Vessiot Connections of Partial Differential Equations[1]

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Abstract

We provide a rigorous formulation of Vessiot's vector field approach to the analysis of general systems of partial differential equations and prove its equivalence to the formal theory.

Keywords: Vessiot distribution, integral element, involution

1 Introduction

Vessiot [12] proposed in the 1920s an approach to deal with general systems of partial differential equations which takes an intermediate position between the formal theory [8, 9] and the Cartan-Kähler theory of exterior differential systems [1, 4]: while still formulated in the language of differential equations (considered as submanifolds of a jet bundle), it represents essentially a dual, vector field based formulation of the Cartan-Kähler theory replacing exterior derivatives by Lie brackets.

The Vessiot theory has not attracted much attention. Presentations in a more modern language are contained in [2, 10]; applications have mainly appeared in the context of the Darboux method for solving hyperbolic equations, see e. g. [11]. While a number of textbooks provide a very rigorous analysis of the Cartan-Kähler theory, the above mentioned references (and also Vessiot's original work) are somewhat lacking in this respect. In particular, the question what assumptions are needed has been ignored.

The purpose of the present article is to close this gap and simultaneously to relate the Vessiot theory with the key concepts of the formal theory like involution and formal integrability. We will show that the Vessiot construction succeeds, if and only if it is applied to an involutive system. This result is not surprising, given the well-known fact that the formal theory and the Cartan-Kähler theory are equivalent. However, to our knowledge an explicit proof has never been given. As a by-product, we will provide a new definition for integral elements based on the contact map making also the relations between the formal theory and the Cartan-Kähler theory more transparent.

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2 Formal Theory

We cannot give here a detailed introduction into the formal theory. Our presentation and notations follow [9]; other general references are [5, 8]. For simplicity, we will mainly work in local coordinates, although the whole theory can be expressed in an intrinsic way.

Let $\pi : \mathcal{E} \to \mathcal{X}$ be a (smooth) fibred manifold. We call coordinates $\mathbf{x} = (x^1, \dots, x^n)$ of $\mathcal{X}$ *independent variables* and fibre coordinates $\mathbf{u} = (u^1, \dots, u^m)$ in $\mathcal{E}$ *dependent variables*. Sections[2] $\sigma : \mathcal{X} \to \mathcal{E}$ correspond locally to functions $\mathbf{u} = \mathbf{s}(\mathbf{x})$. Derivatives are written in the form $u^\alpha_\mu = \partial^{|\mu|} u^\alpha / \partial x_1^{\mu_1} \cdots \partial x_n^{\mu_n}$ where $\mu = [\mu_1, \dots, \mu_n]$. Adding the derivatives u^α_μ up to order q (denoted by $\mathbf{u}^{(q)}$) defines a local coordinate system for the q-th order jet bundle $J_q\pi$ which may be considered as the space of truncated Taylor expansions.

The jet bundle $J_q\pi$ admits a number of fibrations. For us particularly important are $\pi^q_{q-1} : J_q\pi \to J_{q-1}\pi$ and $\pi^q : J_q\pi \to \mathcal{X}$. To each section $\sigma : \mathcal{X} \to \mathcal{E}$ locally defined by $\sigma(\mathbf{x}) = \big(\mathbf{x}, \mathbf{s}(\mathbf{x})\big)$ we may associate its prolongation $j_q\sigma : \mathcal{X} \to J_q\pi$, a section of the fibration π^q given by $j_q\sigma(\mathbf{x}) = \big(\mathbf{x}, \mathbf{s}(\mathbf{x}), \partial_{\mathbf{x}}\mathbf{s}(\mathbf{x}), \partial_{\mathbf{xx}}\mathbf{s}(\mathbf{x}), \dots\big)$.

The geometry of $J_q\pi$ is to a large extent determined by its *contact structure*. It can be described in a number of ways. We will use three different approaches. The *contact codistribution* $\mathcal{C}^{(0)}_q \subseteq T^*(J_q\pi)$ consists of all one-forms such that their pull-back by a prolonged section vanishes. Locally, it is spanned by the contact forms[3]

$$\omega^\alpha_\mu = du^\alpha_\mu - u^\alpha_{\mu+1_i} dx^i\ , \qquad 0 \leq |\mu| < q\ . \tag{1}$$

Dually, we may consider the *contact distribution* $\mathcal{C}_q \subseteq T(J_q\pi)$ consisting of all vector fields annihilated by $\mathcal{C}^{(0)}_q$. One easily verifies that it is generated by the fields

$$\begin{aligned} C^{(q)}_i &= \partial_i + u^\alpha_{\mu+1_i}\partial_{u^\alpha_\mu}\ , &\qquad 1 \leq i \leq n\ ,\\ C^\mu_\alpha &= \partial_{u^\alpha_\mu}\ , &\qquad |\mu| = q\ . \end{aligned} \tag{2}$$

Note that the latter fields span the vertical bundle $V\pi^q_{q-1}$ of the fibration π^q_{q-1}. Thus the contact distribution can be split as $\mathcal{C}_q = V\pi^q_{q-1} \oplus \mathcal{H}$. Here the complement $\mathcal{H}$ is an n-dimensional transversal subbundle of $T(J_q\pi)$ and obviously not uniquely determined (though any local coordinate chart induces via the span of the vectors $C^{(q)}_i$ one possible choice). Note that any such complement $\mathcal{H}$ may be considered as the horizontal bundle of a connection on the fibred manifold $\pi^q : J_q\pi \to \mathcal{X}$ (*not* for the fibration π^q_{q-1}!). Following Fackerell [2], we call any connection on π^q whose horizontal bundle consists of contact fields a *Vessiot connection*[4].

[2] Although we will exclusively consider local sections, we will use throughout a "global" notation in order to avoid the introduction of many local neighbourhoods.
[3] Throughout the article we use the convention that a summation over repeated indices is understood.
[4] In the literature the name *Cartan connection* [6] is more common.

As a third approach to the contact structure we consider, following [7], the *contact map*. It is the unique map $\Gamma_q : J_q\pi \times_{\mathcal{X}} T\mathcal{X} \to T(J_{q-1}\pi)$ such that the diagram

$$\begin{array}{ccc} J_q\pi\times_{\mathcal{X}}T\mathcal{X} & \xrightarrow{\Gamma_q} & T(J_{q-1}\pi) \\ & {\scriptstyle ((j_q\sigma)\circ\tau_{\mathcal{X}})\times\mathrm{id}_{T\mathcal{X}}} \nwarrow \quad \nearrow {\scriptstyle T(j_{q-1}\sigma)} & \\ & T\mathcal{X} & \end{array} \tag{3}$$

commutes for any section σ. Because of its linearity over π^q_{q-1}, we may also consider it as a map $\Gamma_q : J_q\pi \to T^*\mathcal{X} \otimes_{J_{q-1}\pi} T(J_{q-1}\pi)$ with the local coordinate form:

$$\Gamma_q : (\mathbf{x}, \mathbf{u}^{(q)}) \mapsto \bigl(\mathbf{x}, \mathbf{u}^{(q-1)}; dx^i \otimes (\partial_{x^i} + u^\alpha_{\mu+1_i}\partial_{u^\alpha_\mu})\bigr) \;. \tag{4}$$

Now one can see that $\operatorname{im}\Gamma_q = \mathcal{C}_{q-1}$ and hence $\mathcal{C}_q = (T\pi^q_{q-1})^{-1}(\operatorname{im}\Gamma_q)$.

Proposition 2.1. *A section $\gamma : \mathcal{X} \to J_q\pi$ is of the form $\gamma = j_q\sigma$ for a section $\sigma : \mathcal{X} \to \mathcal{E}$, if and only if $\operatorname{im}\Gamma_q\bigl(\gamma(x)\bigr) = T_{\gamma(x)}\pi^q_{q-1}\bigl(T_{\gamma(x)}\operatorname{im}\gamma\bigr)$ for all points $x \in \mathcal{X}$ where γ is defined.*

Thus for any section $\sigma : \mathcal{X} \to \mathcal{E}$ the equality $\operatorname{im}\Gamma_{q+1}\bigl(j_{q+1}\sigma(x)\bigr) = \operatorname{im}T_x(j_q\sigma)$ holds and we may say that knowing the $(q+1)$-jet $j_{q+1}\sigma(x)$ of a section σ at some $x \in \mathcal{X}$ is equivalent to knowing its q-jet $\rho = j_q\sigma(x)$ at x plus the tangent space $T_\rho(\operatorname{im} j_q\sigma)$ at this point. This observation will later be the key for the Vessiot theory.

A *differential equation* of order q is a fibred submanifold $\mathcal{R}_q \subseteq J_q\pi$ locally described as the zero set of some smooth functions on $J_q\pi$:

$$\mathcal{R}_q \;:\; \Bigl\{\; \Phi^\tau(\mathbf{x}, \mathbf{u}^{(q)}) = 0 \;, \quad \tau = 1, \dots, t \;. \tag{5}$$

(Note that we do not distinguish between scalar equations and systems). We denote by $\iota : \mathcal{R}_q \hookrightarrow J_q\pi$ the canonical inclusion map. Differentiating every equation in (5) yields the *prolonged equation* $\mathcal{R}_{q+1} \subseteq J_{q+1}\pi$ defined by the equations $\Phi^\tau = 0$ and $D_i\Phi^\tau = 0$ with the formal derivative D_i. Iteration of this process gives the higher prolongations $\mathcal{R}_{q+r} \subseteq J_{q+r}\pi$. A subsequent projection leads to $\mathcal{R}^{(1)}_q = \pi^{q+1}_q(\mathcal{R}_{q+1}) \subseteq \mathcal{R}_q$ which can be a proper submanifold, if integrability conditions are hidden. $\mathcal{R}_q$ is *formally integrable*, if at any prolongation order $r > 0$ the equality $\mathcal{R}^{(1)}_{q+r} = \mathcal{R}_{q+r}$ holds.

A *solution* is a section $\sigma : \mathcal{X} \to \mathcal{E}$ such that its prolongation satisfies $\operatorname{im} j_q\sigma \subseteq \mathcal{R}_q$. In local coordinates, this obviously coincides with the usual notion of a solution. For formally integrable equations it is straightforward to construct order by order formal power series solutions. Otherwise it is very hard to find solutions. A key insight of Cartan was to introduce *infinitesimal solutions* or *integral elements* at a point $\rho \in \mathcal{R}_q$ as subspaces $\mathcal{U}_\rho \subseteq T_\rho\mathcal{R}_q$ which are potentially part of the tangent space of a prolonged solution.

Definition 2.2. *Let $\mathcal{R}_q \subseteq J_q\pi$ be a differential equation. A linear subspace $\mathcal{U}_\rho \subseteq T_\rho\mathcal{R}_q$ is an* integral element *at the point $\rho \in \mathcal{R}_q$, if a point $\hat\rho \in \mathcal{R}_{q+1}$ exists such that $\pi^{q+1}_q(\hat\rho) = \rho$ and $T\iota(\mathcal{U}_\rho) \subseteq \operatorname{im}\Gamma_{q+1}(\hat\rho)$.*

The above definition of an integral element is not the standard one. Usually, one considers the pull-back $\iota^*\mathcal{C}_q^0$ of the contact codistribution or more precisely the differential ideal $\mathcal{I}[\mathcal{R}_q] = \langle\iota^*\mathcal{C}_q^0\rangle_{\text{diff}}$ generated by it (recall that algebraically $\mathcal{I}[\mathcal{R}_q]$ is thus spanned by a basis of $\iota^*\mathcal{C}_q^0$ and the exterior derivatives of the forms in this basis) and an integral element is a subspace on which this ideal vanishes.

Proposition 2.3. *Let $\mathcal{R}_q$ be a differential equation such that $\mathcal{R}_q^{(1)} = \mathcal{R}_q$. A linear subspace $\mathcal{U}_\rho \subseteq T_\rho\mathcal{R}_q$ is an* integral element *at the point $\rho \in \mathcal{R}_q$, if and only if $T\iota(\mathcal{U}_\rho)$ lies transversal to the fibration π^q_{q-1} and every differential form $\omega \in \mathcal{I}[\mathcal{R}_q]$ vanishes on $\mathcal{U}_\rho$.*

Proof. Assume first that $\mathcal{U}_\rho$ is an integral element. Thus there exists a point $\hat\rho \in \mathcal{R}_{q+1}$ such that $\pi^{q+1}_q(\hat\rho) = \rho$ and $T\iota(\mathcal{U}_\rho) \subseteq \operatorname{im}\Gamma_{q+1}(\hat\rho)$. This implies firstly that $T\iota(\mathcal{U}_\rho)$ is transversal to π^q_{q-1} and secondly that every one-form $\omega \in \iota^*\mathcal{C}_q^0$ vanishes on $\mathcal{U}_\rho$, as $\operatorname{im}\Gamma_{q+1}(\hat\rho) \subset (\mathcal{C}_q)_\rho$. Thus there only remains to show that the same is true for the two-forms $d\omega \in \iota^*(d\mathcal{C}_q^0)$.

Choose a section $\gamma : \mathcal{R}_q \to \mathcal{R}_{q+1}$ such that $\gamma(\rho) = \hat\rho$ and define a distribution $\mathcal{D}$ of rank n on $\mathcal{R}_q$ by setting $T\iota(\mathcal{D}_{\tilde\rho}) = \operatorname{im}\Gamma_{q+1}\bigl(\gamma(\tilde\rho)\bigr)$ for any point $\tilde\rho \in \mathcal{R}_q$. Obviously, by construction $\mathcal{U}_\rho \subseteq \mathcal{D}_\rho$. It follows from the coordinate form (4) of the contact map that locally the distribution $\mathcal{D}$ is spanned by n vector fields X_i such that $\iota_*X_i = C_i^{(q)} + \gamma^\alpha_{\mu+1_i}C^\mu_\alpha$ where the coefficients γ^α_ν are the highest-order components of the section γ. Thus the commutator of two such vector fields satisfies

$$\iota_*\bigl([X_i, X_j]\bigr) = \bigl(C_i^{(q)}(\gamma^\alpha_{\mu+1_j}) - C_j^{(q)}(\gamma^\alpha_{\mu+1_i})\bigr)C^\mu_\alpha + \gamma^\alpha_{\mu+1_j}[C_i^{(q)}, C^\mu_\alpha] - \gamma^\alpha_{\mu+1_i}[C_j^{(q)}, C^\mu_\alpha]\,. \tag{6}$$

The commutators in the second line vanish whenever $\mu_i = 0$ or $\mu_j = 0$, respectively. Otherwise we obtain $-\partial_{u^\alpha_{\mu-1_i}}$ and $-\partial_{u^\alpha_{\mu-1_j}}$, respectively. But this implies that the two sums in the second line cancel each other and we find that $\iota_*\bigl([X_i, X_j]\bigr) \in \mathcal{C}_q$. Thus we find for any contact form $\omega \in \mathcal{C}_q^0$: that

$$\iota^*(d\omega)(X_i, X_j) = d\omega(\iota_*X_i, \iota_*X_j) = \iota_*X_i\bigl(\omega(\iota_*X_j)\bigr) - \iota_*X_j\bigl(\omega(\iota_*X_i)\bigr) + \omega\bigl(\iota_*([X_i, X_j])\bigr)\,. \tag{7}$$

Each summand in the last expression vanishes, as all appearing fields are contact fields. Hence any form $\omega \in \iota^*(d\mathcal{C}_q^0)$ vanishes on $\mathcal{D}$ and in particular on $\mathcal{U}_\rho \subseteq \mathcal{D}_\rho$.

For the converse, note that any subspace $\mathcal{U}_\rho \subseteq T_\rho\mathcal{R}_q$ satisfying the imposed conditions is spanned by linear combinations of vectors v_i such that $T\iota(v_i) = C_i^{(q)}|_\rho + \gamma^\alpha_{\mu,i}C^\mu_\alpha|_\rho$ where $\gamma^\alpha_{\mu,i}$ are real coefficients. Now consider a contact form ω^α_ν with $|\nu| = q-1$. Then $d\omega^\alpha_\nu = dx^i \wedge du^\alpha_{\nu+1_i}$. Evaluating the condition $\iota^*(d\omega^\alpha_\nu)|_\rho(v_i, v_j) = d\omega\bigl(T\iota(v_i), T\iota(v_j)\bigr) = 0$ yields the equation $\gamma^\alpha_{\nu+1_i,j} = \gamma^\alpha_{\nu+1_j,i}$. Hence the coefficients are actually of the form $\gamma^\alpha_{\mu,i} = \gamma^\alpha_{\mu+1_i}$ and a section σ exists such that $\rho = j_q\sigma(x)$ and $T_\rho(\operatorname{im} j_q\sigma)$ is spanned by the vectors $T\iota(v_1), \dots, T\iota(v_n)$. But this implies that $\mathcal{U}_\rho$ is an integral element. □

For many purposes the purely geometric notion of formal integrability is not sufficient and one needs the stronger algebraic concept of involution. An intrinsic definition of involution requires the Spencer cohomology. We give here only a simplified coordinate version requiring that one works in "good", so-called δ-regular, coordinates $\mathbf{x}$ (this is not a strong restriction, as generic coordinates are δ-regular and there are possibilities to construct systematically "good" coordinates – see e. g. [3]).

The *(geometric) symbol* of a differential equation $\mathcal{R}_q$ is $\mathcal{N}_q = V\pi^q_{q-1} \cap T\mathcal{R}_q$, i. e. the vertical part of the tangent space to $\mathcal{R}_q$. Locally, $\mathcal{N}_q$ is the solution space of the following linear system of (algebraic) equations in the unknowns v^α_μ (coordinates on $S_q(T^*\mathcal{X}) \otimes V\mathcal{E}$):

$$\mathcal{N}_q \; : \; \left\{ \sum_{\alpha,|\mu|=q} \left(\frac{\partial \Phi^\tau}{\partial u^\alpha_\mu} \right) v^\alpha_\mu = 0 \, . \right. \tag{8}$$

Thee prolonged symbols $\mathcal{N}_{q+r}$ are simply the symbols of the prolonged equations $\mathcal{R}_{q+r}$.

The *class* of a multi-index $\mu = [\mu_1, \dots, \mu_n]$ is the smallest k for which μ_k is different from zero. The columns of the symbol matrix (8) are labelled by the v^α_μ. After ordering them by class, i. e. a column with a multi-index of higher class is always left of one with lower class, we compute a row echelon form. We denote the number of rows where the pivot is of class k by $\beta^{(k)}_q$ and associate with each such row the *multiplicative variables* $x^1, \dots, x^k$. Prolonging each equation only with respect to its multiplicative variables yields independent equations of order $q+1$, as each has a different leading term. If prolongation with respect to the non-multiplicative variables does not lead to additional independent equations of order $q+1$, in other words if

$$\operatorname{rank} \mathcal{N}_{q+1} = \sum_{k=1}^{n} k\beta^{(k)}_q \; , \tag{9}$$

then the symbol $\mathcal{N}_q$ is *involutive* (Cartan test).

The differential equation $\mathcal{R}_q$ is called *involutive*, if it is formally integrable and its symbol is involutive. Involutive equations possess a number of pleasant properties; for our purposes the most important one is the *Cartan-Kähler theorem* asserting the existence and uniqueness of analytic solutions for the formally well-posed initial value problem with an analytic involutive differential equation.

For notational simplicity, we will consider in our subsequent analysis mainly first-order equations $\mathcal{R}_1 \subseteq J_1\pi$. Furthermore, we will assume that any present algebraic (i. e. zeroth-order) equation has been explicit solved, reducing thus the number of dependent variables. From a theoretical point of view this does not really represent a restriction, as any differential equation $\mathcal{R}_q$ can be transformed into an equivalent first-order one and under some mild regularity assumptions the algebraic equations can always be solved locally.

For later use, we define now a local normal form, the *Cartan normal form*, for such an equation. It arises by solving each equation for a derivative u^α_j, the *principal derivative*, and eliminating this derivative from all other equations. Furthermore, the principal derivatives are chosen in such a manner that their classes are as great as possible. All the remaining derivatives are called *parametric*. Ordering the obtained equations by their class, we can decompose into subsystems:

$$u_n^\alpha = \phi_n^\alpha(\mathbf{x}, \mathbf{u}, u_j^\gamma) \qquad \begin{cases} 1 \leq \alpha \leq \beta_1^{(n)} \\ 1 \leq j \leq n \\ \beta_1^{(j)} < \gamma \leq m \end{cases} \tag{10}$$

$$u_{n-1}^\alpha = \phi_{n-1}^\alpha(\mathbf{x}, \mathbf{u}, u_j^\gamma) \qquad \begin{cases} 1 \leq \alpha \leq \beta_1^{(n-1)} \\ 1 \leq j \leq n-1 \\ \beta_1^{(j)} < \gamma \leq m \end{cases} \tag{11}$$

$$\vdots$$

$$u_1^\alpha = \phi_1^\alpha(\mathbf{x}, \mathbf{u}, u_j^\gamma) \qquad \begin{cases} 1 \leq \alpha \leq \beta_1^{(1)} \\ 1 = j \\ \beta_1^{(j)} < \gamma \leq m \end{cases} \tag{12}$$

Note that the values $\beta_1^{(k)}$ are indeed exactly those appearing in the Cartan test (9), as the symbol matrix of a differential equation in Cartan normal form is automatically triangular with the principal derivatives as pivots. The *Cartan characters* of $\mathcal{R}_1$ are defined as $\alpha_1^{(k)} = m - \beta_1^{(k)}$ and thus equal the number of parametric derivatives of class k.

For a differential equation $\mathcal{R}_1$ in Cartan normal form it is possible to perform an involution analysis in closed form. We remark that an effective test of involution proceeds as follows. Each equation in (10) is prolonged with respect to each of its non-multiplicative variables. The arising second-order equations are simplified modulo the original system and the prolongations with respect to the multiplicative variables. The symbol $\mathcal{N}_1$ is involutive, if and only if after the simplification none of the equations is second-order any more. The equation $\mathcal{R}_1$ is involutive, if and only if all new equations simplify to zero, as any remaining first-order equation would be an integrability condition.

In order to apply this test, we set $\mathcal{B} := \{(\alpha, i) \in \mathbb{N}^m \times \mathbb{N}^n : u_i^\alpha \text{ is a principal derivative}\}$ and for each $(\alpha, i) \in \mathcal{B}$ we define $\Phi_i^\alpha := u_i^\alpha - \phi_i^\alpha$. Now a straightforward calculation yields

$$D_j \Phi_i^\alpha = u_{ij}^\alpha - C_j^{(1)}(\phi_i^\alpha) - \sum_{h=1}^{i} \sum_{\gamma=\beta_1^{(h)}+1}^{m} u_{hj}^\gamma C_\gamma^h(\phi_i^\alpha) \; . \tag{13}$$

For $j > i$, the prolongation $D_j \Phi_i^\alpha$ is non-multiplicative, otherwise it is multiplicative.

Now let $j > i$, so that (13) is a non-multiplicative prolongation. According to our test, the symbol $\mathcal{N}_1$ is involutive, if and only if it is possible to eliminate on the right hand side of (13) all second-order derivatives by adding multiplicative prolongations. As a first step we compute the difference $D_j \Phi_i^\alpha - D_i \Phi_j^\alpha - \sum_{h=1}^{i} \sum_{\gamma=\beta_1^{(h)}+1}^{m} C_\gamma^h(\phi_h^\alpha) D_h \Phi_j^\gamma$ eliminating all second-order derivatives explicitly present in (13). Expansion of the formal derivatives yields after a tedious but straightforward computation:

$$\begin{aligned}
&D_j\Phi_i^\alpha - D_i\Phi_j^\alpha + \sum_{h=1}^{i}\sum_{\gamma=\beta_1^{(h)}+1}^{m} C_\gamma^h(\phi_h^\alpha)D_h\Phi_j^\gamma \\
&\quad = C_i^{(1)}(\phi_j^\alpha) - C_j^{(1)}(\phi_i^\alpha) - \sum_{h=1}^{i}\sum_{\gamma=\beta_1^{(h)}+1}^{m} C_\gamma^h(\phi_h^\alpha)C_h^{(1)}(\phi_j^\gamma) \\
&\quad + \sum_{a=1}^{i-1}\sum_{\delta=\beta_1^{(a)}+1}^{m} u_{aa}^\delta\left[\sum_{\gamma=\beta_1^{(a)}+1}^{\beta_1^{(j)}} C_\gamma^a(\phi_a^\alpha)C_\delta^a(\phi_j^\gamma)\right] \\
&\quad + \sum_{1\le a<b<i}\left\{\sum_{\delta=\beta_1^{(a)}+1}^{\beta_1^{(b)}} u_{ab}^\delta\left[\sum_{\gamma=\beta_1^{(b)}+1}^{\beta_1^{(j)}} C_\gamma^b(\phi_b^\alpha)C_\delta^a(\phi_j^\gamma)+\right]\right. \\
&\qquad \left. + \sum_{\delta=\beta_1^{(b)}+1}^{m} u_{ab}^\delta\left[\sum_{\gamma=\beta_1^{(a)}+1}^{\beta_1^{(j)}} C_\gamma^a(\phi_a^\alpha)C_\delta^b(\phi_j^\gamma) + \sum_{\gamma=\beta_1^{(b)}+1}^{\beta_1^{(j)}} C_\gamma^b(\phi_b^\alpha)C_\delta^a(\phi_j^\gamma)\right]\right\} \\
&\quad + \sum_{\substack{a=1\\ b=i}}^{i}\left\{\sum_{\delta=\beta_1^{(a)}+1}^{\beta_1^{(i)}} u_{ai}^\delta\left[-C_\delta^a(\phi_a^\alpha) + \sum_{\gamma=\beta_1^{(i)}+1}^{\beta_1^{(j)}} C_\gamma^i(\phi_i^\alpha)C_\delta^a(\phi_j^\gamma)\right]\right. \\
&\qquad \left. + \sum_{\delta=\beta_1^{(i)}+1}^{m} u_{ai}^\delta\left[-C_\delta^a(\phi_a^\alpha) + \sum_{\gamma=\beta_1^{(i)}+1}^{\beta_1^{(j)}} C_\gamma^i(\phi_i^\alpha)C_\delta^a(\phi_j^\gamma) + \sum_{\gamma=\beta_1^{(a)}+1}^{\beta_1^{(j)}} C_\gamma^a(\phi_a^\alpha)C_\delta^i(\phi_j^\gamma)\right]\right\} \\
&\quad + \sum_{a=1}^{i-1}\sum_{\delta=\beta_1^{(j)}+1} u_{aj}^\delta\left[C_\delta^a(\phi_a^\alpha) + \sum_{\gamma=\beta_1^{(a)}+1}^{\beta_1^{(j)}} C_\gamma^a(\phi_1^\alpha)C_\delta^j(\phi_j^\gamma)\right] \\
&\quad + \sum_{\substack{a=1\\ i+1\le b<j}}^{i}\sum_{\delta=\beta_1^{(b)}}^{m} u_{ab}^\delta\left[\sum_{\gamma=\beta_1^{(a)}+1}^{\beta_1^{(j)}} C_\gamma^a(\phi_1^\alpha)C_\delta^b(\phi_j^\gamma)\right] \\
&\quad + \sum_{\delta=\beta_1^{(j)}+1}^{m} u_{ij}^\delta\left[C_\delta^i(\phi_i^\alpha) - C_\delta^j(\phi_j^\alpha) + \sum_{\gamma=\beta_1^{(i)}+1}^{\beta_1^{(j)}} C_\gamma^i(\phi_i^\alpha)C_\delta^j(\phi_j^\gamma)\right].
\end{aligned} \tag{14}$$

In the first line we collected all terms of lower than second order. Furthermore, none of the now appearing second-order derivatives is of a form that it could be eliminates by adding some multiplicative prolongation. Hence the symbol $\mathcal{N}_1$ is involutive, if and only if all the expressions in square brackets vanish. The differential equation $\mathcal{R}_1$ is involutive, if and only if in addition the first line vanishes, as it represents an integrability condition. Thus (14) provides us with an explicit form of all obstructions to involution for $\mathcal{R}_1$.

3 The Vessiot distribution

By Proposition 2.1, the tangent spaces $T_\rho(\operatorname{im} j_q\sigma)$ of prolonged sections at points $\rho \in J_q\pi$ are always subspaces of the contact distribution $\mathcal{C}_q|_\rho$. If the section σ is a solution of $\mathcal{R}_q$, it furthermore satisfies by definition $\operatorname{im} j_q\sigma \subseteq \mathcal{R}_q$ and hence $T(\operatorname{im} j_q\sigma) \subseteq T\mathcal{R}_q$. These considerations motivate the following construction.

Definition 3.1. *The* Vessiot distribution *of a differential equation $\mathcal{R}_q \subseteq J_q\pi$ is the distribution $\mathcal{V}[\mathcal{R}_q] \subseteq T\mathcal{R}_q$ defined by*

$$T\iota\big(\mathcal{V}[\mathcal{R}_q]\big) = T\iota\big(T\mathcal{R}_q\big) \cap \mathcal{C}_q|_{\mathcal{R}_q} \,. \tag{15}$$

Again, this is not the usual definition found in the literature. But the equivalence to the standard approach is an elementary exercise in computing with pull-backs:

Proposition 3.2. *The Vessiot distribution satisfies $\mathcal{V}[\mathcal{R}_q] = (\iota^*\mathcal{C}_q^0)^0$.*

The Vessiot distribution is not necessarily of constant rank along $\mathcal{R}_q$; for simplicity, we will assume its rank does not vary over the differential equation. Note that the symbol $\mathcal{N}_q$ as the vertical part of $T\mathcal{R}_q$ is always contained in $\mathcal{V}[\mathcal{R}_q]$. In general, $\mathcal{V}[\mathcal{R}_q]$ is not involutive (an exception are formally integrable equations of finite type), but it may contain involutive subdistributions; among these, those of dimension n which are transversal (to the fibration $\mathcal{R}_q \to \mathcal{X}$) are of special interest for us.

Lemma 3.3. *If the section $\sigma : \mathcal{X} \to \mathcal{E}$ is a solution of the equation $\mathcal{R}_q$, then the tangent bundle $T(\operatorname{im} j_q\sigma)$ is an n-dimensional transversal involutive subdistribution of $\mathcal{V}[\mathcal{R}_q]|_{\operatorname{im} j_q\sigma}$. Conversely, if $\mathcal{U} \subseteq \mathcal{V}[\mathcal{R}_q]$ is an n-dimensional transversal involutive subdistribution, then any integral manifold of $\mathcal{U}$ has locally the form $\operatorname{im} j_q\sigma$ for a solution σ of $\mathcal{R}_q$.*

This simple observation forms the basis of Vessiot's approach to the analysis of $\mathcal{R}_q$: he proposed to search for all n-dimensional, transversal involutive subdistributions of $\mathcal{V}[\mathcal{R}_q]$. Before we do this, we first show how integral elements appear in this program.

Proposition 3.4. *Let $\mathcal{U} \subseteq \mathcal{V}[\mathcal{R}_q]$ be a transversal subdistribution of the Vessiot distribution of constant rank k. Then the spaces $\mathcal{U}_\rho$ are k-dimensional integral elements for all points $\rho \in \mathcal{R}_q$ if, and only if, $[\mathcal{U}, \mathcal{U}] \subseteq \mathcal{V}[\mathcal{R}_q]$.*

Proof. Let $\{\omega_1, \dots, \omega_r\}$ be a basis of the codistribution $\iota^*\mathcal{C}_q^0$. Then an algebraic basis of the ideal $\mathcal{I}[\mathcal{R}_q]$ is $\{\omega_1, \dots, \omega_r, d\omega_1, \dots, d\omega_r\}$. Any vector field $X \in \mathcal{U}$ trivially satisfies $\omega_i(X) = 0$ by Proposition 3.2. For arbitrary fields $X_1, X_2 \in \mathcal{U}$, we have $d\omega_i(X_1, X_2) = X_1\big(\omega_i(X_2)\big) - X_2\big(\omega_i(X_1)\big) + \omega_i\big([X_1, X_2]\big)$. The first two summands on the right hand side vanish trivially and the remaining equation implies our claim. □

For obvious reasons, we call a subdistribution $\mathcal{U} \subseteq \mathcal{V}[\mathcal{R}_q]$ satisfying the conditions of Proposition 3.4 an *integral distribution*[5] for the differential equation $\mathcal{R}_q$. Note that generally an integral distribution is *not* integrable; the name only reflects the fact that it is composed of integral elements.

Since the symbol $\mathcal{N}_q$ of the equation $\mathcal{R}_q$ is contained in the Vessiot distribution, we can split the Vessiot distribution into $\mathcal{V}[\mathcal{R}_q] = \mathcal{N}_q \oplus \mathcal{H}$ where $\mathcal{H}$ is some complement. By analogy to the above discussed decomposition of the full contact distribution, this leads naturally to connections: provided $\dim \mathcal{H} = n$, it may be considered as the horizontal bundle of a connection of the fibred manifold $\mathcal{R}_q \to \mathcal{X}$ and we call any such connection a *Vessiot connection* for $\mathcal{R}_q$. The existence of n-dimensional complements is connected to the absence of integrability conditions.

Proposition 3.5. *If the differential equation $\mathcal{R}_q$ satisfies $\mathcal{R}_q^{(1)} = \mathcal{R}_q$, then its Vessiot distribution possesses locally a decomposition $\mathcal{V}[\mathcal{R}_q] = \mathcal{N}_q \oplus \mathcal{H}$ with an n-dimensional complement $\mathcal{H}$.*

Proof. The assumption $\mathcal{R}_q = \mathcal{R}_q^{(1)}$ implies that to every point $\rho \in \mathcal{R}_q$ at least one point $\hat{\rho} \in \mathcal{R}_{q+1}$ with $\pi_q^{q+1}(\hat{\rho}) = \rho$ exists. We choose such a $\hat{\rho}$ and consider $\operatorname{im}\Gamma_{q+1}(\hat{\rho}) \subset T_\rho(J_q\pi)$. By definition of the contact map Γ_{q+1}, this is an n-dimensional transversal subset of $\mathcal{C}_q|_\rho$. Thus there only remains to show that it is also tangential to $\mathcal{R}_q$, as then we can define a complement by $T\iota(H_\rho) = \operatorname{im}\Gamma_{q+1}(\hat{\rho})$. But this tangency is a trivial consequence of $\hat{\rho} \in \mathcal{R}_{q+1}$; using for example the local coordinates expression (4) for Γ_q and a local representation $\Phi^\tau = 0$ of $\mathcal{R}_q$, one immediately sees that the vector $v_i = \Gamma_{q+1}(\hat{\rho}, \partial_{x^i}) \in T_\rho(J_q\pi)$ satisfies $d\Phi^\tau|_\rho(v_i) = D_i\Phi^\tau(\hat{\rho}) = 0$ and thus is tangential to $\mathcal{R}_q$.

Hence we have proven that it is possible to construct for each point $\rho \in \mathcal{R}_q$ a complement $\mathcal{H}_\rho$ such that $\mathcal{V}_\rho[\mathcal{R}_q] = (\mathcal{N}_q)_\rho \oplus \mathcal{H}_\rho$. Now we must show that these complements can be chosen in such a way that the form a distribution (which by definition is smooth). Our assumption $\mathcal{R}_q = \mathcal{R}_q^{(1)}$ implies that the restricted projection $\hat{\pi}_q^{q+1} : \mathcal{R}_{q+1} \to \mathcal{R}_q$ is a surjective submersion, i. e. it defines a fibred manifold. Thus if we choose a local section $\gamma : \mathcal{R}_q \to \mathcal{R}_{q+1}$ and then always take $\hat{\rho} = \gamma(\rho)$, it follows immediately that the corresponding complements $\mathcal{H}_\rho$ define a smooth distribution as required. □

Any n-dimensional complement $\mathcal{H}$ is obviously a transversal subdistribution of $\mathcal{V}[\mathcal{R}_q]$, but not necessarily involutive. Conversely, any n-dimensional subdistribution $\mathcal{H}$ of $\mathcal{V}[\mathcal{R}_q]$

[5] In the literature the terminology "involution" is common for such distributions which, however, is quite confusing in our opinion.

is a possible choice as complement. Hence we may reformulate Vessiot's goal as the construction of all *flat* Vessiot connections. Choosing a "reference" complement $\mathcal{H}_0$ with a basis $\{X_1, \dots, X_n\}$, a basis for any other complement $\mathcal{H}$ arises by adding some vertical fields to the vectors X_i. We will follow this approach in the next section. For the remainder of this section we turn our attention to the choice of a convenient basis of $\mathcal{V}[\mathcal{R}_q]$ that will facilitate our computations.

Since the symbol $\mathcal{N}_q$ is an involutive distribution, there is a basis $(Y_1, Y_2, \dots Y_r)$ for it with $r = \dim \mathcal{N}_q$ whose Lie brackets vanish: $[Y_k, Y_\ell] = 0$ for all $1 \leq k, \ell \leq r$. Since the vertical bundle $V\pi^q_{q-1}$ is also involutive, we can decompose $V\pi^q_{q-1} = \mathcal{N}_q \oplus \mathcal{Z}$ where $\mathcal{Z}$ is again an involutive distribution. $\mathcal{Z}$ can be spanned by vector fields $Z_1, \dots, Z_s$ where $s = \sum_{k=1}^n \beta_q^{(k)}$ equals the number of principal derivatives which are chosen such that we have $[Z_a, Z_b] = 0$ for all $1 \leq a, b \leq s$. In local coordinates, a particularly convenient choice for the fields Y_k and Z_a exists. We choose for any $1 \leq k \leq r$ a *parametric* derivative u^α_μ such that $(\alpha, \mu) \notin \mathcal{B}$ and $Y_k = Y^\alpha_\mu = \iota_*(\partial_{u^\alpha_\mu})$, and for any $1 \leq a \leq s$ there is a *principal* derivative u^α_μ such that $(\alpha, \mu) \in \mathcal{B}$ and $Z_a = Z^\alpha_\mu = \partial_{u^\alpha_\mu}$.

The reference complement $\mathcal{H}_0$ is chosen as follows. Any basis of it must consist of n transversal contact fields. Since the fields C^μ_α are vertical, we can always use a basis $(\tilde{X}_1, \dots, \tilde{X}_n)$ of the form $\tilde{X}_i = C_1^{(q)} + \xi^\alpha_{i\mu} C^\mu_\alpha$ with some coefficient functions $\xi^\alpha_{i\mu}$ chosen such that $\tilde{X}_i$ is tangential to $\mathcal{R}_q$. The fields C^μ_α also span the vertical bundle $V\pi^q_{q-1}$ and hence we may exploit the above decomposition for a further simplification of the basis. By subtracting from each $\tilde{X}_i$ a suitable linear combination of the fields Y_k spanning the symbol $\mathcal{N}_q$, we arrive at a basis $(X_1, \dots, X_n)$ where $X_i = C_i^{(q)} + \xi^\alpha_i Z_a$.

As already mentioned above, generally, the Vessiot distribution $\mathcal{V}[\mathcal{R}_q]$ is not involutive. Hence it is not very surprising that its structure equations will be of great importance later. Since the only non-vanishing Lie brackets of the contact fields are

$$[C_\alpha^{\nu+1_i}, C_i^{(q)}] = \partial_{u^\alpha_\nu}\,, \qquad |\nu| = q-1\,, \tag{16}$$

we may extend the above chosen basis (X_i, Y_k) of $\mathcal{V}[\mathcal{R}_q]$ to a basis of the derived Vessiot distribution $\mathcal{V}'[\mathcal{R}_q]$ by adding vector fields $Z_{s+1}, \dots, Z_t$ where for each a there exists a derivative u^α_μ of order $q-1$ such that $Z_a = \partial_{u^\alpha_\mu}$. By construction, the non-vanishing structure equations of $\mathcal{V}[\mathcal{R}_q]$ take now the form

$$[X_i, X_j] = A^a_{ij} Z_a \qquad \text{and} \qquad [X_i, Y_k] = B^a_{ik} Z_a \tag{17}$$

with some smooth functions A^a_{ij} and B^a_{ik}.

For a first-order equation $\mathcal{R}_1$ with Cartan normal form (10) satisfying the assumptions of Proposition 3.5 it is possible to perform this process explicitly. We choose as reference complement $\mathcal{H}_0$ the linear span of the vector fields

$$X_i = C_i^{(q)} + \sum_{(\alpha,\mu)\in\mathcal{B}} C_i^{(q)}(\phi^\alpha_\mu) C^\mu_\alpha\,. \tag{18}$$

One easily verifies in a rather straightforward computation that this is a valid choice. Using this reference complement, we can explicitly evaluate the Lie brackets (17) on $\mathcal{R}_1$ and obtain for the coefficient A^a_{ij} with $i < j$ and $Z_a = Z^\alpha_i = Z^\alpha_j = \partial_{u^\alpha}$ that

$$A^a_{ij} = \begin{cases} 0 & : (\alpha,i) \notin \mathcal{B} \text{ and } (\alpha,j) \notin \mathcal{B} \\ C^{(1)}_i(\phi^\alpha_j) & : (\alpha,i) \notin \mathcal{B} \text{ and } (\alpha,j) \in \mathcal{B} \\ C^{(1)}_i(\phi^\alpha_j) - C^{(1)}_j(\phi^\alpha_i) & : (\alpha,i) \in \mathcal{B} \text{ and } (\alpha,j) \in \mathcal{B} \end{cases} \tag{19}$$

and for B^a_{ik} with $Y_k = Y^\beta_j$ and $Z_a = \partial_{u^\alpha}$

$$B^a_{ik} = \begin{cases} 0 & : (\alpha,i) \notin \mathcal{B} \text{ and } (\alpha,i) \neq (\beta,j) \\ -1 & : (\alpha,i) \notin \mathcal{B} \text{ and } (\alpha,i) = (\beta,j) \\ -C^j_\beta(\phi^\alpha_i) & : (\alpha,i) \in \mathcal{B} \end{cases} \quad . \tag{20}$$

We collect these coefficients into vectors A_{ij} and matrices B_i where, for $Z_a = \partial_{u^\alpha}$, the m rows are ordered according to increasing α, and the $\dim \mathcal{N}_1 = r$ columns are ordered, according to increasing j, in n blocks (empty for those j with $m = \beta^{(j)}_1$) and within each block according to increasing β ($\beta^{(j)}_1 < \beta \leq m$). Note that for a differential equation with constant coefficients all the A_{ij} vanish and for a maximally overdetermined equation there are no B_i. The matrices B_i have a special form: for any class i, the matrix B_i has $m - \beta^{(i)}_1$ rows where all entries are zero with only one exception: for each $1 \leq \ell_i < m - \beta^{(i)}_1$ we have $B^{\beta^{(i)}_1 + \ell_i}_{i,k} = -\delta_{\ell k}$ where $\ell := \sum_{h=1}^{i-1}(m - \beta^{(h)}_1) + \ell_i$.

The entries in the remaining $\beta^{(i)}_1$ rows are $-C^j_\beta(\phi^\alpha_i)$. Some of these vanish, too: all of the parametric derivatives on the right side of an equation in the Cartan normal form (10) are of a class lower than that of the equation's left side as otherwise we would solve this equation for the derivative of highest class. This means $-C^j_\beta(\phi^\alpha_i) = 0$ whenever $j = \operatorname{class}(u^\beta_j) > \operatorname{class}(u^\alpha_i) = i$, and it follows that for each i, $1 \leq i \leq n$ the matrix B_i looks like

$$B_i = \begin{pmatrix} -C^1_{\beta^1}(\phi^\alpha_i) & \cdots & -C^{i-1}_{\beta^{i-1}}(\phi^\alpha_i) & -C^i_{\beta^i}(\phi^\alpha_i) & 0\cdots0 \\ 0 & \cdots & 0 & -\mathbb{1}_{\alpha^{(i)}_1} & 0\cdots0 \end{pmatrix} . \tag{21}$$

Here, for $1 \leq j \leq i$, we have $\beta^{(j)}_1 + 1 \leq \beta^j \leq m$. The unit block leads immediately to the estimate

$$\alpha^{(i)}_1 = m - \beta^{(i)}_1 \leq \operatorname{rank} B_i \leq \min\{m, \sum_{j=1}^{i} \alpha^{(j)}_1\} . \tag{22}$$

Since the matrices B_i are made up of block matrices and we are going to calculate with these blocks, we introduce the following notation: let for any i, $1 \leq i \leq n$, ${}^{b}_{a}[B_i]^{d}_{c}$ denote the block in B_i consisting of the entries from the ath row to the bth row and from the cth column to the dth column.

4 Flat Vessiot connections

Recall that our goal is the construction of all n-dimensional transversal involutive subdistributions $\mathcal{U}$ within the Vessiot distribution $\mathcal{V}[\mathcal{R}_1]$. Taking (X_i, Y_k) as a basis for $\mathcal{V}[\mathcal{R}_1]$, we make for the basis $(U_1, \dots, U_n)$ of such a distribution $\mathcal{U}$ the ansatz $U_i = X_i + \zeta_i^k Y_k$ with yet undetermined coefficient functions ζ_i^k. This ansatz follows naturally from our considerations above, as the fields X_i span a reference complement to $\mathcal{N}_1$ and all fields Y_k are vertical. Since the fields U_i are in triangular form, the distribution $\mathcal{U}$ is involutive, if and only if their Lie brackets vanish, and using (17) this means:

$$\begin{aligned} [U_i, U_j] &= [X_i, X_j] + \zeta_i^k [Y_k, X_j] + \zeta_j^k [X_i, Y_k] + (U_i(\zeta_j^k) - U_j(\zeta_i^k))Y_k \\ &= (A_{ij}^a - B_{jk}^a \zeta_i^k + B_{ik}^a \zeta_j^k) Z_a + (U_i(\zeta_j^k) - U_j(\zeta_i^k))Y_k = 0 \;. \end{aligned} \tag{23}$$

By definition of the Y_k and Z_a, these fields are linearly independent, so their coefficients must vanish for $\mathcal{U}$ to be involutive. Thus (23) yields two sets of conditions for the coefficient functions ζ_i^k: a system of algebraic equations

$$A_{ij}^a - B_{jk}^a \zeta_i^k + B_{ik}^a \zeta_j^k = 0 \;, \qquad \begin{cases} 1 \leq a \leq t \;, \\ 1 \leq i < j \leq n \end{cases} \tag{24}$$

and a system of differential equations

$$U_i(\zeta_j^k) - U_j(\zeta_i^k) = 0 \;, \qquad \begin{cases} 1 \leq k \leq r \;, \\ 1 \leq i < j \leq n \;. \end{cases} \tag{25}$$

The vector fields Y_k lie in $\mathcal{V}[\mathcal{R}_1]$. Thus, according to Proposition 3.4, $\mathcal{U}$ is an integral distribution, if and only if the coefficients ζ_k^i satisfy the algebraic conditions (24). This observation permits us immediately to reduce the number of unknowns in our ansatz. Assume that we have values $1 \leq i, j \leq n$ and $1 \leq \alpha \leq m$ such that both (α, i) and (α, j) are contained in $\mathcal{B}$, i. e. u_i^α and u_j^α are both parametric derivatives (and thus obviously the second-order derivative u_{ij}^α, too). Then there exist two symbol fields $Y_k = \iota_*(\partial_{u_i^\alpha})$ and $Y_l = \iota_*(\partial_{u_j^\alpha})$. Now it follows from the coordinate form (4) of the contact map that $\mathcal{U}$ can be an integral distribution, if and only if $\zeta_j^k = \zeta_i^l$.

As the unknowns ζ_k^j may be understood as labels for the columns of the matrices B_h, this identification leads to a contraction of these matrices. We introduce now contracted matrices $\hat{B}_h$ which arise as follows: whenever $\zeta_j^k = \zeta_i^l$ then the corresponding columns of B_h are added. Similarly, we introduce reduced vectors $\hat{\zeta}_h$ where the redundant components are left out. From now on we always understand that in the equations above this reduction has been performed.

Now the question arises, when the combined system (24,25) has solutions. We begin by analysing the algebraic part (24). As a system for the vectors $\hat{\zeta}_i$, we seek to build a solution step by step with i increasing. Thus we begin the construction of the integral distribution $\mathcal{U}$ by first choosing an arbitrary vector field U_1 and then aiming for another vector field U_2 such that $[U_1, U_2] \in \mathcal{V}[\mathcal{R}_q]$. During the construction of U_2 we regard the components of the vector $\hat{\zeta}_1$ as given parameters and the components of $\hat{\zeta}_2$ as the only unknowns of the system

$$\hat{B}_1\hat{\zeta}_2 = \hat{B}_2\hat{\zeta}_1 - A_{12} \ . \tag{26}$$

Since the components of $\hat{\zeta}_1$ are not considered as unknowns, the system (26) must not lead to any restrictions for the coefficients $\hat{\zeta}_1^k$. Obviously, this is the case, if and only if

$$\operatorname{rank} \hat{B}_1 = \operatorname{rank} (\hat{B}_1 \quad \hat{B}_2) \ . \tag{27}$$

Assuming that (27) holds, the system (26) is solvable, if and only if it satisfies the augmented rank condition

$$\operatorname{rank} \hat{B}_1 = \operatorname{rank} (\hat{B}_1 \quad \hat{B}_2 \quad -A_{12}) \ . \tag{28}$$

Now we proceed by iteration. Given $i-1$ vector fields $U_1, U_2, \dots U_{i-1}$ of the required form spanning an involutive subdistribution of $\mathcal{V}[\mathcal{R}_1]$, we construct the next vector field U_i by solving the system

$$\begin{aligned} \hat{B}_1\hat{\zeta}_i &= \hat{B}_i\hat{\zeta}_1 - A_{1i} \\ &\vdots \\ \hat{B}_{i-1}\hat{\zeta}_i &= \hat{B}_i\hat{\zeta}_{i-1} - A_{i-1,i} \ . \end{aligned} \tag{29}$$

Again we consider only the components of the vector $\hat{\zeta}_i$ as unknowns and (29) may not imply any further restrictions on the components of the vectors $\hat{\zeta}_j$ for $1 \leq j < i$. The corresponding rank condition is

$$\operatorname{rank} \begin{pmatrix} \hat{B}_1 \\ \hat{B}_2 \\ \vdots \\ \hat{B}_{i-1} \end{pmatrix} = \operatorname{rank} \begin{pmatrix} \hat{B}_1 & \hat{B}_i & & & \\ \hat{B}_2 & & \hat{B}_i & & 0 \\ \vdots & 0 & & \ddots & \\ \hat{B}_{i-1} & & & & \hat{B}_i \end{pmatrix} . \tag{30}$$

Assuming that it holds, (29) is solvable for the components of $\hat{\zeta}_i$, if and only if it satisfies the augmented rank condition

$$\operatorname{rank} \begin{pmatrix} \hat{B}_1 \\ \hat{B}_2 \\ \vdots \\ \hat{B}_{i-1} \end{pmatrix} = \operatorname{rank} \begin{pmatrix} \hat{B}_1 & \hat{B}_i & & & & -A_{1i} \\ \hat{B}_2 & & \hat{B}_i & & 0 & -A_{2i} \\ \vdots & 0 & & \ddots & & \vdots \\ \hat{B}_{i-1} & & & & \hat{B}_i & -A_{i-1,i} \end{pmatrix} . \tag{31}$$

The following theorem relates the satisfaction of these rank conditions and thus the solvability of the algebraic system (24) by the above described step by step process to intrinsic properties of the differential equation $\mathcal{R}_1$ and its symbol $\mathcal{N}_1$.

Theorem 4.1. *Assume that δ-regular coordinates have been chosen for the differential equation $\mathcal{R}_1$. The rank condition (30) is satisfied for all $1 \le i \le n$, if and only if the symbol $\mathcal{N}_1$ is involutive. The augmented rank condition (31) holds for all $1 \le i \le n$, if and only if the differential equation $\mathcal{R}_1$ is involutive.*

Proof. In order to prove (30), we transform the matrices into row echelon form. Since each matrix $\hat{B}_i$ contains a unit block, there is an obvious way to do this. We describe the procedure using the above introduced notation for subblocks. As we shall see, the relevant entries in this row echelon form are the coefficients of the second-order derivatives u^δ_{ab} in (14) and therefore their vanishing is equivalent to involution of the symbol $\mathcal{N}_1$.

We start with $i = 1$, i. e. with (27). Since $\hat{B}_1$ is a negative unity matrix of $\alpha_1^{(1)}$ rows with a $\beta_1^{(1)} \times \alpha_1^{(1)}$-matrix stacked upon it and only zeros for all other entries, we have $\operatorname{rank}(\hat{B}_1) = \alpha_1$. Next, we transform the matrix $(\hat{B}_1\ \hat{B}_2)$ into row echelon form using the special structure of the matrices $\hat{B}_i$ as given in equation (21); the blocks are replaced in this way:

$$
{}_{1}^{\beta_1^{(1)}}[\hat{B}_1]_1^{\alpha_1^{(1)}} \leftarrow {}_{1}^{\beta_1^{(1)}}[\hat{B}_1]_1^{\alpha_1^{(1)}} + {}_{1}^{\beta_1^{(1)}}[\hat{B}_1]_1^{\alpha_1^{(1)}} \cdot {}_{\beta_1^{(1)}+1}^{m}[\hat{B}_1]_1^{\alpha_1^{(1)}} , \tag{32}
$$

$$
{}_{1}^{\beta_1^{(1)}}[\hat{B}_2]_1^{\alpha_1^{(1)}} \leftarrow {}_{1}^{\beta_1^{(1)}}[\hat{B}_2]_1^{\alpha_1^{(1)}} + {}_{1}^{\beta_1^{(1)}}[\hat{B}_1]_1^{\alpha_1^{(1)}} \cdot {}_{\beta_1^{(1)}+1}^{m}[\hat{B}_2]_1^{\alpha_1^{(1)}} , \tag{33}
$$

$$
{}_{1}^{\beta_1^{(1)}}[\hat{B}_2]_{\alpha_1^{(1)}+1}^{\alpha_1^{(1)}+\alpha_1^{(2)}} \leftarrow {}_{1}^{\beta_1^{(1)}}[\hat{B}_2]_{\alpha_1^{(1)}+1}^{\alpha_1^{(1)}+\alpha_1^{(2)}} + {}_{1}^{\beta_1^{(1)}}[\hat{B}_1]_1^{\alpha_1^{(1)}} \cdot {}_{\beta_1^{(1)}+1}^{m}[\hat{B}_2]_{\alpha_1^{(1)}+1}^{\alpha_1^{(1)}+\alpha_1^{(2)}} . \tag{34}
$$

If, for the sake of simplicity, we use the same names for the changed blocks, then we have

$$
{}_{1}^{\beta_1^{(1)}}[\hat{B}_2]_1^{\alpha_1^{(1)}} = \left(-C^1_\delta(\phi^\alpha_2) + \sum_{\gamma=\beta_1^{(1)}+1}^{\beta_1^{(2)}} C^1_\gamma(\phi^\alpha_1) C^1_\delta(\phi^\gamma_2) \right)_{\substack{1 \le \alpha \le \beta_1^{(1)} \\ \beta_1^{(1)}+1 \le \delta \le m}} , \tag{35}
$$

$$
{}_{1}^{\beta_1^{(1)}}[\hat{B}_2]_{\alpha_1^{(1)}+1}^{\alpha_1^{(1)}+\alpha_1^{(2)}} = \left(-C^2_\delta(\phi^\alpha_2) + \sum_{\gamma=\beta_1^{(1)}+1}^{\beta_1^{(2)}} C^1_\gamma(\phi^\alpha_1) C^2_\delta(\phi^\gamma_2) + C^1_\delta(\phi^\alpha_1) \right)_{\substack{1 \le \alpha \le \beta_1^{(1)} \\ \beta_1^{(2)}+1 \le \delta \le m}} . \tag{36}
$$

A comparison with the obstructions to involution obtained by evaluating (14) for $i = 1$ and $j = 2$ shows that all these entries vanish, if and only if the obstructions vanish. It follows that the first $\beta_1^{(1)}$ rows of the matrix $(\hat{B}_1\ \hat{B}_2)$ are zero. The last $\alpha_1^{(1)}$ rows begin with the block $-\mathbb{1}_{\alpha_1^{(1)}}$ and hence $\operatorname{rank}(\hat{B}_1\ \hat{B}_2) = \alpha_1^{(1)} = \operatorname{rank} \hat{B}_1$. Thus we may conclude that the rank condition (27) holds, if and only if no non-multiplicative prolongation $D_2\Phi^a_1$ leads to an obstruction of involution.

The claim for the augmented condition (28) follows from the explicit expression (19) for the entries A^a_{ij}. Performing the same computations as above described with the augmented system yields as additional relevant entries exactly the integrability conditions arising from (14) evaluated for $i = 1$ and $j = 2$. Hence (28) holds, if and only if no non-multiplicative prolongation $D_2\Phi_1^a$ yields an integrability condition.

As one might expect from the above considerations for $i = 1$, the analysis of (30) for a general $1 \leq i \leq n$ will require the non-multiplicative prolongations $D_i\Phi_1^\alpha, D_i\Phi_2^\alpha, \dots D_i\Phi_{i-1}^\alpha$. It follows trivially from the block form (21) of the matrices B_i that the rank of the matrix on the left hand side of (30) is $\sum_{k=1}^{i-1} \alpha_1^{(k)}$.

For lack of space we skip the details for the general case. We follow the same steps as in the case $i = 1$. The transformation of the matrix on the right hand side of (30) can be described using block matrices, and the resulting matrix in row echelon form has as its entries in the rows where no unit block appears the coefficients of the second-order derivatives in (14). Thus we may conclude again that satisfaction of (30) is equivalent to the fact that in the non-multiplicative prolongations $D_i\Phi_1^\alpha, \dots, D_i\Phi_{i-1}^\alpha$ no obstructions to involution arise. In the case of the augmented conditions (31), it follows again from the explicit expression (19) for the entries A_{ij}^a that the additional relevant entries are identical with the potential integrability conditions produced by the non-multiplicative prolongations $D_i\Phi_1^\alpha, \dots, D_i\Phi_{i-1}^\alpha$.

At this point it becomes apparent why we had to introduce the contracted matrices $\hat{B}_i$. As we are dealing with smooth functions, partial derivatives commute: $u_{ij}^\alpha = u_{ji}^\alpha$. In (14) each obstruction to involution actually consists of two parts: one arises as coefficient of u_{ij}^α, the other one as coefficient of u_{ji}^α. Of course, we do not see this in (14) because of the commutativity of the derivatives. However, in the matrices B_i these two parts appear in different columns and in general the rank condition (30) will not hold, if we replace the contracted matrices $\hat{B}_i$ by the original matrices B_i (see the example below). The effect of the contraction is to combine the two parts in order to obtain the right rank. □

There remains to analyse the solvability, if we add the differential system (25). We first note that one can show in a straightforward computation that (25) alone is again an involutive system. If the original equation $\mathcal{R}_1$ is analytic, then the quasi-linear system (25) is analytic, too. Thus we may apply the Cartan-Kähler theorem to it which guarantees the existence of solutions.

The problem is that the combined system (24,25) is in general not involutive, as the prolongation of the algebraic equations (24) leads to additional differential equations. Instead of analysing the effect of these integrability conditions, we proceed as follows. If we assume that $\mathcal{R}_1$ is involutive, then we know from Theorem 4.1 that the algebraic equations (24) are solvable. In the proof of the theorem we even produced an explicit row echelon form of the system matrix which we can now exploit to eliminate some of the unknowns $\hat{\zeta}_i^k$ as a linear combinations of the remaining ones.

Theorem 4.2. *Assume that δ-regular coordinates have been chosen for the differential equation $\mathcal{R}_1$ and that $\mathcal{R}_1$ is analytic. Then the combined system (24,25) is solvable.*

Proof. Following the strategy outlined above, we eliminate some of the unknowns $\hat{\zeta}_i^k$. Because of the simple structure of (25), it turns out that we must take a closer look only at those equations where the leading derivative is of one of the unknowns we eliminate. A somewhat lengthy but straightforward computation shows that these equations actually vanish. The remaining equations still form an involutive system. Thus we eventually arrive at an analytic involutive differential equation for the coefficient functions $\hat{\zeta}_i^k$ which is solvable according to the Cartan-Kähler theorem. □

Example 4.3. *Consider the first-order equation*

$$\mathcal{R}_1 : \begin{cases} u_t = v_t = w_t = u_s = 0 \ , & v_s = 2u_x + 4u_y \ , \\ w_s = -u_x - 3u_y \ , & u_z = v_x + 2w_x + 3v_y + 4w_y \end{cases} . \tag{37}$$

It is obviously formally integrable, and its symbol is involutive with $\dim \mathcal{N}_1 = 8$*. Thus* $\mathcal{R}_1$ *is an involutive equation. For the matrices* B_i*, all of which are* 3×8*-matrices, we have*

$$\begin{gathered} B_1 = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} , \qquad B_2 = \begin{pmatrix} 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \end{pmatrix} , \\ B_3 = \begin{pmatrix} 0 & -1 & -2 & 0 & -3 & -4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix} , \qquad B_4 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -2 & 0 & 0 & -4 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 3 & 0 & 0 & 0 & 0 \end{pmatrix} , \qquad B_5 = 0_{3\times 8} . \end{gathered} \tag{38}$$

For the first two steps in the construction of the fields U_i*, the rank conditions are trivially satisfied even for the non-contracted matrices. But not so in the third step where we have in the row echelon form of the arising* 9×32*-matrix in the 7th row zero entries throughout except in the 12th column (where we have* -2*) and in the 17th column (where we have* 2*). As a consequence, we obtain the equality* $\zeta_1^4 = \zeta_2^1$ *and the rank condition for this step does not hold. However, since both* u_x *and* u_y *are parametric derivatives and in our ordering* $Y_1 = \iota_*(\partial_{u_x})$ *and* $Y_4 = \iota_*(\partial_{u_y})$*, this equality is already taken into account in our reduced ansatz and for the matrices* $\hat{B}_i$ *the rank condition is satisfied.*

Note that the rank condition is first violated when the rank reaches the symbol dimension (which is 8*). From then on, the rank of the left matrix in (30) stagnates at* $\dim \mathcal{R}_1$ *while the rank of the augmented matrix may rise further. The entries breaking the rank condition only differ by their sign, since they correspond to coefficient sums in (14) that do the same.*

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