for a suitably chosen ordering. It was not really relevant that parts of the system were linear (although it makes the determination of the Thomas decomposition faster). Now we study first only the linear parts of (8) as a parametric linear system in the unknowns $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ and $\mathbf{d}$ with the jet variables $\mathbf{x}$ and $\mathbf{u}^{(q)}$ considered as parameters (appearing in polynomial form).

Parametric Gaussian elimination has been studied for more than 30 years, see e. g. [4,15,36]. A parametric Gaussian elimination returns a finite set of pairs (γ, H) where the *guard* γ describes the conditions for this particular case and H represents the corresponding solution of the linear system. The guard γ is basically a conjunction of equations and inequations describing the choices made for the various pivots arising during the solution process. A key point is the application of advanced logic and decision procedures for an efficient heuristic handling of the potentially exponentially large number of arising cases. We used a reimplementation of the Redlog [10] package PGauss – see [35, Sect. 3]. It applies strong heuristic simplification techniques [11] and quantifier elimination-based decision procedures [19,31,42,43].

The two linear subsystems of (8) are independent of each other. The analysis of the Jacobian criterion is straightforward: changes in the rank of the matrix are automatically delivered by a parametric Gauss algorithm. The analysis of the system for the Vessiot spaces is a bit more involved. For simplicity, we restrict to the case of ordinary differential equations where the vector $\mathbf{a}$ contains only a single entry a. Here we can give pointwise criteria: a point is regular, if the Vessiot space is one- and the symbol space zero-dimensional; it is regular singular, if both spaces are one-dimensional and irregular singular if the Vessiot space has a dimension higher than one. Thus here it does not suffice to look only at the rank of the matrix; one must also analyse the relative position of the Vessiot space to the vertical space of the jet bundle or more prosaically whether there are non-trivial solutions for which $a = 0$. In [35, Sect. 3], we developed a variant of parametric Gaussian elimination which takes as additional input a sublist of variables for which such considerations are taken into account.

Once all the different cases appearing in the solution of the linear systems have been determined, we must check which of them actually occur on our semi-algebraic jet set. Thus for each case (γ, H) obtained we must verify whether or not the conjunction of its guard γ with the semialgebraic description of the jet set possesses a solution. Such a check represents a classical task for *real quantifier elimination* [7]. If the answer is yes, then the conjunction gives a semialgebraic description of one component in our regularity decomposition. For various reasons like a better readability of the results, we always return a disjunctive normal

Algorithm 2: RealSingularities

Input: $\Sigma_\ell = \big((p_a = 0)_{a=1,\dots,A}, (q_b > 0)_{b=1,\dots,B}\big)$ well-prepared, basic semialgebraic system with $p_a, q_b \in \mathcal{D}_\ell \cap \mathbb{Z}[t, \mathbf{u}, \dots, \mathbf{u}^{(\ell)}]$

Output: finite system $(\Gamma_i, \mathrm{H}_i)_{i=1,\dots,I}$ with

(i) each Γ_i is a disjunctive normal form of polynomial equations, inequations, and inequalities over $\mathcal{D}_\ell$ describing a semialgebraic subset $\mathcal{J}_{\ell,i} \subseteq \mathcal{J}_\ell$

(ii) each H_i describes the Vessiot spaces of all points on $\mathcal{J}_{\ell,i}$

(iii) all sets $\mathcal{J}_{\ell,i}$ are disjoint and their union is $\mathcal{J}_\ell$

begin
  set up the matrix A of the second linear part of (8) using the equations $(p_a = 0)_{a=1,\dots,A}$
  $\Pi = \big(\gamma_\tau, \mathrm{H}_\tau\big)_{\tau=1,\dots,t} \leftarrow$ `ParametricGauss`$\big(A, (\mathbf{b}, a), (a), \mathbb{R}\big)$
  for $\tau = 1$ **to** t **do**
    let Γ_τ be a disjunctive normal form of $\gamma_\tau \wedge \bigwedge \Sigma_\ell$
    check satisfiability of Γ_τ using real quantifier elimination on $\exists t\, \exists \mathbf{u} \dots \exists \mathbf{u}^{(\ell)}\, \Gamma_\tau$
    if Γ_τ is unsatisfiable **then**
      delete $(\gamma_\tau, \mathrm{H}_\tau)$ from Π
    else
      replace $(\gamma_\tau, \mathrm{H}_\tau)$ by $(\Gamma_\tau, \mathrm{H}_\tau)$ in Π
  return Π

form of the semialgebraic description (for more details see [35]). In a more formal language we arrive thus at Algorithm 2.

Example 17. We study again the *hyperbolic gather*. In Example 15, the Thomas decomposition yielded unnecessarily many components in the regularity decomposition and some of them contained only complex points making them irrelevant for a real analysis. Using the above outlined approach, one obtains a real regularity decomposition consisting of exactly three components corresponding to the regular, the regular singular and the irregular singular points [35, Ex. 15]. This is an effect of the reversal of the analysis: we first study the different cases arising in the linear systems, then we check where on the differential equation the cases occur. This strategy should generally lead to a lower number of cases.

[35, Ex. 15] considers also the *elliptic gather* given by $(u')^3 - uu' - x = 0$ (i. e. it differs only by the sign of the middle term). In a complex analysis, one obtains for both gathers essentially the same result. Over the real numbers, the elliptic gather has no irregular singularities (they are now complex). Consequently, our real approach yields a regularity decomposition with only two components.

6 Analysis of Geometric Singularities

After the detection of singularities, we will now discuss the local solution behaviour around them – but only for ordinary differential equations of finite type, as not much is known for partial differential equations. We also consider only the real case using methods from dynamical systems theory. Let $\rho = (\bar{x}, \bar{\mathbf{u}}^{(\ell)})$ be a smooth point on an algebraic differential equation $\mathcal{J}_\ell$. We consider it as initial data for an initial value problem: we search for solutions ϕ of $\mathcal{J}_\ell$ such that $\phi(\bar{x}) = \bar{\mathbf{u}}$, $\phi'(\bar{x}) = \bar{\mathbf{u}}'$,..., $\phi^{(\ell)}(\bar{x}) = \bar{\mathbf{u}}_\ell$. We distinguish between *two-sided solutions* which exist in an interval $(\bar{x}-\epsilon, \bar{x}+\epsilon)$, i. e. for which $\operatorname{im} j_\ell\phi$ goes through the point ρ, and *one-sided solutions* which either begin in ρ, i. e. exist on an interval $[\bar{x}, \bar{x}+\epsilon)$, or end in ρ, i. e. exist on an interval $(\bar{x}-\epsilon, \bar{x}]$. We are interested in the existence, (non)uniqueness and regularity of such solutions. Away from irregular singularities, the theory is rather simple, as the following generalisation of standard results for explicit ordinary differential equations shows.

Theorem 18 ([18, Thm. 4.1]). *Let $\mathcal{J}_\ell$ be a smooth algebraic ordinary differential equation of order ℓ such that at every point $\rho \in \mathcal{J}_\ell$ the Vessiot space $\mathcal{V}_\rho[\mathcal{J}_\ell]$ is one-dimensional. If ρ is a regular point, then there exists a unique smooth classical two-sided solution ϕ with $\rho \in \operatorname{im} j_\ell\phi$. More precisely, it can be extended in both directions until $\operatorname{im} j_\ell\phi$ reaches either the boundary of $\mathcal{J}_\ell$ or a regular singular point. If ρ is a regular singular point, then either two smooth classical one-sided solutions ϕ_1, ϕ_2 exist with $\rho \in \overline{\operatorname{im} j_\ell\phi_i}$ which either both start or both end in ρ or only one classical two-sided solution exists whose $(\ell+1)$th derivative blows up at $x = \pi^\ell(\rho)$.*

Proof. By the made assumptions, $\mathcal{V}[\mathcal{J}_\ell]$ can be generated in an open neighbourhood of ρ by a smooth vector field X. The standard existence and uniqueness theorems guarantee for each point $\rho \in \mathcal{J}_\ell$ the existence of a unique integral curve of X defining a unique generalised solution $\mathcal{N}_\rho$ with $\rho \in \mathcal{N}_\rho$. This generalised solution is a smooth curve which can be extended until it reaches the boundary of $\mathcal{J}_\ell$ and around each regular point $\bar{\rho} \in \mathcal{N}_\rho$ it projects onto the graph of a strong solution ϕ, since $\mathcal{V}_{\bar{\rho}}[\mathcal{J}_\ell]$ is transversal to π^ℓ by definition of a regular point.

If ρ is a regular singular point, then X_ρ is vertical for π^ℓ, i. e. its ∂_x-component vanishes. The behaviour of the corresponding geometric solution $\widetilde{\mathcal{N}}_\rho = \pi_0^\ell(\mathcal{N}_\rho)$ depends on whether or not the ∂_x-component changes its sign at ρ. If the sign changes, then $\widetilde{\mathcal{N}}_\rho$ has two branches corresponding to two classical solutions which either both end or both begin at $\hat{\rho} = \pi_0^\ell(\rho)$. Otherwise $\widetilde{\mathcal{N}}_\rho$ is around $\hat{\rho}$ the graph of a classical solution, but Remark 8 implies that the $(\ell+1)$th derivative of this solution at $x = \pi^\ell(\rho)$ must be infinite.

Example 19. We continue our study of the *hyperbolic gather* by looking at the local solution behaviour. Figure 2 shows on the left hand side a number of generalised solutions in cyan and on the right hand side the corresponding geometric solutions in blue. One can see that whenever a generalised solution crosses transversally the white fold line outside of an irregular singularity, then the geometric solution reverses its direction (more precisely, the curve defining it has a cusp there). At these "reversal points", the geometric solution cannot be interpreted as the graph of a function. Hence one obtains in the classical picture two one-sided solutions. The curve in magenta shows the generalised solution that goes through the tip of the fold line. The corresponding geometric solution is still a classical one, but only $\mathcal{C}^1$: one can see that it is not smooth at the origin, as by Theorem 18 its second derivative blows up.

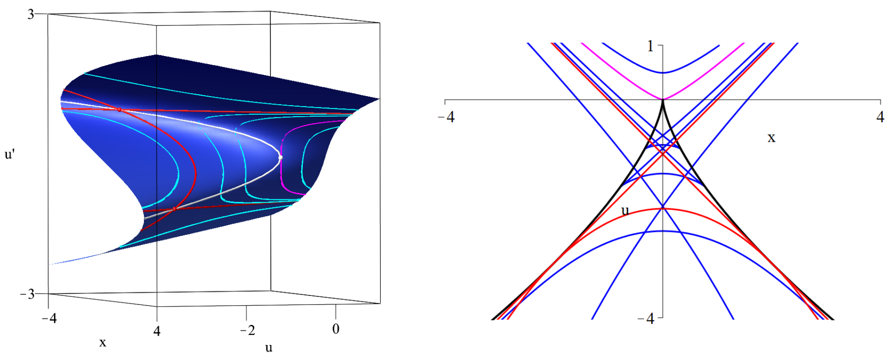

Fig. 2. Generalised solutions of the hyperbolic gather. Left: situation in $J_1(\mathbb{R}, \mathbb{R})$. Right: projection to x-u plane.

At irregular singularities, the solution behaviour can be more complicated. It follows from their definition that they form an algebraic jet set of codimension at least 2. Hence, if $\rho \in \mathcal{J}_\ell$ is an irregular singularity, then we can find an open, simply connected submanifold $\mathcal{U} \subset \mathcal{J}_\ell$ such that $\rho \in \overline{\mathcal{U}}$ and everywhere in $\mathcal{U}$ the Vessiot spaces are one-dimensional. On $\mathcal{U}$ the Vessiot distribution $\mathcal{V}[\mathcal{J}_\ell]$ can be generated by a single smooth vector field X. In principle, it is straightforward to construct such a vector field by solving (6), but one must exclude certain degeneracies appearing e. g. in the presence of singular integrals.

If $\mathcal{J}_\ell$ is locally integrable, then we may assume without loss of generality by (the proof of) [32, Prop. 9.5.10] that $\mathcal{J}_\ell$ is described by a square system $p_\tau = 0$ with as many equations as unknowns. Thus the symbol matrix M_ℓ is square and for ordinary differential equations the matrix D becomes a vector $\mathbf{d}$. Let $M^\dagger = \operatorname{adj}(M_\ell)$ be the adjugate of M. On $\mathcal{U}$, the Vessiot distribution is generated by the vector field $X = \det(M)C_1^{(\ell)} - \sum_{\alpha=1}^{m}(M^\dagger\mathbf{d})^\alpha C_\alpha^\ell$ which can be smoothly extended to a neighbourhood of ρ, as all its coefficients are polynomials.

Proposition 20. *Let $\mathcal{J}_\ell$ be a locally integrable differential equation and assume that on $\mathcal{U}$ the vector $\mathbf{d}$ does not vanish and that $\det(M)$ and the components of $M^\dagger\mathbf{d}$ do not possess a non-trivial joint common divisor. Then any smooth extension of the vector field X vanishes at ρ.*

Proof. The made assumptions ensure that X is a “minimal” generator of the Vessiot distribution on $\mathcal{U}$. At the irregular singularity ρ, the rank of $M(\rho)$ drops and thus $\det(M) = 0$. If it drops by more than one, then $M^\dagger(\rho) = 0$ and the claim is trivial. If the rank drops only by one, then the vector $\mathbf{d}(\rho)$ must lie in the column space of $M(\rho)$ for an irregular singularity. It follows now by Cramer's rule that $M^{\dagger}(\rho)\mathbf{d}(\rho) = 0$ and hence $X_\rho = 0$.

Thus at least generically we can analyse the local solution behaviour by using dynamical systems theory: we are given a smooth vector field X on $\mathcal{J}_\ell$ for which ρ is a stationary point. If ρ is a hyperbolic stationary point, then the eigenvalues of $\mathrm{Jac}\,(X)$ completely determine the local phase portrait. Otherwise, one must resort to more advanced techniques like blow-ups.

Example 21. In the case of the *hyperbolic gather*, the Vessiot distribution is generated by the vector field $X = (3(u')^2 + u)(\partial_x + u'\partial_u) + (1 - (u')^2)\partial_{u'}$. The Jacobian at the irregular singularity $\rho = (2, -3, 1)$ (the analysis of the other irregular singularity proceeds analogously) is $J = \begin{pmatrix} 0 & 1 & -6 \\ 0 & -1 & 6 \\ 0 & 0 & 2 \end{pmatrix}$ with the three eigenvalues 2, -1 and 0. Although $\mathcal{J}_1$ is a two-dimensional submanifold, we are computing here with all three jet coordinates in $J_1(\mathbb{R}, \mathbb{R})$. Thus we must decide which eigenvalue is irrelevant. This is straightforward: we only have to check which eigenvector is not tangential to $\mathcal{J}_1$. It turns out that in our case 0 is irrelevant. Hence ρ is a saddle point of X, as one can also clearly see on the left hand side of Figure 2. The red curves there are two invariant manifolds tangent to the eigenspaces which for us represent two generalised solutions which intersect at the irregular singularity.

If an irregular singularity ρ is a node of the vector field X, then infinitely many (two-sided) generalised solutions intersect there. At a focus, all generalised solutions are one-sided, as they do not possess a well-defined tangent when spiralling into ρ and hence cannot be combined to a smooth curve through ρ. For higher-dimensional equations, the analysis in particular of non-hyperbolic stationary points can be arbitrarily complicated. For scalar first-order equations a complete classification of generic irregular singularities was given in [8,9]. The typical behaviour at an irregular singularity is thus that the usual uniqueness statements break down and several general solutions intersect there. There are, however, also degenerate situations where one still obtains a unique solution (see e. g. [34, Ex. 3.5]); in this case one speaks of an *apparent singularity*.

7 Quasilinear Equations

Quasilinear ordinary differential equations have their own theory, which somewhat surprisingly seems to have been overlooked in the differential topological literature. By contrast, in the context of differential algebraic equations, authors have studied almost exclusively the quasilinear case – see e. g. [26,28,38,41] – using analytic methods. For simplicity, we study here following [34] only the case of a scalar ordinary differential equation ([33] treats first-order systems)

$$g(x, u^{(\ell-1)})u_\ell = f(x, u^{(\ell-1)}) \tag{9}$$

where u_ℓ denotes the ℓth derivative of u and $u^{(\ell)}$ all derivatives up to order ℓ. We further assume that f, g are polynomials of their arguments. Let $\mathcal{J}_\ell$ be the corresponding algebraic jet set.

Whether or not a point $\rho = (\bar{x}, \bar{u}^{(\ell)}) \in \mathcal{J}_\ell$ is a singularity does not depend on the value of $\bar{u}_\ell$ in this special case, as it does not appear in (6). The key property of quasilinear equations is that they can be studied at one order less. More precisely, outside of the irregular singularities the Vessiot distribution $\mathcal{V}[\mathcal{J}_\ell]$ can be generated by a vector field X. Denoting by $C_t^{(\ell)} = \partial_x + \sum_{i=0}^{\ell-1} u_{i+1}\partial_{u_i}$ the transversal contact field on $J_\ell(\mathbb{R},\mathbb{R})$ and by $C_v^{(\ell)} = \partial_{u_\ell}$ the vertical one, we may choose $X = gC_t^{(\ell)} + \big(C_t^{(\ell)}(g)u_\ell - C_t^{(\ell)}(f)\big)C_v^{(\ell)}$. Expanding X, one sees that it is projectable to the field $Y = gC_t^{(\ell-1)} + fC_v^{(\ell-1)}$ on $J_{\ell-1}(\mathbb{R},\mathbb{R})$. Strictly speaking, Y is only defined outside the projections of the irregular singularities of $\mathcal{J}_\ell$. But as we assume that f, g are polynomials, Y can obviously be extended smoothly to the whole jet bundle $J_{\ell-1}(\mathbb{R},\mathbb{R})$.

Definition 22. *A point $\tilde{\rho} \in J_{\ell-1}(\mathbb{R},\mathbb{R})$ is an* impasse point *for $\mathcal{J}_\ell$, if Y is not transversal at $\tilde{\rho}$ (i. e. if $g(\tilde{\rho}) = 0$). Otherwise, it is a* regular point. *An impasse point is* proper*, if Y vanishes there, and improper otherwise. A* weak generalised solution *of $\mathcal{J}_\ell$ is a one-dimensional manifold $\tilde{\mathcal{N}} \subset J_{\ell-1}(\mathbb{R},\mathbb{R})$ such that $Y_{\tilde{\rho}} \in T_{\tilde{\rho}}\tilde{\mathcal{N}}$ for all points $\tilde{\rho} \in \tilde{\mathcal{N}}$. A* weak geometric solution *is the projection $\pi_0^{\ell-1}(\tilde{\mathcal{N}})$ of a weak generalised solution $\tilde{\mathcal{N}}$.*

We use here the terminology “impasse points” to distinguish them from the singularities of $\mathcal{J}_\ell$ which are always points on $\mathcal{J}_\ell$. Singularities always project on impasse points, but there may be impasse points without a point on $\mathcal{J}_\ell$ above them (see [34, Prop. 5.4] for a more precise analysis). This is the deeper reason why quasilinear equations require their own theory. Like a singularity, an impasse point can be only apparent – see [34, Ex. 6.3]. We speak about “weak” generalised solutions, as even in the case that they are the prolongations of a function it is not guaranteed that this function is ℓ times differentiable. Hence it can be considered as a solution only in a weak sense. One can provide an existence and (non)uniqueness theorem analogous to Theorem 18 for equations without proper impasse points. For lack of space, we omit the details and refer to [34, Thm. 6.5].

To indicate the wide variety of phenomena that may appear around impasse points of quasilinear equations, we now specialise to the following class of singular second-order initial value problems

$$g(x)u'' = f(x, u, u')\,, \qquad u(y) = c_0\,,\ u'(y) = c_1 \tag{10}$$

where we assume that y is *simple* zero of g. Liang [23] studied it for the special case $g(x) = x$ and $y = 0$ with analytical techniques. We showed in [34] that all his results can be recovered with geometric means in our slightly more general situation in a much more transparent way. Here we can only sketch some basic ideas of our approach; for all details we refer to [34, Sect. 8].

Key questions are the (non)uniqueness and the regularity of the solutions of (10). For the latter point, it does not suffice to study only the differential equation $\mathcal{J}_2 \subset J_2(\mathbb{R},\mathbb{R})$ corresponding to (10), but one must also analyse its *prolongations* $\mathcal{J}_\ell \subset J_\ell(\mathbb{R},\mathbb{R})$ for all $\ell > 2$ which are obtained by differentiating the given equation (10). We set $F_2(x,u^{(2)}) = g(x)u'' - f(x,u,u')$ and write for any order $\ell > 2$

$$F_\ell(x,u^{(\ell)}) = g(x)u_\ell + \Big[(\ell-2)g'(x) - f_{u'}(x,u^{(1)})\Big]u_{\ell-1} - h_\ell(x,u^{(\ell-2)})$$

where the contributions of the lower-order terms can be recursively computed as $h_3(x,u^{(1)}) = C_t^{(1)} f(x,u^{(1)})$ and for $\ell > 3$ as

$$h_\ell(x,u^{(\ell-2)}) = C_t^{\ell-2}\Big(h_{\ell-1}(x,u^{(\ell-3)}) - \big[(\ell-3)g'(x) - f_{u'}(x,u^{(1)})\big]u_{\ell-2}\Big)\,.$$

Then the equation $\mathcal{J}_\ell$ is the zero set of $F_2,\dots,F_\ell$. If we apply the above idea of projecting the Vessiot distribution to one order less, then $\mathcal{J}_2$ yields the vector field $Y^{(1)} = g(x)\partial_x + g(x)u'\partial_u + f(x,u^{(1)})\partial_{u'}$ on $J_1(\mathbb{R},\mathbb{R})$ and for any $\ell \geq 2$ we get from $\mathcal{J}_{\ell+1}$ the vector field

$$Y^{(\ell)} = g(x)C_t^{(\ell)} + \Big(h_{\ell-1}(x,u^{(\ell-3)}) - \big[(\ell-1)g'(x) - f_{u'}(x,u^{(1)})\big]u_\ell\Big)C_v^{(\ell)} \quad (11)$$

defined on the three-dimensional submanifold $\mathcal{J}_\ell \subset J_\ell(\mathbb{R},\mathbb{R})$.

Our initial data define a point $\rho_1 = (y,c_0,c_1) \in J_1(\mathbb{R},\mathbb{R})$. It turns out that for the analysis of our initial value problem at each relevant prolongation order ℓ there exists a unique irregular singularity $\rho_\ell \in \mathcal{J}_\ell$ above ρ_1, i.e. with $\pi_1^\ell(\rho_\ell) = \rho_1$. We will find that the existence, (non)uniqueness and regularity of solutions depend solely on two values: $\delta = g'(y)$ and $\gamma = f_{u'}(\rho_1)$. Because of our assumption that y is a simple zero, δ cannot vanish. Our initial value problem has a *resonance* at order $q \in \mathbb{N}$, if $q\delta = \gamma$. If this is the case, we consider the *resonance parameter* $A_q = h_{q+2}(\rho_q)$ and speak of a *smooth* resonance for $A_q = 0$ and of a *critical* resonance otherwise.

Our approach consists of analysing the phase portraits of the vector fields $Y^{(\ell)}$ around their stationary points ρ_ℓ. This requires in particular to determine the eigenvalues of the Jacobians $\mathrm{Jac}\,(Y^{(\ell)})(\rho_\ell)$. This is fairly simple except that we face again the problem that (11) is written in all jet variables up to order ℓ, although $Y^{(\ell)}$ lives only on a three-dimensional manifold. Fortunately, it can be overcome with a little trick and one obtains as eigenvalues δ, 0 and $\gamma - (\ell-1)\delta$. If δ and γ have different signs, then we find at any prolongation order one negative, one zero and one positive eigenvalue and thus qualitatively the same phase portrait. If δ and γ have the same sign, then at a certain prolongation order the phase portrait changes qualitatively, as one eigenvalue changes its sign. In the case of a resonance, one finds at a certain prolongation order a double eigenvalue. It depends on the vanishing of the resonance parameter whether or not the Jacobian is diagonalisable. A deeper study of the invariant manifolds of the stationary point leads to the following result.

Theorem 23. *If there is no resonance, then three cases arise:*

$\delta\gamma < 0$**:** *The initial value problem* (10) *possesses a unique smooth two-sided solution and no additional one-sided solutions.*

$\delta\gamma > 0$**:** *The initial value problem* (10) *possesses a one-parameter family of two-sided solutions and no additional one-sided solutions. One member of the family is smooth, all others are in* $\mathcal{C}^k \setminus \mathcal{C}^{k+1}$ *with* $k = \lceil \gamma/\delta \rceil$.

$\gamma = 0$**:** *The initial value problem* (10) *possesses a unique smooth two-sided solution and possibly further additional one-sided solutions.*

If there is a resonance at order $k > 0$*, then the initial value problem* (10) *possesses a one-parameter family of two-sided solutions and no additional one-sided solutions. If the resonance is smooth, all solutions are smooth. For a critical resonance, all solutions are in* $\mathcal{C}^k \setminus \mathcal{C}^{k+1}$.

As demonstrated by an explicit example in [34], there are many possibilities in the case $\gamma = 0$: there could be no one-sided solutions at all or there could be infinitely many which either come from both sides or only from one side. The exact behaviour depends on further values besides δ and γ and no complete classification is known. The situation becomes much more complicated, if one drops the assumption that y is a simple zero. In this case $\delta = 0$ and if in addition $\gamma = 0$, then the Jacobian has a triple eigenvalue 0. The analysis of such a stationary point is rather difficult, as it requires a blow-up in three dimensions. For the subsequent blow-down, one must understand the *global* dynamics of a two-dimensional dynamical system which can be very complicated.

8 Analysis of Algebraic Singularities

Singularities of varieties have been extensively studied in algebraic geometry, but not much is known about their effect on differential equations. As algebraic differential equations are locally Zariski closed sets, we cannot avoid dealing with them. In the complex case, their detection is straightforward using the Jacobian criterion (over the real numbers the situation is somewhat different, as at a singularity the variety may still be locally a manifold). Thus the main point is to analyse the local solution behaviour in their neighbourhood. Ritt provides several examples of algebraic differential equations with singular integrals where the singular integrals consist entirely of algebraic singularities (see e. g. [29, II.§19]). However, he does not comment on this fact.

We will not develop a general theory for handling algebraic singularities, but we will indicate with two concrete examples some phenomena that can show up. We will use a rather ad hoc approach which probably can be extended to more general situations, but we refrain here from any formalities. In the first example, we study the local solution behaviour near an isolated algebraic singularity using the Vessiot spaces of neighbouring points.

Example 24. Let $\mathcal{J}_1$ be the two-dimensional cone in the three-dimensional jet bundle $J_1(\mathbb{R}, \mathbb{R})$ given by $(u')^2 - u^2 - x^2 = 0$. The vertex is an isolated algebraic singularity representing one component of a regularity decomposition while all other points are regular and form the second component. We are interested in how many solutions go through the vertex and their regularity.

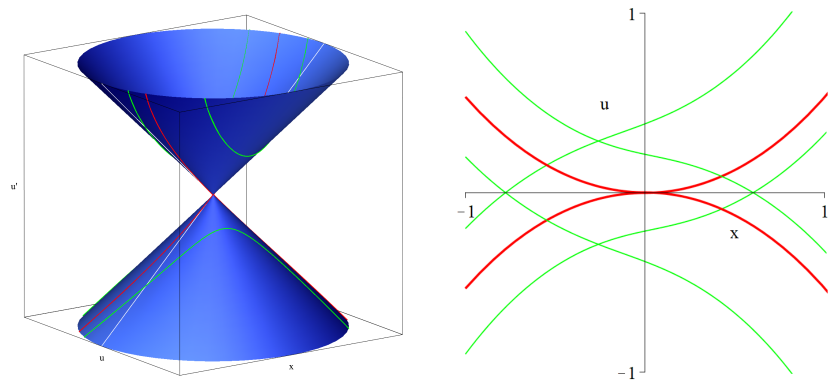

Fig. 3. Generalised solutions going through an algebraic singularity of a real first-order differential equation. Left: situation in $J_1(\mathbb{R}, \mathbb{R})$. Right: projection to x-u plane.

Consider the Vessiot spaces of the regular points. They are generated by the vector field $X = u'\partial_x + (x^2 + u^2)\partial_u + (x - uu')\partial_{u'}$. By restricting to either the lower or the upper half cone, we can express u' by x and u and project to the x-u plane obtaining the vector fields $Y_\pm = \pm\sqrt{x^2 + u^2}\partial_x + (x^2 + u^2)\partial_u$ which can trivially be continued to the origin where they vanish. As they are not differentiable there, the origin cannot be studied using the Jacobian.

By transforming to polar coordinates, i. e. by performing a blow-up of the stationary point in the origin, one can show that the dynamical system defined by Y possesses a unique invariant curve going through the origin and within a sufficiently small neighbourhood of the origin all nearby trajectories look similar to this manifold. Furthermore, on one side of the origin the invariant curve corresponds to a trajectory going into the origin, while on the side we have an outgoing trajectory (hence the stationary point is not really visible).

Recall that such an invariant curve corresponds to a generalised solution and we obtain one such curve for each half cone, i. e. from each of the fields $Y_\pm$ (see the red curves in Figure 3). As the graphs of both solutions possess a horizontal tangent at the origin, it is possible to "switch" at the singularity from one to the other. Hence, we find that our equation possesses exactly four $\mathcal{C}^1$ solutions for the initial condition $u(0) = 0$ and $u'(0) = 0$. By analysing the prolongations of our equation, it is not difficult to verify that the solutions that stay inside of one half cone are even smooth, whereas the "switching" solutions are only $\mathcal{C}^1$, as their second derivative jumps from 1 to -1 or vice versa at $x = 0$. Figure 3 also shows in white the Vessiot cone at the algebraic singularity which consists of two intersecting lines. One sees that they are indeed the tangents to the prolonged solutions through the singularity.

In Algorithm 1, we perform a prime decomposition so that we always work with (subsets of) irreducible varieties. One reason for this in practise rather expensive step is to avoid the algebraic singularities automatically given by the intersection of different irreducible components in the case of a reducible variety. We prefer to deal with such points only in a later stage after we have already analysed each irreducible component separately. Given an algebraic differential equation $\mathcal{J}_q$ which is reducible in this sense, an obvious interesting question is whether solutions exist which "switch" from one component to another and if yes, what is their regularity?

Example 25. We consider over the real numbers the scalar first-order ordinary differential equation $\mathcal{J}_1$ given by $(u'-c)\big((u')^2+u^2+x^2-1\big)=0$ with a constant $c\in[-1,1]$. As we have written the equation in factored form, one immediately recognises that $\mathcal{J}_1$ is simply the unit sphere $\mathcal{J}_{1,1}$ in the jet bundle $J_1(\mathbb{R},\mathbb{R})$ united with a horizontal plane $\mathcal{J}_{1,2}$ at height c (see Figure 4). For $|c|\neq 1$, the intersection is a circle $\mathcal{C}$, otherwise simply a point.

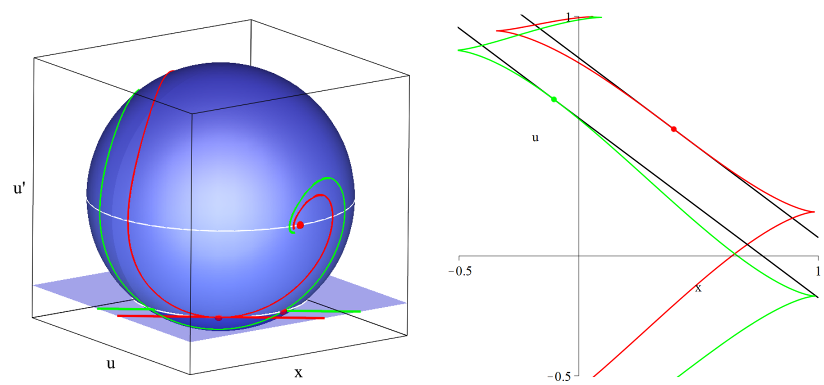

Fig. 4. First-order differential equation with two irreducible components. Left: generalised solutions in $J_1(\mathbb{R},\mathbb{R})$. Right: solution graphs in x-u plane.

The differential equation $\mathcal{J}_{1,2}\,:\,u'=c$ is of course trivial to analyse: as it is explicit, all points on the corresponding plane are regular and all generalised solutions are straight lines. The differential equation $\mathcal{J}_{1,1}\,:\,(u')^2+u^2+x^2=1$ has already been studied at many places (see e. g. [32, Ex. 9.1.12] or [35, Ex. 10]) and its singularities form the equator. A point $\rho=(\bar{x},\bar{u},\bar{p})$ lies on the intersection and thus is an algebraic singularity of $\mathcal{J}_1$, if $\bar{p}=c$ and $\bar{u}^2+\bar{x}^2=1-c^2$.

Consider first the case $c\neq 0$. Then ρ is for each component $\mathcal{J}_{1,i}$ a regular point and we have on each component a unique generalised solution curve γ_i through ρ. Without loss of generality, assume $\gamma_i(0)=\rho$. We may form a further generalised solution like $\gamma(t)=\gamma_1(t)$ for $t\leq 0$ and $\gamma(t)=\gamma_2(t)$ for $t\geq 0$ (and yet another by swapping the indices 1 and 2). While this curve γ is trivially continuous at 0, it is in general not differentiable there, as the tangent vectors $\gamma_1'(0)$ and $\gamma_2'(0)$ disagree. However, the corresponding geometric solution, i. e. the projection of $\operatorname{im}\gamma$ to the x-u plane, is the graph of an everywhere differentiable function $u=f(x)$ with $f'(x)=\bar{p}$. This can be seen on the right hand side of

Figure 4. The green and the red curve represent geometric solutions of $\mathcal{J}_{1,1}$; the black lines the corresponding ones of $\mathcal{J}_{1,2}$. One can see that the latter ones are exactly the tangents of the former one and hence connecting "half" a curve with "half" a line yields still the graph of a function which is at least $\mathcal{C}^1$.

The tangent vectors $\gamma_1'(0)$ and $\gamma_2'(0)$ generate the Vessiot spaces $\mathcal{V}_\rho[\mathcal{J}_{1,1}]$ and $\mathcal{V}_\rho[\mathcal{J}_{1,2}]$. As their slopes correspond to the second derivatives of the solutions leading to the curves γ_1 and γ_2, our "composed" solutions can be $\mathcal{C}^2$, if and only if these Vessiot spaces coincide. A simple computation shows that $\mathcal{V}_\rho[\mathcal{J}_{1,1}]$ is generated by the vector $\bar{p}(\partial_x + \bar{p}\partial_u) - (\bar{x} + \bar{u}\bar{p})\partial_p$ while $\mathcal{V}_\rho[\mathcal{J}_{1,2}]$ is spanned by $\partial_x + c\partial_u$. It is straightforward to show that the Vessiot spaces coincide only at two special points $\rho_\pm$ on the intersection $\mathcal{C}$, namely at

$$\rho_\pm = \left(\mp c\sqrt{\frac{1-c^2}{1+c^2}}, \pm\sqrt{\frac{1-c^2}{1+c^2}}, c\right).$$

In Figure 4 this corresponds to the red curve, as one can see on the right hand side that the intersection of the red and the black graph happens at an inclination point of the red graph. By analysing the next prolongation, one can show that even for these two special points the "composed" solutions are only $\mathcal{C}^2$. We conclude that our differential equation $\mathcal{J}_1$ possesses four solutions through any point $\rho \in \mathcal{C}$. Two of them are smooth (the ones corresponding to γ_1 and γ_2), the two "composed" ones are only $\mathcal{C}^1$ respectively $\mathcal{C}^2$, if ρ is one of the points $\rho_\pm$.

In the case $c = 0$, the intersection $\mathcal{C}$ is the equator and thus the singular locus of $\mathcal{J}_{1,1}$. Here we have an example where the classification of a point lying on several irreducible components differs for the different components. $\mathcal{J}_{1,1}$ has two irregular singularities, namely the points $(0, \pm 1, 0)$, and both are a folded focus. This means that no generalised solution of $\mathcal{J}_{1,1}$ approaches them with a well-defined tangent and this would be necessary for going through them. Thus through each of these two points there exists only one generalised solution of $\mathcal{J}_1$, namely the one of $\mathcal{J}_{1,2}$. Any other point $\rho \in \mathcal{C}$ is a regular singularity of $\mathcal{J}_{1,1}$. Thus on $\mathcal{J}_{1,1}$ there are only two one-sided solutions starting or ending at ρ. However, we can combine each of them with "half" a solution of $\mathcal{J}_{1,2}$ as discussed for $c \neq 0$ to generate two additional $\mathcal{C}^1$ solutions so that $\mathcal{J}_1$ has three solutions through ρ one of which is smooth. As at all regular singularities the Vessiot space is vertical, we do not find any special points where the "composed" solutions possess a higher regularity than $\mathcal{C}^1$.

9 Conclusions

We presented a mixture of geometric and algebraic techniques for studying singularities of differential equations. For the basic concepts, we followed essentially the differential topological approach to geometric singularities and extended it also to differential equations which are not of finite type. We augmented this approach by algebraic ideas to extend its range of applicability, as many differential equations in applications do not lead to manifolds, but only to varieties. This implies that we must furthermore deal with algebraic singularities.

In the first half of this article, we concentrated on the algorithmic detection of singularities. We used the differential Thomas decomposition from differential algebra to obtain simple differential systems from which we can extract in a well-defined manner an algebraic differential equation to which the geometric theory can be applied. The actual detection of the singularities is then performed with an algebraic Thomas decomposition. Although there is still a gap in the theory for differential equations which are not of finite type, as it is not clear whether there might appear further types of singularities, it is remarkable that the classification can be performed completely algorithmically.

This is possible only, because we searched for singularities at a prescribed order. For lack of space, we did not discuss here the question how singularities at different prolongation orders are related. It is easy to see that if we prolong, then every point on the prolonged equation lying over a singularity of the original equation must be again a singularity. Furthermore, there can never be a point over a regular singularity. For the existence of (formal) power series solutions, it is therefore necessary that we can construct an infinite tower of irregular singularities lying above each other. An example due to Lange-Hegermann [20, Ex. 2.93] (see also the discussion in [35, Ex. 16]) shows that generally it is not possible to decide the existence of such an infinite tower at any finite order.

It should have become apparent that for the study of singularities it makes a great difference whether we work over the real or the complex numbers. The detection of singularities over the complex numbers is simpler, as they form an algebraically closed field which is algorithmically a great advantage. Any analysis of singularities was performed in this work over the real numbers, as it was based on techniques from dynamical systems theory. One should note that also the questions studied differ considerably in dependence of the base field. The regularity of solutions or the difference between one- and two-sided solutions is an issue only over the real numbers. Over the complex numbers, there exists already an extensive theory of singularities of *linear* differential equations going back at least to Fuchs and Frobenius which is nowadays often considered as a part of differential Galois theory. Here the determination of monodromy or the Stokes phenomenon are of great importance and have no real counterpart.

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References

1. Arnold, V.: Geometrical Methods in the Theory of Ordinary Differential Equations, Grundlehren der mathematischen Wissenschaften, vol. 250. Springer-Verlag, New York, 2^{nd} edn. (1988)

2. Bächler, T., Gerdt, V., Lange-Hegermann, M., Robertz, D.: Algorithmic Thomas decomposition of algebraic and differential systems. J. Symbolic Comput. **47**(10), 1233–1266 (2012)
3. Bächler, T., Lange-Hegermann, M.: AlgebraicThomas and DifferentialThomas: Thomas Decomposition for algebraic and differential systems (2008-2012), (http://www.mathb.rwth-aachen.de/go/id/rnab/lidx/1)
4. Ballarin, C., Kauers, M.: Solving parametric linear systems: An experiment with constraint algebraic programming. ACM SIGSAM Bulletin **38**, 33–46 (2004)
5. Becker, E., Neuhaus, R.: Computation of real radicals of polynomial ideals. In: Eyssette, F., Galligo, A. (eds.) Computational Algebraic Geometry, pp. 1–20. Progress in Mathematics 109, Birkhäuser, Basel (1993)
6. Bochnak, J., Conte, M., Roy, M.: Real Algebraic Geometry. Ergebnisse der Mathematik und ihrer Grenzgebiete 36, Springer-Verlag, Berlin (1998)
7. Caviness, B., Johnson, J. (eds.): Quantifier Elimination and Cylindrical Algebraic Decomposition. Texts and Monographs in Symbolic Computation, Springer-Verlag, Vienna (1998)
8. Dara, L.: Singularités génériques des équations différentielles multiformes. Bol. Soc. Bras. Mat. **6**, 95–128 (1975)
9. Davydov, A., Ishikawa, G., Izumiya, S., Sun, W.Z.: Generic singularities of implicit systems of first order differential equations on the plane. Japan. J. Math. **3**, 93–119 (2008)
10. Dolzmann, A., Sturm, T.: REDLOG: Computer algebra meets computer logic. ACM SIGSAM Bulletin **31**, 2–9 (1997)
11. Dolzmann, A., Sturm, T.: Simplification of quantifier-free formulae over ordered fields. J. Symb. Comput. **24**, 209–231 (1997)
12. Falkensteiner, S., Sendra, J.: Solving first order autonomous algebraic ordinary differential equations by places. Math. Comput. Sci. **14**, 327–337 (2020)
13. Gerdt, V.: On decomposition of algebraic PDE systems into simple subsystems. Acta Appl. Math. **101**, 39–51 (2008)
14. Gerdt, V., Lange-Hegermann, M., Robertz, D.: The Maple package TDDS for computing Thomas decompositions of systems of nonlinear PDEs. Comp. Phys. Comm. **234**, 202–215 (2019)
15. Grigoriev, D.: Complexity of deciding Tarski algebra. J. Symb. Comput. **5**, 65–108 (1988)
16. Hubert, E.: Notes on triangular sets and triangulation-decomposition algorithms. I: Polynomial systems. In: Winkler and Langer [44], pp. 1–39
17. Hubert, E.: Notes on triangular sets and triangulation-decomposition algorithms. II: Differential systems. In: Winkler and Langer [44], pp. 40–87
18. Kant, U., Seiler, W.: Singularities in the geometric theory of differential equations. In: Feng, W., Feng, Z., Grasselli, M., Lu, X., Siegmund, S., Voigt, J. (eds.) Dynamical Systems, Differential Equations and Applications (Proc. 8th AIMS Conference, Dresden 2010), vol. 2, pp. 784–793. AIMS (2012)
19. Košta, M.: New Concepts for Real Quantifier Elimination by Virtual Substitution. Doctoral dissertation, Saarland University, Germany (2016)
20. Lange-Hegermann, M.: Counting Solutions of Differential Equations. Ph.D. thesis, RWTH Aachen, Germany (2014), available at http://publications.rwth-aachen.de/record/229056
21. Lange-Hegermann, M.: The differential counting polynomial. Found. Comp. Math. **18**, 291–308 (2018)
22. Lange-Hegermann, M., Robertz, D., Seiler, W., Seiß, M.: Singularities of algebraic differential equations. Preprint Kassel University (arXiv:2002.11597) (2020)

23. Liang, J.: A singular initial value problem and self-similar solutions of a nonlinear dissipative wave equation. J. Diff. Eqs. **246**, 819–844 (2009)
24. Neuhaus, R.: Computation of real radicals of polynomial ideals II. J. Pure Appl. Alg. **124**, 261–280 (1998)
25. Plesken, W.: Counting solutions of polynomial systems via iterated fibrations. Arch. Math. (Basel) **92**(1), 44–56 (2009)
26. Rabier, P., Rheinboldt, W.: Theoretical and numerical analysis of differential-algebraic equations. In: Ciarlet, P., Lions, J. (eds.) Handbook of Numerical Analysis, vol. VIII, pp. 183–540. North-Holland, Amsterdam (2002)
27. Remizov, A.: Multidimensional Poincaré construction and singularities of lifted fields for implicit differential equations. J. Math. Sci. **151**, 3561–3602 (2008)
28. Riaza, R.: Differential-Algebraic Systems. World Scientific, Hackensack (2008)
29. Ritt, J.: Differential Algebra. Dover, New York (1966), (Original: AMS Colloquium Publications, Vol. XXXIII, 1950)
30. Robertz, D.: Formal Algorithmic Elimination for PDEs, Lecture Notes in Mathematics, vol. 2121. Springer, Cham (2014)
31. Seidl, A.: Cylindrical Decomposition Under Application-Oriented Paradigms. Doctoral dissertation, Universität Passau, Germany (2006)
32. Seiler, W.: Involution — The Formal Theory of Differential Equations and its Applications in Computer Algebra, Algorithms and Computation in Mathematics, vol. 24. Springer-Verlag, Berlin (2010)
33. Seiler, W.: Singularities of implicit differential equations and static bifurcations. In: Gerdt, V., Koepf, W., Mayr, E., Vorozhtsov, E. (eds.) Computer Algebra in Scientific Computing — CASC 2013, Lecture Notes in Computer Science, vol. 8136, pp. 355–368. Springer-Verlag, Berlin (2013)
34. Seiler, W., Seiß, M.: Singular initial value problems for scalar quasi-linear ordinary differential equations. Preprint Kassel University (arXiv:2002.06572) (2018)
35. Seiler, W., Seiß, M., Sturm, T.: A logic based approach to finding real singularities of implicit ordinary differential equations. Math. Comp. Sci. (2020), to appear (arXiv: 2003.00740)
36. Sit, W.: An algorithm for solving parametric linear systems. J. Symb. Comput. **13**, 353–394 (1992)
37. Spang, S.: On the Computation of the Real Radical. Diploma thesis, Technical University Kaiserslautern, Department of Mathematics (2007)
38. Thomas, G.: The problem of defining the singular points of quasi-linear differential-algebraic systems. Theor. Comp. Sci. **187**, 49–79 (1997)
39. Thomas, J.: Differential Systems. Colloquium Publications XXI, American Mathematical Society, New York (1937)
40. Thomas, J.: Systems and Roots. W. Byrd Press (1962)
41. Tuomela, J.: On singular points of quasilinear differential and differential-algebraic equations. BIT **37**, 968–977 (1997)
42. Weispfenning, V.: The complexity of linear problems in fields. J. Symb. Comput. **5**, 3–27 (1988)
43. Weispfenning, V.: Quantifier elimination for real algebra—the quadratic case and beyond. Appl. Algebr. Eng. Comm. **8**, 85–101 (1997)
44. Winkler, F., Langer, U. (eds.): Symbolic and Numerical Scientific Computation, Lecture Notes in Computer Science, vol. 2630. Springer-Verlag, Berlin (2003)