δ- and Quasi-Regularity for Polynomial Ideals[1]

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Abstract

We consider the effective construction of δ-regular coordinates for polynomial ideals. Special attention is given to quasi-stable ideals, i. e. monomial ideals possessing a Pommaret basis. Finally, we show that δ-regularity for an ideal $\mathcal{I}$ is equivalent to quasi-regularity for $\mathcal{P}/\mathcal{I}$ (in the sense of Serre).

Keywords: polynomial ideal, Pommaret basis, δ-regularity, quasi-regularity

1 Introduction

Involutive bases [3, 5, 9] are a special kind of Gröbner bases [1] with additional combinatorial properties. The underlying ideas originated in the theory of differential equations. In particular, Pommaret bases are closely related to the involution analysis of symbols in the formal theory of differential equations [11]. It is a well-known problem that a polynomial ideal $\mathcal{I}$ possesses only in suitable, so-called δ-regular, coordinates a Pommaret basis (note, however, that generic coordinates are δ-regular). In [10] it is shown that coordinates regular in this sense are very useful for a number of applications; e. g. regular sequences or Noether normalisations of $\mathcal{P}/\mathcal{I}$ take a particularly simple form.

The traditional approach to obtain δ-regular coordinates consists of applying a random transformation (see e. g. [14] for a discussion in the context of differential equations). This method has at least two disadvantages. While random coordinates are δ-regular with probability 1, they still may be singular. More importantly, random transformations usually destroy any sparsity present in a basis of the ideal $\mathcal{I}$ making any subsequent computation much more expensive.

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In [8] we presented a deterministic solution for the related problem of δ-regularity in partial differential equations based on a comparison of the Janet and Pommaret multiplicative variables. Here we first adapt this solution to polynomial ideals. Then we show that our criterion for singular coordinates is closely related to the algebraic theory of a class of monomial ideals studied by Bermejo and Gimenez [2]. Finally, we relate the theory of Pommaret bases to Serre's dual version of the Cartan test (see the letter by Serre appended to [7]). We prove that Serre's notion of quasi-regular coordinates for the factor ring $\mathcal{P}/\mathcal{I}$ coincides with δ-regularity for the ideal $\mathcal{I}$.

2 Involutive Bases

Identifying the Abelian monoid $(\mathbb{N}_0^n, +)$ with the set of power products x^μ in a polynomial ring $\mathcal{P} = \Bbbk[x_1, \dots, x_n]$ over a field $\Bbbk$, we have the usual divisibility relation: $\mu|\nu$ if $\nu \in C(\mu) := \mu + \mathbb{N}_0^n$. An *involutive division* is a rule L (satisfying certain conditions, see [5, 9] for details) restricting this relation by assigning to each member μ of every finite subset $\mathcal{N} \in \mathbb{N}_0^n$ a set $N_{L,\mathcal{N}}$ of allowed (*multiplicative*) indices, resulting in restricted *involutive* cones $C_{L,\mathcal{N}}(\mu) := \mu + \{\nu \mid \nu_i = 0 \,\text{for}\, i \notin N\}$. For this new relation, we write $\mu|_{L,\mathcal{N}}\nu$ if $\nu \in C_{L,\mathcal{N}}(\mu)$ (μ *involutively divides* ν). In this article, we only need the following two involutive divisions which we denote by P and J, respectively:[2]

***Pommaret division*:** $N_{P,\mathcal{N}}(\mu) := \{i \mid i \leq \operatorname{cls}\mu\}$, where $\operatorname{cls}\mu := \min\{i \mid \mu_i \neq 0\}$.[3]

***Janet division*:** $N_{J,\mathcal{N}}(\mu) := \{i \mid \mu_i \geq \nu_i \,\text{for all}\, \nu \in \mathcal{N} \,\text{with}\, \mu_j = \nu_j \,\text{for}\, j > i\}$.

The *(involutive) span* $\langle\mathcal{N}\rangle$ (resp. $\langle\mathcal{N}\rangle_L$) of $\mathcal{N}$ is the union of the (involutive) cones of its elements. $\mathcal{N}$ is *(involutively) autoreduced*, if no member is contained in the (involutive) cone of another element. A finite subset $\hat{\mathcal{N}} \subset \langle\mathcal{N}\rangle$ is a *weak involutive basis* of $\langle\mathcal{N}\rangle$, if $\langle\hat{\mathcal{N}}\rangle_L = \langle\mathcal{N}\rangle$, and a *(strong) involutive basis*, if furthermore $\hat{\mathcal{N}}$ is autoreduced. We refer to $\hat{\mathcal{N}}$ as a *(weak/strong) involutive completion* of the set $\mathcal{N}$, if $\mathcal{N} \subseteq \hat{\mathcal{N}}$. One can show that to every finite set $\mathcal{N}$ there exists a Janet basis of $\langle\mathcal{N}\rangle$, but not necessarily a Pommaret basis. By contrast, a basis minimal among all the Pommaret bases of $\mathcal{N}$ is unique, whereas the same does not hold for the Janet division.

While the definitions of the Pommaret and Janet division, respectively, look very different, the two divisions are actually closely related. The following result gives a first indication of this fact; more will become evident in the next section.

Proposition 2.1 ([4])**.** *Let the finite set $\mathcal{N} \subset \mathbb{N}_0^n$ be involutively autoreduced with respect to the Pommaret division. Then $N_P(\nu) \subseteq N_{J,\mathcal{N}}(\nu)$ for all $\nu \in \mathcal{N}$.*

[2] We follow here the original convention of Janet which are the usual ones in the theory of differential equations; [5] uses a reverse ordering of the variables.

[3] This is independent of the set $\mathcal{N}$, so we will drop the reference to it in the sequel.

An involutive basis $\mathcal{N}$ leads via the involutive cones to a disjoint decomposition of the ideal $\mathcal{I} = \langle \mathcal{N} \rangle$ as a $\Bbbk$-linear space (a so-called *Stanley decomposition* [12, 13]). For many applications it is also of interest to decompose the complement $\mathcal{I}^c := \mathbb{N}_0^n \setminus \mathcal{I}$; both Janet and Pommaret bases induce such *complementary Stanley decompositions*. In the latter case, we have the following result.

Proposition 2.2 ([10]). *The monoid ideal $\mathcal{I} \subseteq \mathbb{N}_0^n$ possesses a weak Pommaret basis of degree q, if and only if the sets $\bar{\mathcal{N}}_0 = \{\nu \in \mathcal{I}^c \mid |\nu| < q\}$ and $\bar{\mathcal{N}}_1 = \{\nu \in \mathcal{I}^c \mid |\nu| = q\}$ define the complementary decomposition*

$$\mathcal{I}^c = \bar{\mathcal{N}}_0 \cup \bigcup_{\nu \in \bar{\mathcal{N}}_1} C_P(\nu)\ . \tag{1}$$

The notion of an involutive basis can now be easily lifted to polynomial ideals. Choosing a term order $\prec$ determines for each $f \in \mathcal{P}$ its *leading term* $\mathrm{lt}_\prec f$ with *leading exponent vector* $\mathrm{le}_\prec f$. Let $\mathcal{F} \subset \mathcal{P}$ be a finite set. Then we assign to each element $f \in \mathcal{F}$ the multiplicative variables

$$X_{L,\mathcal{F},\prec}(f) = \{x_i \mid i \in N_{L,\mathrm{le}_\prec \mathcal{F}}(\mathrm{le}_\prec f)\}\ ; \tag{2}$$

the involutive span of $\mathcal{F}$ is then the set

$$\langle \mathcal{F} \rangle_{L,\prec} = \sum_{f \in \mathcal{F}} \Bbbk[X_{L,\mathcal{F},\prec}(f)] \cdot f \subseteq \langle \mathcal{F} \rangle\ . \tag{3}$$

A polynomial $g \in \mathcal{P}$ is *involutively reducible* with respect to $\mathcal{F}$, if it contains a term x^μ such that $\mathrm{le}_\prec f \,|_{L,\mathrm{le}_\prec \mathcal{F}}\, \mu$ for some $f \in \mathcal{F}$; g is *involutively head reducible*, if $x^\mu = \mathrm{lt}_\prec g$ in the previous definition. The set $\mathcal{F}$ is *involutively autoreduced*, if no polynomial $f \in \mathcal{F}$ contains a term x^μ such that another polynomial $f' \in \mathcal{F} \setminus \{f\}$ exists with $\mathrm{le}_\prec f' \,|_{L,\mathrm{le}_\prec \mathcal{F}}\, \mu$; the definition of *involutively head autoreduced* is similar. A finite set $\mathcal{H} \subset \mathcal{P}$ is a *weak involutive basis* of $\mathcal{I}$ for an involutive division L if $\mathrm{le}_\prec \mathcal{H}$ is a weak involutive basis of $\mathrm{le}_\prec \mathcal{I}$; it is a *(strong) involutive basis*, if $\mathrm{le}_\prec \mathcal{H}$ is a strong involutive basis of $\mathrm{le}_\prec \mathcal{I}$ and no two elements of $\mathcal{H}$ have the same leading exponents. As above, any finite set $\mathcal{F}$ can be *completed* to a Janet basis of $\langle \mathcal{F} \rangle$, while this is not necessarily true for Pommaret bases.

For the remainder of the article, all ideals $\mathcal{I}$ considered will be homogeneous. If $\mathcal{M}$ is a graded $\mathcal{P}$-module, we write $\mathcal{M}_q$ for the homogeneous component of degree q and $\mathcal{M}_{\geq q} := \bigoplus_{q' \geq q} \mathcal{M}_{q'}$ for the truncated module (similar for $\mathcal{M}_{<q}$). Let $\mathfrak{m} = \langle x_1, \dots, x_n \rangle$ denote the irrelevant ideal of $\mathcal{P}$; as usual, we call $\mathcal{I}^{\mathrm{sat}} := \mathcal{I} : \mathfrak{m}$ the saturation of $\mathcal{I}$.

Remark 2.3. If a Pommaret basis $\mathcal{H}$ of the ideal $\mathcal{I}$ exists, then a number of important invariants of the factor algebra $\mathcal{P}/\mathcal{I}$ can be immediately read off of $\mathcal{H}$ [10]:

- If $\deg \mathcal{H} := \max_{h \in \mathcal{H}} \deg h = q$, then the *dimension* D of the algebra $\mathcal{P}/\mathcal{I}$ is given by $D = \min\{i \mid \langle \mathcal{H}, x_1, \dots, x_i \rangle_q = \mathcal{P}_q\}$ and $\{x_1, \dots, x_D\}$ is a maximal independent set modulo $\mathcal{I}$ (in fact, the complementary Stanley decomposition of Proposition 2.2 yields at once the whole *Hilbert series* of $\mathcal{P}/\mathcal{I}$).

- If $\operatorname{cls}\mathcal{H} := \min_{h\in\mathcal{H}} \operatorname{cls} h = d$ (where $\operatorname{cls} h := \operatorname{cls}\operatorname{le}_\prec h$), then $\operatorname{depth}\mathcal{P}/\mathcal{I} = d-1$ and $(x_1,\dots,x_{d-1})$ is a maximal regular sequence for $\mathcal{P}/\mathcal{I}$ (combined with the result above, this observation yields a simple proof of the well-known Hironaka criterion for Cohen-Macaulay rings).
- If $\prec$ is the *degree reverse lexicographic order*[4], then $\deg\mathcal{H}$ equals the *Castelnuovo-Mumford regularity* of $\mathcal{I}$ (this is a consequence of the interesting syzygy theory of Pommaret bases leading to a free resolution of minimal length).
- The isomorphism $\mathcal{P}/\mathcal{I} \cong \bigoplus_{\nu\in\bar{\mathcal{N}}_0} \Bbbk\cdot x^\nu \oplus \bigoplus_{\nu\in\bar{\mathcal{N}}_1} \Bbbk[x_1,\dots,x_{\operatorname{cls}\nu}]\cdot x^\nu$ as $\Bbbk$-linear spaces is a *Rees decomposition* of $\mathcal{P}/\mathcal{I}$, where the sets $\bar{\mathcal{N}}_0$ and $\bar{\mathcal{N}}_1$ are defined for the Pommaret basis of $\operatorname{le}_\prec\mathcal{H}$ as in Proposition 2.2.

Together with Theorem 3.10 below, the first two items show that if a Pommaret basis exists, then the chosen coordinates are particularly adapted to the ideal $\mathcal{I}$ and considerably simplify the analysis of the algebra $\mathcal{P}/\mathcal{I}$. In the next section we will see how one can systematically construct such coordinates for any ideal $\mathcal{I}$.

3 δ-Regularity and Systems of Parameters

An ideal $\mathcal{I} \subseteq \mathcal{P}$ may be interpreted as an ideal in the symmetric algebra $S\mathcal{V}$ over an n-dimensional $\Bbbk$-linear space $\mathcal{V} \cong \mathcal{P}_1$ after having chosen a basis $(x_1,\dots,x_n)$ of $\mathcal{V}$. As we will show now, the existence of a Pommaret basis for $\mathcal{I}$ depends only on this choice.

Definition 3.1. The variables $\mathbf{x} = (x_1,\dots,x_n)$ are *δ-regular* for the ideal $\mathcal{I} \subseteq \mathcal{P}$ and the term order $\prec$, if $\mathcal{I}$ possesses a Pommaret basis for $\prec$.

As in practice one defines an ideal $\mathcal{I} \subseteq \mathcal{P}$ by some finite generating set $\mathcal{F} \subset \mathcal{I}$, we introduce a concept of δ-regularity for such sets. Assume that $\mathcal{F}$ is involutively head autoreduced with respect to an involutive division L. We call the total number of multiplicative variables of its elements its *involutive size* and denote it by

$$|\mathcal{F}|_{L,\prec} = \sum_{f\in\mathcal{F}} |X_{L,\prec,\mathcal{F}}(f)|\,. \tag{4}$$

Let $\tilde{\mathbf{x}} = A\mathbf{x}$ be a linear change of coordinates with a regular matrix $A \in \Bbbk^{n\times n}$, i.e. a change of basis in the vector space $\mathcal{V}$. It transforms each polynomial $f \in \mathcal{P}$ into a polynomial $\tilde{f} \in \tilde{\mathcal{P}} = \Bbbk[\tilde{x}_1,\dots,\tilde{x}_n]$ of the same degree. Thus $\mathcal{F}$ is transformed into a set $\tilde{\mathcal{F}} \subset \tilde{\mathcal{P}}$ which generally is no longer involutively head autoreduced.[5] Performing an involutive head autoreduction yields a set $\tilde{\mathcal{F}}^\triangle$. The leading exponents of $\tilde{\mathcal{F}}^\triangle$ may be very different from those of $\mathcal{F}$ and thus $|\mathcal{F}|_{L,\prec}$ may differ from $|\tilde{\mathcal{F}}^\triangle|_{L,\prec}$.

[4] Note that we use the ordering $x_n > \ldots > x_1$ on the variables.
[5] We consider here the involutive division and the term order as being defined on the exponent vectors. Thus after the transformation we can still use the same division and order as before.

Definition 3.2. Let the finite set $\mathcal{F} \subset \mathcal{P}$ be involutively head autoreduced with respect to the Pommaret division. The coordinates $\mathbf{x}$ are *δ-regular* for $\mathcal{F}$, if after any linear change of coordinates $\tilde{\mathbf{x}} = A\mathbf{x}$ the inequality $|\mathcal{F}|_{P,\prec} \geq |\tilde{\mathcal{F}}^{\triangle}|_{P,\prec}$ holds.

Note that generally δ-regularity of variables $\mathbf{x}$ for a set $\mathcal{F}$ according to Definition 3.2 and for the ideal $\mathcal{I} = \langle \mathcal{F} \rangle$ according to Definition 3.1 are independent properties. We will see later that if $\mathcal{F}$ is actually a Gröbner basis of $\mathcal{I}$, then the two definitions coincide.

Example 3.3. One of the simplest instances where the definitions differ is not for an ideal but for a submodule of the free $\Bbbk[x, y]$-module with basis $\{\mathbf{e}_1, \mathbf{e}_2\}$. Consider the set $\mathcal{F} = \{y^2\mathbf{e}_1, xy\mathbf{e}_1 + \mathbf{e}_2, x\mathbf{e}_2\}$ and any term order for which $xy\mathbf{e}_1 \succ \mathbf{e}_2$. The used coordinates are not δ-regular for $\mathcal{F}$, as any transformation of the form $x = \bar{x} + a\bar{y}$ with $a \neq 0$ will increase the involutive size. Nevertheless, the used coordinates are δ-regular for the submodule $\langle \mathcal{F} \rangle$. Indeed, adding the generator $y\mathbf{e}_2$ (the S-"polynomial" of the first two generators) makes $\mathcal{F}$ to a reduced Gröbner basis which is simultaneously a minimal Pommaret basis. Examples of this type are critical for the algorithmic determination of Pommaret bases: although a finite basis exist, some completion algorithms may loop infinitely in such a situation, as they implicitly try to construct a Pommaret basis for $\langle \mathrm{le}_\prec \mathcal{F} \rangle$ as an intermediate step.

It is, however, straightforward to prove the following statement.

Proposition 3.4. *Let $\mathcal{H}$ be a Pommaret basis of an ideal $\mathcal{I} \subseteq \mathcal{P}$. Then the given coordinates $\mathbf{x}$ are δ-regular for $\mathcal{H}$.*

Most coordinates are δ-regular for a given set $\mathcal{F}$. Choosing an arbitrary reference coordinate system, we may identify every system of coordinates with the regular matrix $A \in \Bbbk^{n \times n}$ defining the linear transformation from our reference system to it.

Proposition 3.5. *The coordinate systems that are δ-singular for a given finite involutively head autoreduced set $\mathcal{F} \subset \mathcal{P}$ form a Zariski closed set in $\Bbbk^{n \times n}$.*

Proof. We perform first a linear coordinate transformation with an undetermined matrix $A = (a_{ij}) \in \Bbbk^{n \times n}$, i.e. we treat its entries as parameters. This obviously leads to a δ-regular coordinate system, as each polynomial in $\tilde{\mathcal{F}}^{\triangle}$ will get its maximally possible class. δ-singular coordinates are defined by the vanishing of certain (leading) coefficients. Since these coefficients are polynomials in the entries a_{ij} of A, the set of all δ-singular coordinate systems can be described as the zero set of an ideal of $\Bbbk[a_{11}, \ldots, a_{nn}]$. $\square$

Theorem 3.6. *Let the finite set $\mathcal{F} \subset \mathcal{P}$ be involutively head autoreduced for the Pommaret division and a class respecting term order*[6] *$\prec$. Furthermore assume that the underlying field $\Bbbk$ is infinite. If $|\mathcal{F}|_{J,\prec} > |\mathcal{F}|_{P,\prec}$, then the coordinates $\mathbf{x}$ are δ-singular for $\mathcal{F}$.*

[6] This means that for $\deg t_1 = \deg t_2$ and $\mathrm{cls}\, t_1 < \mathrm{cls}\, t_2$ we always have $t_1 \prec t_2$. The degree reverse lexicographic order has this property.

Proof. By Proposition 2.1, we have $X_P(f) \subseteq X_{J,\mathcal{F}}(f)$ for all $f \in \mathcal{F}$. Assume that for a polynomial $h \in \mathcal{F}$ the strict inclusion $X_P(h) \subset X_{J,\mathcal{F}}(h)$ holds. Thus at least one variable $x_\ell \in X_{J,\mathcal{F}}(h)$ with $\ell > k = \operatorname{cls} h$ exists. We perform the linear change of variables $x_i = \tilde{x}_i$ for $i \neq k$ and $x_k = \tilde{x}_k + a\tilde{x}_\ell$ with a yet arbitrary parameter $a \in \Bbbk \setminus \{0\}$. This induces the following transformation of the terms:

$$x^\mu = \sum_{j=0}^{\mu_k} \binom{\mu_k}{j} a^j \tilde{x}^{\mu - j_k + j_\ell} . \tag{5}$$

Let $\operatorname{le}_\prec h = \mu$. Thus $\mu = [0, \dots, 0, \mu_k, \dots, \mu_n]$ with $\mu_k > 0$. Consider the multi index $\nu = \mu - (\mu_k)_k + (\mu_k)_\ell$; obviously, $\operatorname{cls} \nu > k$. Applying our transformation to h leads to a polynomial $\tilde{h}$ containing the term $\tilde{x}^\nu$. Note that ν cannot be an element of $\operatorname{le}_\prec \mathcal{F}$. Indeed, if it was, it would be an element of the same set $(\mu_{\ell+1}, \dots, \mu_n)$ as μ. But this contradicts our assumption that ℓ is multiplicative for the multi index μ with respect to the Janet division, as by construction $\nu_\ell > \mu_\ell$.

Transforming all polynomials $f \in \mathcal{F}$ yields the set $\tilde{\mathcal{F}}$ on which we perform an involutive head autoreduction in order to obtain the set $\tilde{\mathcal{F}}^\triangle$. Since we assume that the ground field $\Bbbk$ is infinite, we can always choose the parameter a such that after the transformation each polynomial $\tilde{f} \in \tilde{\mathcal{F}}$ has at least the same class as the corresponding polynomial $f \in \mathcal{F}$, as our term order respects classes. This is a simple consequence of (5): cancellations of terms may occur only, if the parameter a is a zero of some polynomial (possibly one for each member of $\mathcal{F}$) with a degree not higher than $\deg \mathcal{F}$. By the definition of the Pommaret division, if $\operatorname{le}_\prec f_2 \,|_P \operatorname{le}_\prec f_1$, then $\operatorname{cls} \operatorname{le}_\prec f_2 \geq \operatorname{cls} \operatorname{le}_\prec f_1$. Hence even after the involutive head autoreduction the involutive size of $\tilde{\mathcal{F}}^\triangle$ cannot be smaller than that of $\mathcal{F}$.

Consider again the polynomial h. The leading term of the transformed polynomial $\tilde{h}$ must be greater than or equal to $\tilde{x}^\nu$. Thus its class is greater than k. This remains true even after an involutive head autoreduction with all those polynomials $\tilde{f} \in \tilde{\mathcal{F}}$ that are of class greater than k, as $x^\nu \notin \operatorname{lt}_\prec \mathcal{F}$. Hence the only possibility to obtain a leading term of class less than or equal to k consists of an involutive reduction with respect to a polynomial $\tilde{f} \in \tilde{\mathcal{F}}$ with $\operatorname{cls} f \leq k$. But this implies that $\operatorname{cls} \operatorname{le}_\prec \tilde{f} > k$. So we may conclude that after the transformation we have at least one polynomial more whose class is greater than k. So the coordinates $\mathbf{x}$ cannot be δ-regular. ☐

Corollary 3.7. *If the coordinates* $\mathbf{x}$ *are* δ*-regular for the finite Pommaret head autoreduced set* $\mathcal{F}$*, then* $\langle \mathcal{F} \rangle_{J,\prec} = \langle \mathcal{F} \rangle_{P,\prec}$ *for any class respecting term order* $\prec$*.*

It is important to note that this corollary provides us only with a necessary but *not* with a sufficient criterion for δ-regularity. In other words, even if the Janet and the Pommaret size are equal for a given set $\mathcal{F} \subset \mathcal{P}$, this fact does not imply that the used coordinates are δ-regular for $\mathcal{F}$.

Example 3.8. Let $\mathcal{F} = \{\underline{z^2}+y^2-2x^2,\ \underline{xz}+xy,\ \underline{yz}+y^2+x^2\}$. The underlined terms are the leaders with respect to the degree reverse lexicographic order. One easily checks that the Janet and the Pommaret division yield the same multiplicative variables. If we perform the transformation $\tilde{x} = z$, $\tilde{y} = y + z$ and $\tilde{z} = x$, then we obtain after an autoreduction the set $\tilde{\mathcal{F}}^\triangle = \{\underline{\tilde{y}^2},\ \underline{\tilde{y}\tilde{z}},\ \underline{\tilde{z}^2} - \tilde{y}\tilde{x}\}$. Again the Janet and the Pommaret division yield the same multiplicative variables, but $|\tilde{\mathcal{F}}^\triangle|_{P,\prec} > |\mathcal{F}|_{P,\prec}$. Thus the coordinates (x, y, z) are not δ-regular for $\mathcal{F}$.

The explanation of this phenomenon is very simple. Obviously our criterion depends only on the leading terms of the set $\mathcal{F}$. In other words, it analyses the monomial ideal $\langle \mathrm{lt}_\prec \mathcal{F}\rangle$. Here we find $\langle \mathrm{lt}_\prec \mathcal{F}\rangle = \langle xz, yz, z^2\rangle$ and one easily verifies that the used generating set is already a Pommaret basis. However, for $\mathcal{I} = \langle \mathcal{F}\rangle$ the leading ideal is $\mathrm{lt}_\prec \mathcal{I} = \langle x^3, xz, yz, z^2\rangle$ (one obtains a Janet basis for $\mathcal{I}$ by adding x^3 to $\mathcal{F}$) and obviously it does not possess a Pommaret basis, as such a basis would have to contain all monomials x^3y^k with $k \in \mathbb{N}$ (or we exploit our criterion noting that y is a Janet but not a Pommaret multiplicative variable for x^3). Obviously, we have here just the opposite situation to Example 3.3: there $\mathrm{lt}_\prec \mathcal{I}$ had a Pommaret basis but $\langle \mathrm{lt}_\prec \mathcal{F}\rangle$ not.

Theorem 3.9. *In suitably chosen coordinates* $\mathbf{x}$ *every ideal* $\mathcal{I} \subseteq \mathcal{P}$ *has a Pommaret basis.*

Proof. We only sketch a proof here, as a rigorous demonstration requires the detailed formulation of a completion algorithm for the construction of an involutive basis [5, 6], which we omit here for lack of space.

Each iteration of the completion algorithm consists of selecting an element from the current basis, multiplying it with one of its non-multiplicative variables, performing an involutive reduction and then adding the result (if it is different from zero) to the basis. Only if this action has enlarged the ideal spanned by the leading terms, a test for δ-regularity and, if necessary, a coordinate change according to the proof of Theorem 3.6 are carried out. By regarding all the bases in a fixed reference coordinate system (or alternatively the coordinate invariant form of the basis in the symmetric algebra $S\mathcal{V}$), the ideals spanned by the leading terms form an ascending chain that eventually becomes stationary at $\mathrm{lt}_\prec \mathcal{I}$. On the other hand, if the ideal spanned by the leading terms remains unchanged (which it does especially after we have reached $\mathrm{lt}_\prec \mathcal{I}$), we have performed one step in the monomial completion of this ideal, which has to terminate under the assumption that, after suitable transformations, we are in δ-regular coordinates. □

Note that the considerations above imply that after performing a finite number of coordinate transformations in the form used in the proof of Theorem 3.6, we obtain a δ-regular coordinate system for the ideal $\mathcal{I}$. Thus the effective construction of such a system can be automatically done during the determination of a Pommaret basis for $\mathcal{I}$. As the following result shows, the search for δ-regular coordinates corresponds to putting the ideal $\mathcal{I}$ into *Noether position.*

Theorem 3.10. *If the coordinates* $\mathbf{x}$ *are* δ*-regular for the ideal* $\mathcal{I} \subseteq \mathcal{P}$*, the restriction of the canonical projection* $\pi : \mathcal{P} \to \mathcal{P}/\mathcal{I}$ *to* $\mathbb{k}[x_1, \ldots, x_D]$ *is a Noether normalisation of* $\mathcal{P}/\mathcal{I}$ *(or equivalently,* $x_1, \ldots, x_D$ *form a homogeneous system of parameters).*

Proof. $\{x_1, \ldots, x_D\}$ is a maximal independent set modulo $\mathcal{I}$, so the restriction of the projection π to $\mathbb{k}[x_1, \ldots, x_D]$ is injective. Proposition 2.2 gives the complementary decomposition for $\mathrm{le}_\prec \mathcal{I}$ which is defined by a finite set $\mathcal{N} \subset \mathbb{N}_0^n$. As for each generator in $\mathcal{N}$ the associated multiplicative indices form a subset of $\{1, \ldots, D\}$ and since the complement of $\mathrm{lt}_\prec \mathcal{I}$ is a basis of $\mathcal{P}/\mathcal{I}$ as a vector space over $\mathbb{k}$, the finite set $\{\pi(x^\nu) \mid \nu \in \mathcal{N}\}$ generates $\mathcal{P}/\mathcal{I}$ as a $\mathbb{k}[x_1, \ldots, x_D]$-module. □

The converse of this theorem is generally not true: even if $\mathbb{k}[x_1, \ldots, x_D]$ defines a Noether normalisation of $\mathcal{P}/\mathcal{I}$, this is not sufficient to conclude that $\mathcal{I}$ possesses a Pommaret basis. In the next section, we will show for monomial ideals that the existence of a Pommaret basis is equivalent to a stronger property.

4 Quasi-stable Ideals

Theorem 3.9 asserts that any polynomial ideal possesses a Pommaret basis after a suitable coordinate transformation. However, this result is not really useful for monomial ideals, as the transformed ideal is in general no longer monomial. Hence it is something particular, if a monomial ideal has a Pommaret basis, and we give these ideals a special name.

Definition 4.1. A monomial ideal $\mathcal{I} \subseteq \mathcal{P}$ is *quasi-stable*, if it has a Pommaret basis.

Remark 4.2. Recall that a (possibly infinite) set $\mathcal{N} \subseteq \mathbb{N}_0^n$ is called *stable*, if for each multi index $\nu \in \mathcal{N}$ all multi indices $\nu - 1_k + 1_j$ with $k = \operatorname{cls} \nu < j \leq n$ are also contained in $\mathcal{N}$. A monomial ideal $\mathcal{I} \subseteq \mathcal{P}$ is stable, if the exponent vectors of the monomials contained in it form a stable set. If $\mathcal{I}$ is a quasi-stable ideal in the sense of the definition above, then one can easily show that for a sufficiently high degree q the truncated ideal $\mathcal{I}_{\geq q}$ is stable (one may e. g. take $q = \deg \mathcal{H}$ with $\mathcal{H}$ the Pommaret basis of $\mathcal{I}$).

Proposition 4.3. *Let* $\mathcal{I} \subseteq \mathcal{P}$ *be a monomial ideal with* $\dim \mathcal{P}/\mathcal{I} = D$*. Then the following five statements are equivalent.*

(i) *$\mathcal{I}$ is quasi-stable.*
(ii) *The variable x_1 is not a zero divisor for $\mathcal{P}/\mathcal{I}^{\mathrm{sat}}$ and for all $1 \leq j < D$ the variable x_{j+1} is not a zero divisor for $\mathcal{P}/\langle \mathcal{I}, x_1, \ldots, x_j\rangle^{\mathrm{sat}}$.*
(iii) *We have $\mathcal{I} : \langle x_1\rangle^\infty \subseteq \mathcal{I} : \langle x_2\rangle^\infty \subseteq \cdots \subseteq \mathcal{I} : \langle x_D\rangle^\infty$ and for all $D < j \leq n$ an exponent $k_j \geq 1$ exists such that $x_j^{k_j} \in \mathcal{I}$.*
(iv) *For all $1 \leq j \leq n$ the equality $\mathcal{I} : \langle x_j\rangle^\infty = \mathcal{I} : \langle x_j, \ldots, x_n\rangle^\infty$ holds.*
(v) *For every associated prime ideal $\mathfrak{p} \in \operatorname{Ass}(\mathcal{P}/\mathcal{I})$ an integer $1 \leq j \leq n$ exists such that $\mathfrak{p} = \langle x_j, \ldots, x_n\rangle$.*

Proof. The equivalence of the statements (ii)–(v) was proven by Bermejo and Gimenez [2, Proposition 3.2] who called ideals satisfying one of these conditions *monomial ideals of nested type*.[7] We now show that this concept is identical with quasi-stability by proving the equivalence of (i) and (iii).

Assume first that the ideal $\mathcal{I}$ is quasi-stable with Pommaret basis $\mathcal{H}$. The existence of a term $x_j^{k_j} \in \mathcal{I}$ for all $D < j \leq n$ follows then immediately from Remark 2.3. Consider a term $x^\mu \in \mathcal{I} : \langle x_k \rangle^\infty \setminus \mathcal{I}$ for some $1 \leq k \leq n$. There exists an integer ℓ such that $x_k^\ell x^\mu \in \mathcal{I}$ and hence a generator $x^\nu \in \mathcal{H}$ such that $x^\nu \,|_P\, x_k^\ell x^\mu$. If $\operatorname{cls} \nu > k$, then ν would also be an involutive divisor of μ contradicting the assumption $x^\mu \notin \mathcal{I}$. Thus we find $\operatorname{cls} \nu \leq k$ and $\nu_k > \mu_k$. Next we consider for arbitrary exponents $m > 0$ the terms $x_{k+1}^m x^\nu \in \mathcal{I}$. For each m a generator $x^{\rho^{(m)}} \in \mathcal{H}$ exists which involutively divides $x_{k+1}^m x^\nu$. By the same reasoning as above, $\operatorname{cls} x^{\rho^{(m)}} > k+1$ is not possible for an involutively autoreduced basis $\mathcal{H}$ yielding the estimate $\operatorname{cls} \nu \leq \operatorname{cls} x^{\rho^{(m)}} \leq k+1$.

We claim now that there exists an integer m_0 such that $\rho^{(m)} = \rho^{(m_0)}$ for all $m \geq m_0$ and $\operatorname{cls} x^{\rho^{(m_0)}} = k+1$. Indeed, if $\operatorname{cls} x^{\rho^{(m)}} < k+1$, then we must have $\rho_{k+1}^{(m)} = v_{k+1} + m$, since x_{k+1} is not multiplicative for $x^{\rho^{(m)}}$. Hence $x^{\rho^{(m)}}$ cannot be an involutive divisor of $x_{k+1}^{m+1} x^\nu$ and $\rho^{(m+1)} \notin \{\rho^{(1)}, \dots, \rho^{(m)}\}$. As the Pommaret basis $\mathcal{H}$ is a finite set, $\operatorname{cls} x^{\rho^{(m_0)}} = k+1$ for some value $m_0 > 0$. But then x_{k+1} is multiplicative for $x^{\rho^{(m_0)}}$ and thus $x^{\rho^{(m_0)}}$ is trivially an involutive divisor of $x_{k+1}^m x^\nu$ for all values $m \geq m_0$.

By construction, the generator $x^{\rho^{(m_0)}}$ is also an involutive divisor of $x_{k+1}^{m_0} x^\mu$, as x_k is multiplicative for it. Hence this term must lie in $\mathcal{I}$ and consequently $x^\mu \in \mathcal{I} : \langle x_{k+1} \rangle^\infty$. Thus we can conclude that $\mathcal{I} : \langle x_k \rangle^\infty \subseteq \mathcal{I} : \langle x_{k+1} \rangle^\infty$. This proves (iii).

For the converse assume that (iii) holds and let $\mathcal{B}$ be the minimal basis of the ideal $\mathcal{I}$. Let $x^\mu \in \mathcal{B}$ be an arbitrary term of class k. Then $x^\mu / x_k \in \mathcal{I} : \langle x_k \rangle^\infty$. By assumption, this means that also $x^\mu / x_k \in \mathcal{I} : \langle x_\ell \rangle^\infty$ for any non-multiplicative index ℓ. Hence for each term $x^\mu \in \mathcal{B}$ and for each value $\operatorname{cls}(x^\mu) < \ell \leq n$ there exists an integer $q_{\mu,\ell}$ such that $x_\ell^{q_{\mu,\ell}} x^\mu / x_k \notin \mathcal{I}$ but $x_\ell^{q_{\mu,\ell}+1} x^\mu / x_k \in \mathcal{I}$. For the values $1 \leq \ell \leq \operatorname{cls} x^\mu$ we set $q_{\mu,\ell} = 0$. Observe that if $x^\nu \in \mathcal{B}$ is a minimal generator dividing $x_\ell^{q_{\mu,\ell}+1} x^\mu / x_k$, then $x^\nu \prec_{\text{invlex}} x^\mu$, since $\operatorname{cls}(x^\nu) \geq \operatorname{cls}(x^\mu)$ and $\nu_k < \mu_k$.

Consider now the set

$$\mathcal{H} = \left\{ x^{\mu+\rho} \mid x^\mu \in \mathcal{B},\ \forall 1 \leq \ell \leq n : 0 \leq \rho_\ell \leq q_{\mu,\ell} \right\} . \tag{6}$$

We claim that it is a weak involutive completion of $\mathcal{B}$ and thus a weak Pommaret basis of $\mathcal{I}$. In order to prove this assertion, we must show that each term $x^\lambda \in \mathcal{I}$ lies in the involutive cone of a member of $\mathcal{H}$.

As x^λ is assumed to be an element of $\mathcal{I}$, we can factor it as $x^\lambda = x^{\sigma^{(1)}} x^{\rho^{(1)}} x^{\mu^{(1)}}$ where $x^{\mu^{(1)}} \in \mathcal{B}$ is a minimal generator, $x^{\sigma^{(1)}}$ contains only multiplicative variables for $x^{\mu^{(1)}}$

[7] One must revert the ordering of the variables in order to recover the statements in [2].

and $x^{\rho^{(1)}}$ only non-multiplicative ones. If $x^{\mu^{(1)}+\rho^{(1)}} \in \mathcal{H}$, then we are done, as obviously $\operatorname{cls}\left(x^{\mu^{(1)}+\rho^{(1)}}\right) = \operatorname{cls}\left(x^{\mu^{(1)}}\right)$ and hence all variables contained in $x^{\sigma^{(1)}}$ are multiplicative for $x^{\mu^{(1)}+\rho^{(1)}}$, too.

Otherwise there exists a non-multiplicative variables x_ℓ such that $\rho^{(1)}_\ell > q_{\mu^{(1)},\ell}$. Any minimal generator $x^{\mu^{(2)}} \in \mathcal{H}$ dividing $x_\ell^{q_{\mu^{(1)},\ell}+1} x^{\mu^{(1)}}/x_k$ is also a divisor of x^λ and we find a second factorisation $x^\lambda = x^{\sigma^{(2)}} x^{\rho^{(2)}} x^{\mu^{(2)}}$ where again $x^{\sigma^{(2)}}$ consists only of multiplicative and $x^{\rho^{(2)}}$ only of non-multiplicative variables for $x^{\mu^{(2)}}$. If $x^{\mu^{(2)}+\rho^{(2)}} \in \mathcal{H}$, then we are done by the same argument as above; otherwise we iterate.

According to the observation made above, the sequence $x^{\mu^{(1)}}, x^{\mu^{(2)}}, \dots$ of minimal generators constructed this way is strictly descending with respect to the inverse lexicographic order. However, the minimal basis $\mathcal{B}$ is a finite set and thus the iteration cannot go on infinitely. As the iteration only stops, if there exists an involutive cone containing x^λ, the involutive span of $\mathcal{H}$ is indeed $\mathcal{I}$ and thus $\mathcal{I}$ quasi-stable. $\square$

Note that we actually proved that (iii) may be replaced by the equivalent statement $\mathcal{I} : \langle x_1\rangle^\infty \subseteq \mathcal{I} : \langle x_2\rangle^\infty \subseteq \cdots \subseteq \mathcal{I} : \langle x_n\rangle^\infty$ requiring no a priori knowledge of D (the dimension D arises then obviously as the smallest value k such that $\mathcal{I} : \langle x_k\rangle^\infty = \mathcal{P}$). In this formulation it is straightforward to verify (iii) effectively: bases of the colon ideals $\mathcal{I} : \langle x_k\rangle^\infty$ are obtained by setting $x_k = 1$ in a basis of $\mathcal{I}$ and for monomial ideals it is trivial to check inclusion.

For the sequel, we make the convention that the formal expression $\mathcal{I} : x_0^\infty$ equals $\mathcal{I}$. The following technical results will be useful later.

Lemma 4.4. *For a quasi-stable ideal $\mathcal{I}$ and for all $0 \le i \le n$, we have:*

(i) *$\mathcal{I} : \langle x_i\rangle^\infty$ can be minimally generated by elements having class at least $i+1$.*

(ii) *$\left(\mathcal{I} : \langle x_i\rangle^\infty\right) : \langle x_j\rangle^\infty = \mathcal{I} : \langle x_j\rangle^\infty$ for all $i < j \le n$.*

Proof. No element of a minimal basis of $\mathcal{H}$ of $\mathcal{I} : \langle x\rangle_i^\infty$ can depend on x_j. Now assume that $x^\nu \in \mathcal{H}$ satisfies $\operatorname{cls}\nu = \ell < k$. Then $x_j^m x^\nu$ is a minimal generator of $\mathcal{I}$ for some suitable exponent $m \in \mathbb{N}_0$. This in turn implies that $x_j^m x^\nu / x_\ell^{\nu_\ell} \in \mathcal{I} : \langle x_\ell\rangle^\infty \subseteq \mathcal{I} : \langle x_j\rangle^\infty$ and hence $x^\nu/x_\ell^{\nu_\ell} \in \mathcal{I} : \langle x_j\rangle^\infty$ which contradicts our assumption that x^ν was a minimal generator. This proves Part (i); Part (ii) follows directly from the definition of the saturation and Proposition 4.3 (iii). $\square$

From the next proposition it follows that for a *monomial* set $\mathcal{H}$, equality of the Pommaret and the Janet size entails quasi-stability of the ideal $\langle\mathcal{H}\rangle$; thus in this case a converse to Theorem 3.6 can be obtained.

Proposition 4.5. *Let $\mathcal{I} \subseteq \mathcal{P}$ be a monomial ideal and $\mathcal{H}$ a finite, Pommaret autoreduced monomial basis of it. If I is not quasi-stable, then $|\mathcal{H}|_J > |\mathcal{H}|_P$, i. e. for some generator in $\mathcal{H}$ a variable exists which is Janet but not Pommaret multiplicative.*

Proof. By Proposition 2.1 we have $|\mathcal{H}|_J \geq |\mathcal{H}|_P$. As $\mathcal{I}$ is not quasi-stable, there exists a minimal value k such that $\mathcal{I} : \langle x_k \rangle^\infty \nsubseteq \mathcal{I} : \langle x_{k+1} \rangle^\infty$. Let x^μ be a minimal generator of $\mathcal{I} : \langle x_k \rangle^\infty$ which is not contained in $\mathcal{I} : \langle x_{k+1} \rangle^\infty$. Then for a suitable exponent $m \in \mathbb{N}_0$ the term $x^{\bar{\mu}} = x_k^m x^\mu$ is a minimal generator of $\mathcal{I}$ and hence contained in $\mathcal{H}$.

We claim now that $\mathcal{H}$ contains a generator for which x_{k+1} is Janet but not Pommaret multiplicative. If $x_{k+1} \in X_{J,\mathcal{H}}(x^{\bar{\mu}})$, then we are done, since according to Lemma 4.4 (i) $\operatorname{cls} \bar{\mu} = k$ and hence $x_{k+1} \notin X_P(x^{\bar{\mu}})$. Otherwise $\mathcal{H}$ contains a term x^ν such that $\nu_\ell = \mu_\ell$ for $k+1 < \ell \leq n$ and $\nu_{k+1} > \mu_{k+1}$. If several generators with this property exist in $\mathcal{H}$, we choose one for which ν_{k+1} takes a maximal value so that we have $x_{k+1} \in X_{J,\mathcal{H}}(x^\nu)$ by definition of the Janet division. If $\operatorname{cls} \nu < k+1$, we are again done, since in this case $x_{k+1} \notin X_P(x^\nu)$. Finally assume that $\operatorname{cls} \nu = k+1$ and consider the term $x^\rho = x^\nu / x_{k+1}^{\nu_{k+1}}$. Obviously, $x^\rho \in \mathcal{I} : \langle x_{k+1} \rangle^\infty$ contradicting our assumption $x^\mu \notin \mathcal{I} : \langle x_{k+1} \rangle^\infty$ since $x^\rho \mid x^\mu$. Hence this case cannot arise. ∎

From Condition (v) in Proposition 4.3 we see that (modulo a permutation of the variables) quasi-stable ideals are precisely those monomial ideals with a single minimal associated prime ideal. Going one step further, one can even read off a primary decomposition from the ascending chain of ideals in Condition (iii).

As above, let D denote the dimension of $\mathcal{I}$. We restrict ourselves to the case that x_1 (and hence each variable) occurs in some minimal generator of $\mathcal{I}$; otherwise, we change $\mathcal{P}$ accordingly. $\mathcal{I}$ contains pure powers of exactly the variables $x_{D+1}, \dots, x_n$; thus $\mathcal{I} : \langle x_D \rangle^\infty$ is $\langle x_{D+1}, \dots, x_n \rangle$-primary. For $1 \leq i \leq D$, let $s_i := \min\{s \mid \mathcal{I} : \langle x_i \rangle^s = \mathcal{I} : \langle x_i \rangle^{s+1}\}$; this is just the the maximal x_i-degree of a minimal generating set of $\mathcal{I}$. Furthermore, let $\mathcal{S}_i := \mathcal{I} + \langle x_{i+1}^{s_{i+1}}, \dots, x_D^{s_D} \rangle$ and $\mathfrak{q}_i := \mathcal{S}_i : \langle x_i \rangle^\infty = \mathcal{I} : \langle x_i \rangle^\infty + \langle x_{i+1}^{s_{i+1}}, \dots, x_D^{s_D} \rangle$ for $0 \leq i \leq D$. Because of Lemma 4.4, $\mathfrak{q}_i$ is $\langle x_{i+1}, \dots, x_n \rangle$-primary. By repeatedly applying the well-known identity $\mathcal{I} = \mathcal{I} + \langle x^s \rangle \cap \mathcal{I} : \langle x \rangle^s$ (provided that $\mathcal{I} : \langle x \rangle^s = \mathcal{I} : \langle x \rangle^\infty$) and Lemma 4.4 (ii), we can decompose each ideal $\mathcal{I} : \langle x_i \rangle^\infty$ $(0 \leq i \leq D)$ as:

$$
\begin{aligned}
\mathcal{I} : \langle x_i \rangle^\infty &= \bigl(\mathcal{I} : \langle x_i \rangle^\infty + \langle x_{i+1}^{s_{i+1}} \rangle\bigr) \cap \underbrace{\bigl(\mathcal{I} : \langle x_i \rangle^\infty\bigr) : \langle x_{i+1} \rangle^\infty}_{= \mathcal{I} : \langle x_{i+1} \rangle^\infty} \\
&= \bigl(\mathcal{I} : \langle x_i \rangle^\infty + \langle x_{i+1}^{s_{i+1}}, x_{i+2}^{s_{i+2}} \rangle\bigr) \cap \underbrace{\bigl(\mathcal{I} : \langle x_i \rangle^\infty + \langle x_{i+1}^{s_{i+1}} \rangle\bigr) : \langle x_{i+2} \rangle^\infty}_{= \mathcal{I} : \langle x_{i+2} \rangle^\infty + \langle x_{i+1}^{s_{i+1}} \rangle} \cap \mathcal{I} : \langle x_{i+1} \rangle^\infty \\
&\dots \\
&= \mathfrak{q}_i \cap \bigl(\mathcal{I} : \langle x_D \rangle^\infty + \langle x_{i+1}^{s_{i+1}}, \dots, x_{D-1}^{s_{D-1}} \rangle\bigr) \cap \dots \cap \mathcal{I} : \langle x_{i+1} \rangle^\infty
\end{aligned}
\tag{7}
$$

Because of the quasi-stability of $\mathcal{I}$, the last ideal in this decomposition is always contained in all the preceding ones except $\mathfrak{q}_i$, so these can be dropped. Since $\mathfrak{q}_D = \mathcal{I} : \langle x_D \rangle^\infty$, we thus get a primary decomposition $\mathcal{I} = \bigcap_{i=0}^{D} \mathfrak{q}_i$, where the radicals of the primary components are pairwise different.

Proposition 4.6. *In the decomposition $\mathcal{I} = \bigcap_{i=0}^{D} \mathfrak{q}_i$, the primary component $\mathfrak{q}_j$ is redundant if and only if $\mathcal{I} : \langle x_k \rangle^\infty = \mathcal{I} : \langle x_{k+1} \rangle^\infty$.*

Proof. The above construction immediately yields $\mathcal{I} : x_k^\infty = \bigcap_{i=k}^{D} \mathcal{I} : x_i^\infty$. An elementary computation involving sums and intersection of ideals furthermore shows that $\mathcal{S}_k = \bigcap_{i=0}^{k} \mathfrak{q}_i$. Therefore, $\mathcal{I} = \mathcal{S}_{k-1} \cap \mathcal{I} : x_k^\infty$ (we set $\mathcal{S}_{-1} = \mathcal{P}$). From that, we see at once that $\mathfrak{q}_k$ is redundant if $\mathcal{I} : x_k^\infty = \mathcal{I} : x_{k+1}^\infty$. For the other direction, assume that $\mathcal{I} : x_k^\infty \subsetneq \mathcal{I} : x_{k+1}^\infty$. For $k = 0$, this immediately implies that $\mathfrak{q}_0$ cannot be redundant. For $k > 0$, take a minimal generator m of $\mathcal{I} : \langle x_{k+1} \rangle^\infty$ (having class at least $k+2$) which is not in $\mathcal{I} : \langle x_k \rangle^\infty$ and consider the monomial $x_k^{s_k} m$. It is obviously contained both in $\mathcal{S}_{k-1}$ and in $\mathcal{I} : \langle x_{k+1} \rangle^\infty$, but not in $\mathcal{I} : \langle x_k \rangle$ and thus also not in $\mathfrak{q}_k$ (since from $\mathcal{I} : \langle x_k \rangle^\infty = \mathfrak{q}_k \cap \mathcal{I} : \langle x_{k+1} \rangle^\infty$ we have that $\mathfrak{q}_k \cap (\mathcal{I} : \langle x_{k+1} \rangle^\infty \setminus \mathcal{I} : \langle x_k \rangle^\infty) = 0$); therefore, $\mathfrak{q}_k \not\supseteq \mathfrak{q}_0 \cap \ldots \cap \mathfrak{q}_{k-1} \cap \mathfrak{q}_{k+1} \cap \ldots \cap \mathfrak{q}_D$. □

The following result on the Noether normalisation of a quasi-stable ideal is also due to Bermejo and Gimenez [2, Proposition 3.6].

Corollary 4.7. *Let $\mathcal{I} \subseteq \mathcal{P}$ be a monomial ideal with $\dim \mathcal{P}/\mathcal{I} = D$. Furthermore, let $\mathcal{I} = \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_r$ be an irredundant monomial primary decomposition with $D_j = \dim \mathcal{P}/\mathfrak{q}_j$ for $1 \leq j \leq r$. The ideal $\mathcal{I}$ is quasi-stable, if and only if $\mathbb{k}[x_1, \ldots, x_D]$ defines a Noether normalisation of $\mathcal{P}/\mathcal{I}$ and $\mathbb{k}[x_1, \ldots, x_{D_j}]$ one of $\mathcal{P}/\mathfrak{q}_j$ for each primary component $\mathfrak{q}_j$.*

Proof. By assumption, each ideal $\mathfrak{q}_j$ is a monomial primary ideal. This implies that $\mathbb{k}[x_1, \ldots, x_{D_j}]$ defines a Noether normalisation of $\mathcal{P}/\mathfrak{q}_j$, if and only if the associated prime ideal is $\sqrt{\mathfrak{q}_j} = \langle x_{D_j+1}, \ldots, x_n \rangle$. Now the assertion follows from Condition (v) in Proposition 4.3. □

5 δ-Regularity vs. Quasi-Regularity

Definition 5.1. A linear form $v = a_1x_1 + \ldots a_nx_n \in \mathcal{P}_1$ is *quasi-regular* at degree q for the $\mathcal{P}$-module $\mathcal{M}$, if $v \cdot m = 0$ entails $m \in \mathcal{M}_{<q}$. A finite sequence $(v_1, \ldots, v_k)$ of linear forms in $\mathcal{P}$ is *quasi-regular* at degree q for $\mathcal{M}$, if each v_i is quasi-regular at degree q for the factor module $\mathcal{M}/\langle v_1, \ldots, v_{i-1} \rangle \mathcal{M}$.

This generalisation of the classical notion of a regular sequence appears in the dual formulation of Cartan's test for an involutive polynomial module due to Serre (see his letter appended to [7]). Recall that a polynomial module $\mathcal{M}$ is involutive at a degree q, if no minimal generator of the Koszul homology $H_\bullet(\mathcal{M})$ has a symmetric degree greater than or equal to q.

Theorem 5.2 (Dual Cartan Test [7])**.** *Let $\mathcal{M}$ be a polynomial module finitely generated in degree less than $q > 0$. The module $\mathcal{M}$ is involutive at degree q, if and only if for* *generic coordinates* $\{x_1, \dots, x_n\}$ *the maps*

$$\mu_k : \mathcal{M}_r/\langle x_1, \dots, x_{k-1}\rangle\mathcal{M}_{r-1} \longrightarrow \mathcal{M}_{r+1}/\langle x_1, \dots, x_{k-1}\rangle\mathcal{M}_r \tag{8}$$

induced by multiplication with x_k *are injective for all* $r \geq q$ *and* $1 \leq k \leq n$*, i. e. if and only if* $(x_1, \dots, x_n)$ *is a quasi-regular sequence at degree* q *for* $\mathcal{M}$*.*

The goal of this section is to show that the notion of δ-regularity for an ideal $\mathcal{I}$ as discussed in Section 3 is equivalent to the above introduced concept of quasi-regularity for the polynomial module $\mathcal{P}/\mathcal{I}$.

Lemma 5.3. *Let* $\mathcal{I} \subseteq \mathcal{P}$ *be a homogeneous ideal and* $\prec$ *the degree reverse lexicographic order. The sequence* $(x_1, \dots, x_n)$ *is quasi-regular at degree* q *for the module* $\mathcal{M} = \mathcal{P}/\mathcal{I}$*, if and only if it is quasi-regular at degree* q *for* $\mathcal{M}' = \mathcal{P}/\mathrm{lt}_\prec\mathcal{I}$*.*

Proof. Let $\mathcal{G}$ be a Gröbner basis of $\mathcal{I}$ for $\prec$, so that the normal form with respect to $\mathcal{G}$ defines an isomorphism between the vector spaces $\mathcal{M}$ and $\mathcal{M}'$. One direction is trivial, as an obvious necessary condition for $m = [f] \in \mathcal{M}$ to satisfy $x_1 \cdot m = 0$ is that $x_1 \cdot [\mathrm{lt}_\prec f] = 0$ in $\mathcal{M}'$. Hence quasi-regularity of x_1 for $\mathcal{M}'$ implies quasi-regularity of x_1 for $\mathcal{M}$ and by iteration the same holds true for the whole sequence.

For the converse let $r \geq q$ be an arbitrary degree. We may choose for the vector space $\mathcal{M}_r$ a basis where each member is represented by a monomial, i. e. the representatives simultaneously induce a basis of $\mathcal{M}'_r$. Let x^μ be one of these monomials. As x_1 is quasi-regular for $\mathcal{M}$, we have $x_1 \cdot [x^\mu] \neq 0$ in $\mathcal{M}$. Suppose that $x_1 \cdot [x^\mu] = 0$ in $\mathcal{M}'$ so that x_1 is not quasi-regular for $\mathcal{M}'$. Thus $x^{\mu+1_1} \in \mathrm{lt}_\prec\mathcal{I}$ and $\mathcal{G}$ contains a polynomial g with $\mathrm{lt}_\prec g \mid x^{\mu+1_1}$. Because of the assumption $x^\mu \notin \mathrm{lt}_\prec\mathcal{I}$, we must have $\mathrm{cls}(\mathrm{lt}_\prec g) = 1$. By definition of the reverse lexicographic order, this implies that every term in g is of class 1. Iteration of this argument shows that the normal form of $x^{\mu+1_1}$ with respect to $\mathcal{G}$ is divisible by x_1, i. e. it can be written as $x_1 f$ with $f \in \mathcal{P}_r$ and $\mathrm{lt}_\prec f \prec x^\mu$. Consider now the polynomial $\bar{f} = x^\mu - f \in \mathcal{P}_r \setminus \{0\}$. As it consists entirely of terms not contained in $\mathrm{lt}_\prec\mathcal{I}$, we have $[\bar{f}] \neq 0$ in $\mathcal{M}_r$. However, by construction $x_1 \cdot [\bar{f}] = 0$ contradicting the injectivity of multiplication by x_1 on $\mathcal{M}_r$.

For the remaining elements of the sequence $(x_1, \dots, x_n)$ we note for each $1 \leq k < n$ the isomorphism $\mathcal{M}^{(k)} = \mathcal{M}/\langle x_1, \dots, x_k\rangle\mathcal{M} \cong \mathcal{P}^{(k)}/\mathcal{I}^{(k)}$ where $\mathcal{P}^{(k)} = \mathbb{k}[x_{k+1}, \dots, x_n]$ and $\mathcal{I}^{(k)} = \mathcal{I} \cap \mathcal{P}^{(k)}$. It implies that we may iterate the arguments above so that indeed quasi-regularity of $(x_1, \dots, x_n)$ for $\mathcal{M}'$ is equivalent to quasi-regularity for $\mathcal{M}'$. ☐

Theorem 5.4. *The coordinates* $\mathbf{x}$ *are* δ*-regular for the homogeneous ideal* $\mathcal{I} \subseteq \mathcal{P}$ *in the sense that* $\mathcal{I}$ *possesses a Pommaret basis* $\mathcal{H}$ *for the degree reverse lexicographic term order with* $\deg \mathcal{H} = q$*, if and only if the sequence* $(x_1, \dots, x_n)$ *is quasi-regular for the factor algebra* $\mathcal{M} = \mathcal{P}/\mathcal{I}$ *at degree* q *but not at any lower degree.*

Proof. By the definition of a Pommaret basis and by Lemma 5.3, it suffices to consider monomial ideals $\mathcal{I}$. Assume first that the basis $\{x_1,\dots,x_n\}$ is δ-regular. By Proposition 2.2, the leading terms $\mathrm{lt}_{\prec}\mathcal{H}$ induce a complementary decomposition of $\mathcal{M}$ where all generators are of degree $q=\deg\mathcal{H}$ or less. Thus, if $\mathcal{M}_q\neq 0$ (otherwise there is nothing to show), then we can choose a vector space basis of it as part of the complementary decomposition and the variable x_1 is multiplicative for all its members. But this observation immediately implies that multiplication with x_1 is injective from degree q on, so that x_1 is quasi-regular for $\mathcal{M}$ at degree q.

For the remaining elements of the basis $\{x_1,\dots,x_n\}$ we proceed as in the proof of Lemma 5.3 and use the isomorphism $\mathcal{M}^{(k)}\cong\mathcal{P}^{(k)}/\mathcal{I}^{(k)}$. A Pommaret basis of $\mathcal{I}^{(k)}$ is obtained by setting $x_1=\cdots=x_k=0$ in the subset $\mathcal{H}^{(k)}=\{h\in\mathcal{H}\mid \operatorname{cls}h>k\}$. Thus we can again iterate for each $1<k\leq n$ the argument above so that indeed $(x_1,\dots,x_n)$ is a quasi-regular sequence for $\mathcal{M}$ at degree q.

For the converse, we first show that quasi-regularity of $(x_1,\dots,x_n)$ implies the existence of a Rees decomposition for $\mathcal{M}$. Exploiting again the isomorphism $\mathcal{M}^{(k)}\cong\mathcal{P}^{(k)}/\mathcal{I}^{(k)}$, one easily sees that a vector space basis of $\mathcal{M}^{(k)}_q$ is induced by all terms $x^\mu\notin\mathcal{I}$ with $|\mu|=q$ and $\operatorname{cls}\mu\geq k$. By the definition of quasi-regularity, multiplication with x_k is injective on $\mathcal{M}^{(k)}$, hence we take $\{x_1,\dots,x_k\}$ as multiplicative variables for such a term (which is exactly the assignment used in the Rees decomposition induced by a Pommaret basis according to Remark 2.3).

We claim now that this assignment yields a Rees decomposition of $\mathcal{M}_{\geq q}$ (and hence one of $\mathcal{M}$, since we only have to add all terms $x^\mu\notin\mathcal{I}$ with $|\mu|<q$ without any multiplicative variables). The only thing to prove is that our decomposition covers all of $\mathcal{M}_{\geq q}$. If $x^\mu\notin\mathcal{I}$ is an arbitrary term with $|\mu|=q+1$ and $\operatorname{cls}\mu=k$, then we can write $x^\mu=x_k\cdot x^{\mu-1_k}$. Obviously, $x^\mu\notin\mathcal{I}$ implies $x^{\mu-1_k}\notin\mathcal{I}$ and $\operatorname{cls}(\mu-1_k)\geq k$ so that x_k is multiplicative for it. Hence all of $\mathcal{M}_{q+1}$ is covered and an easy induction shows that we have indeed a decomposition of $\mathcal{M}_{\geq q}$.

Proposition 2.2 entails now that $\mathcal{I}$ has a *weak* Pommaret basis of degree q. As the autoreduction of a weak basis to a strong one can only decrease the degree, $\mathcal{I}$ has a strong Pommaret basis of degree at most q. However, if the degree of the basis actually decreased, then, by the converse statement already proven, $(x_1,\dots,x_n)$ would be a quasi-regular sequence for $\mathcal{M}$ at a lower degree than q contradicting our assumptions.

The same "reverse" argument shows that if $\mathcal{I}$ has a Pommaret basis of degree q, then the sequence $(x_1,\dots,x_n)$ cannot be quasi-regular for $\mathcal{M}$ at a lower degree, as otherwise a Pommaret basis of lower degree would exist which is not possible by the uniqueness of strong Pommaret bases. ☐

For monomial ideals $\mathcal{I}\subseteq\mathcal{P}$ a much stronger statement is possible. Using again the isomorphism $\mathcal{M}^{(k)}\cong\mathcal{P}^{(k)}/\mathcal{I}^{(k)}$, we identify elements of $\mathcal{M}^{(k)}$ with linear combinations of the terms $x_\nu\notin\mathcal{I}$ satisfying $\operatorname{cls}x_\nu>k$. Then we obtain the following simple relationship between the Pommaret basis of $\mathcal{I}$ and the kernels of the maps μ_k in Theorem 5.2.

Proposition 5.5. *Let $x_1, \dots, x_n$ be δ-regular coordinates for the quasi-stable ideal $\mathcal{I}$. Furthermore, let $\mathcal{H}$ be the Pommaret basis of $\mathcal{I}$ and set $\mathcal{H}_k = \{x_\nu \in \mathcal{H} \mid \operatorname{cls}\nu = k\}$ for any $1 \leq k \leq n$. Then the set $\{x_{\mu-1_k} \mid x^\mu \in \mathcal{H}_k\}$ is a basis of $\ker \mu_k$.*

Proof. Assume that $x_\nu \in \mathcal{H}_k$. Then $x_{\nu-1_k} \notin \mathcal{I}$, as otherwise the Pommaret basis $\mathcal{H}$ was not involutively autoreduced, and hence we find $x_{\nu-1_k} \in \ker \mu_k$.

Conversely, suppose that $x_\nu \in \ker \mu_k$. Obviously, this implies $x_{\nu+1_k} \in \mathcal{I}$ and the Pommaret basis $\mathcal{H}$ must contain an involutive divisor of $x_{\nu+1_k}$. If this divisor was not $x_{\nu+1_k}$ itself, the term x_ν would have to be an element of $\mathcal{I}$ which is obviously not possible. Since $x_\nu \in \ker \mu_k$ entails $\operatorname{cls}(\nu + 1_k) = k$, we thus find $x_{\nu+1_k} \in \mathcal{H}_k$. □

Remark 5.6. These results also lead to a simple proof of the characterisation (ii) of a quasi-stable ideal in Proposition 4.3. If $\mathcal{I}$ is quasi-stable, then the coordinates $x_1, \dots, x_n$ are δ-regular for it, hence by Theorem 5.4 they form a quasi-regular sequence for $\mathcal{P}/\mathcal{I}$ at a suitably chosen degree. By Proposition 4.3, Condition (iv), we have that $\mathcal{I}^{\text{sat}} = \mathcal{I} : \langle x_1 \rangle^\infty$ and hence multiplication by x_1 is injective on $\mathcal{P}/\mathcal{I}^{\text{sat}}$. As obviously $\mathcal{P}/\langle \mathcal{I}, x_1, \dots, x_j \rangle^{\text{sat}} \cong \mathcal{P}^{(j)}/(\mathcal{I}^{(j)})^{\text{sat}}$, we can apply the same argument also for all $1 \leq j < D$.

Conversely, if $x_1 \cdot f \in \mathcal{I}$ for a polynomial $f \in \mathcal{P} \setminus \mathcal{I}$, then $f \in \mathcal{I}^{\text{sat}} \setminus \mathcal{I}$ and hence $\deg f < \operatorname{sat} \mathcal{I}$. Thus x_1 is quasi-regular for $\mathcal{P}/\mathcal{I}$ at the degree $\operatorname{sat} \mathcal{I}$. Using again the isomorphisms $\mathcal{P}/\langle \mathcal{I}, x_1, \dots, x_j \rangle^{\text{sat}} \cong \mathcal{P}^{(j)}/(\mathcal{I}^{(j)})^{\text{sat}}$, we can apply the same argument for all $1 \leq j < D$, so that $(x_1, \dots, x_D)$ is a quasi-regular sequence for $\mathcal{P}/\mathcal{I}$ at a sufficiently high degree.

The characterisation (ii) of Proposition 4.3 obviously implies that the set $\{x_1, \dots, x_D\}$ is maximally independent modulo $\mathcal{I}^{\text{sat}}$. Hence $\dim \mathcal{P}/\langle \mathcal{I}^{\text{sat}}, x_1, \dots, x_D \rangle = 0$ entailing that $(x_1, \dots, x_n)$ is a quasi-regular sequence for the algebra $\mathcal{P}/\mathcal{I}$ at a sufficiently high degree. By Theorem 5.4, the ideal $\mathcal{I}$ is thus quasi-stable.

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