Errata and Comments

Nobody is perfect and thus my book *Involution* (Springer-Verlag 2010) contains of course errors. Those that have been found before the date at the bottom are contained in the following list together with some additional comments. The list is ordered by chapter. A negative line number is taken from the bottom; displayed formulae are not counted as lines.

Chapter 2: *(Formal Geometry of Differential Equations)*

Example 2.3.7: An unpleasant "feature" of the given example is that the case $u_{xx} = (x/y)^2$ yields an inconsistent equation (further computations lead to the equation $x = 0$). Here is a modified version leading to two distinct consistent cases:

$$y^5 u_{yy} + \frac{1}{2}u_{xx}^2 = 0\,, \tag{1a}$$

$$y u_{xy} - x u_{xx} = 0\,. \tag{1b}$$

The following linear combination of differential consequences yields then an equation that factors:

$$xy^3 D_y(1\text{b}) + y^4 D_x(1\text{b}) - D_x(1\text{a}) = (u_{xx} - x^2y^3)u_{xxx} = 0\,.$$

Completion of the original equations augmented by one of the factors yields two involutive equations of finite type. The first one is of third order and given by the following equations:

$$u_{xxx} = 0\,, \qquad y^5 u_{yy} + \frac{1}{2}u_{xx}^2 = 0\,,$$
$$y u_{xxy} - u_{xx} = 0\,, \qquad y u_{xy} - x u_{xx} = 0\,.$$

Its general solution depends on four parameters:

$$u(x,y) = ax^2y + \frac{a^2}{2} + bx + cy + d\,.$$

The second case yields a second-order equation:

$$u_{xx} = x^2y^3\,, \quad u_{xy} = x^3y^2\,, \quad u_{yy} = \frac{1}{2}x^4y$$

with a general solution depending only on three parameters:

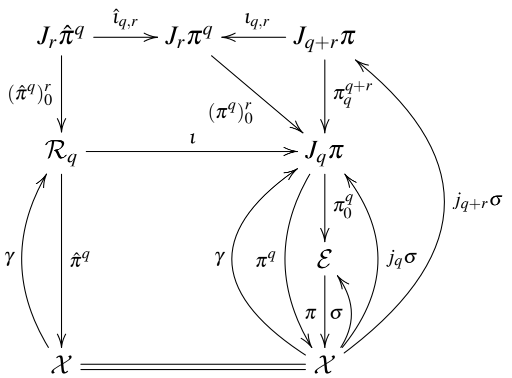
$$u(x,y) = \frac{1}{12}x^4y^3 + ex + fy + g\,.$$

Thus the two cases differ not only in the order at which one obtains an involutive equation, but also in the size of the solution space.

In general, the treatment of fully nonlinear equations, i. e. equations which are nonlinear in the derivatives of maximal order, is highly nontrivial. If the nonlinearity is of a polynomial form (which is the case in most practically occuring differential equations), the *Thomas decomposition* allows to combine the completion to involution (in the form of determining a passive system for the Janet division) with an algorithmic treatment of the arising case distinctions. A modern description of this theory can be found in [10]; an implementation in MAPLE is described in [2].

Definition 2.3.15: I have been repeatedly asked whether it is obvious that one always has the inclusion $\mathcal{R}_{q+r}^{(1)} \subseteq \mathcal{R}_{q+r}$? There are (at least) two ways to see this. The simplest approach consists of studying the corresponding local systems. By definition, $\mathcal{R}_{q+r}^{(1)} = \pi_{q+r}^{q+r+1}(\mathcal{R}_{q+r+1})$. Assume that we are given some local system describing $\mathcal{R}_{q+r}$; then a local system for $\mathcal{R}_{q+r+1}$ is obtained by *adding* all formal derivatives of the equations in this local system. Obviously, the original equations "survive" the projection back to order $q+r$, as they are of order $q+r$ or less, and thus the arising local system of $\mathcal{R}_{q+r}^{(1)}$ contains at least these equations (and potentially some more, if integrability conditions exist) which proves the claimed inclusion.

One can also give an intrinsic argument, although it is a bit more awkward due to the fact that the intrinsic definition (2.51) of a prolonged differential equation is rather cumbersome. For notational simplicity, we will now show that $\mathcal{R}_q^{(r)} \subseteq \mathcal{R}_q$. The following diagram exhibits all needed maps.



As the projection $\hat{\pi}^q : \mathcal{R}_q \to \mathcal{X}$ is just a restriction of the canonical projection $\pi^q : J_q\pi \to \mathcal{X}$, we may consider any section $\gamma \in \Gamma_{loc}(\hat{\pi}^q)$ simultaneously as a section in $\Gamma_{loc}(\pi^q)$ and consequently the jet bundle $J_r\hat{\pi}^q$ as a subset of $J_r\pi^q$ (making $\hat{\iota}_{q,r}$ a simple inclusion map). Now we take a point $[\gamma]_{x_0}^{(r)} \in J_r\hat{\pi}^q$ and assume that it is also contained in $\operatorname{im}\iota_{q,r}$ so that it lies in the intersection defining $\mathcal{R}_{q+r} \subseteq J_{q+r}\pi$. This assumption implies that a section $\sigma \in \Gamma_{loc}(\pi)$ exists with $[\gamma]_{x_0}^{(r)} = [j_q\sigma]_{x_0}^{(r)}$ and thus $[\gamma]_{x_0}^{(r)} = \iota_{q,r}\big([\sigma]_{x_0}^{(q+r)}\big)$. Now we find on one hand that $[\sigma]_{x_0}^{(q+r)} \in \mathcal{R}_{q+r}$. On the other hand, it follows from $\operatorname{im}\gamma \subseteq \mathcal{R}_q$ that $[\sigma]_{x_0}^{(q)} \in \mathcal{R}_q$. As the section γ was arbitrary, this observation entails that every point in $\mathcal{R}_{q+r}$ lies over a point in $\mathcal{R}_q$ and hence our assertion $\pi_q^{q+r}(\mathcal{R}_{q+r}) \subseteq \mathcal{R}_q$.

Example 2.4.2: In order to be dimensionally consistent, one should include in the incompressible Navier-Stokes equations (2.88) the constant fluid density ρ. Equation (2.88a) then reads

$$\rho \mathbf{u}_t + \rho(\mathbf{u} \cdot \nabla)\mathbf{u} = \nu \triangle \mathbf{u} - \nabla p \,.$$

However, in mathematics it is common to set $\rho = 1$, as mathematicians simply ignore dimensions.

Chapter 3: *Involution I: Algebraic Theory*

Figure 3.1: In the left half of the diagramm, there should also be a vertical arrow starting at the multi index $[2,0]$, as for the full cones every direction is "allowed".

Lemma 3.1.19: In the proof the notations are not fully consistent with the use of multi indices instead of terms. In the third line, a rigorous formulation is $x^\mu = \operatorname{lcm}\{x^\nu \mid \nu \in \mathcal{B}\}$. In (3.4) the intersection with $\mathbb{N}_0^n$ is unnecessary; it suffices to write $\tilde{\nu} \in \langle \mathcal{B} \rangle$.

Lemma 3.4.14: The lemma claims an equivalence between two statements; however, only one direction is correct. The direction considered as "obvious" in the proof is in fact wrong, as the following simple counterexample demonstrates. Consider the set $\mathcal{F} = \{1 + x_1x_2,\ x_1x_2\}$. Obviously, it is *not* involutively head autoreduced for any term order or involutive division. Nevertheless, it induces a direct sum decomposition of the involutive span $\langle \mathcal{F} \rangle_{L,\prec}$.

Chapter 4: *(Completion to Involution)*

Lemma 4.1.5: In the fifth line of the proof wrong variable names are used. The correct definition of the position k is $k = \max\{\ell \mid \nu_\ell^{(i)} \neq \nu_\ell^{(i+1)}\}$.

Lemma 4.1.8: As constructivity includes continuity, the correct statement of the lemma is that any globally defined *continuous* division is constructive.

Lemma 4.3.2: This lemma is very useful in many theoretical considerations, as it implies that for any ideal $\mathcal{I}$ with a Pommaret basis a degree q exists such that $\mathcal{I}_{\geq q}$ possesses a Pommaret basis where all generators are of degree q. This property seems to be particular for the Pommaret division and does not hold for the Janet division. The following simple counterexample is due to Mario Albert and Matthias Fetzer (private communication).

Consider the monomial ideal $\mathcal{I} = \langle x_1, x_2^2 \rangle \lhd \Bbbk[x_1,x_2,x_3]$. Then for any degree $q \geq 2$ the minimal basis of $\mathcal{I}_{\geq q}$ consists simply of all terms of degree q contained in $\mathcal{I}$. Two elements of this set are the generators $h_1 = x_1x_3^{q-1}$ and $h_2 = x_2^2x_3^{q-2}$. The first one is the only term in $\mathcal{I}_q$ with x_3-degree $q-1$; all other generators are of lower degree in x_3. Thus with respect to the Janet division x_3 is multiplicative only for h_1. But this observation immediately implies that no term of degree q can be an involutive divisor of the non-multiplicative product x_3h_2, as its x_3-degree is $q-1$ and h_1 is not a divisor. Hence, it is not possible to find a Janet basis consisting only of elements of the same degree.

Corollary 4.3.10: In the definition of the set $\mathcal{B}_P$, we must of course require that $\mu \neq \nu$, i. e. ν must have a *proper* Pommaret divisor in $\mathcal{B}$.

Example 4.3.18: As one can see in Example 4.3.14, the transformed and autoreduced basis is $\tilde{\mathcal{F}}^{\triangle} = \{\tilde{z}^2 - \tilde{x}\tilde{y},\ \tilde{y}\tilde{z},\ \tilde{y}^2\}$. Obviously, it is not possible that linear terms arise in the generators as printed.

Section 4.3: We discuss here the problem of δ-regularity only in the context of polynomial ideals. But it is of course also highly relevant for differential equations. In a purely algebraic setting, this concerns for example the computation of Pommaret bases for ideals in a ring of linear differential operators $\mathcal{D} = \mathbb{F}[\partial_{x^1}, \dots, \partial_{x^n}]$ where $\mathbb{F}$ is some field of functions in the variables $x^1, \dots, x^n$. More generally, it is shown in Proposition 7.1.19 how for arbitrary differential equations the notion of an involutive symbol is related to Pommaret bases.

Any solution of the problem of δ-regularity for polynomial ideals can immediately also be applied to differential equations (and vice versa). The only point is that one must use the *contragredient* transformation: if $\bar{\mathbf{x}} = A\mathbf{x}$ represents a transformation to δ-regular coordinates for an ideal in the polynomial ring $\mathcal{P} = \Bbbk[x^1, \dots, x^n]$, then the right transformation for the corresponding ideal in the ring $\mathcal{D}$ is $\bar{\mathbf{x}} = A^{-t}\mathbf{x}$, i. e. we must take the transposed of the inverse matrix of A. This fact follows immediately from the different transformation properties of a monomial x^μ and the corresponding differential operator $\partial^{|\mu|}/\partial x^\mu$ (or the derivative u_μ, respectively).

For a concrete instance consider Example 4.3.18 where we saw that the transformation $\tilde{x} = z$, $\tilde{y} = y + z$ and $\tilde{z} = x$ yields δ-regular coordinates for the polynomial ideal $\langle \underline{z^2} - y^2 - 2x^2,\ \underline{xz} + xy,\ \underline{yz} + y^2 + x^2 \rangle \subset \Bbbk[x, y, z]$. If instead we had considered the linear differential system $u_{zz} - u_{yy} - 2u_{xx} = 0$, $u_{xz} + u_{xy} = 0$ and $u_{yz} + u_{yy} + u_{xx} = 0$ obtained by substituting derivatives for the monomials, then we would have to use the transformation $\tilde{x} = -y + z$, $\tilde{y} = y$ and $\tilde{z} = x$ in order to find exactly the same transformation behaviour as in the polynomial case.

Algorithm 4.6: Line /10/ of this algorithm for computing a minimal involutive basis must be replaced by the following line:

$$\mathcal{Q} \leftarrow \mathcal{Q} \cup \mathcal{H}' \cup \{x \star h \mid h \in \mathcal{H},\ x \in \overline{X}_{L,\mathcal{H},\prec}(h)\}\ .$$

Only then the proof of Theorem 4.4.4. is correct: in order to ensure local involution upon termination we must add to $\mathcal{Q}$ the non-multiplicative products of *all* elements of $\mathcal{H}$ and not only of the new element g. Because of the changes of $\mathcal{H}$ made in the previous line, we must generally expect that for the elements that remain in $\mathcal{H}$ variables which previously were multiplicative become now non-multiplicative and hence the corresponding product must be added to $\mathcal{Q}$.

While this correction makes the algorithm more expensive, as it enlarges the set $\mathcal{Q}$, one can also achieve a simple optimisation by modifying Line /9/. In its current form, it moves all elements of $\mathcal{H}$ with a leading exponent larger than $\mathrm{le}_\prec g$ to $\mathcal{Q}$. However, a closer look at the proof of Theorem 4.4.4 quickly reveals that it suffices to move only those elements with a leading exponent which is a multiple of $\mathrm{le}_\prec g$, i. e. we may replace the first statement in Line /9/ by

$$\mathcal{H}' \leftarrow \left\{ h \in \mathcal{H} \mid \ \mathrm{le}_\prec g \mid \mathrm{le}_\prec h \right\} .$$

Generally, this modification will have the effect that much less generators are moved which should significantly increase the efficiency of the algorithm. The correctness follows from the following simple considerations. It is not necessary to have $\max_\prec \mathcal{H} \prec \min_\prec \mathcal{Q}$. By the structure of the algorithm, the set $\mathcal{H}$ is always involutive up to the leading exponent of $\min_\prec \mathcal{Q}$ and this suffices to invoke Lemma 4.2.6 in the termination proof. No other part of the proof of Theorem 4.4.4 is affected by this modification.

Chapter 5: *(Structure Analysis of Polynomial Module)*

Algorithm 5.1: The given algorithm has two minor problems. Opposed to what is written in the main text, it is not guaranteed that the set $\mathcal{B}'_q$ constructed in Line /6/ is a minimal basis of the monoid ideal generated by it. Hence it must be minimised before the recursive call in the next line. Furthermore, it may happen that $\mathcal{B}'_q = \emptyset$ and thus the algorithm must be adapted to handle this trivial case. A corrected version is given in Algorithm 1 where `Minimise` denotes a procedure for extracting the minimal basis from an arbitrary monomial basis.

Proposition 5.1.4/Algorithm 5.2: The proof and the algorithm based on it can be slightly simplified. If we define the numbers q_{k-1} not as the maximum but as the minimum $\min_{\mu \in (d_k,\dots,d_n)} \mu_{k-1}$, then we can simply say that in each step the set $\overline{\mathcal{B}}$ is enlarged by all multi indices $\nu = [0,\dots,0,q,d_k,\dots,d_n]$ with $0 \le q < q_{k-1}$. This leads to the same result as the given description. Indeed, if this minimum is smaller than the maximum $\bar{q}_{k-1}$, then for all values $q_{k-1} \le q \le \bar{q}_{k-1}$ the subset $(q,d_k,\dots,d_n)$ must be non-empty, as for any multi index $\mu \in (q_{k-1},d_k,\dots,d_n)$ the index $k-1$ is non-multiplicative by definition of the Janet division and hence $\mu + 1_{k+1}$ must be an element of the Janet basis $\mathcal{B}_J$. Iteration proves the claim.

Applying this simplification to Algorithm 5.2 is trivial. In Line /6/ max must be replaced by min and the iteration must be ended one step before this minimum. Furthermore, Line /7/ is simply erased.

There is a typo in Line /8/: n must be replaced by d_n.

Proposition 5.1.6: In the statement of the proposition "Pommaret basis of degree q" must be replaced by "Pommaret basis of degree $\le q$". Otherwise, the "only if" part is not true, as one can see from its proof where it cannot be guaranteed that the final Pommaret basis is still of degree q.

Algorithm 1 Complementary decomposition (from minimal basis)

Input: minimal basis $\mathcal{B}$ of monoid ideal $\mathcal{I} \subset \mathbb{N}_0^n$

Output: finite complementary decomposition $\overline{\mathcal{B}}$

if $\mathcal{B} = \emptyset$ **then** $\{\mathcal{I}$ is the zero ideal and thus the complement the whole ring$\}$
  $\overline{\mathcal{B}} \leftarrow \Big\{ \big([0,\dots,0], \{1,2,\dots,n\}\big) \Big\}$
else if $n = 1$ **then** $\{$in this case $\mathcal{B} = \{\nu\}\}$
  $q_0 \leftarrow \nu_1$
  if $q_0 = 0$ **then** $\{\mathcal{I}$ is the whole ring and thus the complement empty$\}$
    $\overline{\mathcal{B}} \leftarrow \emptyset$
  else
    $\overline{\mathcal{B}} \leftarrow \Big\{ \big([0], \emptyset\big), \dots, \big([q_0 - 1], \emptyset\big) \Big\}$
  end if
else
  $q_0 \leftarrow \max_{\nu \in \mathcal{B}} \nu_n; \quad \overline{\mathcal{B}} \leftarrow \emptyset$
  for q **from** 0 **to** q_0 **do**
    $\mathcal{B}'_q \leftarrow \mathtt{Minimise}\big(\{\nu' \in \mathbb{N}_0^{n-1} \mid \nu \in \mathcal{B},\ \nu_n \leq q\}\big)$
    $\overline{\mathcal{B}}'_q \leftarrow \mathtt{ComplementaryDecomposition}(\mathcal{B}'_q)$
    if $\overline{\mathcal{B}}'_q \neq \emptyset$ **then**
      if $q < q_0$ **then**
        $\overline{\mathcal{B}} \leftarrow \overline{\mathcal{B}} \cup \Big\{ \big([\nu', q], N_{\nu'}\big) \mid (\nu', N_{\nu'}) \in \overline{\mathcal{B}}'_q \Big\}$
      else
        $\overline{\mathcal{B}} \leftarrow \overline{\mathcal{B}} \cup \Big\{ \big([\nu', q], N_{\nu'} \cup \{n\}\big) \mid (\nu', N_{\nu'}) \in \overline{\mathcal{B}}'_q \Big\}$
      end if
    end if
  end for
end if
return $\overline{\mathcal{B}}$

A further comment: the simple complementary Rees decomposition given here is usually highly redundant (see Example 5.1.7). Hironaka [12, §4] provided a computational description of a complementary Rees decomposition which generally is much more compact. Given a Pommaret basis $\mathcal{H}$ of the monoid ideal $\mathcal{I}$, one can straightforwardly determine this decomposition algorithmically and it exists only, if the ideal is quasi-stable. The key is to apply the projections

$$\mathrm{pr}^{(k)} : \mathbb{N}_0^n \longrightarrow \mathbb{N}_0^n, \quad [\mu_1, \dots, \mu_n] \longmapsto [0, \dots, 0, \mu_{k+1}, \dots, \mu_n]$$

for $0 \leq k < n$ to the given monoid ideal $\mathcal{I} \lhd \mathbb{N}_0^n$. Given a multi index $\nu \in \mathbb{N}_0^n$ we denote $\mathcal{C}_k(\nu)$ the cone defined by the vertex ν and the multiplicative indices $1, \dots, k$, i. e.

$$\mathcal{C}_k(\nu) = \nu + (\mathbb{N}_0)_1 + \cdots + (\mathbb{N}_0)_k \,.$$

Given a finite set $\mathcal{N} \subset \mathbb{N}_0^n$, we write $\mathcal{C}_k(\mathcal{N})$ for the union of all cones $\mathcal{C}_k(\nu)$ with $\nu \in \mathcal{N}$. Finally, we introduce for $0 \leq k < n$ the sets

$$\overline{\mathcal{N}}_k = \big(\mathrm{pr}^{(k+1)}(\mathcal{I}) + (\mathbb{N}_0)_{k+1}\big) \setminus \mathrm{pr}^{(k)}(\mathcal{I})$$

We claim now that the complement $\overline{\mathcal{I}}$ can be written as the disjoint union of all the sets $\mathcal{C}_k(\overline{\mathcal{N}}_k)$, i. e. these sets induce a complementary Rees decomposition. Indeed, it is not difficult to see that

$$\begin{aligned} \mathcal{C}_k(\overline{\mathcal{N}}_k) &= \big\{\nu \in \overline{\mathcal{I}} \mid \nu + (\mathbb{N}_0^n)_{k+1} \cap \mathcal{I} \neq \emptyset\big\} \setminus \mathcal{C}_{k-1}(\overline{\mathcal{N}}_{k-1}) \\ &= \big\{\nu \in \overline{\mathcal{I}} \mid \exists \ell \in \mathbb{N} : \nu + \ell_{k+1} \in \mathcal{I} \wedge \forall 1 \leq j \leq k : \nu + (\mathbb{N}_0)_j \subseteq \overline{\mathcal{I}}\big\} \,. \end{aligned}$$

Of course, we can really speak of a Rees decomposition only, if all the sets $\overline{\mathcal{N}}_k$ are finite. Hironaka only showed that this is generically the case. However, it is not difficult to show that this is the case, if and only if the ideal $\mathcal{I}$ is quasi-stable, i. e. if it possesses a Pommaret basis.

Proof. Let us assume first that all the sets $\overline{\mathcal{N}}_k$ are finite and denote by $\tilde{q}$ the maximal degree of an element of one of these sets. We study now an induced decomposition of $\overline{\mathcal{I}}_{>\tilde{q}}$. If $\nu \in \overline{\mathcal{N}}_k$, then by construction $\mathrm{cls}\,\nu > k$. Assume that $|\nu| = q$. Then we obtain a disjoint decomposition

$$\mathcal{C}_k(\nu)_{>q} = \mathcal{C}_k(\nu + 1_k) \cup \mathcal{C}_{k-1}(\nu + 1_{k-1}) \cup \cdots \cup \mathcal{C}_1(\nu + 1_1) \,. \qquad (2)$$

Note that $\mathrm{cls}\,(\nu + 1_i) = i$ for all $1 \leq i \leq k$ and hence $\mathcal{C}_i(\nu + 1_i)$ equals the Pommaret cone $\mathcal{C}_P(\nu + 1_i)$. Iterating the decomposition step (2) sufficiently often until all vertices are of degree $\tilde{q} + 1$, we obtain a disjoint decomposition of $\overline{\mathcal{I}}_{>\tilde{q}}$ consisting entirely of Pommaret cones with vertices of degree $\tilde{q} + 1$. According to Proposition 5.1.6, this implies the existence of a Pommaret basis of $\mathcal{I}$ of degree at most $\tilde{q} + 1$.

The converse follows immediately from Algorithm 2 below, as it produces only finite sets. □

Hironaka's approach translates immediately into an algorithm; in fact, one simply obtains a special form of Janet's Algorithm 5.2 assuming that the input is not an arbitrary Janet basis but a Pommaret basis. In the way we present it in Algorithm 2, the order of the outer loop is reversed. This has the benefit that after the kth iteration of the loop the set $\mathcal{H}$ is a Pommaret basis of the ideal[1] $\mathcal{I} : \langle x^1, \dots, x^k\rangle^\infty = \mathcal{I} : (x^k)^\infty$.

Algorithm 2 Complementary Rees decomposition (from Pommaret basis)

Input: Pommaret basis $\mathcal{H}$ of monoid ideal $\mathcal{I} \subset \mathbb{N}_0^n$
Output: complementary Rees decomposition

for k **from** 1 **to** n **do**
  $\overline{\mathcal{N}}_{k-1} \leftarrow \bigcup_{\substack{\mu\in\mathcal{H}\\ \operatorname{cls}\mu=k}} \bigcup_{\ell=1}^{\mu_k} \{\mu - \ell_k\}$
  $\mathcal{H} \leftarrow \{\mu \in \mathcal{H} \cup \overline{\mathcal{N}}_{k-1} \mid \operatorname{cls}\mu > k\}$
  involutively autoreduce $\mathcal{H}$
end for
return $\overline{\mathcal{N}}_0, \dots, \overline{\mathcal{N}}_{n-1}$

Remark 5.1.8: Replace $\mathcal{B}$ by $\overline{\mathcal{B}}$.

Remark 5.1.8: Identifying the appearing monoid ideals in $\mathbb{N}_0^n$ with monomial ideals in the polynomial ring $\mathcal{P}$, it is easy to see that Stanley filtrations as defined here provide a simple example for the *prime filtrations* introduced by Eisenbud [7, Prop. 3.7]. Indeed, setting $\mathcal{M}_j = \mathcal{P}/\mathcal{I}_j$, we find that

$$\mathcal{M}_j/\mathcal{M}_{j-1} \cong \Bbbk[x^k \mid k \in N_{\nu(j)}] \cong \mathcal{P}/\langle x^k \mid k \notin N_{\nu(j)}\rangle\ .$$

Proposition 5.2.1: In the proposition only an expression for the Hilbert *series* is given. Of course, one obtains equally easily an explicit formula for the Hilbert *function* of the algebra $\mathcal{A}$:

$$h_{\mathcal{A}}(q) = \sum_{t\in\mathcal{T}} [q_t \le q] \binom{q - q_t + k_t - 1}{q - q_t}\ .$$

Here $[\cdot]$ denotes the Kronecker-Iverson symbol which yields 1, if the condition in the bracket is satisfied, and 0 otherwise. The Hilbert *polynomial* of $\mathcal{A}$ is obtained by simply omitting the Kronecker-Iverson terms. Thus the Hilbert function is always polynomial beyond the degree $\max_{t\in\mathcal{T}} q_t$ (but it is possible that it becomes polynomial already at some lower degree; the Hilbert regularity cannot be deduced from the above representation).

Page 177, Line 1: Read $X \neq X_{\bar{t}}$ instead of $X \neq X_t$.

Proposition 5.2.7: The proof of this proposition makes essential use of the observation that we can always find a maximal regular sequence lying in $\mathcal{P}_1$. The given reference [18, Lemma 4.1] refers for the crucial point to a result

[1] As for monoid ideals the notion of a colon ideal makes no sense, we identify now each multi index μ with the term x^μ and $\mathcal{I}$ with the corresponding monomial ideal.

of Baclawski and Garsia [3, Lemma 2.2] who in turn credit the book by Kaplansky [14, Sect. 2.2]. If one follows the proof until the end, one notices that it is actually more or less a (variation of a) side product of Kaplansky's proof of the fact that a finitely generated module over a Noetherian ring has only finitely many associated prime ideals. Thus, assuming this standard theorem considerably simplifies the proof.[2]

Proposition 1. *Let $\Bbbk$ be an infinite field and $\mathcal{R}$ a standard $\Bbbk$-algebra (i. e. $\mathcal{R}$ is a graded $\Bbbk$-algebra such that $\mathcal{R}_0 = \Bbbk$ and $\mathcal{R}$ is generated as an algebra by $\mathcal{R}_1$ with $\dim_{\Bbbk} \mathcal{R}_1 = n < \infty$). We write $\mathcal{R}_+ = \oplus_{q>0}\mathcal{R}_q$. If $\mathcal{M}$ is a finitely generated, graded $\mathcal{R}$-module such that $\operatorname{Ann}_{\mathcal{R}}(m) \neq \mathcal{R}_+$ for all non-vanishing homogeneous elements $m \in \mathcal{M}$, then a generic element of $\mathcal{R}_1$ is a non zero divisor on $\mathcal{M}$ (i. e. identifying $\mathcal{R}_1 \cong \Bbbk^n$, the set of non zero divisors contains a Zariski open set).*

Proof. By definition, every zero divisor on $\mathcal{M}$ lies in some annihilator $\operatorname{Ann}_{\mathcal{R}}(m)$. Since $\mathcal{R}$ is Noetherian and thus satisfies the ascending chain condition, every annihilator is contained in a maximal one. By a simple standard argument (see e. g. [14, Thm. 6]), any maximal annihilator is a prime ideal and thus an associated prime. Under the made assumptions on the ring $\mathcal{R}$ and the module $\mathcal{M}$, $\operatorname{Ass}_{\mathcal{R}} \mathcal{M}$ is a finite set. Thus there exists a finite number of elements $m_1, \dots, m_r \in \mathcal{M}$ such that all zero divisors on $\mathcal{M}$ lie in $\operatorname{Ann}_{\mathcal{R}}(m_1) \cup \cdots \cup \operatorname{Ann}_{\mathcal{R}}(m_r)$. Since no annihilator equals $\mathcal{R}_+$, all intersections $\operatorname{Ann}_{\mathcal{R}}(m_1) \cap \mathcal{R}_1$ define proper $\Bbbk$-linear subspaces of $\mathcal{R}_1$. Since the field $\Bbbk$ is infinite, the claim follows. □

Proposition 5.2.7: Half of the proof can be omitted by refering to a standard result in commutative algebra, namely that over a Noetherian ring all maximal regular sequences have the same length. Thus, once we have established that $x^1, \dots, x^d$ is a maximal regular sequence, we are done and there is no need to study longer sequences $y^1, \dots, y^{d+1}$.

Proposition 5.3.4: Following some ideas contained in the thesis of Caviglia [4, Sect. 4.1], one can derive yet another algebraic characterisation of quasi-stable ideals (called weakly stable by him) leading even to an explicit description of the Pommaret basis of such an ideal. Let $\mathcal{B} = \{t_1, \dots, t_r\}$ be the *minimal* basis of the monomial ideal $\mathcal{I} \subset \mathcal{P}$. We assume that the generators are sorted according to the (pure not degree!) reverse lexicographic order: $t_1 \succ t_2 \succ \cdots \succ t_r$. For each index $1 \le i \le r$, we introduce the monomial colon ideal $\mathcal{J}_i = \langle t_1, \dots, t_{i-1} \rangle : t_i$. Setting $k_i = \operatorname{cls} t_i$, we consider the set $\mathcal{C}_i$ of all terms in $\mathcal{P}_i = \Bbbk[x^{k_i+1}, \dots, x^n]$, i. e. of terms in the non-multiplicative variables of the generator t_i, which are *not* contained in $\mathcal{J}_i$.

[2] In fact, the following result is just a variation of Lemma 6.2.8 where only the special case $\mathcal{R} = S\mathcal{V}$ is treated. There one allows on one side that more generally below a prescribed degree q some module elements have $\mathcal{R}_+$ as annihilator; on the other side one must impose the restriction that the module is generated in degree $q-1$ (see below). However, the key argument is the same in both proofs.

Proposition 2. *The ideal $\mathcal{I}$ is quasi-stable, if and only if all the sets $\mathcal{C}_i$ are finite and thus all the ideals $\hat{\mathcal{J}}_i = \mathcal{J}_i \cap \mathcal{P}_i \trianglelefteq \mathcal{P}_i$ zero-dimensional. In this case the Pommaret basis of $\mathcal{I}$ is given by*

$$\mathcal{H} = \mathcal{B} \cup \bigcup_{i=1}^{r} \{st_i \mid s \in \mathcal{C}_i\}\,. \tag{3}$$

Proof. Assume first that all sets $\mathcal{C}_i$ and thus $\mathcal{H}$ is finite. Obviously, $\mathcal{H}$ generates $\mathcal{I}$ and thus we only have to prove that it is involutive for the Pommaret division. Consider a term $r \in \bar{\mathcal{C}}_i = \{t_i\} \cup \{st_i \mid s \in \mathcal{C}_i\}$; obviously, $\operatorname{cls} r = k_i$. We choose an index $k_i < j \leq n$ which is thus non-multiplicative for r. If $x^j r \in \bar{\mathcal{C}}_i$, then there is nothing to prove. Otherwise write $r = st_i$ with $s = 1$ or $s \in \mathcal{C}_i$. Then $x^j r \notin \bar{\mathcal{C}}_i$ is equivalent to $x^j s \notin \mathcal{C}_i$ which in turn implies that $x^j r \in \langle t_1, \dots, t_{i-1}\rangle$. Let $1 \leq \ell < i$ be the smallest index such that $t_\ell \mid x^j st_i$ and write $x^j r = r_m r_{nm} t_\ell$ with terms $r_m \in \Bbbk[x^1, \dots, x^{k_\ell}]$ and $r_{nm} \in \Bbbk[x^{k_\ell+1}, \dots, x^n]$. Because of the minimality of the index ℓ, we must have that $r_{nm} \in \mathcal{C}_\ell$. Hence $r_{nm}t_\ell$ is an element of $\mathcal{H}$ and an involutive divisor of $x^j r$ so that we are done.

For the opposite direction, assume that $\mathcal{I}$ is quasi-stable and hence possesses a finite Pommaret basis $\mathcal{H}$. Obviously, this basis can be written in the form (3) with the sets $\mathcal{C}_i$ replaced by some finite sets $\hat{\mathcal{C}}_i \subset \Bbbk[x^{k_i+1}, \dots, x^n]$ of terms. We first show that $\hat{\mathcal{C}}_i \cap \mathcal{J}_i = \emptyset$ and thus $\hat{\mathcal{C}}_i \subseteq \mathcal{C}_i$. Assume that there existed a term $s \in \hat{\mathcal{C}}_i \cap \mathcal{J}_i$. Then $st_i \in \langle t_1, \dots, t_{i-1}\rangle$. As above choose the minimal index $1 \leq \ell < i$ such that $t_\ell \mid st_i$ and write $st_i = r_m r_{nm} t_\ell$ with terms $r_m \in \Bbbk[x^1, \dots, x^{k_\ell}]$ and $r_{nm} \in \Bbbk[x^{k_\ell+1}, \dots, x^n]$. The minimality of ℓ implies that $r_{nm}t_\ell$ cannot have a proper involutive divisor in $\mathcal{H}$[3] and thus must itself be an element of $\mathcal{H}$. But then $r_{nm}t_\ell \,|_P\, st_i$ in contradiction to the fact that a monomial involutive basis is always involutively autoreduced.

Let s be an arbitrary term in $\mathcal{P}_i \setminus \hat{\mathcal{C}}_i$. Then $st_i \notin \mathcal{H}$, but it must possess an involutive divisor $s_j t_j \in \mathcal{H}$. Using again the argument in the footnote, we find $j < i$ and thus $s \in \mathcal{J}_i$. Hence we also have the inclusion $\mathcal{C}_i \subseteq \hat{\mathcal{C}}_i$ which finishes the proof. □

Definition 5.4.8: It is somewhat pointless to define that a division is of Schreyer type *with respect to a term order*. Obviously, it suffices to consider in the definition monomial involutive bases $\mathcal{H}$ and thus the property of being of Schreyer type is independent of the used term order.

Example 5.4.11: In Equation (5.37a) the non-existent syzygy $\mathbf{S}_{4;2}$ must be replaced by $\mathbf{S}_{3;2}$.

Theorem 5.4.12: In the resolution (5.38) the first term should be $\mathcal{P}^{t_0}$ and not $\mathcal{P}^{t_1}$.

[3] Assume that some term $s_j t_j \in \mathcal{H}$ was a proper involutive divisor of $r_{nm}t_\ell$. Since s_j contains only non-multiplicative variables for t_j, this immediately implies that $k_j \geq k_\ell$. If $k_j > k_\ell$, the reverse lexicographic order implies $j < \ell$. If $k_j = k_\ell = k$, then we must have $\deg_{x^k} t_j < \deg_{x^k} t_\ell$ which again yields $j < \ell$.

Theorem 5.4.12: The derivation of (5.40) speaks just of a "simple induction" and a "well-known identity." May be it is worth while giving a few more details. The case $i = 1$ is trivial. For the induction step we first note that obviously $\beta_i^{(\ell)} = 0$, if $\ell < i+1$. Exploiting this observation, we can write with the help of the induction assumption

$$\beta_{i+1}^{(k)} = \sum_{\ell=i+1}^{k-1} \beta_i^{(\ell)} = \sum_{\ell=i+1}^{k-1} \sum_{j=1}^{\ell-1} \binom{\ell-j-1}{i-1} \beta_0^{(j)}$$
$$= \sum_{j=1}^{k-i-1} \left[\sum_{\ell=i+j}^{k-1} \binom{\ell-j-1}{i-1} \right] \beta_0^{(j)} .$$

Shifting the index in the inner sum by $j+1$ yields our claim via the following identity obtained by summing over one column in the Pascal triangle:

$$\sum_{m=i-1}^{k-j-2} \binom{m}{i-1} = \binom{k-j-1}{i} .$$

The final expression for the ranks t_i is obtained from the same identity in a very similar computation:

$$t_i = \sum_{j=1}^{n} \sum_{k=1}^{j-i} \binom{j-k-1}{i-1} \beta_0^{(k)} = \sum_{k=1}^{n-i} \left[\sum_{j=i+k}^{n} \binom{j-k-1}{i-1} \right] \beta_0^{(k)}$$
$$= \sum_{k=1}^{n-i} \binom{n-k}{i} \beta_0^{(k)} .$$

Theorem 5.4.12: In the given form the theorem gives only an upper bound t_i for the total Betti numbers β_i. But of course one can perform in the graded case the same calculations degree by degree and obtains then an upper bound for the *bigraded* Betti numbers: if $\beta_{0,j}^{(k)}$ denotes the number of generators in the Pommaret basis $\mathcal{H}$ of class k and degree j, then

$$t_{i,j} = \sum_{k=1}^{n-i} \binom{n-k}{i} \beta_{0,j-i}^{(k)}$$

is an upper bound of the Betti number $\beta_{i,j}$.

Page 200, Line 11: Read t_i instead of r_i.

Remark 5.4.13: There is a typo in (5.41): of course, we must use the induced Schreyer order $\prec_{\mathcal{H}}$ instead of the original order $\prec$ for selecting the leading terms of syzygies.

Theorem 5.5.7: It is perhaps worth while to note explicitly the following result which is implicitly hidden in the proof of the theorem (and which was already explicitly mentioned by Mall [15, Thm 2.15]): The reduced Gröbner basis $\mathcal{G}$ of an ideal $\mathcal{I}$ for a term order $\prec$ is a Pommaret basis, if and only if the leading ideal $\operatorname{lt}_\prec \mathcal{I}$ is stable.

Proof. Since $\mathcal{G}$ is a reduced Gröbner basis, its leading terms $\mathrm{lt}_\prec \mathcal{G}$ form the minimal basis of $\mathrm{lt}_\prec \mathcal{I}$. According to Proposition 3.4.16, the leading ideal $\mathrm{lt}_\prec \mathcal{I}$ is now stable, if and only if $\mathrm{lt}_\prec \mathcal{G}$ is its Pommaret basis. □

Example 5.5.9: The ideal should be called $\mathcal{U}$ instead of $\mathcal{I}$.

Theorem 5.5.11: One of the decisive arguments in the proof is very short and in fact some non-trivial considerations are missing. In particular, from the given proof it is not apparent why we need the assumption that a class respecting term order is used.

Consider the chosen generator $\mathbf{h}_\gamma$ of minimal class d and maximal degree in its class. Because of the use of a class respecting term order, the support of $\mathbf{h}_\gamma$ is contained in the submodule $\langle x^1,\dots,x^d\rangle^m \subset \mathcal{P}^m$, as any other term is of higher class. If $\mathbf{S}_{\gamma;k} = x^k\mathbf{e}_\gamma - \sum_\beta P_\beta^{(\gamma;k)}\mathbf{e}_\beta \in \mathcal{P}^s$ is one of the first syzygies induced by $\mathbf{h}_\gamma$, then we again find that all coefficients satisfy $P_\beta^{(\gamma;k)} \in \langle x^1,\dots,x^d\rangle$. This follows for all generators $\mathbf{h}_\beta$ with $\mathrm{cls}\,\mathbf{h}_\beta > d$ from class considerations and for the generators $\mathbf{h}_\beta$ with $\mathrm{cls}\,\mathbf{h}_\beta = d$ from degree considerations. Thus the syzygy $\mathbf{S}_{\gamma;d+1}$ is of minimal class and maximal degree in its class in the Pommaret basis of the first syzygy module. Furthermore, its support lies by the considerations above in the submodule $\langle x^1,\dots,x^{d+1}\rangle^s \subset \mathcal{P}^s$ (with x^{d+1} required only for the leading term $x^{d+1}\mathbf{e}_\gamma$). Now a simple iteration shows that the crucial syzygy $\mathbf{S}_{\gamma;d+1,\dots,n}$ has indeed no constant term, as its support lies in a submodule $\langle x^1,\dots,x^{n-1}\rangle^t \subset \mathcal{P}^t$ for some rank t.

This more detailed argument furthermore shows that the assumptions of the theorem can be slightly relaxed. We only need that the Pommaret basis contains a generator $\mathbf{h}_\gamma$ of minimal class d and maximal degree in its class such that its support lies in $\langle x^1,\dots,x^d\rangle^m \subset \mathcal{P}^m$. If the term order is class respecting, the existence of such a generator is guaranteed. If we consider for example the first syzygy module in our resolution, then we use there a Pommaret basis for a term order which is not class respecting (in general, a Schreyer order does not respect classes, even if it is induced by a class respecting term order). However, we showed above that it nevertheless contains a generator with the required properties and thus we may conclude that the resolution of the syzygy module induced by this Pommaret basis is of minimal length.

Corollary 5.5.31: The given proof is a bit short. As trivial consequences of Proposition 5.5.28 and Corollary 5.5.29, we only obtain the inequalities

$$\mathrm{sat}\,\mathcal{I} = \deg \mathcal{H}_1 \leq \deg \mathcal{H} = \mathrm{reg}\,\mathcal{I}\,,$$
$$\mathrm{reg}\,\mathcal{I}^{\mathrm{sat}} = \deg \bar{\mathcal{H}} \leq \deg \mathcal{H} = \mathrm{reg}\,\mathcal{I}\,.$$

It requires a little more work to prove that in at least one of them actually equality holds. The following proof has been published in [17].

We first note that in δ-regular coordinates all involved quantities are already determined by the leading ideal and therefore we may restrict for notational simplicity to a monomial ideal $\mathcal{I}$. If $\operatorname{sat}\mathcal{I} = \operatorname{reg}\mathcal{I}$, then we are done. Thus assume that $\operatorname{sat}\mathcal{I} < \operatorname{reg}\mathcal{I}$; we show now that then it is not possible that $\operatorname{reg}\mathcal{I}^{\text{sat}} < \operatorname{reg}\mathcal{I}$. Consider an element $h_{\max} \in \mathcal{H}$ with $\deg h_{\max} = \deg \mathcal{H}$; by assumption, we must have $\operatorname{cls} h_{\max} > 1$. Now the inequality $\operatorname{reg}\mathcal{I}^{\text{sat}} < \operatorname{reg}\mathcal{I}$ may only hold, if $\mathcal{H}_1$ contains an element $h_1 = (x^1)^\ell \bar{h}_1$ with $\bar{h}_1 \mid_P h_{\max}$.

It follows from our assumptions that $\bar{h}_1$ is a proper divisor of $h_{\max}$. Thus we can find a variable x^j with $j > 1$ which divides $h_{\max}/\bar{h}_1$. Since x^j is non-multiplicative for h_1, the Pommaret basis $\mathcal{H}$ must contain an involutive divisor h_2 of $x^j h_1$. If $\operatorname{cls} h_2 = 1$, then we have either $\deg h_2 > \deg h_1$ (if $h_2 = x^j h_1$) or $\deg_{x^1} h_2 < \deg_{x^1} h_1$. In both cases we may replace h_1 by h_2 and start again; after a finite number of such restarts, we will have $\operatorname{cls} h_2 > 1$ by degree reasons. But now the Pommaret basis $\mathcal{H}$ contains an involutive divisor h_2 of $h_{\max}$ in contradiction to the definition of a strong involutive basis. Hence we must have $\operatorname{reg}\mathcal{I}^{\text{sat}} = \operatorname{reg}\mathcal{I}$.

Proposition 5.5.33: The reference at the beginning of the proof should be to Remark 3.1.17 instead of Example 3.1.16.

Chapter 6: *(Involution II: Homological Theory)*

Remark 6.1.15: In the last but first line of the remark read $\beta_{q,p}(\mathcal{M})$ instead of $\beta_{q,p}\mathcal{M}$.

Example 6.1.24: In line 6 the representative of the generator of $H_{2q-1,1}(\mathcal{I})$ should read $x^q y^{q-1} \otimes y - x^{q-1} y^q \otimes x$ (there is a spurious exponent 1 in the last factor).

Lemma 6.2.8: (This error also concerns the subsequent material up to **Proposition 6.2.11**.) All the results here are only correct, if we assume that the module $\mathcal{M}$ is finitely generated by elements *of degree* $q-1$. Otherwise we cannot expect in the proof of Lemma 6.2.8 that $\overline{\mathcal{M}}$ is a module, as the multiplication of elements of $\mathcal{K}$ by ring elements may lead to elements of $\mathcal{A}$. This additional assumption slightly simplifies the proof of **Lemma 6.2.10**, as we do not need to consider elements of degree less than $q-1$.

Fortunately, this restriction does not pose any problems for our derivation of the dual Cartan test. Indeed, in the proof of the Proposition 6.2.12, we may simply apply the previous results to the module $\mathcal{M}_{\geq q-1}$ which by assumption is indeed finitely generated by elements of degree $q-1$.

Definition 6.2.9: Following Serre's letter appended to [11], I use here the terminology *quasi-regular* (at a degree q). In the commutative algebra community, the terminology *filter-regular* (usually without the specification of a degree) introduced by Schenzel et al. [16] for local rings has become more common (Aramova and Herzog [1] speak of an *almost regular* sequence). Applied to the polynomial ring, both are equivalent. One should also note that many modern textbooks on commutative algebra introduce a concept of quasi-regularity which is not related to our use of this notion.

Lemma 6.3.1: (This error also concerns **Theorem 6.3.2** and the **Remarks 6.3.5** and **6.3.7**). Given the Pommaret basis $\mathcal{H}$ of the ideal $\mathcal{I} \subseteq \mathcal{P}$, the ideal $\mathcal{I}^{(k)}$ should not be defined as $\mathcal{I} \cap \mathcal{P}^{(k)}$ but as $\mathcal{I}^{(k)} = \langle \mathcal{H}|_{x^1=\cdots=x^k=0} \rangle \subseteq \mathcal{P}^{(k)}$. All statements are correct then.

Remark 6.3.5: In (6.36) the middle term should be $H_{q+1,p}(\mathcal{P}/\mathcal{I})$. This isomorphism follows from the fact that any minimal resolution of $\mathcal{I}$ trivially induces a minimal resolution of $\mathcal{P}/\mathcal{I}$ where all terms are simply moved by one position. This also implies that in (6.37) also the indices in the middle term must be changed to $\beta_{q+1,p}(\mathcal{P}/\mathcal{I})$. In the argument leading to (6.36), the reference to Proposition 6.1.18 is superfluous.

Chapter 7: *(Involution III: Differential Theory)*

Theorem 7.1.6: There is a typo in the formula giving $\mathcal{R}_q^{(1)}$, namely an s too much. The correct definition is $\mathcal{R}_q^{(1)} = \pi_q^{q+1}(\mathcal{R}_{q+1})$.

Remark 7.1.9: Read *integral* instead of *integal*.

Example 7.2.12: As the name "Frobenius Theorem" indicates, this result is usually attributed to Frobenius [9]. Actually, Frobenius' contribution consists "only" of transferring the theorem to the language of Pfaffian systems (differential one-forms). The here used formulation via first-order linear differential operators is older: Deahna [6] proved the sufficiency of involution for the existence of solutions; the necessity was shown by Clebsch [5] based on results by Jacobi [13] (posthumously published by Clebsch).

Section 7.3: For the special case of linear differential equations with constant coefficients, Wong [19] gave a simple description of the completion process. Assume that we are given the equation $E\mathbf{u}' = A\mathbf{u}$ where E, A are $t \times m$ matrices. Set $\mathcal{V}_0 = \mathbb{R}^m$ and recursively $\mathcal{V}_{k+1} = A^{-1}\big(E\mathcal{V}_k\big)$. This sequence trivially stabilises at some index $0 \leq k \leq n$. It is not difficult to see that $\mathcal{V}_k$ is just the constraint manifold after the kth completion step. This fact stems from the simple observation that if $\mathbf{u}$ must satisfy the algebraic equations $B\mathbf{u} = 0$ for some matrix B, then differentiation shows that the derivatives must satisfy the identical equation $B\mathbf{u}' = 0$. Obviously, this only holds because of the constant coefficients!

Example 7.4.3: The general solution presented in (7.71) is false. Computing explicitly the Taylor expansion of the solution around the origin, one obtains the following polynomial of degree 5

$$\begin{aligned} u(x,y,z) = {} & a_1 + a_2x + a_3y + a_4z + \frac{1}{2}a_5x^2 + a_6xy + a_7xz + a_8yz + \\ & \frac{1}{6}a_9x^3 + \frac{1}{2}a_{10}x^2z + a_{11}xyz + \frac{1}{2}a_5yz^2 + \\ & \frac{1}{6}a_{12}x^3z + \frac{1}{2}a_9xyz^2 + \frac{1}{6}a_{10}yz^3 + \frac{1}{6}a_{12}xyz^3 \,. \end{aligned}$$

Chapter 9: *(Existence and Uniqueness of Solutions)*

Section 9.1: This section considers only the existence and uniqueness theory for involutive ordinary differential equations. It is, however, also completely straightforward and quite instructive to extend the usual elementary, i. e. linear, stability theory of equilibria of normal ordinary differential equations to arbitrary, not underdetermined involutive ones (some of this material has now been published in [8]).

We begin with the case of a linear system with constant coefficients:

$$\mathbf{u}' = M\mathbf{u}\,, \quad 0 = N\mathbf{u}\,.$$

We claim that this system is involutive, if and only if the kernel of N is an M-invariant subspace. If the system is involutive, then the algebraic equations obtained by prolonging the algebraic subsystem must be linearly dependent of the algebraic subsystem. More precisely, a matrix L must exist such that $N\mathbf{u}' = NM\mathbf{u} = LN\mathbf{u}$. For any vector $\mathbf{u} \in \ker N$ we thus obtain $NM\mathbf{u} = LN\mathbf{u} = 0$ and $\ker N$ is M-invariant. The converse follows from reverting the argument: the M-invariance gives now the existence of the matrix L guaranteeing the involution of the system.

We split the vector $\mathbf{u}$ into two components $\mathbf{u} = (\mathbf{v}, \mathbf{w})$ such that the algebraic subsystem takes the form $0 = C\mathbf{v} + D\mathbf{w}$ with an *invertible* matrix D. Obviously, this is only possible, if the matrix N has maximal rank which we can always achieve by eliminating redundant constraints. The splitting induces a decomposition

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}\,.$$

Solving the algebraic equations, we obtain $\mathbf{w} = -D^{-1}C\mathbf{v}$ and the normal ordinary differential equation $\mathbf{v}' = S\mathbf{v}$ where $S = A - BD^{-1}C$ is the Schur complement of the matrix

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

which arises naturally as structure matrix of the equivalent linear system $\mathbf{v}' = A\mathbf{v} + B\mathbf{w}$, $0 = C\mathbf{v} + D\mathbf{w}$ (which is obviously *not* involutive, but of index 1 in the DAE jargon).

Any homogeneous linear system admits the zero solution as an equilibrium. In order to decide its stability we should not look at all eigenvalues of M but only at those which possess eigenvectors lying in $\ker N$; in other words we should analyse the linear map obtained by restricting M to this kernel. In the decomposed form, this map is given by the Schur complement S. Hence its eigenvalues decide the stability of the origin.

The extension to nonlinear, semi-explicit autonomous systems of the form

$$\mathbf{u}' = \boldsymbol{\phi}(\mathbf{u})\,, \quad 0 = \boldsymbol{\psi}(\mathbf{u})$$

is very simple. We assume that $\boldsymbol{\phi}(0) = \boldsymbol{\psi}(0) = 0$ so that again the origin is an equilibrium. We denote by

$$M = \frac{\partial \boldsymbol{\phi}}{\partial \mathbf{u}}(0)\,, \quad N = \frac{\partial \boldsymbol{\psi}}{\partial \mathbf{u}}(0)$$

the Jacobians at the origin. We claim that if the system is involutive, then $\ker N$ is M-invariant. As above, involution entails the existence of a matrix-valued function $\boldsymbol{\zeta}$ such that

$$\frac{\partial \boldsymbol{\psi}}{\partial \mathbf{u}} \cdot \boldsymbol{\phi} = \boldsymbol{\zeta} \cdot \boldsymbol{\psi}\,.$$

Differentiating this condition and evaluating the result at the origin yields again $NM = LN$ where $L = \boldsymbol{\zeta}(0)$. Now we can apply the same arguments as in the linear case. Thus again not all eigenvalues of M are relevant but only those possessing eigenvectors lying in $\ker N$.

Obviously, we do not change the solution space of our system, if we add a linear combination of the constraints to the right hand side of its differential part. One readily checks that the invariance of $\ker N$ is not affected by such a transformation. Also all stability relevant eigenvalues remain unchanged whereas the eigenvalues with eigenvectors transversal to $\ker N$ will generally change. Indeed, the remaining eigenvalues describe the stability of the constraint manifold under the flow of the underlying vector field and our transformation yields a different underlying vector field. Thus for the numerical integration of our system, these eigenvalues are highly relevant—but not for the stability of the equilibrium!

Page 412, Line 14: Read $tf'(x+ut) = 1$ instead of $tf'(x+ut) \neq 1$.

Appendix A: *(Miscellaneous)*

Lemma A.1.2: In the last line of the proof omit the erroneous $\in \mathcal{B}_i$ after the definition of the multi index μ.

Appendix B: *(Algebra)*

Page 544, Line -2: In the definition of the differential of a tensor product complex the crucial sign factor is missing. The correct expression is

$$\partial_k = \bigoplus_{i+j=k} \left(d_i \otimes \mathrm{id}_{\mathcal{N}_j} + (-1)^i \mathrm{id}_{\mathcal{M}_i} \otimes \delta_j \right)\,.$$

Only with this factor, the complex condition $\partial \circ \partial = 0$ is satisfied.

Page 552, Line 17: Read *contracting homotopy* instead of *contracting1 homotopy*.

Lemma B.2.33: There is a small gap in the proof of the exactness at the second term of the sequence, as it is not shown that the element $m_1 = \iota(n) \in \mathcal{M}_1$

has a preimage under $(\mathrm{id}_{\mathcal{M}_1} - \psi \circ \phi)$. This preimage is given by $\tilde{m}_1 = m_1 - \psi(m_2)$. Indeed,

$$(\mathrm{id}_{\mathcal{M}_1} - \psi \circ \phi)\big(m_1 - \psi(m_2)\big) = \\ m_1 - \psi\big(\phi(m_1) + (\mathrm{id}_{\mathcal{M}_2} - \phi \circ \psi)(m_2)\big) = m_1$$

where in the last line the argument of ψ vanishes, since we come from an element of $\ker \rho_2$.

Definition B.2.35: Read $0 \to \mathcal{M} \to \mathcal{I}$ for the injective coresolution instead of $0 \to \mathcal{I} \to \mathcal{M}$.

References:

[167] This article by Goldschmidt appeared in 1967 (not 1965) and in Volume 86 (not 82).

[168] And this article by Goldschmidt also appeared in 1967 (not in 1969).

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(Last update: August 23, 2025)